ARE ALL SCALES OPTIMAL IN DEA?

THEORY AND EMPIRICAL EVIDENCE*

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Abstract: Policy recommendations concerning optimal scale of production units often have serious implications for the restructuring of a sector, while tests of natural monopoly have important implications for regulatory structure. The piecewise linear frontier production function framework is becoming the most popular one for assessing not only technical efficiency of operations, but also for scale efficiency and calculation of optimal scale sizes. The main purpose of the present study is to check if neoclassical production theory gives any guidance as to the nature of scale properties in the DEA model, and to empirically investigate such properties. The empirical results indicate that optimal scale may be found over almost the entire size variations in outputs and inputs, thus making policy recommendations about scale efficiency dubious. It is necessary to establish the nature of optimal scale before any practical use can be made. Proposals for such indexes that should be calculated are provided.

Keywords: Optimal scale, scale elasticity, data envelopment analysis (DEA), frontier production functions, duals.

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1. Introduction

The optimal scale level of an economic activity is usually of great interest both from a productivity point of view and from a market point of view. The issue of economies of scale is not limited to manufacturing industries. It is now especially topical in previously regulated or state-owned industries, like electricity, water, telecom, etc, but also in many traditional public sector activities like hospitals and schools. For example in electricity distribution, the debate has been lively about the minimum efficient scale and the potential for increased productivity by further exploitation of economies of scale, while in electricity generation important issues are whether minimum efficient scales will allow competitive markets to be established and if the existing size distribution of firms is consistent with a competitive market outcome. From a policy point of view, examination of scale properties and scale efficiency of production units is, therefore, paramount.

Studies of scale economies are traditionally based on the neoclassical cost function approach. However, the surge in production frontier-based analyses of productive efficiency of all kinds of economic activities during the last decade has also stimulated examination of scale economies and scale efficiency of production units within this framework. Empirical research on production frontiers is largely dominated by two approaches, viz., the parametric stochastic frontier analysis (SFA) approach, and the non-parametric deterministic data envelopment analysis (DEA) approach[1]. While the scale properties of parametric production and cost functions are relatively well known, the corresponding properties of the non-parametric functions are less explored. The main objective of this study is an empirical exploration of scale issues within the DEA model.

1 For a recent survey on SFA, see Kumbhakar and Lovell (2000), and for a brief survey of the evolution of DEA and a bibliography of about 700 published articles and dissertations applying DEA during the period 1978-1995, see Seiford (1996).

With the emergence of a large number of user-friendly software packages, the DEA model has now become easily accessible for practitioners. It offers a seemingly simple method for estimation of efficiency, and it accommodates easily multiple-output multiple-input technologies. Moreover, it provides a lot of useful information – not only about efficiency but also, for example, about optimal scale. Indeed, one of the most frequently conducted investigations concerns returns to scale and the optimal size of decision-making units (DMUs in DEA terminology), (see e.g. Førsund (1996) and Førsund and Hjalmarsson, 1996). Against this background, it is not surprising that we now see the emergence of an international consulting industry doing benchmarking and calculating efficiency based on the DEA- model.

Policy recommendations concerning optimal scale of production units (like electricity network service areas) often have serious implications for the restructuring of a sector, while tests of natural monopoly have important implications for regulatory structure. Because DEA has become such a widespread and important analytical tool in practical evaluations of productive efficiency (including scale efficiency) all over the world, especially for public services and publicly regulated sectors, an investigation of the use of DEA for the purpose of revealing scale properties is indeed warranted. While there are several theoretical contributions within DEA framework on estimation and classification of scale properties, we lack a thorough understanding of the relevance of scale properties for inefficient units and a discussion of the empirical usefulness or applicability of knowledge about scale properties.

The main purpose of the present study is to check on the theoretical restrictions on the nature of scale properties in the DEA model and empirically investigate them for electricity distribution utilities. More specifically we will address the question whether optimal scale is at all a meaningful concept for policy recommendations in DEA. The exploration of that issue is the major contribution of this paper. Our main message is that information about optimal scale levels generated by the DEA model may be useless in applied efficiency research, and that it is necessary to investigate the scope for adjustment to optimal scale. We offer calculations of range of output mix and input mix as diagnostic devices to ascertain the nature of optimal scale in each empirical application.

Some basic definitions and relationships of neoclassical production functions are presented in Section 2, together with the derived concepts of optimal scale curve, efficiency frontier and M-locus to be illustrated empirically. An extended definition of the Regular Ultra Passum Law is introduced, and its existence within the DEA model analysed. The data used to calculate and explore optimal scale properties are presented in Section 3, and the empirical results are given in Section 4. Tentative policy conclusions are offered in Section 5.

2. The Neoclassical underpinnings

The starting point is a standard neoclassical production function for multiple outputs, multiple inputs. The output-vector is $y = (y_1,..,y_M) \in R_+^M$ and the input-vector $x = (x_1,..,x_N) \in R_+^N$:

$$F(y,x) = 0 \, , \; \frac{\partial F(y,x)}{\partial y_m} > 0 \, , \; m = 1,..,M \, , \; \frac{\partial F(y,x)}{\partial x_n} < 0 \, , \; n = 1,..,N \qquad (1)$$

The general transformation function $F(y,x) = 0$ represents the efficient output-input combinations, and it is assumed to be continuously differentiable and strictly increasing in outputs and decreasing in inputs.

The Passus Coefficient

The returns to scale, or scale elasticity, or the *Passus Coefficient* (here denoted by *e*) in the terminology of Frisch (1965), is a measurement of the increase in output relative to a proportional increase in all inputs, evaluated as marginal changes at a point in output – input space. In a multi-output setting the scale elasticity definition is based on the relationship between the proportional expansion of outputs, b, that for a proportional expansion, m, of inputs satisfies the production function; see Hanoch (1970), Starrett

(1977) and Panzar and Willig (1977). Following Starrett (1977) the procedure is to expand inputs proportionally with factor *m*, and then pick the proportional expansion, *b*, that yields the maximal expansion,

$b(m, y, x)y, mx) = Max\{b : F(by, mx) = 0\}$ (*we have* $b(1, y, x) = 1$),

of outputs allowed by the transformation function:

$$F(b(m, y, x)y, mx) = 0 \qquad (2)$$

The scale elasticity, *e*, as a function of outputs and inputs is obtained by differentiating (2) with respect to the input scaling factor:

$$\sum_{m=1}^{M} \frac{\partial F(by,mx)}{\partial y_m} y_m \frac{\partial b}{\partial m} + \sum_{n=1}^{N} \frac{\partial F(by,mx)}{\partial x_n} x_n = 0$$

$$\frac{\partial b}{\partial m} = e(y,x) = -\frac{\sum_{n=1}^{N} \frac{\partial F(y,x)}{\partial x_n} x_n}{\sum_{m=1}^{M} \frac{\partial F(y,x)}{\partial y_m} y_m} \qquad (3)$$

evaluating the function, without loss of generality, at *b* = *m* = 1. Equation (3) is the generalisation of Frisch`s *Passus Equation*, or sometimes called the generalised Euler equation, with regard to multiple outputs; see Frisch (1965), Hanoch (1970), Starrett (1977) and Panzar and Willig (1977).

The Regular Ultra Passum Law

A question is now if there are any restrictions on the shape of the scale elasticity function ε(y,x) within the neoclassical framework. For a traditional "S-shaped" production function, the *Regular Ultra Passum Law* in the terminology of Frisch (1965), the elasticity of scale varies from values larger than one for suboptimal output levels, through one at the optimal scale level, to values less than one for superoptimal output levels (and to negative values if the production function has a peak, i.e. no fee disposal) when moving "outwards" in the output-input space, i.e. all inputs and outputs non-decreasing and at least one input and one output strictly increasing.

Definition:

A production function $F(y,x) = 0$ defined by (1) obeys the Regular Ultra Passum Law if ¶e/¶$y_k < 0$, $k = 1,..,m$, ¶e/¶$x_r < 0$, $r = 1,..,n$, where the scale elasticity function $e(x,y)$ is defined in (3) , and for some point (x_1,y_1) we have $e(y_1, x_1) > 1$, and for some point (x_2,y_2), , where $x_2 > x_1$, $y_2 > y_1$, we have $e(y_2, x_2) < 1$.

What can we say about the shape of the contour curves of the scale elasticity function? In the case of single output and two inputs the contour curves will have negative slopes within the substitution region, but they may be either concave or convex, even if the production function is quasi-concave (see Førsund, 1971). This means that in general, in the traditional S -shaped neoclassical single output production function, the output level varies *monotonically* along the curve in the input space. The situation is illustrated in Figure1 [2]. Only in the case of a homothetic production function will isoquants of the production function coincide with contour curves of the scale elasticity function.

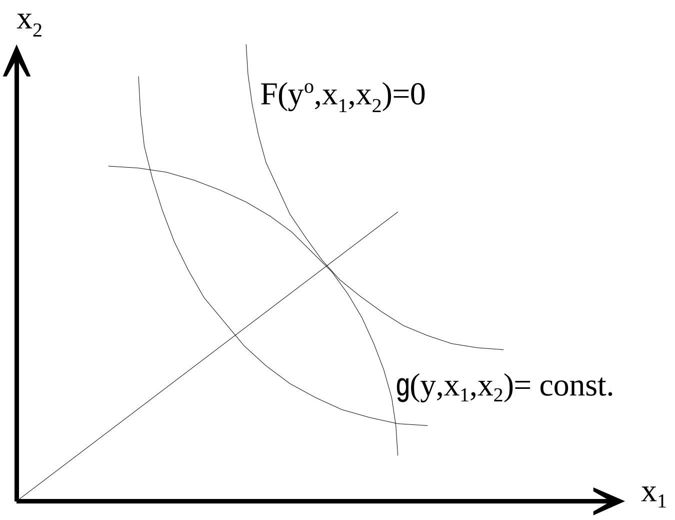


Figure 1. Contour curves of the scale elasticity function

2 As an example of a production function with classical neoclassical properties, Frisch (1965) suggested the following: $y = 100\frac{(x_1x_2x_3)^2}{x_1^4 + x_2^4 + x_3^4}(\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3})$, which is homogeneous of degree one in three inputs but S-shaped (regular ultra passum) for one of the inputs constant. Bramness (1975) plotted several aspects of this function, among them the non-convex-towards-the-origin optimal scale curve.

The optimal scale curve

It is one contour curve of the scale elasticity function that is of special importance. The locus of *e(y,x)* =1 in the input space was introduced by Frisch (1965) as the *techncially optimal scale curve (TOPS)*:

$$TOPS = \{(y,x) : e(y,x) = 1, F(y,x) = 0\} \quad (4)$$

For movements along factor rays the productivities are maximal on the curve in the single output case (illustrated in Figure 1 for $e(y, x_1, x_2) = 1$), or in the general case of multiple outputs the ratio *b/m* is maximised. Notice from Figure 1 that there is no limit on the variation of the optimal scale value in the general case. So what would be the recommendation for optimal scale? The point is that this can only be a relevant question when the factor prices are known. The point of intersection between the expansion path and the TOPS curve is the text-book long-run equilibrium point for a unit in a competitive market (disregarding any problems with a finite number of units), assuming that it is relevant to operate with constant factor prices. Thus a recommondation of adjusting to optimal scale has a relevant frame of reference.

In the case of a single output transforming the optimal scale curve into the input coefficient space it becomes the *efficiency frontier (EFF)*:

$$EFF = \left\{ \left(\frac{x_1}{y}, .., \frac{x_n}{y}\right) : e(y, x_1, .., x_n) = 1, F(y, x_1, .., x_n) = 0 \right\} \quad (5)$$

This is made up of all points where the input coefficients reach their minimum along rays from the origin. Since the optimal scale curve is a contour curve we have from the section above that the shape of the technically optimal scale curve may vary between production functions. Even if the production function is quasi-concave, the elasticity of scale function may not have this property, i.e. the optimal scale curve may or may not be convex towards the origin. But we note that in the neoclassical world the output level in the single output case varies monotonically along the curve and then also along the efficiency frontier. In the case of the production function (1) being *simultaneous homothetic* (Hanoch, 1970) we have that the optimal scale contour in input space for fixed outputs coincide with an input isoquant, and for fixed inputs it coincides with an output isoquant (in the terminology of Frisch, 1965). Simultaneous homotheticity in the case of variable returns to scale implies that the production function is separable; *F(y,x)* = *f(y)g(x)* (Hanoch (1970), p. 425). In the single output case there is then a unique optimal scale level independent of the factor ratio elaboration.

The M - locus

The concept of the *M - locus* in the case of multi output was introduced in Baumol et al. (1982) to designate the set of all output vectors that minimise average ray costs along their own ray. Thus, in our setting, the M - locus corresponds to the *technically optimal scale* extended to the multi output case, i.e. the geometric locus in output space for all points where the scale elasticity equals one (see Baumol et al. (1982), p. 58):

$$M = \{y : e(y,x) = 1, F(y,x) = 0\} \quad (6)$$

The shape of the M - locus is an important diagnostic for determining the number of firms in an industry and thereby the market structure. The crucial information is the difference between industry outputs and the output levels at the M - locus. It is conjectured that the shape may be irregular (pp. 58-59) and that the distance from the origin in output space may differ substantially between rays. This is the problem with determining the number of firms in an industry: the number may be dependent on the output mix. But one main conjecture in the two-dimensional illustration in Baumol et al. (Figure 3D1, p. 58) is that there is a trade-off between efficient output levels, similar to a traditional transformation curve in output space. In the case of two outputs and one input the M-locus must be a falling curve in the output space.

Introducing inefficiency

So far efficient operations have been assumed. We need a production technology where both feasible efficient and inefficient point can be identified. A production possibility set S is in general defined by:

$$S = \{(y,x) : x \text{ can produce } y)\} \quad (7)$$

We then need to distinguish between efficient and inefficient poins as subsets of the production set S. The connection between the neoclassical production function (1) and the production set formulation (7) is as follows (see Hanoch (1970), and McFadden (1978), which states conditions for a unique connection) , with standard properties of S:

$$S = \{(y,x) : x\, can\;\; produce\;\; y)\} \equiv \{(y,x) : F(y,x) \leq 0\} \qquad (8)$$

The subset of efficient point is then defined by *F(y,x)* = 0.

It should be born in mind that returns to scale is a local property and applies only to *efficient* points, i.e. points satisfying *F(y,x)* = 0. To associate an *inefficient* point with a scale elasticity value is at best ambiguous, because the existence of inefficieny means that the local increase in output when inputs are increased cannot be separated from the increase due to a reduction in inefficiency[3]. Therefore, a very basic observation for the discussion of scale properties using the DEA model is that inefficient observations must first be represented by efficient points. Thus the discussion of scale properties for inefficient units must be conditional on a meaningful and interesting representation.

The DEA model

The efficient subset in the DEA model corresponding to *F(y,x)* = 0 maintains the convexity of isoquants, but in the case of variable returns to sale (VRS) the origin is not assumed to be in the set, and it is convex. The surface is made up of facets, thus we do not have differentiability at corners or along ridges. Rates of substitution and rates of transformation are constant on a facet, and changes from facet to facet. Although we have to take these features into account we can use the basic definition (2) of the scale elasticity (see e.g. Banker et al. (1984), Førsund, 1996). Specificially, the optimal scale curve, the efficiency frontier and the M - locus all exist in the DEA model. These concepts all belong to a VRS frontier function.

It has become a common practice in the field of non-parametric efficiency analysis to name the linear programme for the calculation of all Farrell (1957) technical efficiency

[3] Banker (1984) and Banker et al. (1984) are clear on this point. However, notice that a set is usually defined as having constant returns to scale if all finite points on rays belong to the set, i.e. the set is a cone. The definition of economies of scale in Panzar and Willig (1977) as a property of the production *set* in general is rather awkward.

scores for the DEA model. The efficiency scores for the VRS input- and output oriented DEA models, E_{1i} and E_{2i} respectively for unit i, are found by solving the following two linear programmes:

$$
\begin{aligned}
& E_{1i} = Min\ q_i \\
& s.t. \\
& \sum_{j=1}^{J} l_j y_{mj} - y_{mi} \geq 0 \ ,\ m = 1,..,M \\
& q_i x_{ni} - \sum_{j=1}^{J} l_j x_{nj} \geq 0 \ ,\ n = 1,..,N \\
& \sum_{j=1}^{J} l_j = 1 \\
& l_j \geq 0 \ ,\ j = 1,..,J
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
& \frac{1}{E_{2i}} = Max\ f_i \\
& s.t. \\
& \sum_{j=1}^{J} l_j y_{mj} - f_i y_{mi} \geq 0 \ ,\ m = 1,..,M \\
& x_{ni} - \sum_{j=1}^{J} l_j x_{nj} \geq 0 \ ,\ n = 1,..,N \\
& \sum_{j=1}^{J} l_j = 1 \\
& l_j \geq 0 \ ,\ j = 1,..,J
\end{aligned}
\tag{10}
$$

The constraints in (9) and (10) represent the definition of the piecewise linear technology relevant for unit *i*. This unit may be inefficient in e.g. its use of inputs. The input vector in (9) is adjusted by the efficiency score, q_i, and then compared with the *reference point*, $\sum_{j=1}^{J} l_j x_{nj}$, on the frontier. To find the optimal scale units, the simplest procedure is to use either model (9) or (10) without the constraint that the sum of weights add up to one, i.e. the CRS envelopment. The optimal scale units are then identified by having no slacks on the input (or output) constraints and an efficiency score of 1[4].

The Regular Ultra Passum Law and the DEA model

We want to investigate whether the DEA model fulfills the Regular Ultra Passum Law or not. One way of doing this is to use the scale elasticity function for a DEA model based on the approach first introduced in Banker et al. (1984) (see also Førsund (1996), Førsund and Hjalmarsson, 1996). We then need the dual programmes to the problems (9) and (10). Let u_{mi} and v_{ni} be the non-negative shadow prices on the output- and input constraints respectively in the optimisation problem (9), and u_i^{in} the (unrestricted) shaow price on the convexity constraint. The dual problem is then:

$$\begin{aligned} & Max \sum_{m=1}^{M} u_{mi} y_{mi} + u_i^{in} \\ & subject\ to \\ & \sum_{n=1}^{N} v_{ni} x_{ni} = 1 \\ & \sum_{m=1}^{M} u_{mj} y_{mj} - \sum_{n=1}^{N} v_{nj} x_{nj} + u_i^{in} \leq 0,\ j = 1,..,J \end{aligned} \tag{11}$$

Using the same symbols for the shadow prices on the constraints in problem (10) and calling the (unrestricted) shadow price on the convexity contraint for u_i^{out} we have the dual problem:

$$\begin{aligned} & Min \sum_{n=1}^{N} v_{ni} x_{ni} + u_i^{out} \\ & subject\ to \\ & \sum_{m=1}^{M} u_{mj} y_{mj} = 1 \\ & -\sum_{m=1}^{M} u_{mj} y_{mj} + \sum_{n=1}^{N} v_{nj} x_{nj} + u_i^{out} \leq 0,\ j = 1,..,J \end{aligned} \tag{12}$$

The values of the shadow prices u_i^{in} and u_i^{out} determine the scale property. In the case of a input-oriented (output- oriented) reference point we have increasing returns when u_i^{in} ($-u_i^{out}$) > 0, constant when u_i^{in} (u_i^{out}) = 0 and decreasing returns when u_i^{in} ($-u_i^{out}$) < 0.

It is shown in Førsund and Hjalmarsson (1996) that the scale elasticity function in (3)

4 We will not go into details about how to deal with multiple solutions.

can be written in the following two equivalent ways for the two reference points corresponding to an inefficient unit, *i*:

$$e(y_i, E_{1i}x_i) = \frac{E_{1i}}{E_{1i} - u_i^{in}}$$
$$e(\frac{1}{E_{2i}} y_i, x_i) = 1 - E_{2i}u_i^{out} \tag{13}$$

We will assume that the reference points are in the interior of frontier function facets so we can differentiate the scale elasticity function at the reference point. Differentiating the first expression in (13) w.r.t. the ouput type *m* for unit *i* yields:

$$\frac{\partial e(y_i, E_{1i}x_i)}{\partial y_{mi}} = -\frac{u_i^{in} \frac{\partial E_{1i}}{\partial y_{mi}}}{(E_{1i} - u_i^{in})^2} = -\frac{u_i^{in} u_{mi}}{(E_{1i} - u_i^{in})^2} \quad , m = 1,.., M \tag{14}$$

The last expression is obtained using the Envelope Theorem on the Lagrangian function for the problem (9), yielding $\partial E_{1i} / \partial y_{mi} = u_{mi}$[5]. We see from (14) that the sign of the partial derivative of the scale elasticity function w.r.t. an output depends on the sign of the shadow price u_i^{in}. For increasing returns, u_i^{in} > 0, we have a falling value of the scale elasticity in accordance with the requirement of the Regular Ultra Passum Law, but for decreasing returns, u_i^{in} < 0, we have an increasing value of the scale elasticity in *contradiction* of the law.

Differentiating the second expression in (13) w.r.t. the ouput type *m* for unit *i* yields:

$$\frac{\partial e(\frac{1}{E_{2i}} y_{mi}, x_{ni})}{\partial x_{ni}} = -u_i^{out} \frac{\partial E_{2i}}{\partial x_{ni}} = u_i^{out} v_{ni} E_{2i}^2 \quad , n = 1,.., N \tag{15}$$

The last expression is again obtained by using the Envelope Theorem on the Lagrangian function for problem (10) for investigating the impact of a parameter change, yielding $\partial E_{2i} / \partial x_{ni} = -E_{2i}^2 v_{ni}$. Increasing returns to scale, u_i^{out} < 0, yields a decreasing scale

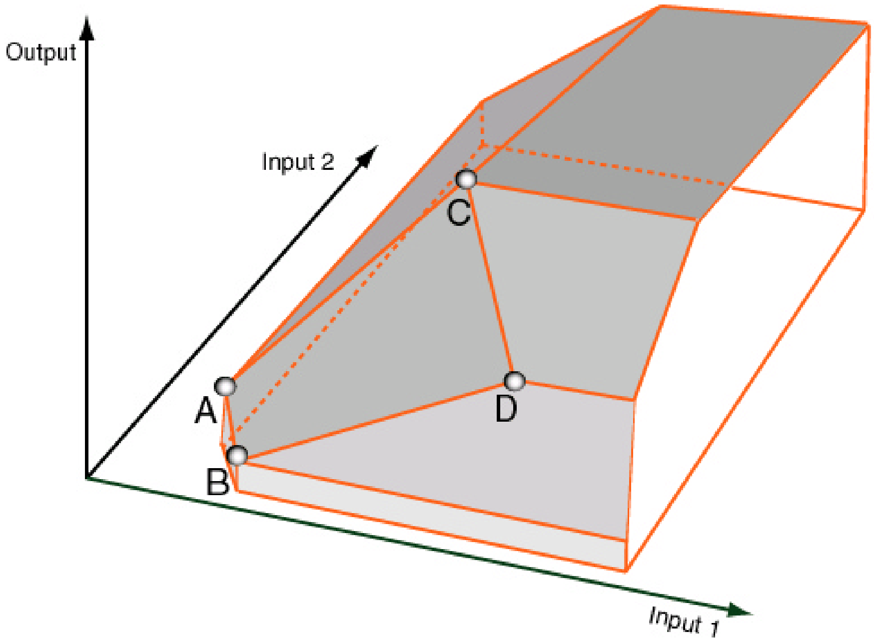
5 We assume that the basis in the LP solution for (9) does not change, so the shadow prices remain unchanged.

elasticity in accordance with the Regular Ultra Passum Law, while decreasing returns to scale, $u_i^{out} > 0$, yields an increasing scale elasticity, violating the law.

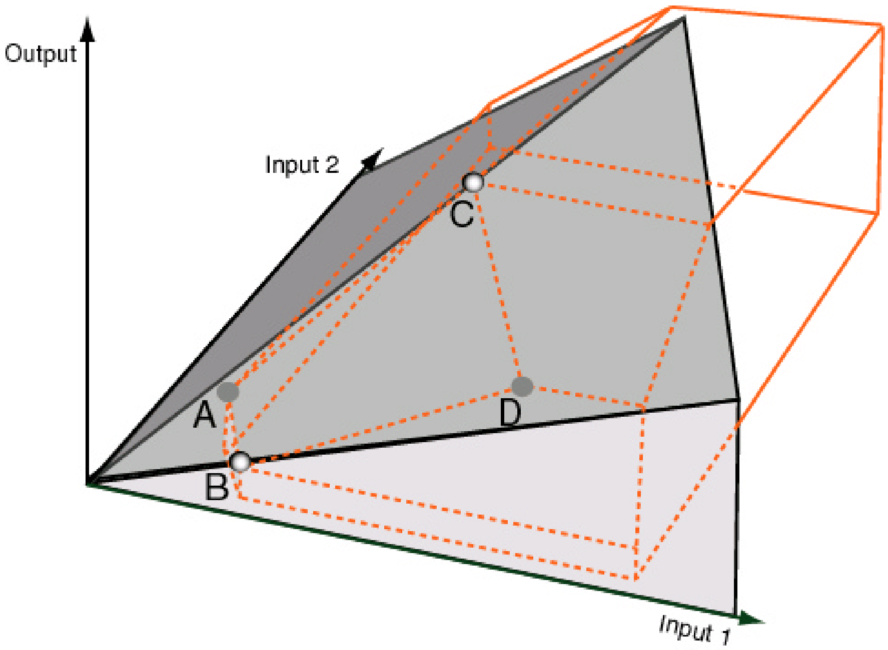
An illustration

Before reporting the empirical results, a stylised figure (Figure 2) based on only four units may enhance our understanding of the character of returns to scale in DEA[6]. All points are efficient in the case of a variable returns to scale (VRS) envelopment shown in Panel a. Panel b shows the difference in frontier surfaces between VRS and constant returns to scale (CRS) envelopment. We see that of the four units two of them, *B* and *C*, are optimal scale units. Unit *C* has maximal output level, while unit *B* has the minimal. The lesson from the stylised figure is that it may be quite normal, even in the case of a single output, to have both the maximal and minimal output level as the optimal scale. The central facet has the points *A,B,C, D* as corners. The technically optimal scale curve will be a line from *B* to *C* in Panel a, with a corresponding variation in the factor ratio. Notice that it seems easy to calibrate the points *B* and *C* such that the optimal scale curve will be a rising curve in the input plane, in violation of the Regular Ultra Passum Law. If we cut the surface of the production function in Panel a with a plane, parallel with the output axis, along a factor ray between the values for points *A* and *B*, then the scale elasticity is infinite at the left-hand side of the intersection point of the plane and the facet border betwen *A* and *B*, and has a value greater than one at the right-hand side of the intersection point. When the plane intersects the optimal scale curve , a line from *B* to *C*, the scale elasticity obtains the value of one, and on the left-hand side of the intersection point with the facet border between *C* and *D* the scale elasticity obtains a value less than the value on the left-hand side, but will end up at the left-hand side of the north- east border of the facet with a *higher* value than the starting value, although this will be smaller than the left-hand value on the facet border between *C* and *D*. On the ”flat ” facet North-East of point *C* the value of the scale elasticity will be zero.

6 We are indebted to Dag Fjeld Edvardsen for making the figure.



Panel a. The VRS production function



Panel b. The CRS envelopment.

Figure 2. The VRS production function and its CRS envelopment

3. Data and model specification

In the empirical application we will select our data from a set of data which have been utilised in previous work[7]. The set constitutes a four output - four input model covering Swedish electric distribution utilities, and was earlier applied in Hjalmarsson and Veiderpass (1992a) and (1992b), Kumbhakar and Hjalmarsson (1997) and also in Zhang and Bartels (1998). The data applied in this study cover 163 Swedish electricity retail distributors in 1987. Only distributors who supply more than 500 low voltage customers are included. The data are constructed based on information obtained from the Association of Swedish Electric Utilities (SEF), Statistics Sweden (SCB) and different retail distributors.

Modelling of electric utilities varies (see Jamasb and Pollitt (2001) for a review of model specifications). The maximal disaggregation our data allows is to specify four outputs and four inputs. As regards choice of output measure we consider the total amount of low and high voltage electricity in MWhs received by the customers (Y_1, Y_2) and the number of low and high voltage customers served (Y_3, Y_4) as the four outputs. On the input side we use kilometers of low and high voltage power lines (K_1, K_2) and total transformer capacity (K_3) in kVa as the capital variables. Labour L is measured in full time equivalent employees. Max, min and mean statistics are shown in Table 1.

However, more aggregate models can also be found in the literature. We will therefore specify different models that can be used to study the derived economic concepts such as technically optimal scale curve, the efficiency frontier and the M - locus. We will use the optimal scale results from three different models. Model 1 is a single-output, two-input model with total electricity ($Y = Y_1+Y_2$) as output and labour and transformer capacity as inputs. Model 2 is a two-output two-input model with total electricity (Y_1+Y_2) and total number of customers

[7] In Førsund and Hjalmarsson (1996) we also use a single output, two input data set for Swedish dairies.

Table 1.List of variables and key statistics

	Y1 MWh low voltage	Y2 MWh high voltage	Y3 Customers low voltage	Y4 Customers high voltage	L Labour	K1 Lines in Km low voltage	K2 Lines in Km high voltage	K3 Transformers in kVa
Mean	286057	665979	22841	36	133	1168	989	155434
Stdev	3454887	46644285	225909	641	6493	21159	40783	1801496
Min	9190	0	695	0	2	21	8	4000
Max	4895138	65966223	422793	908	9189	30033	57733	2554000

(Y_3+Y_4) as outputs and labour and transformer capacity as inputs. Model 3 contains all four outputs and all four inputs; see Table 2.

4. Empirical results

Because of the many outputs and inputs dealt with, between 10 and 53 units out of 163 are on the frontier in the case of variable returns to scale. Among these between 3 and 25 are optimal scale units; see Table 2. We have also added the number of units within the models. Notice that the largest unit is of optimal scale in all models. In Model 1 about 25% of the units are within the range of optimal scale sizes, in Model 2 about 60% and in Model 3 about 90%.

Table 2. The number of optimal scale and frontier units

Model	Outputs	Inputs	Frontier units, VRS	Optimal scale units	Units within the optimal scale range	Total sample
1	$Y = Y_1 + Y_2$	L, K_3	10	3	39	163
2	Y_1+Y_2, $Y_3 + Y_4$	L, K_3	15	4	97,99*	163
3	Y_1, Y_2, Y_3, Y_4	L, K_1, K_2, K_3	53	25	146	163

*) Both output- and input orientations are run

Optimal scale curves

All the optimal scale values in the input space with labour, *L*, and transformer capacity, K_3, in Model 1 is plotted in Figure 3. This is the *technically optimal scale curve* introduced in Frisch (1965). The most striking feature in our DEA case is the *positive* slope of the curve. In the case of a classical S-shaped production function, the Regular Ultra Passum Law introduced in Section 2, the optimal scale curve has a negative slope. Moreover, when moving along the locus of optimal scale in the input space, the optimal scale values either increase or decrease monotonically with the factor ratio, or remain constant in the homothetic case. Although the opposite may occur for other sets of data, this is also the case in Figure 3, where the smallest unit has a size about 10% of the second to smallest unit, which in turn has a size about 5% of the largest unit. However, the slope is in contradiction of the Regular Ultra Passum Law. But as indicated in Figure 2, a positive slope of the curve may well occur.

In the case of DEA, some times small changes in factor ratio “cause” large changes in optimal scale. This is obvious from Figure 3, where a small change in the factor ratio causes a large change in optimal scale. Moreover, the largest optimal scale level occurs at a relatively low capital-labour ratio. A priori, one might expect the opposite, namely that more capital-intensive technologies coincide with large optimal scale levels.

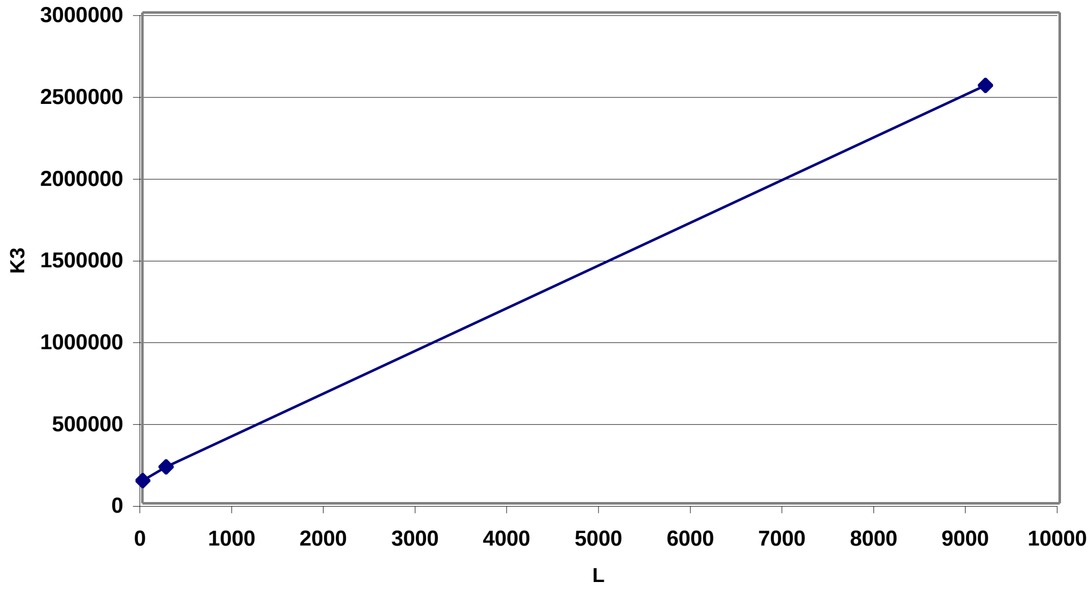


Figure 3. The optimal scale curve

Efficiency frontiers

By dividing all optimal scale input values by output we get the efficiency frontier. It represents the boundary of the feasible production set. In the neoclassical case, the efficiency frontier is convex towards the origin, and, except in the homothetic case, with monotonically changing output level along the frontier. The nature of optimal scale in DEA is further illustrated for Model 1 in Figure 4, where all optimal scale units are plotted in the input coefficient space and connected with straight lines to the efficiency frontier. Model 1 yields a traditional efficiency frontier convex towards the origin, and, consistent with the variation along the optimal scale curve, the output level varies from the smallest in the capital-intensive corner to the largest in the labour-intensive corner. A small change in its factor ratio may pass a certain scale efficient unit into the set of highly scale inefficient units.

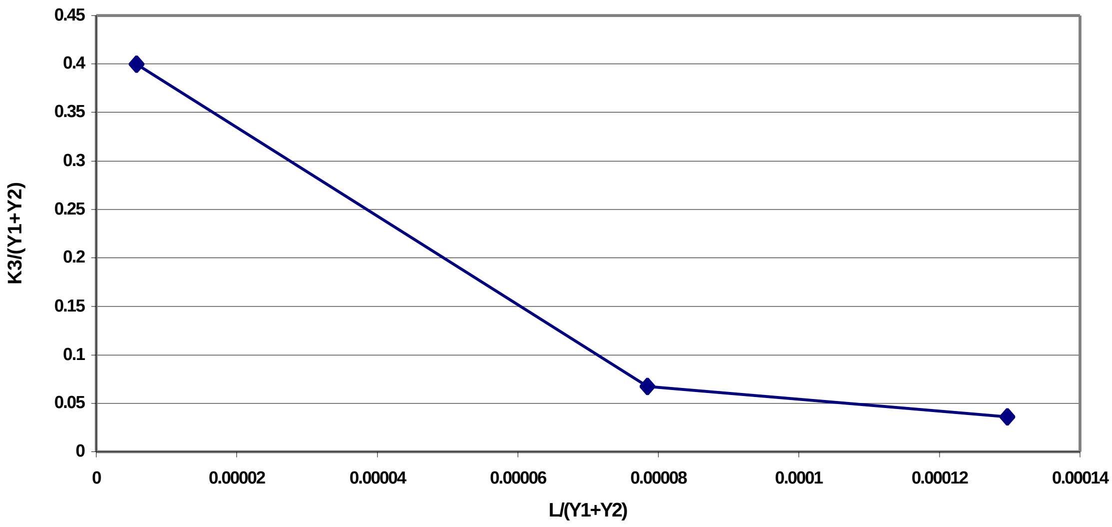


Figure 4. The efficiency frontier

The M - locus

In the only empirical application we are aware of, Kim (1987) uses the M - locus (found by estimating an average translog cost function) to illustrate his findings for water utilities producing two outputs. The form of the locus bears some resemblance to the shape illustrated in Baumol et al. (1982) as to a trade-off between efficient output levels, and shows a wide variation in optimal scale.

The M - locus in our two-output DEA model (Model 2) is illustrated in Figure 5 with the total energy on the abscissa axis and total number of customers on the ordinate axis. Since all our optimal scale firms have positive amounts of both outputs, we have no observation on stand-alone production used for anchoring the M - locus in Baumol et al. The shape is irregular in accordance with the conjectures in Baumol et al. However, there is one crucial difference: Our M - locus is an *increasing* curve in the output space. This means that there are no signs of any specialisation effect along the locus.

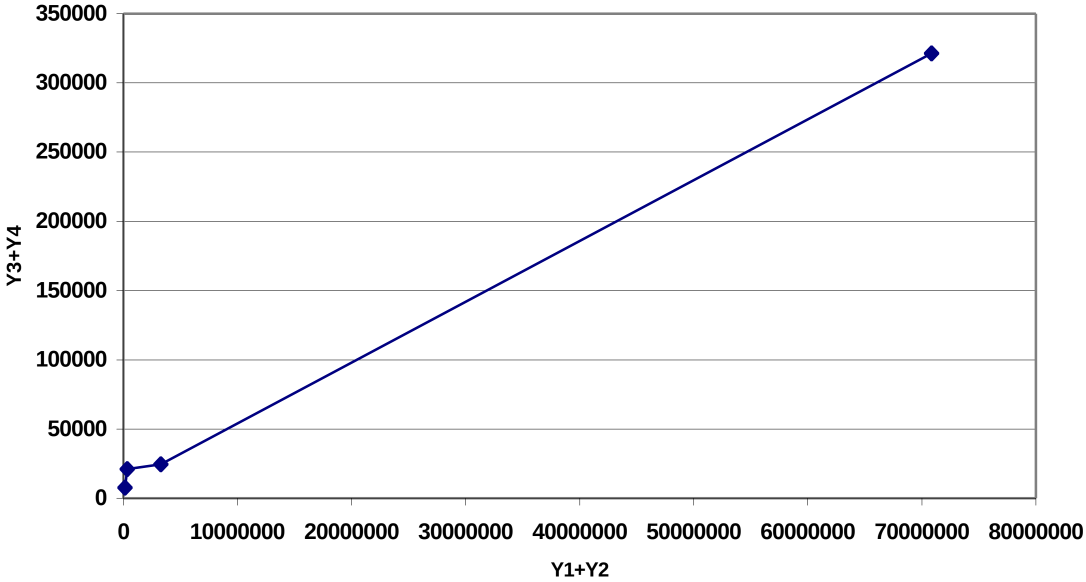


Figure 5. The M- locus

The range of optimal scale values

While Table 2 shows the number of units within and outside the range of optimal scale values, Table 3 compares, for all three models, the amount of overlapping in output and input ratios of optimal scale units to the entire sample. These optimal scale ratios on sample ratios are our *output- and input mix-indexes*[8]. The most amazing result is the frequency of ones in the case of the input mix-index i.e. the almost identical range of factor ratios for the optimal scale units and the entire sample. The amount of overlapping is less on the output side and varies substantially among output ratios in Model 3. The way of using the information of the indices for this specification is to observe that optimal scale may be obtained for any observed factor ratio, with the exception of the ratio between lines for low voltage and lines for high voltage, where the range within optimal scale is more limited. As for output ratios there is no restriction for the ratio between high- and low voltage energy, but varying restrictions on other output

8 The output mix-index is, of course, meaningless in the case of single output and in Table 3, Model 2 the ratio Y_2/Y_1 stands for $(Y_3+Y_4)/(Y_1+Y_2)$.

Table 3. Range of optimal scale mixes of outputs and inputs

Model	Output	Y2/Y1	Y3/Y1	Y4/Y1	Y3/Y2	Y4/Y2	Y4/Y3
1	Y1+Y2	1					
2	Y1+Y2,Y3+Y4	0.94					
3	Y1,Y2,Y3,Y4	1.00	0.24	0.37	0.92	0.46	0.34
Model	**Input**	**L/K1**	**L/K2**	**L/K3**	**K1/K2**	**K1/K3**	**K2/K3**
1	L, K3			1			
2	L, K3			1			
3	L, K1, K2, K3	1	1	1	0.62	1	1

ratios. The conclusion is that without any price (or relative value) information on outputs and inputs, the range of ratios realizing optimal scale is too wide to provide any insights for policy purposes.

5. Conclusions

Substantial effort has been devoted to the definition, analytical derivation, classification and measurement of optimal scale in the DEA models, while few have raised the question wether measures of optimal scale is of any practical use in efficiency analysis.

First of all a relevant representation of inefficient units on the frontier must be established. Too often an adjustment in either input- or output direction is used without questioning the relevance of such an adjustment for a unit managing to become efficient.

The smooth neoclassical production function gives rise to useful concepts such as the elasticity of scale, optimal scale size, the optimal scale curve, the efficiency frontier and the M - locus. In this paper we have studied the empirical aspects of these concepts in the DEA model. While the theoretical concepts carry over to the piecewise linear frontier production function, the properties of the optimal scale curve and the M - locus do not. Neither is the Regular Ultra Passum Law obeyed. The efficiency frontier behaves as in neo-classical production theory because it is based on the basic convexity of the producion set.

The empirical application illustrates some problems with these concepts in applied DEA analyses. The range of optimal scale levels may be extremely wide, as may be the range of factor ratios for the set of optimal scale units. Inclusion or exclusion of a few DMUs may have a large effect on the set of optimal scale units and their size. In a technical sense the scale properties revealed by a DEA study is correct, provided the outputs or the inputs are changed in a strictly proportional fashion. But this is not very comforting for policy recommendations when optimal scale changes dramatically from one output- or input ray to the other. A fundamental problem with DEA applications arise in the case when there are no output- or input prices, which often is the case, at least for outputs, for public sector applications. Without expansions paths in input- and output space as reference change of input-and output mix may be as relevant as proportional scaling up or down along observed proportions.

What about scale efficiency? Scale efficiency is a relative concept tied to optimal scale. The scale efficiency of a certain unit depends on its benchmark or yardstick unit - not on its absolute size. This benchmark may vary substantially for small changes in input and output mix of a specific unit. Therefore, scale efficiency is as ambiguous empirically as a basis for recommendations for change as optimal scale.

What about input- and output oriented Farrell technical efficiency measures, the *raison d'etre* of DEA studies? The situation is different for these measures, because the key question here is the distance to the frontier, according to some common rule of measuring distance. One is not pursuing a recommendation for a specific change, just to point out a *potential* for an improvement.

A general problem with estimating production functions is whether the specification of the production relations is sufficiently close to what we want to model. A feature often regarded as a strenght of the non-parametric DEA model is that it reflects just the observations and no preconceived functional form. However, the DEA model may be *too* data dependent; the model as it is usually specified may lack enough structure to generate credible information about optimal scale levels. We should recall that the scale elasticity and optimal scale level are derived along rays.The general nature of the requirements for properties *across* rays, i.e. requirements about shapes of isoquants and transformation curves may not be enough to sufficiently mirror the real life engineering restrictions on substitution and transformation not captured by the DEA model specification[9].

Our recommendation to the dilemmas for policy conclusions based on DEA models as to optimal scale is to show the empirical scope for output- and input mixes of optimal scale. In the case of wide scopes there is no way around using price data for outputs and inputs to establish a frame of reference for scale adjustments. The prices may be observed or of a shadow price nature, especially for outputs.

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9 Note that the neoclassical formulation F(y,x) = 0 is a very general formulation too, may be too general, cf. the considerably more elaborate formulation in Frisch (1965). See also Førsund (1999) for a review of Frisch's production theory and engineering production functions.

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