Are there approximate Fast Fourier Transform on graphs ?

Luc Le Magoarou, Rémi Gribonval

Inria, Rennes, France

Objective

Goal: Enable Fast Fourier Transform (FFT) and fast filtering on large graphs.
Approach: Provide a general method for approximating the graph Fourier matrix $\mathbf{U}$, giving approximations $\hat{\mathbf{U}}$ that can be applied rapidly.

Graph Fourier transform

Let $\mathbf{L} \in \mathbb{R}^{n\times n}$ be the laplacian matrix of a graph, and $\mathbf{U} \in \mathbb{R}^{n\times n}$ its eigenvectors matrix. Let $\mathbf{x} \in \mathbb{R}^n$ be a signal on the graph, and $\mathbf{y} \in \mathbb{R}^n$ its Fourier transform, we have:

$$\mathbf{y} = \mathbf{U}^T\mathbf{x}$$
$$\mathbf{x} = \mathbf{U}\mathbf{y}.$$

The matrix $\mathbf{U}$ being dense in general, the Fourier transform costs $\mathcal{O}(n^2)$ arithmetic operations.

Fast transforms

Many widely used transforms (classical Fourier, wavelets, DCT, etc.) are paired with a fast algorithm, exploiting the factorizability of the associated matrix $\mathbf{A}$ into sparse factors,

$$\mathbf{A} = \prod_{j=1}^{J} \mathbf{S}_j.$$

This factorizability is necessary and sufficient for a fast linear algorithm to exist. In the case of the classical Fourier transform, $\mathbf{A}$ can be factorized into $J = \log_2(n)$ factors, each having $2n$ nonzero entries.

FAμST approximations

We approximate $\mathbf{U}$ using Flexible Approximate MUlti-layer Sparse Transforms (FAμST) [1]:

$$\mathbf{U} \approx \hat{\mathbf{U}} = \prod_{j=1}^{J} \mathbf{S}_j,$$

allowing to compute approximate Fourier transformations ($\hat{\mathbf{U}}^T\mathbf{x}$ and $\hat{\mathbf{U}}\mathbf{y}$) in only $\mathcal{O}\Big(\sum_{j=1}^{J} \|\mathbf{S}_j\|_0\Big)$ arithmetic operations.

Optimization problems

We consider two optimization problems:

- Approximate factorization of $\mathbf{U}$ (giving $\hat{\mathbf{U}}_{\text{fact}}$):

$$\begin{aligned} &\underset{\mathbf{S}_1,\ldots,\mathbf{S}_J}{\text{minimize}} && \tfrac{1}{2}\|\mathbf{U} - \mathbf{S}_J \ldots \mathbf{S}_1\|_F^2 \\ &\text{subject to} && \mathbf{S}_j \in \mathcal{S}_j,\ \forall j \in \{1,\ldots,J\}, \end{aligned} \tag{P1}$$

- Approximate diagonalization of $\mathbf{L}$ (giving $\hat{\mathbf{U}}_{\text{diag}}$):

$$\begin{aligned} &\underset{\mathbf{S}_1,\ldots,\mathbf{S}_J,\mathbf{D}}{\text{minimize}} && \tfrac{1}{4}\|\mathbf{L} - \mathbf{S}_J \ldots \mathbf{S}_1\mathbf{D}\mathbf{S}_1^T \ldots \mathbf{S}_J^T\|_F^2 \\ &\text{subject to} && \mathbf{S}_j \in \mathcal{S}_j,\ \forall j \in \{1,\ldots,J\} \\ &&& \mathbf{D} \in \mathcal{D}, \end{aligned} \tag{P2}$$

both tackled with the hierarchical strategy of [1].

Main Contribution

A flexible approach that allows to get FAμSTs with computational complexities $\mathcal{O}(n^\alpha)$, $1 < \alpha < 2$, approximating well the Fourier transform of many classical families of graphs.

Experimental validation

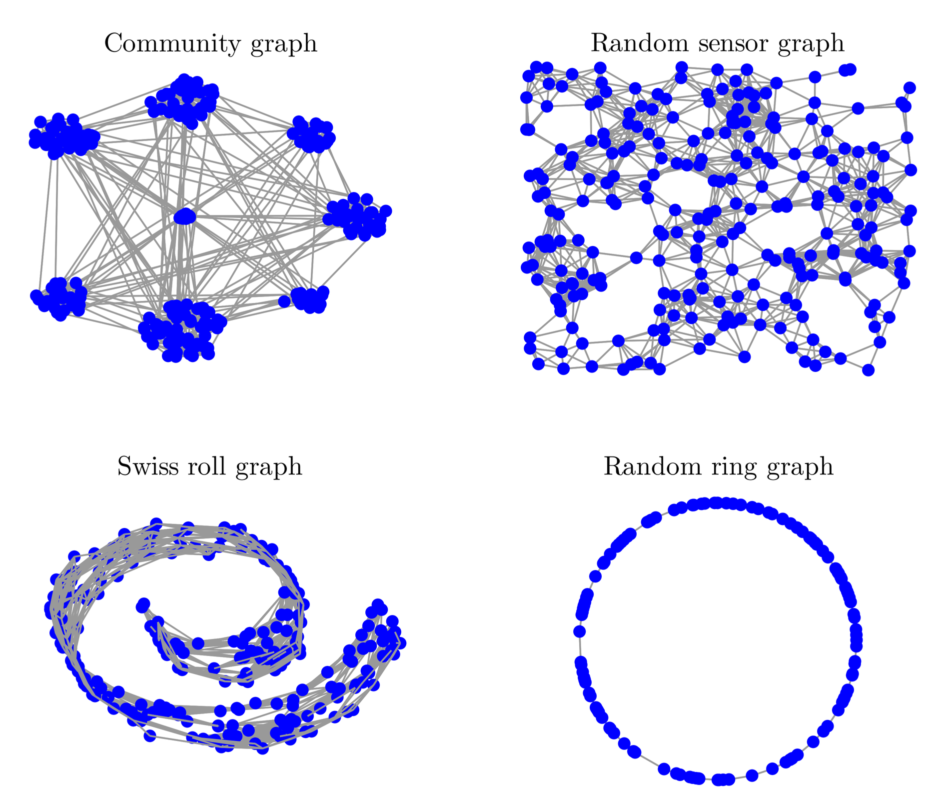

Figure 1: Different graphs used.

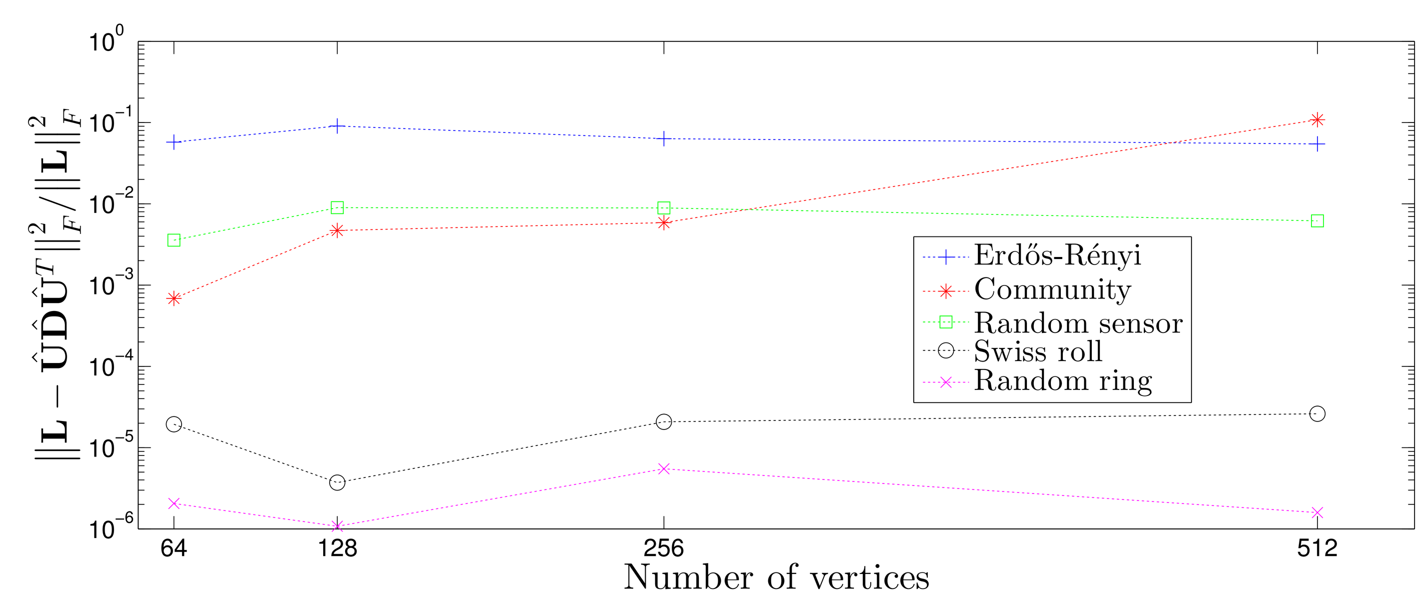

Figure 2: Approximation error for (P2), various graphs of different dimensions $n \in \{64, 128, 256, 512\}$, and FA$\mu$STs of complexity $\mathcal{O}(n^{1.26})$. The mean over 10 independent trials is shown.

Filtering experiment

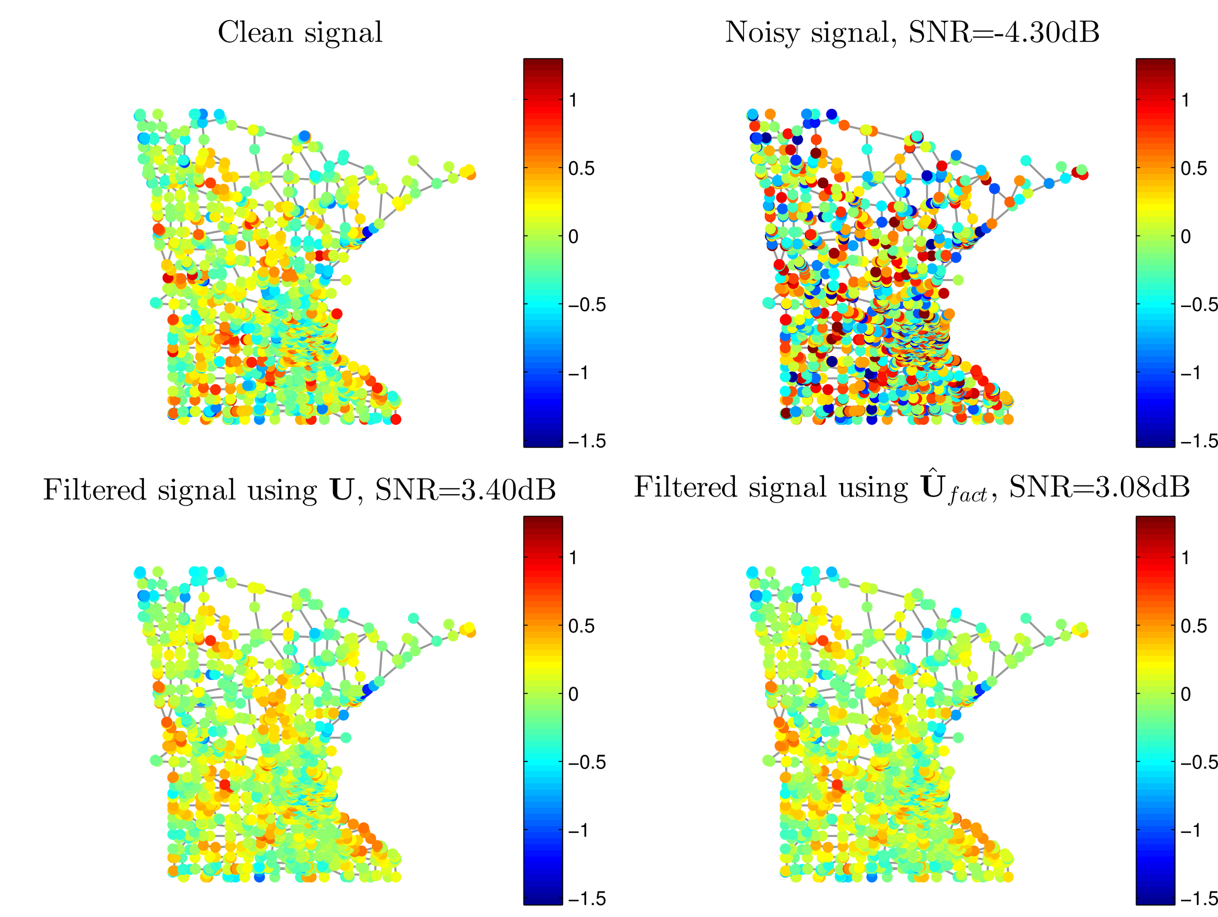

Figure 3: Example of filtering on the Minnesota road graph. Filtering using $\mathbf{U}$ and filtering using a FAμST $\hat{\mathbf{U}}_{\text{fact}}$ (eight times more computationally efficient) are shown.

	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$
Noisy	1.82	-0.68	-2.65
Filtered using $\mathbf{U}$	5.11	4.57	3.89
Filtered using $\hat{\mathbf{U}}_{\text{diag}}$	4.04	3.62	3.11
Filtered using $\hat{\mathbf{U}}_{\text{fact}}$	4.70	4.23	3.59

Table 1: Filtering results, the SNRs in dBs and in average over 100 independently drawn signals for each noise level are given.

Future work

- Designing a method that does not require a pre-computed diagonalization of the Laplacian $\mathbf{L}$.
- Imposing orthogonal FAμSTs, to ensure perfect reconstruction ($\hat{\mathbf{U}}^T\hat{\mathbf{U}} = \mathbf{Id}$).

Contact Information

luc.le-magoarou@inria.fr

Acknowledgments

This work was supported in part by the European Research Council, PLEASE project (ERC-StG-2011-277906).

References

[1] Luc Le Magoarou and Rémi Gribonval.
Flexible multi-layer sparse approximations of matrices and applications. *CoRR*, abs/1506.07300, 2015.