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Archimedes revisited: a faster, better, cheaper method of accurately measuring the volume of small objects

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Abstract

A little-known method of measuring the volume of small objects based on Archimedes principle is described, which involves suspending an object in a water-filled container placed on electronic scales. The suspension technique is a variation on the hydrostatic weighing technique used for measuring volume. The suspension method was compared with two other traditional water displacement methods of measuring volume – i.e. placing an object in a measuring cylinder and recording the rise in the water level and immersing the object in a water-filled container with an overflow spout to record the volume of overflow. The accuracy and precision of the three methods was compared using 10 accurately machined PVC cylinders ranging in volume from 1.5 to 15.7 ml. The mean difference between the actual and measured volumes was 3.3 ± 7.3%, -1.6 ± 7.2% and 0.03 ± 0.45%, for the level, overflow and suspension methods respectively. Each measurement was repeated twice to obtain the reproducibility of the three displacement techniques. The reproducibility was –1.7 ± 8.5%, 0.09 ± 3% and –0.04 ± 0.43% for the level, overflow and suspension techniques respectively. The results show that the suspension technique is more accurate and precise than the traditional water displacement methods and is more accurate than measuring volume using Vernier calliper measurements.

Introduction

The purpose of this article is to present an adaptation of the hydrostatic technique for measuring the volume of small objects. Hydrostatic weighing is familiar to those in metrology laboratories but from the author's experience does not appear well-known in education circles (i.e. in schools and universities). After a fairly extensive search of physics text books and the scientific literature over a number of years, no direct references to the technique have been discovered[1].

In hydrostatic weighing an object is weighed in air and then in a fluid (usually water). The volume of the object is given by the difference between the weight in air and water divided by the difference between the density of air and water. The water that the object is immersed in supports the object and therefore the object weighs less than in air.

The technique described in this article is a slightly simpler version of the classic hydrostatic weighing method. Rather than weighing an object in air and then in water it involves suspending the object in a container of water placed on an electronic

[1] However, I have seen a hint of the technique in the manual of a well-known make of electronic balance.

balance. An advantage of this technique is that any electronic balance can be used – there is no need for any special attachments to the balance, nor does it require the use of an under-pan hook. For example a laboratory retort stand can be used to suspend the object in the container of water.

The technique could be useful in a wide variety of experimental sciences. For example, it could be used to measure the volume of rock samples, bones, teeth, seeds, leaves etc. The experiment described here forms the basis of an undergraduate physics laboratory experiment at Queensland University of Technology. The experiment has been found to be useful for teaching students about measurement errors and the difference between accuracy and precision, and for introducing students to the utility of the Bland-Altman technique for assessing the agreement between two different techniques of measuring the same quantity.

When authors state that they have used Archimedes' method/law/principle to measure the volume of an object it is generally understood that they have used some form of water displacement. Archimedes' principle states that an object fully or partially immersed in a fluid is buoyed up by a force equal to the weight of the fluid that the object displaces (Halliday *et al*, 1997). Most of us are familiar with the various manifestations of Archimedes' principle. For example, if an object is placed in a measuring cylinder, the rise in water level indicates the volume of the added object. The volume of an object can also be measured by placing the object in a container of water with a spout projecting out of the side and measuring the volume of water that overflows. The weight of water displaced by a floating object, a ship for example, is equal to the weight of the object. We are also familiar with the fact that an object wholly or partially immersed in water is easier to lift, for example someone standing in a swimming pool.

According to history, Archimedes measured his body volume by climbing into a bath filled with water (Heath 2002). The water that Archimedes displaced flowed over the side of the bath. Archimedes then climbed out of the bath and measured the volume of water required to refill up to the brim. Archimedes is also said to have determined that a golden crown commissioned by Hiero II, king of Syracuse, was made from an alloy of gold and silver rather than pure gold. He did this by comparing the water displaced by the crown with that displaced by a gold ingot of the same weight as the crown. The crown displaced more water than the 'gold standard' and therefore was proven to be an alloy of gold.

The experiments described in this paper were designed to answer two questions. (1) How do the accuracy and precision of the suspension method compare with the two more conventional methods of measuring volume? (2) How does the error in measuring volume using the three Archimedes methods compared with volumes calculated from Vernier calliper measurements?

The suspension technique described here involves suspension of an object below the surface of the water in a container placed on electronic scales. To a first approximation, the volume of the immersed object is simply the increase in weight divided by the density of the fluid, i.e.

$$V = \frac{\Delta w}{\rho}$$

where ρ is the measured density of the fluid, Δw is the change in weight recorded by the balance when the object is suspended in the fluid and V is the unknown volume.

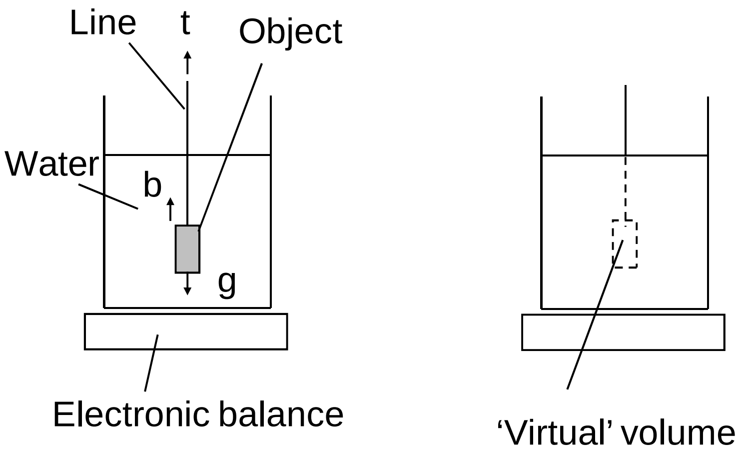

Figure 1. Schematic representation of the suspension method of measuring volume. Since the immersed object is stationary, the downward gravitational force (*g*) is balanced by the upward buoyancy (*b*) and line tension (*t*). The immersed object is equivalent to a 'virtual' volume of water of exactly the same size and shape.

At first sight this technique may seem to present some difficulties. For example, some questions that might be asked are: How can an object simply be suspended in water and the change in weight translated directly into a volume? How does the density affect the measurement? What about the variation in pressure around the object? What about surface tension?

Diameter (cm)	Height (cm)	Volume (cm^3)	Absolute error (mm^3)	% error
20	50.02	15.714	69.1	0.440
20.01	45.02	14.158	62.9	0.444
20.06	39.96	12.629	56.7	0.449
19.97	35.05	10.978	50.2	0.458
20.03	29.94	9.434	44.0	0.466
19.97	25.00	7.830	37.6	0.481
20.03	20.03	6.312	31.5	0.499
19.97	15.02	4.705	25.1	0.534
19.97	10.02	3.138	18.8	0.600
19.97	4.98	1.560	12.5	0.802

Table 1. Dimensions and volume of PVC cylinders. The errors in the diameter and height were ± 0.02 mm in each case. The last two columns show the absolute and percentage error in the measured volume.

From a physics point of view, the analysis of the situation is quite simple. We have an object suspended in water that is stationary. If an object is stationary then we know that the forces on the object are balanced. In the case of an object suspended in water at the end of a line, the downward force due to gravity (*g*) is balanced by the upward buoyancy (*b*) and tension (*t*) of the line (figure 1).

If the forces were not balanced then the object would move. Therefore a stationary object suspended in water is equivalent to a volume of water of exactly the same size and shape at the same position as the object. We can take any arbitrary surface within a static volume of fluid and the surface integral of the force will be zero – i.e. the forces are balanced.

As an example, consider a 10 cm^3 object suspended in a beaker containing 100 ml of water. When placed on the balance, the beaker of water with the suspended object will weigh 110 g – exactly the same as if there were 110 g of water in the beaker and no suspended object.

Materials and Methods

The three volume measuring techniques are summarised in figure 2. Ten accurately machined PVC cylinders ranging in volume between 1.5 and 15.7 ml were obtained (table 1). All cylinders were nominally 20 mm in diameter and 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 mm in length respectively. Vernier callipers accurate to ±0.02 mm were used to measure the cylinder dimensions more precisely. The cylinder dimensions, volumes, and absolute and percentage errors are shown in table 1. A standard method was used to calculate the error in calculating the volume from the Vernier measurements (Kirkup 1994). The volumes calculated from the Vernier dimensions were taken as the 'gold standard'.

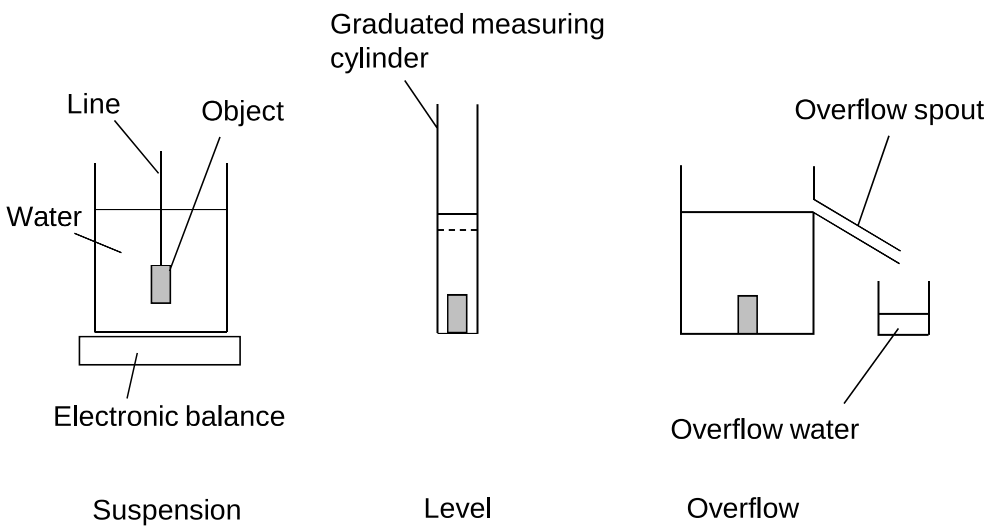

Figure 2. Schematic diagram of the suspension, level, and overflow methods of measuring volume.

A hole, approximately 0.35 mm in diameter was drilled into the end of each PVC cylinder. A piece of 0.22 mm diameter monofilament fishing line approximately was inserted into each hole and attached with Super Glue Gel. A felt-tipped ink pen was used to place a mark on the line 5 mm above the surface of each cylinder.

Each cylinder with line attached was immersed to this point. A 150 ml polythene beaker, approximately 55 mm in diameter, was filled with about 110 ml of water. This was enough to allow suspension of the longest PVC cylinder. The water temperature was measured with a Fluke (www.fluke.com) electronic thermometer with an 80PK-1 immersion probe. The stated accuracy and resolution of the thermometer are ± [0.2% + 0.3°C] and 0.1 ºC respectively. Temperature measurements were made at various times during the course of the experiments. The water temperature was 23 ± 0.5 ºC.

The beaker of water was placed on the pan of a Mettler AE260 Analytical balance (www.mt.com). This balance has a full scale range of 200 g and a precision of 1 mg. However, the object on the balance can be 'tared' off so that the reading is set to zero. In this case, the balance can read up to 60 g with a precision of 0.1 mg if the tared weight is less than 140 g (so that the total weight does not exceed 200 g). Each PVC cylinder was suspended beneath the surface of the water and the weight recorded after a few seconds of stabilisation. The volume of each cylinder was calculated by dividing the measured weight by the density of water at 23ºC which is given as 0.997569 g ml^{-1} in the *Handbook of Chemistry and Physics* (1971). Each volume measurement was performed twice to determine reproducibility by the method of Bland and Altman method (1986).

The volume of each cylinder was also measured using a plastic 50 ml measuring cylinder with an internal diameter of 2.3 cm and a measuring accuracy of ± 0.5 ml. The position of the meniscus was recorded before and after placement of each object in the measuring cylinder. Although the graduations on the cylinder were 1 ml apart, the level of the meniscus was estimated to the nearest 0.1 ml. The volume was taken as the difference in the volume before and after placement of each PVC cylinder in the water. The measuring cylinder was emptied and refilled before inserting each PVC cylinder.

A third method was used to measure the volume of each PVC cylinder, which involved placing the cylinders in a metal container with a downward-pointing spout. The container was filled with water so that the level was just above the bottom of the spout. Two minutes were allowed for the water to overflow through the spout and drop down to the base level. A cylinder was then dropped into the container and the overflow water collected in a 60 ml polythene cup. The cup was weighed using the same Mettler balance described above. The container was re-filled between each volume measurement and the polythene cup emptied and dried. The volume of each PVC cylinder was calculated by dividing the weight of water collected by the density of water at 23°C.

The precision of this method of measuring volume was assessed by measuring the average volume of each drop. This technique cannot be more precise than the volume of the individual drops. A total of ten drops were collected in the plastic cup and then weighed. This was repeated.

Bland-Altman analysis

Bland-Altman analysis is the best way to compare two techniques of measuring the same quantity. For example, we might have a new method of measuring a quantity and want to compare it to an existing method, considered to be a gold standard. A method often used is to plot a scattergram and calculate the *correlation coefficient*. The closer the correlation coefficient is to unity, the better the fit. However, this is not the best way of assessing the agreement between two methods of measuring a quantity. For example, a method of measuring a quantity may result in a reading, that is exactly a factor of two different from another method. In this case the two sets of readings will be perfectly correlated but are not in agreement.

Bland-Altman analysis is an excellent method of comparing one method of measuring a quantity with another. The difference between each pair of measurements is plotted against the mean of the same pair of measurements. Three lines are also plotted – the overall mean and plus and minus 1.96 standard deviations. (In practice $\pm 2\sigma$ can be used as the positions of the lines of a graph would be indistinguishable from $\pm 1.96\sigma$). 95% of the data points are expected to fall within the mean $\pm 1.96\sigma$. The level of the mean above or below zero gives the degree of systematic error, and the distance of the 1.96σ lines the degree of precision.

Results

The basic results are shown in table 2, and table 3 shows an overall comparison of the accuracy and precision of the level, overflow and suspension methods. Table 4 shows the reproducibility based on the repeat measurements.

actual	level	overflow	suspension
15.714	15.5	15.274	15.716
14.158	13.8	14.010	14.157
12.629	12.5	12.599	12.613
10.978	11	11.054	10.971
9.434	9.2	9.302	9.417
7.830	7.6	7.603	7.866
6.312	6.3	6.409	6.305
4.705	4.8	4.742	4.702
3.138	3.1	3.109	3.151
1.560	1.2	1.909	1.552

Table 2. Level, overflow and suspension volumes compared with the actual cylinder volumes. All volumes are in ml.

Method	**Absolute error (ml)**	**% error**	r^2
Level	-0.14 ± 0.16	3.3 ± 7.1	0.9989
Overflow	-0.04 ± 0.21	-1.6 ± 7.4	0.999
Suspension	0.001 ± 0.015	0.01 ± 0.27	1.0

Table 3. Absolute and percentage errors for the three methods of measuring volume. r^2 is the correlation coefficient.

Method	**Absolute error (ml)**	**% error**	r^2
Level	0.01 ± 0.2	-1.7 ± 8.5	0.9997
Overflow	-0.02 ± 0.18	0.09 ± 3.0	0.9994
Suspension	-0.007 ± 0.03	-0.04 ± 0.43	0.9998

Table 4. Reproducibility of the three methods of measuring volume. r^2 is the correlation coefficient.

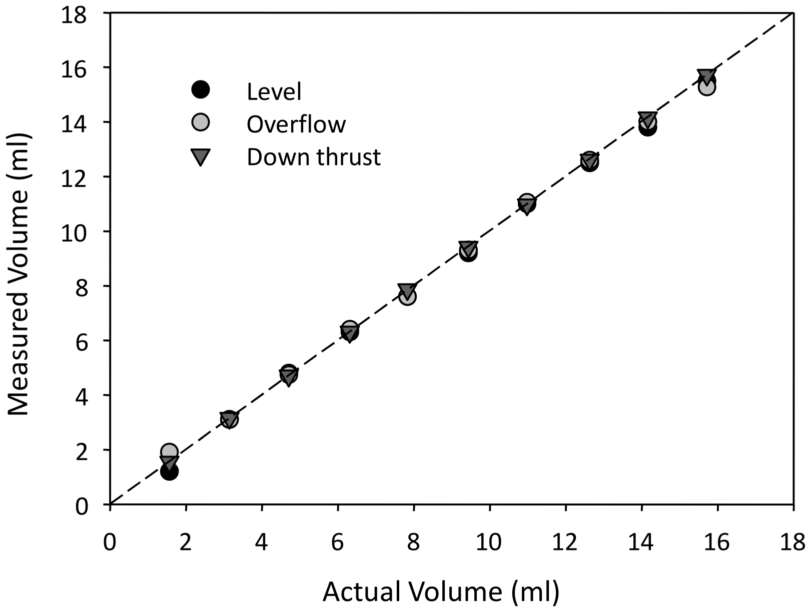

Figure 3. (a) Scatter plot of measured volume versus actual volume for the suspension, level and overflow techniques. The line of identity is shown.

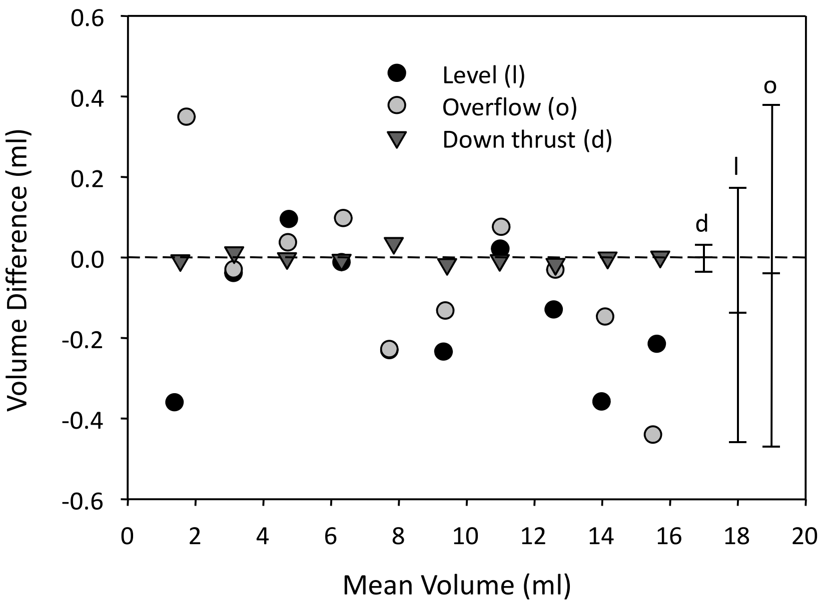

Figure 3. (b) Bland-Altman plots of the data showing the level of agreement between measured and actual volumes. The 'error bars' on the right hand side indicate the accuracy (middle bar) and precession (top and bottom bars) of the three techniques. 95% of the data are expected to fall within ± 1.96σ.

Figure 3(a) shows a scatter plot of the data shown in table 1, and figure 3(b) shows a Bland-Altman plot of the same data. Figure 4(a) shows a scatter plot of the repeat measurements, and figure 4(b) shows a Bland-Altman plot of the data plotted in figure 4(a). For the overflow method, the average drop volume was found to be 0.092 ± 0.005 ml. The correlation coefficient was also calculated as shown in tables 2 and 3.

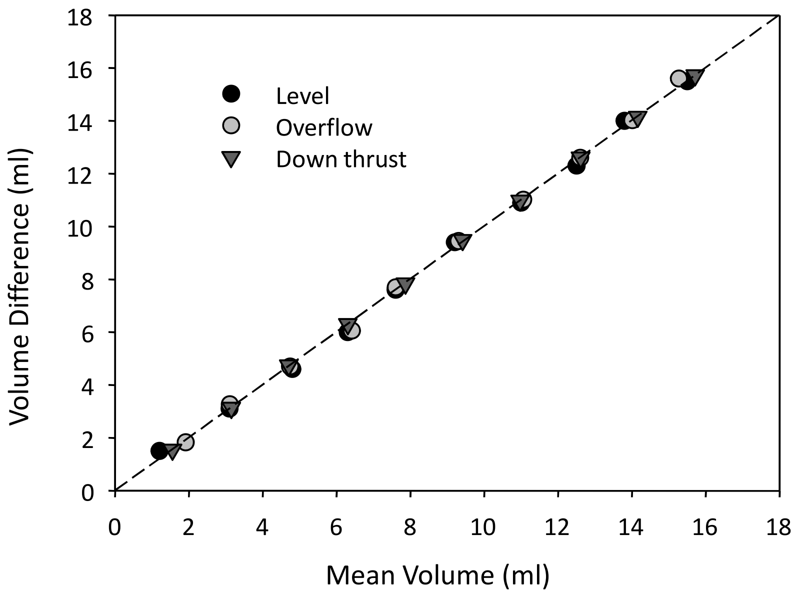

Figure 4. (a) Scatter plot of repeat measurements of the level, overflow and suspension volumes.

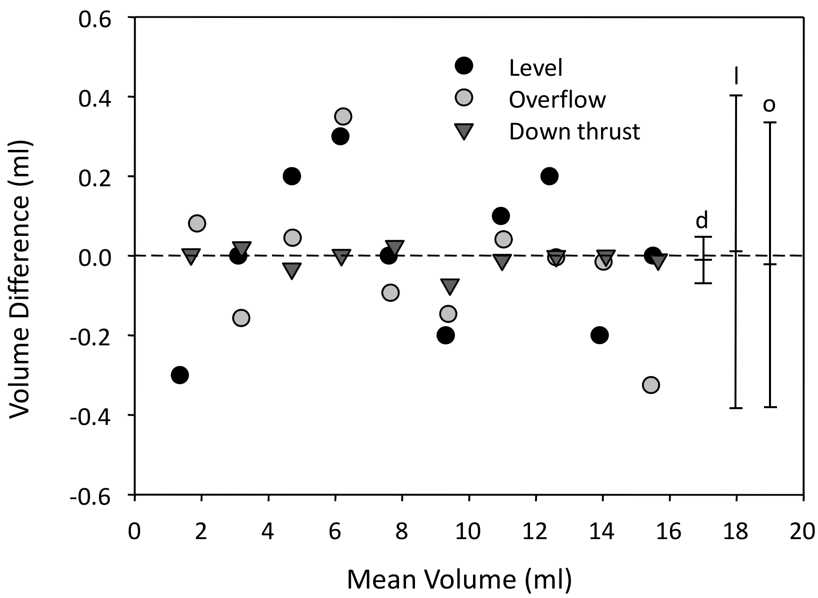

Figure 4. (b) Bland-Altman plot of the data.

Discussion
The results clearly show that the suspension method is superior to the overflow and level methods. In the Bland-Altman plots, the $\pm 1.96\sigma$ limits of agreement are much narrower than the other two methods of measuring volume. Table 3 shows that the absolute error in measuring the cylinder volumes using the suspension method is always smaller than the absolute error in measuring the cylinder volumes by the other methods. On average the difference between the actual and suspension volume expressed as a fraction of the error in the actual volume was 0.33 ± 0.25. However, the fractions for the other two methods are very much larger. The difference between the suspension method and the other two is most noticeable for the 1.5 cm^3 cylinder. The suspension technique is also much faster than the other two methods.

The results also demonstrate the inadequacy of the correlation coefficient in assessing agreement between the methods. For example, the Bland-Altman technique shows that there is a clear difference in the accuracy and precision of the overflow and level techniques and yet the difference in the correlation coefficients is only 0.0001! Also the value of unity (as calculated by Microsoft Excel) for the correlation coefficient for the suspension method could lead one to assume that there is perfect agreement, which is not true. Also, the Bland-Altman technique shows that there are clear differences in the reproducibility of the different methods which cannot be seen by looking at the correlation coefficients.

Although the suspension technique has the greatest accuracy and precision there are of course limitations. For example, electronics scales capable of weighing down to 0.1 mg can generally only weigh objects up to 200 g. This places restrictions on the weight of the water container and the volume of water within the container. Ideally the container should be made from low-density plastic[2] so that the combined weight of the container, water and object is below the limit of the balance.

An inconvenience is the need to attach a supporting line. In some cases, this might not be possible. If a line cannot be attached, then some kind of stage or cage could be constructed to support the object beneath the surface of the water. In this case the volume of the cage would have to be measured accurately, for example, by using the suspension method described in this paper.

A number of factors affect the overall accuracy of the technique, for example, absorbed air bubbles, line surface tension, air currents, water absorption by the object, surface evaporation of water. The theoretical precision of the suspension technique is given by the precision of the scales, which is ± 0.1 mg, the error in measuring the volume will be ± 0.1 mm^3. As can be seen from the absolute error column in table 1, this is far smaller than the error in calculating the cylinder volumes from the Vernier calliper measurements.

The suspension technique could be adapted for measuring the volume of objects less dense that water. This would be a difficult task using the more traditional displacement methods of measuring volume. For example a rigid rod could be

[2] Plastic medicine dispensers are good

attached to the object to hold it under the water, or some kind of rigid metal cage could be constructed for holding the object under the water.

It is interesting to compare the accuracy and precision obtained in this experiment with what is achievable with more sophisticated equipment such as the pycnometer. A survey of the specifications of pycnometers shows that in the volume range 0 – 50 ml they have a typical accuracy of 0.03 % and reproducibility of 0.015%. The results presented in this paper show that the suspension technique achieved the same level of accuracy as pycnometry (0.01%), but the values were not so reproducible (a reproducibility of 0.04% for pycnometry compared with 0.015% for suspension). However, this is an impressive result achieved using an electronic balance costing a few hundred dollars as opposed to a pycnometer costing several thousand dollars.

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