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Area theorem and energy quantization for dissipative optical solitons

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Abstract

Soliton area theorems express the pulse energy as a function of the pulse shape and the system parameters. From an analytical solution to the cubic-quintic Ginzbug-Landau equation, we derive an area theorem for dissipative optical solitons. In contrast to area theorems for conservative optical solitons, the energy does not scale inversely with the pulse duration, and in addition there is an upper limit to the energy. Energy quantization explains the existence of, and conditions for, multiple-pulse solutions. The theoretical predictions are confirmed with numerical simulations and experiments in the context of dissipative soliton fiber lasers.

1. Introduction

Coherent structures play an important role in the dynamics of pattern-forming systems. These localized states are studied extensively in numerous systems ranging from second-order phase transitions to nonlinear optics and the evolution of flame fronts. Specifically, uniformly-translating structures known as solitary waves or solitons have been the object of significant study in the physical sciences in the past few decades. Solitary-waves can be divided into two categories: conservative and dissipative. Systems supporting conservative solitons are Hamiltonian, and may be integrable. The nonlinear Schrodinger equation (NLSE) is a well-studied example. Dissipative solitons are characterized by internal energy flow, which underlies the balance of amplitude and phase modulations needed for soliton formation. Interest in dissipative solitons has grown in the past decade, but experimental studies are still quite limited [1].

Optical soliton research has been dominated by the conservative soliton class of the one-dimensional NLSE because the equation contains only the fundamental two physical effects that govern the propagation of short pulses of light in a cubic nonlinear medium: group-velocity dispersion (GVD) and the electronic Kerr nonlinearity. In addition to their fundamental importance, solitons of the NLSE are now being exploited in long-distance communications networks, and modern picosecond and femtosecond lasers are based largely on soliton-like pulse shaping, which occurs when the GVD is anomalous. The essential behavior of the soliton solutions is contained in the well-known "area theorem" [2,3], which expresses the energy of the pulse as a function of the pulse duration and the system parameters.

NLSE solitons exist at large anomalous dispersion and have been practically limited to ~ 0.1 nJ in single-mode fiber (SMF) lasers by the onset of multiple pulsing. With a dispersion map

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and net GVD near zero, breathing solutions known as dispersion-managed solitons exist [4]. In normal dispersion systems, the existence of solutions is predicted by both the cubic and the quintic complex Ginzburg-Landau equations (CGLE and CQGLE) that model short pulse evolution in fiber [5, 6]. Proctor *et al.* experimentally verified the prediction based on the CGLE that highly-chirped pulses are stable in a solid-state laser with net normal GVD [7]. Multiple-pulsing is a common phenomenon in mode-locked lasers. Within the frameworks of the NLSE or the CGLE, there is no direct mechanism to account for multi-pulsing. Solutions exist for any pulse energy, and multiple-pulsing in these cases is generally understood in an *ad hoc* way as a consequence of soliton fission *via* perturbations to the system.

Recently, Chong *et al.* introduced a new type of laser in which the pulse is shaped primarily by spectral filtering [8]. The pulses are dissipative solitons of the cubic-quintic Ginzburg-Landau equation (CQGLE) [5]. The CQGLE is one of the most studied equations in physics and is one of the simplest models which gives rise to a wealth of behavior (*e.g.*, fronts, pulses, sources, and sinks). Experimental studies of dissipative optical solitons are in their infancy [9], so experimental demonstrations of aspects of these new solitons are highly desirable.

Here we show that an area theorem exists for dissipative solitons of the CQGLE, and that the scaling of energy and pulse duration is completely different from that of previously-studied optical solitons. In contrast to prior area theorems, there is an upper limit to the energy of dissipative solitons. With numerical simulation and experiment, we verify the theorem in the context of fiber lasers, and show that it leads to energy quantization. Furthermore, the existence and conditions for multi-pulsing are explained as a consequence of the upper energy limit, and the implications of these for practical ultrashort-pulse lasers are discussed.

In addition to its fundamental importance, understanding the limits to pulse energy will enable significant improvements in the design and performance of mode-locked lasers. For a given mode-field area, normal-dispersion fiber lasers produce the highest pulse energies [10], which offers major advantages for applications such as nonlinear microscopy, precision material processing, and ocular surgery, among others. In this context, multi-pulsing states are undesirable. However, controlled generation of multiple pulses may allow for scaling of the oscillator repetition rate to otherwise-impossible values. Such harmonic mode-locking based on dissipative solitons has not been investigated. Gigahertz repetition rates will be relevant to the production of frequency combs [11].

2. Area Theorem

Optical pulse evolution in the presence of the electronic Kerr nonlinearity, GVD, and dissipative processes can be modeled by the complex Ginzburg-Landau equation with cubic and quintic saturable absorber terms:

$$U_z = gU + \left(\frac{1}{\Omega} - i\frac{D}{2}\right)U_{tt} + (\alpha + i\gamma)|U|^2U + \delta|U|^4U. \tag{1}$$

U is the electric field envelope, z is the propagation coordinate, t is the retarded time, D is the GVD, g is the net gain, Ω is the net filter bandwidth squared, α is a cubic saturable absorber coefficient, δ is the quintic saturable absorber coefficient, and γ refers to the cubic refractive nonlinearity of the medium. The energy is the integral of the intensity profile, $E = \int_{-\infty}^{\infty} |U[t, z]|^2 dt$.

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Area theorems are simple relations that express the conditions that must be satisfied for a particular pulse solution to exist. For the NLSE, soliton solutions obey a simple area theorem relating the product of pulse energy and pulse duration to the dispersion and nonlinearity (Table 1a). Dispersion-managed soliton solutions are modeled by a gaussian *ansatz* to the NLSE with position-dependent dispersion. Most of the terms of Eq. 1 are retained and treated as perturbations to the fundamental solution, and a similar area theorem results (Table 1b) [12].

The CGLE is Eq. 1 with $\delta = 0$; this equation has an exact chirped hyberbolic-secant solution. However, there is only one solution for a given set of system parameters. In this case we can define an area theorem as the relation between energy and pulse duration when the net gain varies (Table 1c). Considering the different systems and pulses, the area theorems of Table 1 express remarkably similar ideas. The right hand side of each expression is a constant set by the system, and the left hand side expresses how the pulse can scale to satisfy the equation. In all three cases the pulse energy is inversely proportional to the pulse duration, τ.

An exact particular solution to Eq. 1 [13] that accurately models the pulses in normal-dispersion fiber lasers [5] is

$$U[t,z]=\sqrt{\frac{A}{\cosh(\frac{t}{\tau})+B}}e^{-i\frac{\beta}{2}\ln(\cosh(\frac{t}{\tau})+B)+i\theta z}. \tag{2}$$

A, B, τ, β and θ are real constants. Unique features of the pulse profiles are distinguished by the parameter B. The pulses mathematically and conceptually divide into two categories: those with $|B| < 1$ and those with $B > 1$. The pulse shape for $|B| < 1$ is always close to the sech profile, but there are notable features in the spectrum (see *e.g.*, [5] and Fig. 1). For $B > 1$, the pulse is long and flat-topped and the spectrum is narrow. The class of solution that exists depends on the sign of the quintic nonlinear term δ. For $\delta > 0$, $|B| < 1$ and for $\delta < 0$, $B > 1$.

By inserting the solution into Eq. 1 and solving the six resultant algebraic equations, we find that the pulse energy can be expressed as

$$E=F(B)G(D,\Omega,\delta), \tag{3}$$

where

$$F(B)=\begin{cases}\cos^{-1}(B) & \text{for } |B|<1\\ \cosh^{-1}(B) & \text{for } B>1,\end{cases} \tag{4}$$

$$G(D,\Omega,\delta)=\frac{\sqrt{\frac{2}{3}}(\Delta+2)\sqrt{D^2(\Delta-8)\Omega^2+12(\Delta-4)}}{D\sqrt{|\delta|}\Omega\sqrt{\Omega(D^2\Omega^2+4)}}, \tag{5}$$

and

$$\Delta = \sqrt{3D^2\Omega^2 + 16}. \tag{6}$$

The full-width at half-maximum pulse duration, T, can also be written as the product of a constant set by the system and a function of B,

$$T = \left(\frac{|B|\cosh^{-1}(2+B)}{\sqrt{|B^2-1|}} \right) \frac{D\Omega|\delta|G(D,\Omega,\delta)}{2(\Delta+2)\gamma}. \tag{7}$$

The net gain parameter varies B, which in turn varies both the energy and pulse duration. With the elimination of B, Eqs. 3 and 7 combine into one piecewise expression which can be considered the area theorem. Here we leave the theorem in two equations (Eqs. 3 and 7) to make clear the distinctions between pulses with different values of B.

The area theorem provides insight into the particular analytical solution of the CQGLE and exhibits some remarkable features. In the expression for pulse energy, $G(D, \Omega, \delta)$ contains the information about the system, and F(B) contains information about the pulse. The two classes of solutions exhibit qualitatively different features. For $B > 1$, as with other solitons, any pulse energy can satisfy the relation for a fixed system given the appropriate value of B, as expressed by Eqs. 3 and 4 (Fig. 1). However, for $|B| < 1$ there is an upper limit to the pulse energy at $B = -1$ where the *ansatz* diverges and $F(B) = \pi$. This upper limit implies the quantization of pulse energy, and distinguishes the CQGLE pulse solutions from other soliton solutions; for a fixed system modeled by the CQGLE, a pulse has an energy maximum determined by Eqs. 3–5. For $0 < B < 2.217$, the energy scales inversely with pulse duration, as in the other area theorems of Table 1. This is not surprising because the pulse solution reduces to the CGLE solution when $B = 1$, and the area theorem follows. For all other values of B ($B < 0$ and $B > 2.217\ldots$) the energy scales directly with the pulse duration.

3. Experimental Results

A normal-dispersion fiber oscillator built with the design in Refs. [5,8] was used to investigate the area theorem experimentally. The gain fiber is Yb-doped, is pumped with a 980 nm diode laser and emits at a wavelength of 1 μm. A longer (183 cm) segment of single-mode fiber precedes 60 cm of gain fiber, and a 125-cm segment follows it in a unidirectional ring cavity. All components of the laser have normal GVD. Three wave plates and a polarizing beam splitter convert nonlinear polarization evolution into amplitude modulation. A birefringent plate surrounded by polarizers acts as an adjustable spectral filter with a sinusoidal transmission curve with 8 nm bandwidth. We experimentally access distinct operating states of the laser (*i.e.*, distinct solutions of the CQGLE) *via* adjustments to the wave plates, the pump power, and the cavity length. These adjustments effectively vary the cubic and quintic saturable absorber terms, the pulse energy, and the GVD, respectively. We will focus on the class of solutions with $B < 0$ because they offer useful short durations and high peak powers and to demonstrate the new qualitative features that arise in the CQGLE: the dependence of the pulse energy on the pulse duration, and the quantization of pulse energy.

Controlled measurements were made by holding the laser parameters constant while increasing the pump power from zero. For a given setting of the wave plates, at low pump power, the laser operates in continuous-wave mode, which corresponds to a plane-wave solution to Eq. 1. At an initial threshold, mode-locking occurs and a single pulse traverses

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the cavity. Increasing the pump power produces a pulse with higher energy and a broader spectral bandwidth (Fig. 2). This evolution is predicted by the analytical spectra as F(B) approaches the energy limit at π (Fig. 2). To exaggerate the unique features, the analytical spectra were plotted with $\beta = 10$, a factor of 7 from the theoretical value; this is typical of the overall quantitative agreement to the exact particular solution. The characteristic two-peaked spectrum of normal-dispersion lasers develops more structure.

With further increase of the pump power, a new pulse appears and the spectral shape returns to the narrower spectrum of Fig. 2d, which is essentially identical to the low-energy spectrum of Fig. 2a. This trend is summarized in Fig. 3, and it continues up to the maximum available pump power [14]. Up to four pulses have been observed in the cavity. The system arranges itself in such a way that the minimum number of pulses that can satisfy the area theorem exist. Knowledge of the conditions for multiple-pulsing could be used to design a laser with an arbitrary number of identical pulses in the cavity. However, while Eq. 3 explains the number of pulses, the temporal location of these pulses with respect to each other in the cavity is poorly understood; further research will be required to design a laser that operates at an arbitrary harmonic frequency.

4. Numerical Simulations

Numerical simulations were performed to verify and refine the analytical predictions. Gain saturation is added to the CQGLE, with $g = g_o/(1 + E/E_{sat})$, g_o corresponds to 30 dB of small-signal gain, and E_{sat} is the saturation nergy. The fiber lengths are chosen to match the experiment, with $\beta_2 = 230$ fs^2/cm and $\gamma = 0.0047$ (W m)$^{-1}$. The fiber is followed by a monotonic saturable absorber given by $T = 1-l_o/[1+P(\tau)/P_{sat}]$ where l_o=0.7 is the unsaturated loss, $P(\tau)$ is the instantaneous pulse power and $P_{sat} = 0.2$ kW is the saturation power. The gain is assumed to have a gaussian spectral profile with a 40-nm bandwidth, the output coupling is 60%, and an additional gaussian filter with 8 nm bandwidth is implemented after the saturable absorber. The results of these simulations as E_{sat} increases from 400 to 900 pJ exhibit the features and trends of experiment and analytical predictions (Fig. 2).

Within each range of pulse energy for which the number of pulses is constant, the pulse energy is proportional to the pulse duration, as predicted. Measured intensity autocorrelations of single pulses in the cavity are shown in Fig. 4(a), and the variation of energy *vs.* duration is presented in Fig. 4(b), alongside the simulated trend. These measurements highlight unique features of dissipative solitons of the CQGLE. The reasonable agreement between the analytical and numerical calculations implies that the solutions are not influenced dramatically by the discrete, inhomogeneous nature of the laser cavity.

5. Conclusion

In summary, dissipative optical solitons are governed by an area theorem with features that contrast sharply with those of other optical solitons: the energy scales directly with the pulse duration and there is an upper limit to the energy. The predictions are verified in the context of dissipative fiber lasers and energy quantization is shown to explain the generation of multiple pulses. In addition to adding to fundamental understanding of dissipative solitons, this work is immediately applicable to the design of high-energy and harmonically mode-locked lasers.

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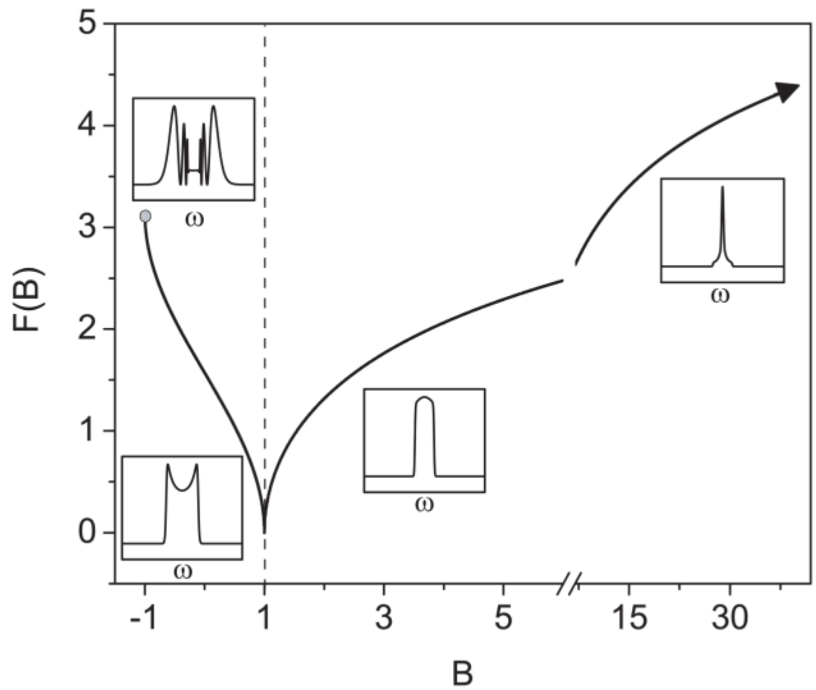

Fig. 1.

Variation of the pulse energy as a function of the pulse parameter, B. The dotted line separates solutions with $|B| < 1$ for $\delta > 0$ from those with $B > 1$ for $\delta < 0$. Insets: spectral profiles plotted for the respective values of B.

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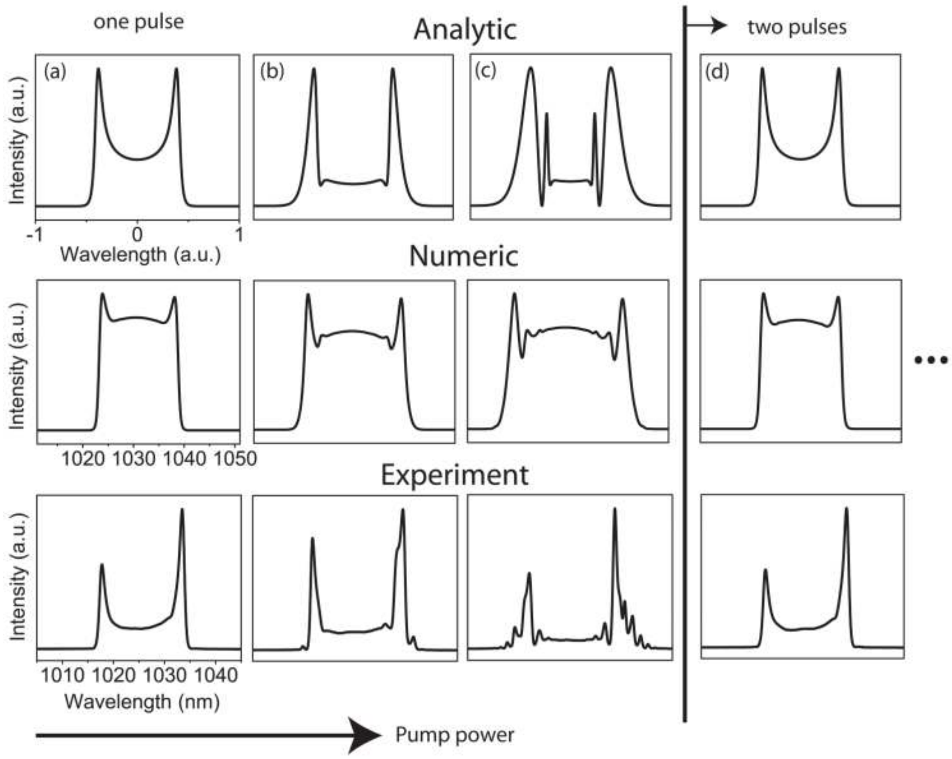

Fig. 2.

Top: theoretical spectra for increasing pulse energy, as B approaches −1; middle: simulated spectra with increasing saturation energy; bottom: measured spectra with increasing pump power. The rightmost spectra correspond to the birth of the second pulse in the cavity.

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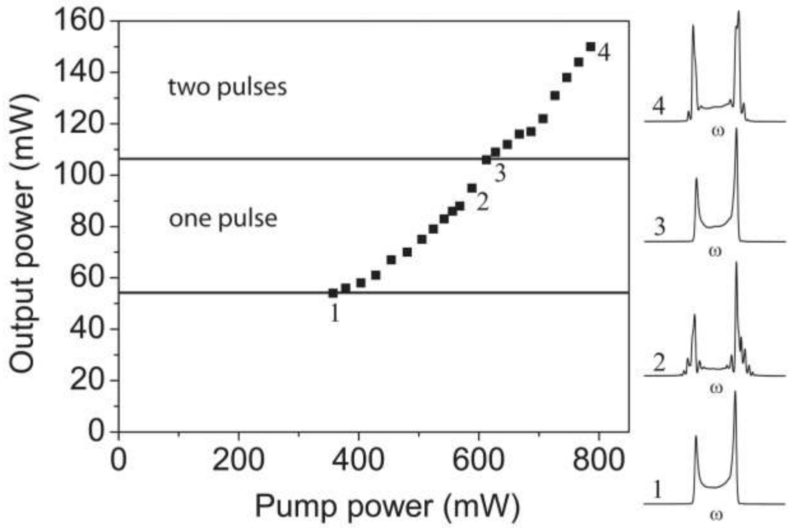

Fig. 3.
Mode-locked output power *vs.* pump power. The spectra on the right are for the corresponding pump levels.

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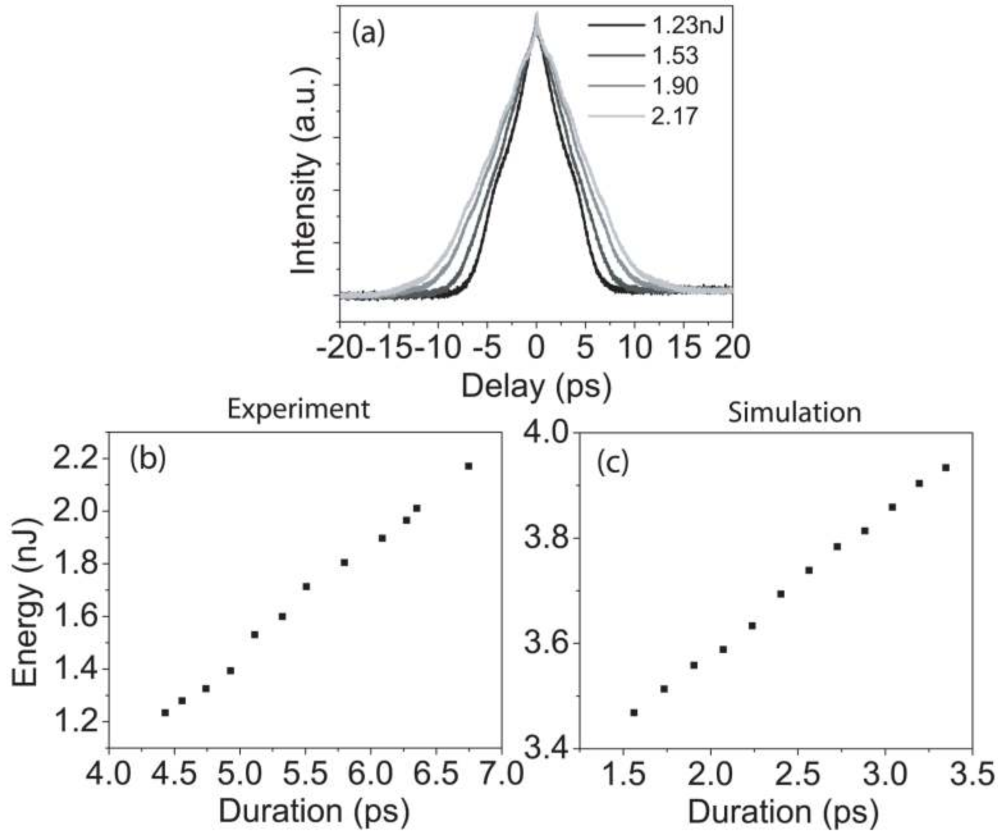

Fig. 4.
Top: single-pulsing intensity autocorrelation for increasing pulse energies; bottom: output energy *vs.* pulse duration from experiment (left) and simulation (right)..

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Table 1

Area theorems: a) soliton of the NLSE; b) DM-soliton of the CGLE; and c) chirped soliton of the CGLE.

	Area theorems
a	$E\tau=\frac{2\vert D\vert}{\gamma}$
b	$E\tau^3=\sqrt{\pi}\sqrt{\frac{\frac{1}{\Omega^2}+4D^2}{\alpha^2+\gamma^2}}$
c	$E\tau=\frac{3(D^2\Omega^2+4)}{2\Omega(2\gamma+D\alpha\Omega)^2}\left(3D\gamma\Omega-6\alpha\pm\sqrt{4(2D^2\Omega^2+9)\alpha^2-4D\gamma\Omega\alpha+\gamma^2(9D^2\Omega^2+32)}\right)$