

Area-universal rectangular layouts

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Area-Universal Rectangular Layouts*

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Abstract

A rectangular layout is a partition of a rectangle into a finite set of interior-disjoint rectangles. A layout is *area-universal* if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout. We identify a simple necessary and sufficient condition for a rectangular layout to be area-universal: a rectangular layout is area-universal if and only if it is *one-sided*. More generally, given any rectangular layout \mathcal{L} and any assignment of areas to its regions, we show that there can be at most one layout (up to horizontal and vertical scaling) which is combinatorially equivalent to \mathcal{L} and achieves a given area assignment. We also investigate similar questions for perimeter assignments. The adjacency requirements for the rectangles of a rectangular layout can be specified in various ways, most commonly via the dual graph of the layout. We show how to find an area-universal layout for a given set of adjacency requirements whenever such a layout exists.

1 Introduction

Motivation. Raisz [7] introduced *rectangular cartograms* in 1934 as a way of visualizing spatial information, such as population or economic strength, of a set of regions like countries or states. Rectangular cartograms represent geographic regions by rectangles; the positioning and adjacencies of the rectangles are chosen to suggest their geographic locations, while their areas are chosen to represent the numeric values being communicated by the cartogram.

Often more than one numeric quantity should be displayed as a cartogram for the same set of geographic regions. To make the visual comparison of multiple related cartograms easier, it is desirable that the arrangement of rectangles be combinatorially equivalent in each cartogram, although the relative sizes of the rectangles will differ. This naturally raises the question: when is this possible?

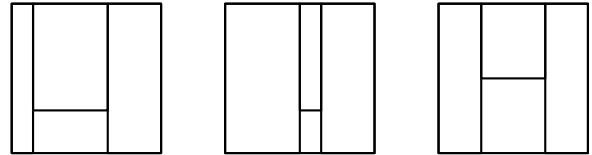
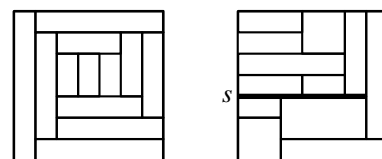


Figure 1: Three area assignments.

Mathematically, a rectangular cartogram is a *rectangular layout*: a partition of a rectangle into finitely many interior-disjoint rectangles. We call a layout \mathcal{L} *area-universal* if, for any area requirement for its regions, some combinatorially equivalent layout \mathcal{L}' has regions with the specified areas. For instance, the four-region rectangular layout shown above with three different area assignments is area-universal: any four numbers can be used as the areas of the rectangles in a combinatorially equivalent layout.

Area-universal rectangular layouts are useful not only for displaying multiple side-by-side cartograms for different sets of data on the same regions, but also for dynamically morphing from one cartogram into another. Additionally, rectangular layouts have other applications in which being able to choose a layout first and then later assigning varying areas while keeping the combinatorial type of the layout fixed may be an advantage: in circuit layout applications of rectangular layouts [10], each component of a circuit may have differing implementations with differing tradeoffs between area, energy use, and speed; and in building design it is desirable to be able to determine the areas of different rooms according to their function [3].

Results. We identify a simple necessary and sufficient condition for a rectangular layout to be area-universal: a rectangular layout is area-universal if and only if it is *one-sided*. One-sided layouts are characterized via their *maximal line segments*. A line segment of a layout \mathcal{L} is formed by a sequence of consecutive inner edges of \mathcal{L} . A segment of \mathcal{L} that is not contained in any other segment is maximal. In a one-sided layout every maximal line segment s must be the

Figure 2: The left layout is one-sided, but the right one is not: s is not the side of any rectangle.

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side of at least one rectangle R ; any vertices interior to s are T-junctions that all have the same orientation, pointing away from R (Fig. 2). Given an area-universal layout \mathcal{L} and an assignment of areas for its regions, we describe a numerical algorithm that finds a combinatorially equivalent layout \mathcal{L}' whose regions have a close approximation to the specified areas.

More generally, given any rectangular layout \mathcal{L} and any assignment of areas to its regions, we show that there can be at most one layout (up to horizontal and vertical scaling) which is combinatorially equivalent to \mathcal{L} and achieves the given area assignment. This result was previously known only for two special classes of rectangular layouts, namely *sliceable layouts* (layouts that can be obtained by recursively partitioning a rectangle by horizontal and vertical lines) and *L-shape destructible layouts* [9] (layouts where the rectangles can be iteratively removed such that the remaining rectangles form an L-shaped polygon).

We also investigate *perimeter cartograms* in which the perimeter of each rectangle is specified rather than its area. Again, any rectangular layout can have at most one combinatorially equivalent layout for a given perimeter assignment; it is possible in polynomial time to find this equivalent layout, if it exists.

The rectangles of a rectangular cartogram should have the same adjacencies as the regions of the underlying map. Hence, the dual graph of the cartogram should be the same as the dual graph of the map. The dual of a rectangular cartogram or layout must be a triangulated plane graph satisfying certain additional conditions. We call such graphs *proper graphs*. Every proper graph \mathcal{G} has at least one *rectangular dual*: a rectangular layout \mathcal{L} whose dual graph is \mathcal{G} . However, not every proper graph has an area-universal rectangular dual; Rinsma [8] described an outerplanar proper graph \mathcal{G} and an assignment of weights to the vertices of \mathcal{G} such that no rectangular dual of \mathcal{G} can have these weights as the areas of its regions. We describe algorithms that, given a proper graph \mathcal{G} , find an area-universal rectangular dual of \mathcal{G} if it exists. These algorithms are not fully polynomial, but are fixed-parameter tractable for a parameter related to the number of separating four-cycles in \mathcal{G} .

In the following we can only sketch our results, a full version of the paper can be found here [2].

2 Preliminaries

A rectangular layout is a partition of a rectangle into a finite set of interior-disjoint rectangles, where no four regions meet in a single point. We denote the dual graph of a layout \mathcal{L} by $\mathcal{G}(\mathcal{L})$. A layout \mathcal{L} such that $\mathcal{G} = \mathcal{G}(\mathcal{L})$ is called a *rectangular dual* of graph \mathcal{G} . $\mathcal{G}(\mathcal{L})$ is a plane triangulated graph and is unique for any layout \mathcal{L} . Not every plane triangulated graph has a rectangular dual, and if it does, then the rectan-

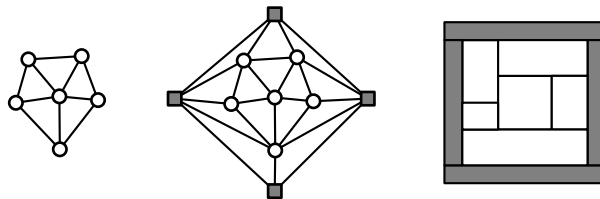


Figure 3: A proper graph \mathcal{G} , an extended graph $E(\mathcal{G})$, and a rectangular dual \mathcal{L} of $E(\mathcal{G})$.

gular dual is not necessarily unique. Kozminski and Kinnen [6] proved that a plane triangulated graph \mathcal{G} has a rectangular dual if and only if we can augment \mathcal{G} with four external vertices in such a way that the *extended graph* $E(\mathcal{G})$ has the following two properties: (i) every interior face is a triangle and the exterior face is a quadrangle; (ii) $E(\mathcal{G})$ has no separating triangles. If a plane triangulated graph \mathcal{G} allows such an augmentation, then we say that \mathcal{G} is a *proper graph*. A rectangular dual of an extended graph of a proper graph \mathcal{G} can be constructed in linear time [5] and it immediately implies a rectangular dual for \mathcal{G} (Fig. 3).

An extended graph $E(\mathcal{G})$ determines uniquely which vertices of a proper graph \mathcal{G} are associated with the corner rectangles of every rectangular dual of \mathcal{G} that corresponds to $E(\mathcal{G})$. For a given proper graph there might be several possible extended graphs and hence several possible *corner assignments*. In many cases we assume that a corner assignment, and hence an extended graph, has already been fixed, but if this is not the case then it is possible to test all corner assignments in polynomial time.

A rectangular layout \mathcal{L} naturally induces a labeling of its extended dual graph $E(\mathcal{G})$. If two rectangles of \mathcal{L} share a vertical segment, then we color the corresponding edge in $E(\mathcal{G})$ blue (solid) and direct it from left to right. Correspondingly, if two rectangles of \mathcal{L} share a horizontal segment, then we color the corresponding edge in $E(\mathcal{G})$ red (dashed) and direct it from bottom to top (Fig. 4). This labeling has the following properties: (i) around each inner vertex in clockwise order we have four contiguous sets of incoming blue edges, outgoing red edges, outgoing blue edges, and incoming red edges; (ii) the left exterior vertex has

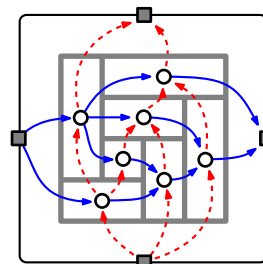


Figure 4: A rectangular layout and the regular edge labeling of its extended dual.

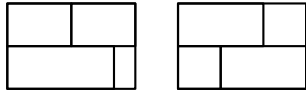


Figure 5: Two inequivalent but order-equivalent rectangular layouts.

only blue outgoing edges, the top exterior vertex has only red incoming edges, the right exterior vertex has only blue incoming edges, and the bottom exterior vertex has only red outgoing edges.

Such a labeling is called a *regular edge labeling*. It was introduced by Kant and He [5] who showed that every regular edge labeling of an extended graph $E(\mathcal{G})$ uniquely defines an equivalence class of rectangular duals of a proper graph \mathcal{G} . Given any extended graph $E(\mathcal{G})$, a regular edge labeling for $E(\mathcal{G})$ can be found in linear time and the rectangular dual defined by it can also be constructed in linear time [5].

Two layouts \mathcal{L} and \mathcal{L}' are *equivalent*, denoted by $\mathcal{L} \sim \mathcal{L}'$, if they induce the same regular edge labeling of the same dual graph. We say that a rectangular layout \mathcal{L} with n rectangles R_1, \dots, R_n realizes a weight function $w : R_1, \dots, R_n \rightarrow \mathbb{R}, w(i) > 0$ as a *rectangular cartogram* if there exists a layout $\mathcal{L}' \sim \mathcal{L}$ such that for any $1 \leq i \leq n$ the area of rectangle R_i equals $w(r_i)$. Correspondingly, we say that a layout \mathcal{L} realizes w as a *perimeter cartogram* if there exists a layout $\mathcal{L}' \sim \mathcal{L}$ such that the perimeter of each rectangle of \mathcal{L}' equals the prescribed weight. A layout \mathcal{L} is *area-universal* if it realizes every possible weight function.

It is convenient to define a weaker equivalence relation on layouts than equivalence, which we call *order-equivalence*. For a layout \mathcal{L} , we define a partial order on the vertical maximal segments, in which $s_1 \leq s_2$ if there exists an x -monotone curve that has its left endpoint on s_1 , its right endpoint on s_2 , and that does not cross any horizontal maximal segments. We define a partial order on the horizontal segments in a symmetric way. \mathcal{L} and \mathcal{L}' are order-equivalent if their rectangles and maximal segments correspond one-for-one in a way that preserves these partial orders.

Observation 1 *A rectangular layout with n rectangular regions has $n - 1$ maximal segments.*

3 There can be only one

We first show that for any combination of layout and weight function there can be at most one rectangular cartogram or perimeter cartogram. More generally, if two geometrically different but order-equivalent layouts share the same bounding box, there is a rectangle in one of the layouts that is larger in both of its dimensions than the corresponding rectangle in the other layout. Thus, let \mathcal{L} and \mathcal{L}' be two geometrically different order-equivalent layouts with the same

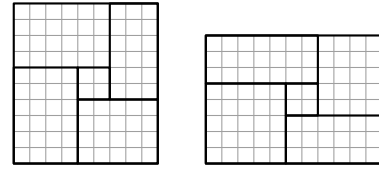


Figure 6: Two equivalent layouts in which corresponding rectangles have the same perimeter.

bounding box. The *push graph* \mathcal{H} of \mathcal{L} and \mathcal{L}' is a directed graph that has a vertex for each rectangle in \mathcal{L} and an edge from vertex R_i to vertex R_j if the rectangles R_i and R_j are adjacent and the maximal segment in \mathcal{L} that separates R_i from R_j is shifted in \mathcal{L}' towards R_j and away from R_i .

Lemma 1 *The push graph for \mathcal{L} and \mathcal{L}' contains a node with no incoming or no outgoing edges.*

Theorem 2 *For any layout \mathcal{L} and any weight function w there is at most one layout \mathcal{L}' (up to affine transformations) that is order-equivalent to \mathcal{L} and that realizes w as a rectangular cartogram.*

For perimeter, such strong uniqueness does not hold: there are equivalent layouts that are not affine transformations of each other in which the perimeters of corresponding rectangles are equal (Fig. 6). However, if we fix the outer bounding box of the layout, the same proof method works:

Theorem 3 *For any layout \mathcal{L} and any weight function w there is at most one layout \mathcal{L}' that is order-equivalent to \mathcal{L} with the same bounding box and that realizes w as a perimeter cartogram.*

More generally the same result holds for any type of cartogram in which rectangle sizes are measured by any strictly monotonic function of the height and width of the rectangles.

4 Area-universality and one-sidedness

All layouts are area-universal in a weak sense involving order-equivalence in place of equivalence. The proof of Lemma 4 uses Theorem 2 to invert the map W from vectors of positions of segments in a layout to vectors of rectangle areas, along a line segment from the area vector of \mathcal{L} to the desired area vector.

Lemma 4 *For any layout \mathcal{L} and weight function w , there exists a layout \mathcal{L}' that has a square outer rectangle, is order-equivalent to \mathcal{L} , and realizes w as a rectangular cartogram.*

One may find \mathcal{L}' by hill-climbing to reduce the Euclidean distance between the current weight function and the desired weight function. No layout \mathcal{L} can

be locally but not globally optimal, because within any neighborhood of \mathcal{L} the inverse image of the line segment connecting its weight vector to the desired weight vector contains layouts that are closer to w . Alternatively, one can find \mathcal{L}' by a numerical procedure that follows this inverse image by inverting the Jacobean matrix of W at each step. We do not know whether it is always possible to find \mathcal{L}' exactly by an efficient combinatorial algorithm (as may easily be done for the subclass of sliceable layouts), or whether the general solution involves roots of high-degree polynomials that can be found only numerically.

Theorem 5 *The following three properties of a layout \mathcal{L} are equivalent:*

1. \mathcal{L} is area-universal.
2. Every layout that is order-equivalent to \mathcal{L} is equivalent to \mathcal{L} .
3. \mathcal{L} is one-sided.

5 Finding perimeter cartograms

Although our proof of uniqueness for rectangular cartograms generalizes to perimeter, our proof that any layout and weight function have a realization as an order-equivalent cartogram does not generalize: there exist one-sided layouts and weight functions that cannot be realized as a perimeter cartogram (Fig. 7).

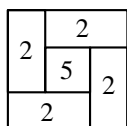


Figure 7: The outer rectangles contribute at most one unit of shared boundary to the perimeter of the central rectangle, which is too large to be realized.

Nevertheless, one can test in polynomial time whether a solution exists for any layout and weight function. The technique involves describing the constraints on the perimeters of rectangles as linear equalities that reduce the dimension of the space of layouts to at most two, and forming a low-dimensional linear program from inequality constraints expressing the equivalence to \mathcal{L} of the other layouts within this low-dimensional space.

Theorem 6 *For any layout \mathcal{L} and any weight function w we can find a layout \mathcal{L}' that is equivalent to \mathcal{L} and that realizes w as a perimeter cartogram, if one exists.*

The same algorithm can be used to find an order-equivalent layout rather than an equivalent layout, by restricting the inequality constraints to the subset that determine order-equivalence.

6 Finding one-sided layouts

Recall that every proper triangulated plane graph has a rectangular dual, but not necessarily a one-sided rectangular dual. Since one-sided duals are area-universal, it is of interest to find a one-sided dual for a proper graph if one exists. Our overall approach is, first, to partition the graph on its separating four-cycles; second, to represent the family of all layouts for a proper graph as a distributive lattice, following Fusy [4]; third, to represent elements of the distributive lattice as partitions of a partial order according to Birkhoff's theorem [1]; fourth, to characterize the ordered partitions that correspond to one-sided layouts; and fifth, to search in the partial order for partitions of this type. This approach does not yield polynomial time algorithms, but they are polynomial whenever the number of separating four-cycles in the given proper graph is bounded by a fixed constant, or more generally when such a bound can be given separately within each of the pieces found in the partition we find in the first stage of our algorithms.

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