

ARENS REGULARITY AND AMENABILITY OF LAU PRODUCT OF BANACH ALGEBRAS DEFINED BY A BANACH ALGEBRA MORPHISM

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Abstract

Given a morphism T from a Banach algebra \mathcal{B} to a commutative Banach algebra \mathcal{A} , a multiplication is defined on the Cartesian product space $\mathcal{A} \times \mathcal{B}$ perturbing the coordinatewise product resulting in a new Banach algebra $\mathcal{A} \times_T \mathcal{B}$. The Arens regularity as well as amenability (together with its various avatars) of $\mathcal{A} \times_T \mathcal{B}$ are shown to be stable with respect to T .

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1. Introduction

Let \mathcal{A} and \mathcal{B} be algebras, let $T : \mathcal{B} \rightarrow \mathcal{A}$ be an algebra homomorphism, and let \mathcal{A} be assumed commutative throughout. (It suffices to assume that the range of T is contained in the centre of \mathcal{A} .) We define a product on $\mathcal{A} \times \mathcal{B}$ as follows:

$$(a, b)(a', b') = (aa' + T(b)a' + T(b')a, bb') \quad ((a, b), (a', b') \in \mathcal{A} \times \mathcal{B}).$$

Then the Cartesian product space $\mathcal{A} \times \mathcal{B}$ is an associative, not necessarily commutative, algebra with this product. We denote $\mathcal{A} \times \mathcal{B}$ with this product by $\mathcal{A} \times_T \mathcal{B}$. If \mathcal{A} and \mathcal{B} are Banach algebras and if $\|T\| \leq 1$, then $\mathcal{A} \times_T \mathcal{B}$ is a Banach algebra with the norm

$$\|(a, b)\| = \|a\| + \|b\| \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

We note that \mathcal{A} is a closed ideal of $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to \mathcal{B} . When $T = 0$, this gives the coordinatewise product. Thus \times_T is the perturbation of the coordinatewise product induced by T . Besides giving a new

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method of constructing Banach algebras, the product \times_T has relevance in at least the following two situations.

(a) Let \mathcal{A} be unital, and let $\theta: \mathcal{B} \rightarrow \mathbb{C}$ be a multiplicative linear functional. Define $T: \mathcal{B} \rightarrow \mathcal{A}$ as $T(x) = \theta(x)e$ ($x \in \mathcal{B}$). Then the above product coincides with the product investigated by Lau [11]. This product is of relevance in Lau algebras arising in harmonic analysis and providing an abstract setting for several Banach algebras of harmonic analysis. A *Lau algebra* (called an F-algebra in [11]) is a pair (\mathcal{A}, M) , where \mathcal{A} is a complex Banach algebra; M is a W^* -algebra and \mathcal{A} is the predual of M such that the identity e of M is a multiplicative linear functional on \mathcal{A} . Examples of Lau algebras include the group algebra $L^1(G)$, the measure algebra $M(G)$ and the Fourier algebra $A(G)$ of a locally compact group G [11]; the Fourier–Stieltjes algebra of a topological group [12]; the measure algebra $M(S)$ of a locally compact semigroup or a hypergroup; as well as the predual algebra of a Hopf von Neumann algebra [16]. The Banach algebra $M \times_T \mathcal{A}$, $T: \mathcal{A} \rightarrow M$ being $T(x) = e(x)e$, encodes a Lau algebra (\mathcal{A}, M) .

(b) In the framework of Brown–Douglas–Fillmore theory [4], given an extension of compact operators $\mathcal{K}(H)$ on a separable Hilbert space H by the abelian C^* -algebra $C(X)$ of continuous functions on a compact space X manifested by a short exact sequence $0 \rightarrow \mathcal{K}(H) \xrightarrow{i} \mathcal{A} \xrightarrow{\varphi} C(X) \rightarrow 0$ of C^* -algebras, the above product produces a Banach $*$ -algebra $C(X) \times_{\varphi} \mathcal{A}$ encoding the extension. This aspect of the extension is yet to be explored.

The purpose of the present paper is to determine the Gel'fand space of $\mathcal{A} \times_T \mathcal{B}$ which turns out to be nontrivial even though $\mathcal{A} \times_T \mathcal{B}$ need not be commutative and to discuss the Arens regularity as well as the amenability of $\mathcal{A} \times_T \mathcal{B}$. These topics are central to the general theory of Banach algebras [3]; and are of current relevance [3, 5, 6]. Arens [1, 2] showed that the given product on a Banach algebra \mathcal{A} induces two canonical products on the second dual \mathcal{A}'' of \mathcal{A} ; and \mathcal{A} is *Arens regular* if these two products coincide. We prove in Theorem 3.1 that for an Arens regular commutative Banach algebra \mathcal{A} , $\mathcal{A} \times_T \mathcal{B}$ is Arens regular if and only if \mathcal{B} is Arens regular, thereby showing Arens regularity to be independent of T . A Banach algebra is *amenable* if any bounded derivation of \mathcal{A} into the dual E' of a Banach \mathcal{A} -module E is *inner* in the sense that it is of the form δ_x , $\delta_x(a) = a \cdot x - x \cdot a$ ($a \in \mathcal{A}$), for some $x \in E'$. Replacing E' by the dual \mathcal{A}' (which is an \mathcal{A} -module) results in *weak amenability*. There are closely related notions like approximate amenability, weak approximate amenability, and cyclic amenability. It is shown in Theorem 4.1 that $\mathcal{A} \times_T \mathcal{B}$ is amenable (weakly amenable, approximately amenable, weak approximately amenable, cyclic amenable, approximate cyclic amenable, respectively) if and only if \mathcal{A} and \mathcal{B} also are. A localised version is also discussed in Theorem 4.2. These results provide analogues for the perturbed product \times_T of the results for the Lau product proved recently in [13]. In fact, the present paper and the arguments herein are inspired by [13]. The message of the paper is that the notions of Arens regularity and amenability are fairly stable with respect to Cartesian product.

2. Gel'fand space and geometrisation of \times_T

THEOREM 2.1. *Let \mathcal{A} be a commutative Banach algebra with the Gel'fand space $\Delta(\mathcal{A})$, let \mathcal{B} be a Banach algebra, and let $T : \mathcal{B} \rightarrow \mathcal{A}$ be a homomorphism with $\|T\| \leq 1$. Then $\Delta(\mathcal{A} \times_T \mathcal{B}) = \{(\varphi, \varphi \circ T) : \varphi \in \Delta(\mathcal{A})\} \dot{\cup} \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}$, a disjoint union, and $E := \{(\varphi, \varphi \circ T) : \varphi \in \Delta(\mathcal{A})\}$ and $F := \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}$ are closed in $\Delta(\mathcal{A} \times_T \mathcal{B})$.*

PROOF. It is easy to see that

$$\{(\varphi, \varphi \circ T) : \varphi \in \Delta(\mathcal{A})\} \cup \{(0, \psi) : \psi \in \Delta(\mathcal{B})\} \subset \Delta(\mathcal{A} \times_T \mathcal{B}).$$

Conversely, let $(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$. Then

$$(\varphi, \psi)[(a, b)(a', b')] = (\varphi, \psi)(a, b)(\varphi, \psi)(a', b')$$

gives

$$\varphi(aa' + T(b')a + T(b)a') + \psi(bb') = \varphi(a)\varphi(a') + \varphi(a)\psi(b') + \varphi(a')\psi(b) + \psi(b)\psi(b').$$

Taking $b = b' = 0$, we get $\varphi(aa') = \varphi(a)\varphi(a')$, and taking $a = a' = 0$, we get $\psi(bb') = \psi(b)\psi(b')$.

First let $\varphi \neq 0$. Then

$$\varphi(T(b))\varphi(a') + \varphi(T(b'))\varphi(a) = \varphi(a')\psi(b) + \varphi(a)\psi(b').$$

Taking $a = a'$ and $b = b'$, we get $\varphi(T(b)) = \psi(b)$. Hence $\psi = \varphi \circ T$.

Suppose that $\varphi = 0$. Then $(0, \psi) \in \Delta(\mathcal{A} \times \mathcal{B})$.

Thus $E \dot{\cup} F = \Delta(\mathcal{A} \times_T \mathcal{B})$. Let $(\varphi_0, \varphi_0 \circ T) \in E$. Then there exists $a \in \mathcal{A}$ such that $\varphi_0(a) \neq 0$. Let $\epsilon = |\varphi_0(a)|/2$, and $U = U((\varphi_0, \varphi_0 \circ T), \epsilon, (a, 0))$. Then

$$\begin{aligned} U &= \{(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B}) : |(\varphi, \psi)(a, 0) - (\varphi_0, \varphi_0 \circ T)(a, 0)| < \epsilon\} \\ &= \{(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B}) : |\varphi(a) - \varphi_0(a)| < \epsilon\}. \end{aligned}$$

If $(0, \psi) \in U$, then $|\varphi_0(a)| < \epsilon$, which is not possible. Hence $U \subset E$. This shows that E is open in $\Delta(\mathcal{A} \times_T \mathcal{B})$ and hence F is closed in $\Delta(\mathcal{A} \times_T \mathcal{B})$.

Let $(0, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$ be in the closure of E . Then there is a net $((\varphi_\alpha, \varphi_\alpha \circ T)) \subset E$ converging to $(0, \psi)$, that is,

$$\varphi_\alpha(a) + \varphi_\alpha \circ T(b) \rightarrow \psi(b) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

In particular, taking $b = 0$, $\varphi_\alpha(a) \rightarrow 0$ ($a \in \mathcal{A}$). Taking $a = 0$, $\varphi_\alpha \circ T(b) \rightarrow \psi(b)$ ($b \in \mathcal{B}$). Since $\varphi_\alpha \rightarrow 0$, we have $\varphi_\alpha \circ T \rightarrow 0$, that is, $\psi = 0$. This is a contradiction. Hence E is closed in $\Delta(\mathcal{A} \times_T \mathcal{B})$. \square

COROLLARY 2.2. *Let \mathcal{A} and \mathcal{B} be commutative Banach algebras, and let $T : \mathcal{B} \rightarrow \mathcal{A}$ be an algebra homomorphism with $\|T\| \leq 1$. Then $\mathcal{A} \times_T \mathcal{B}$ is semisimple if and only if both \mathcal{A} and \mathcal{B} are semisimple.*

PROOF. Let $\mathcal{A} \times_T \mathcal{B}$ be semisimple. Let $a \in \mathcal{A}$ be such that $\varphi(a) = 0$ ($\varphi \in \Delta(\mathcal{A})$). Then $(\varphi, \varphi \circ T)(a, 0) = 0$ ($\varphi \in \Delta(\mathcal{A})$) and $(0, \psi)(a, 0) = 0$ ($\psi \in \Delta(\mathcal{B})$). Since $\mathcal{A} \times_T \mathcal{B}$ is semisimple, $a = 0$. Therefore \mathcal{A} is semisimple. It also follows by an analogous argument that \mathcal{B} is semisimple.

Let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ be such that $(\varphi, \psi)(a, b) = 0$ ($(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$). In particular, $\psi(b) = (0, \psi)(a, b) = 0$ ($\psi \in \Delta(\mathcal{B})$). Since \mathcal{B} is semisimple, it follows that $b = 0$. Since $b = 0$, we have $\varphi(a) = 0$ ($\varphi \in \Delta(\mathcal{A})$). Since \mathcal{A} is semisimple, $a = 0$. Hence $\mathcal{A} \times_T \mathcal{B}$ is semisimple. \square

Since \mathcal{A} is a closed ideal of $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to \mathcal{B} , it follows from Theorems 4.2.6 and 4.3.8 in [8] that $\mathcal{A} \times_T \mathcal{B}$ is regular if and only if both \mathcal{A} and \mathcal{B} are regular.

Theorem 2.1 shows that given locally compact Hausdorff spaces X and Y and a proper continuous map $h : X \rightarrow Y$ (that is, a continuous map for which the preimage under h of any compact set in Y is compact in X), the product $X \times_h Y$ defined below gives a geometrisation of the perturbed product \times_T :

$$X \times_h Y := \{(x, h(x)) : x \in X\} \cup \{(0, y) : y \in Y\}.$$

3. Arens regularity

Let \mathcal{A} be a Banach algebra. Let \mathcal{A}' and \mathcal{A}'' be the dual and second dual Banach spaces, respectively. Let $a \in \mathcal{A}$, $\lambda \in \mathcal{A}'$ and $\Phi, \Psi \in \mathcal{A}''$. Then $\lambda \cdot a$ and $a \cdot \lambda$ are defined as $\lambda \cdot a(x) = \lambda(ax)$ ($x \in \mathcal{A}$) and $a \cdot \lambda(x) = \lambda(xa)$ ($x \in \mathcal{A}$), making \mathcal{A}' an \mathcal{A} -module. Now \mathcal{A}'' is an \mathcal{A}' -module by

$$\langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle, \quad \langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle.$$

This defines two Arens products \square and \diamond on \mathcal{A}'' as

$$\langle \Phi \square \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle, \quad \langle \Phi \diamond \Psi, \lambda \rangle = \langle \Psi, \lambda \cdot \Phi \rangle,$$

making \mathcal{A}'' a Banach algebra with each. For each $\Psi \in \mathcal{A}''$, the maps $L_\Psi : \Phi \mapsto \Psi \diamond \Phi$ and $R_\Psi : \Phi \mapsto \Phi \square \Psi$ are continuous on (\mathcal{A}'', σ) , where $\sigma \equiv \sigma(\mathcal{A}'', \mathcal{A}')$ denotes the weak*-topology on \mathcal{A}'' by the duality $\langle \mathcal{A}', \mathcal{A}'' \rangle$. The products \square and \diamond are respectively the *first* and *second Arens products* on \mathcal{A}'' . The algebra \mathcal{A} is *Arens regular* if these products coincide on \mathcal{A}'' .

The *left* and *right topological centres* of \mathcal{A}'' are defined by

$$\begin{aligned} \mathfrak{Z}_l^{(\ell)}(\mathcal{A}'') &= \{\Phi \in \mathcal{A}'' : \Phi \square \Psi = \Phi \diamond \Psi (\Psi \in \mathcal{A}'')\}, \\ \mathfrak{Z}_r^{(r)}(\mathcal{A}'') &= \{\Phi \in \mathcal{A}'' : \Psi \square \Phi = \Psi \diamond \Phi (\Psi \in \mathcal{A}'')\}. \end{aligned}$$

The Banach algebra \mathcal{A} is *left strongly Arens irregular* if $\mathfrak{Z}_l^{(\ell)}(\mathcal{A}'') = \mathcal{A}$, *right strongly Arens irregular* if $\mathfrak{Z}_r^{(r)}(\mathcal{A}'') = \mathcal{A}$, and *strongly Arens irregular* if it is both left and right strongly Arens irregular.

Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be a continuous algebra homomorphism. Define $T' : \mathcal{B}' \rightarrow \mathcal{A}'$ as $T'(\lambda) = \lambda \circ T$, and $T'' : \mathcal{A}'' \rightarrow \mathcal{B}''$ as $T''(F) = F \circ T'$. Then by [3, p. 251] both $T'' : (\mathcal{A}'', \square) \rightarrow (\mathcal{B}'', \square)$ and $T'' : (\mathcal{A}'', \diamond) \rightarrow (\mathcal{B}'', \diamond)$ are continuous homomorphisms. If $\|T\| \leq 1$, then $\|T''\| \leq 1$ in both the cases.

Let \mathcal{A} be a commutative Banach algebra. Then \mathcal{A} is Arens regular if and only if (\mathcal{A}'', \square) is commutative [3].

THEOREM 3.1. *Let \mathcal{A} and \mathcal{B} be Banach algebras, and let \mathcal{A} be commutative and Arens regular. Let $T : \mathcal{B} \rightarrow \mathcal{A}$ be an algebra homomorphism with norm at most 1.*

- (1) *Suppose that $\mathcal{A}'', \mathcal{B}'',$ and $(\mathcal{A} \times_T \mathcal{B})''$ are equipped with their first (respectively, second) Arens products. Then*

$$(\mathcal{A} \times_T \mathcal{B})'' \cong \mathcal{A}'' \times_{T''} \mathcal{B}'' \quad (\text{isometric isomorphism}).$$

- (2) *Let \mathfrak{Z}_i be either a left or a right topological centre of \mathcal{A}'' . Then $\mathfrak{Z}_i((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{A}'' \times_{T''} \mathfrak{Z}_i(\mathcal{B}'')$. In particular, $\mathcal{A} \times_T \mathcal{B}$ is Arens regular if and only if \mathcal{B} is Arens regular.*

PROOF. (1) Since \mathcal{A} is commutative and Arens regular, (\mathcal{A}'', \square) is commutative. The first Arens product on $\mathcal{A}'' \times_{T''} \mathcal{B}''$ is given as follows. Let $(\Phi, \Psi), (\Phi', \Psi') \in \mathcal{A}'' \times_{T''} \mathcal{B}''$. Then

$$(\Phi, \Psi)(\Phi', \Psi') := (\Phi \square \Phi' + T''(\Psi') \square \Phi + T''(\Psi) \square \Phi', \Psi \square \Psi'). \tag{3.1}$$

We compute the Arens product \square on $(\mathcal{A} \times_T \mathcal{B})''$. For this purpose let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$, $(\phi, \psi) \in \mathcal{A}' \times \mathcal{B}'$, and $(\Phi, \Psi), (\Phi', \Psi') \in \mathcal{A}'' \times \mathcal{B}''$. Let $(a', b') \in \mathcal{A} \times_T \mathcal{B}$. Then

$$\begin{aligned} ((\phi, \psi) \cdot (a, b))(a', b') &= (\phi, \psi)((a, b)(a', b')) \\ &= (\phi, \psi)(aa' + T(b)a' + T(b')a, bb') \\ &= \phi(aa' + T(b)a' + T(b')a) + \psi(bb') \\ &= (\phi \cdot a + \phi \cdot T(b))(a') + (T'(\phi \cdot a) + \psi \cdot b)(b') \\ &= (\phi \cdot a + \phi \cdot T(b), T'(\phi \cdot a) + \psi \cdot b)(a', b'). \end{aligned}$$

Therefore

$$(\phi, \psi) \cdot (a, b) = (\phi \cdot a + \phi \cdot T(b), T'(\phi \cdot a) + \psi \cdot b).$$

Also

$$\begin{aligned} ((\Phi, \Psi) \cdot (\phi, \psi))(a, b) &= (\Phi, \Psi)((\phi, \psi) \cdot (a, b)) \\ &= (\Phi, \Psi)(\phi \cdot a + \phi \cdot T(b), T'(\phi \cdot a) + \psi \cdot b) \\ &= \Phi(\phi \cdot a + \phi \cdot T(b)) + \Psi(T'(\phi \cdot a) + \psi \cdot b) \\ &= \Phi(\phi \cdot a + \phi \cdot T(b)) + \Psi \circ T'(\phi \cdot a) + \Psi(\psi \cdot b) \\ &= \Phi(\phi \cdot a + \phi \cdot T(b)) + T''(\Psi)(\phi \cdot a) + \Psi(\psi \cdot b) \\ &= (\Phi \cdot \phi + T''(\Psi) \cdot \phi)(a) + (T'(\Phi \cdot \phi) + \Psi \cdot \psi)(b) \\ &= (\Phi \cdot \phi + T''(\Psi) \cdot \phi, T'(\Phi \cdot \phi) + \Psi \cdot \psi)(a, b). \end{aligned}$$

Therefore

$$(\Phi, \Psi) \cdot (\phi, \psi) = (\Phi \cdot \phi + T''(\Psi) \cdot \phi, T'(\Phi \cdot \phi) + \Psi \cdot \psi).$$

Now

$$\begin{aligned} ((\Phi, \Psi) \square (\Phi', \Psi'))(\phi, \psi) &= (\Phi, \Psi)((\Phi', \Psi') \cdot (\phi, \psi)) \\ &= (\Phi, \Psi)(\Phi' \cdot \phi + T''(\Psi') \cdot \phi, T'(\Phi' \cdot \phi) + \Psi' \cdot \psi) \\ &= \Phi(\Phi' \cdot \phi + T''(\Psi') \cdot \phi) + \Psi(T'(\Phi' \cdot \phi) + \Psi' \cdot \psi) \\ &= \Phi(\Phi' \cdot \phi + T''(\Psi') \cdot \phi) + T''(\Psi)(\Phi' \cdot \phi) + (\Psi' \cdot \psi) \\ &= (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi))(\phi) + (\Psi \square \Psi')(\psi) \\ &= (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi), \Psi \square \Psi')(\phi, \psi). \end{aligned}$$

Therefore

$$(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi), \Psi \square \Psi'),$$

which is the same as (3.1). Calculations for the second Arens product are analogous.

(2) Since \mathcal{A} is commutative and Arens regular, $\mathcal{A}'' = \mathcal{Z}^{(\ell)}(\mathcal{A}'') = \mathcal{Z}^{(r)}(\mathcal{A}'')$. Let $(\Phi, \Psi) \in \mathcal{Z}^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'')$. Then for any $(\Phi', \Psi') \in \mathcal{A}'' \times_{T''} \mathcal{B}''$, $(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi, \Psi) \diamond (\Phi', \Psi')$, that is,

$$\begin{aligned} (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi), \Psi \square \Psi') \\ = (\Phi \diamond \Phi' + \Phi \diamond T''(\Psi') + \Phi' \diamond T''(\Psi), \Psi \diamond \Psi'). \end{aligned}$$

In particular, $\Psi \square \Psi' = \Psi \diamond \Psi'$ for every $\Psi' \in \mathcal{B}''$. Therefore

$$\mathcal{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'') \subset \mathcal{A}'' \times_{T''} \mathcal{Z}^{(\ell)}(\mathcal{B}'').$$

Conversely, assume that $(\Phi, \Psi) \in \mathcal{A}'' \times_{T''} \mathcal{Z}^{(\ell)}(\mathcal{B}'')$. Since $\mathcal{A}'' = \mathcal{Z}^{(\ell)}(\mathcal{A}'')$, it follows that $(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi, \Psi) \diamond (\Phi', \Psi')$ for every $(\Phi', \Psi') \in \mathcal{A}'' \times_{T''} \mathcal{B}''$, that is, $(\Phi, \Psi) \in \mathcal{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'')$. Therefore $\mathcal{A}'' \times_{T''} \mathcal{Z}^{(\ell)}(\mathcal{B}'') \subset \mathcal{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'')$. This means that $\mathcal{Z}^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{A}'' \times_{T''} \mathcal{Z}^{(\ell)}(\mathcal{B}'')$. It also follows that $\mathcal{A} \times_T \mathcal{B}$ is Arens regular if and only if \mathcal{B} is regular. \square

4. Amenability

Let \mathcal{A} be a Banach algebra, and let E be a Banach \mathcal{A} -module. A *bounded E-derivation* is a bounded linear map $D : \mathcal{A} \rightarrow E$ such that

$$D(ab) = (Da) \cdot b + a \cdot (Db) \quad (a, b \in \mathcal{A}).$$

The set of all bounded E -derivations on \mathcal{A} is denoted by $\mathcal{Z}^1(\mathcal{A}, E)$.

Given $x \in E$, let $\delta_x : \mathcal{A} \rightarrow E$ be given by $\delta_x(a) = a \cdot x - x \cdot a$ ($a \in \mathcal{A}$). Then $\delta_x \in \mathcal{Z}^1(\mathcal{A}, E)$. The derivation δ_x is called an *inner E-derivation*. Let $\mathcal{B}^1(\mathcal{A}, E)$ be the set of all inner E -derivations. Let

$$\mathcal{H}^1(\mathcal{A}, E) = \mathcal{Z}^1(\mathcal{A}, E) / \mathcal{B}^1(\mathcal{A}, E).$$

Then $\mathcal{H}^1(\mathcal{A}, E)$ is the *first cohomology group of \mathcal{A} with coefficients in E* .

A Banach algebra \mathcal{A} is *amenable* if $\mathcal{H}^1(\mathcal{A}, E') = \{0\}$ for every Banach \mathcal{A} -module E and it is *weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}') = \{0\}$. By [3, Proposition 2.8.59], a Banach algebra \mathcal{A} is amenable if and only if for each Banach \mathcal{A} -module E and each $D \in \mathcal{Z}^1(\mathcal{A}, E)$, there exists a bounded net (x_α) in E such that

$$D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

If I is a closed ideal of \mathcal{A} , then, by [3, Proposition 2.8.66], \mathcal{A} is amenable if I and \mathcal{A}/I are amenable and \mathcal{A} is weakly amenable if I and \mathcal{A}/I are weakly amenable.

A derivation $D : \mathcal{A} \rightarrow E$ is *approximately inner* if there exists a net $(x_\alpha) \subset E$ such that $D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a)$ ($a \in \mathcal{A}$). The algebra \mathcal{A} is *approximately amenable* if for each Banach \mathcal{A} -module E every bounded derivation $D : \mathcal{A} \rightarrow E$ is approximately inner and \mathcal{A} is *approximately weakly amenable* if every bounded derivation $D : \mathcal{A} \rightarrow \mathcal{A}'$ is approximately inner. By [6, Proposition 2.2 (iii)], if I is a closed ideal in a Banach algebra \mathcal{A} , I is weakly amenable and \mathcal{A}/I is approximately weakly amenable, then \mathcal{A} is approximately weakly amenable.

A derivation $D : \mathcal{A} \rightarrow \mathcal{A}'$ is *cyclic* if $D(a)(b) + D(b)(a) = 0$ ($a, b \in \mathcal{A}$). Every inner derivation from \mathcal{A} to \mathcal{A}' is cyclic. A Banach algebra \mathcal{A} is *cyclic amenable* if every cyclic derivation is inner.

A Banach algebra \mathcal{A} is *approximately cyclic amenable* if every cyclic derivation $D : \mathcal{A} \rightarrow \mathcal{A}'$ is approximately inner.

The following theorem exhibits the stability of amenability with respect to the product \times_T . Looking to the elementary nature of product \times_T , the proof is quite elementary, avoiding deeper results in amenability such as its functorial properties [15].

THEOREM 4.1. *Let \mathcal{A} be a commutative Banach algebra, let \mathcal{B} be a Banach algebra, and let $T : \mathcal{B} \rightarrow \mathcal{A}$ be an algebra homomorphism with $\|T\| \leq 1$. Then:*

- (1) $\mathcal{A} \times_T \mathcal{B}$ is amenable if and only if both \mathcal{A} and \mathcal{B} are amenable;
- (2) $\mathcal{A} \times_T \mathcal{B}$ is weakly amenable if and only if both \mathcal{A} and \mathcal{B} are weakly amenable;
- (3) $\mathcal{A} \times_T \mathcal{B}$ is approximately weakly amenable if and only if both \mathcal{A} and \mathcal{B} are approximately weakly amenable;
- (4) $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable if and only if both \mathcal{A} and \mathcal{B} are cyclic amenable;
- (5) $\mathcal{A} \times_T \mathcal{B}$ is approximately cyclic amenable if and only if both \mathcal{A} and \mathcal{B} are approximately cyclic amenable.

PROOF. (1) Assume that both \mathcal{A} and \mathcal{B} are amenable. Since I is a closed ideal in $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/I \cong \mathcal{B}$, $\mathcal{A} \times_T \mathcal{B}$ is amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is amenable. Let E be a Banach \mathcal{A} -module, and let $d : \mathcal{A} \rightarrow E$ be a bounded derivation. We may consider this map as $d : \mathcal{A} \times \{0\} \rightarrow E \times \{0\}$. We note that $E \times \{0\}$ is a Banach $\mathcal{A} \times_T \mathcal{B}$ -module. Let $P : \mathcal{A} \times_T \mathcal{B}$ be defined as

$$P(a, b) = (a + T(b), 0) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Then $D = d \circ P : \mathcal{A} \times_T \mathcal{B} \rightarrow E \times \{0\}$ is a bounded derivation on $\mathcal{A} \times_T \mathcal{B}$. Since $\mathcal{A} \times_T \mathcal{B}$ is amenable, there exists a bounded net $((x_\alpha, 0))$ in $E \times \{0\}$ such that

$$D(a, b) = \lim_{\alpha} ((a, b) \cdot (x_\alpha, 0) - (x_\alpha, 0) \cdot (a, b)) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

In particular,

$$\begin{aligned} d(a) &= d(a, 0) = D(a, 0) = \lim_{\alpha} ((a, 0) \cdot (x_\alpha, 0) - (x_\alpha, 0) \cdot (a, 0)) \\ &= \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a). \end{aligned}$$

Hence \mathcal{A} is amenable.

Similarly, by taking $P : \mathcal{A} \times_T \mathcal{B} \rightarrow \{0\} \times E$ as $P(a, b) = (0, b)$ it follows that \mathcal{B} is amenable.

(2) Assume that both \mathcal{A} and \mathcal{B} are weakly amenable. Since I is a closed ideal in $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/I \cong \mathcal{B}$, $\mathcal{A} \times_T \mathcal{B}$ is weakly amenable.

Conversely, let $\mathcal{A} \times_T \mathcal{B}$ be weakly amenable. Let $d : \mathcal{A} \rightarrow \mathcal{A}'$ be a continuous derivation. Let $P : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}$ be defined as

$$P(a, b) = a + T(b) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $D = P' \circ d \circ P$. Then D is a derivation of $\mathcal{A} \times_T \mathcal{B}$ to $\mathcal{A}' \times \mathcal{B}'$. Since $\mathcal{A} \times_T \mathcal{B}$ is weakly amenable, there exists (φ, ψ) in $\mathcal{A}' \times \mathcal{B}'$ such that $D = \delta_{(\varphi, \psi)}$.

Let $a, a' \in \mathcal{A}$. Then

$$\begin{aligned} \langle d(a), a' \rangle &= \langle P' \circ dP(a, 0), (a', 0) \rangle = \langle (a, 0) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (a, 0), (a', 0) \rangle \\ &= \langle a \cdot \varphi - \varphi \cdot a, a' \rangle = 0 \quad (\text{as } \mathcal{A} \text{ is commutative}). \end{aligned}$$

Hence $\mathcal{Z}^1(\mathcal{A}, \mathcal{A}') = \{0\}$, that is, \mathcal{A} is weakly amenable.

Let $d : \mathcal{B} \rightarrow \mathcal{B}'$ be a continuous derivation. Let $P : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{B}$ be defined as

$$P(a, b) = b \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $D = P' \circ d \circ P$. Then D is a derivation of $\mathcal{A} \times_T \mathcal{B}$ to $\mathcal{A}' \times \mathcal{B}'$. Since $\mathcal{A} \times_T \mathcal{B}$ is weakly amenable, there exists $(\varphi, \psi) \in \mathcal{A}' \times \mathcal{B}'$ such that $D = \delta_{(\varphi, \psi)}$.

Let $b, b' \in \mathcal{B}$. Then

$$\begin{aligned} \langle d(b), b' \rangle &= \langle P' \circ d \circ P(0, b), (0, b') \rangle \\ &= \langle (0, b) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (0, b), (0, b') \rangle \\ &= \langle b \cdot \psi - \psi \cdot b, b' \rangle. \end{aligned}$$

Hence $d = \delta_\psi$. Therefore \mathcal{B} is weakly amenable.

(3) Assume that both \mathcal{A} and \mathcal{B} are approximately weakly amenable. Since \mathcal{A} is commutative, it is weakly amenable. Since \mathcal{A} is a closed ideal in $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A} \cong \mathcal{B}$, $\mathcal{A} \times_T \mathcal{B}$ is approximately weakly amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is approximately weakly amenable. Let $d : \mathcal{A} \rightarrow \mathcal{A}'$ be a continuous derivation. Then $D = P' \circ d \circ P$ is a continuous derivation of $\mathcal{A} \times_T \mathcal{B}$. Since $\mathcal{A} \times_T \mathcal{B}$ is weakly approximately amenable, there exists a net $((\varphi_\alpha, \psi_\alpha)) \subset \mathcal{A}' \times \mathcal{B}'$ such that

$$D(a, b) = \lim_{\alpha} ((a, b) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, b)) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Now

$$\begin{aligned} \langle d(a), a' \rangle &= \langle D(a, 0), (a', 0) \rangle \\ &= \langle \lim_{\alpha} ((a, 0) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, 0)), (a', 0) \rangle \\ &= \langle \lim_{\alpha} (a \cdot \varphi_\alpha - \varphi_\alpha \cdot a), a' \rangle = 0. \end{aligned}$$

Therefore $d = 0$. Hence \mathcal{A} is approximately weakly amenable.

The proof of approximately weak amenability of \mathcal{B} is analogous to the above.

(4) Let \mathcal{A} and \mathcal{B} be cyclic amenable. Let $D : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}' \times \mathcal{B}'$ be a bounded cyclic derivation. Then $D|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}'$ and $D|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}'$ are cyclic derivations. Since \mathcal{A} is commutative, $D|_{\mathcal{A}} = 0$, and since \mathcal{B} is cyclic amenable, there exists $\psi \in \mathcal{B}'$ such that $D|_{\mathcal{B}}(0, b) = (0, b) \cdot (0, \psi) - (0, \psi) \cdot (0, b)$ ($b \in \mathcal{B}$). Let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$. Then

$$\begin{aligned} D(a, b) &= D(a, 0) + D(0, b) \\ &= b \cdot \psi - \psi \cdot b \\ &= (a, b) \cdot (0, \psi) - (0, \psi) \cdot (a, b). \end{aligned}$$

Therefore $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable. Let $d : \mathcal{A} \rightarrow \mathcal{A}'$ be a cyclic derivation. Then d can be considered as a map $d : \mathcal{A} \times \{0\} \rightarrow \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$. Let $P : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}$ be defined as $P(a, b) = (a + T(b), 0)$ ($(a, b) \in \mathcal{A} \times_T \mathcal{B}$). Then $d \circ P : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$ is a cyclic derivation. Since $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable, there exists $(\varphi, \psi) \in \mathcal{A}' \times \mathcal{B}'$ such that

$$d \circ P(a, b) = (a, b) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (a, b) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $a \in \mathcal{A}$. Then

$$\begin{aligned} d(a) &= d \circ P(a, 0) = (a, 0) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (a, 0) \\ &= a \cdot \varphi - \varphi \cdot a = 0 \quad (\text{as } \mathcal{A} \text{ is commutative}). \end{aligned}$$

Therefore \mathcal{A} is cyclic amenable. Similarly, one can show that \mathcal{B} is cyclic amenable.

(5) Let \mathcal{A} and \mathcal{B} be approximately cyclic amenable. Let $D : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}' \times \mathcal{B}'$ be a bounded cyclic derivation. Then $D|_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}'$ and $D|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}'$ are cyclic

derivations. Since \mathcal{A} is commutative, $D|_{\mathcal{A}} = 0$, and since \mathcal{B} is approximately cyclic amenable, there exists a net (ψ_α) in \mathcal{B}' such that $D|_{\mathcal{B}}(0, b) = \lim_\alpha ((0, b) \cdot (0, \psi_\alpha) - (0, \psi_\alpha) \cdot (0, b))$ ($b \in \mathcal{B}$). Let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$. Then

$$\begin{aligned} D(a, b) &= D(a, 0) + D(0, b) \\ &= \lim_\alpha (b \cdot \psi_\alpha - \psi_\alpha \cdot b) \\ &= \lim_\alpha ((a, b) \cdot (0, \psi_\alpha) - (0, \psi_\alpha) \cdot (a, b)). \end{aligned}$$

Therefore $\mathcal{A} \times_T \mathcal{B}$ is approximately cyclic amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is approximately cyclic amenable. Let $d : \mathcal{A} \rightarrow \mathcal{A}'$ be a cyclic derivation. Then d can be considered as a map $d : \mathcal{A} \times \{0\} \rightarrow \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$. Let $P : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}$ be defined as $P(a, b) = (a + T(b), 0)$ ($(a, b) \in \mathcal{A} \times_T \mathcal{B}$). Then $d \circ P : \mathcal{A} \times_T \mathcal{B} \rightarrow \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$ is a cyclic derivation. Since $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable, there exists a net $(\varphi_\alpha, \psi_\alpha)$ in $\mathcal{A}' \times \mathcal{B}'$ such that

$$d \circ P(a, b) = \lim_\alpha ((a, b) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, b)) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $a \in \mathcal{A}$. Then

$$\begin{aligned} d(a) &= d \circ P(a, 0) = \lim_\alpha ((a, 0) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, 0)) \\ &= \lim_\alpha (a \cdot \varphi_\alpha - \varphi_\alpha \cdot a) = 0 \quad (\text{as } \mathcal{A} \text{ is commutative}). \end{aligned}$$

Therefore \mathcal{A} is approximately cyclic amenable. Similarly, one can show that \mathcal{B} is approximately cyclic amenable. □

There is a localised version of amenability of much recent interest, namely character amenability [7, 9, 10, 14]. Let $\varphi \in \Delta(\mathcal{A})$. Following [5], \mathcal{A} is φ -inner amenable if there exists $m \in \mathcal{A}''$ such that $m(\varphi) = 1$ and $m \square a = a \square m$ ($a \in \mathcal{A}$). Such an m is called φ -inner mean. A Banach algebra \mathcal{A} is character inner amenable if \mathcal{A} is φ -inner amenable for all $\varphi \in \Delta(\mathcal{A})$. We note that every commutative Banach algebra is character inner amenable.

THEOREM 4.2. *Let \mathcal{A} be a commutative Banach algebra, \mathcal{B} be a Banach algebra, and let $T : \mathcal{B} \rightarrow \mathcal{A}$ be an algebra homomorphism with $\|T\| \leq 1$.*

- (1) $\mathcal{A} \times_T \mathcal{B}$ is $(\varphi, \varphi \circ T)$ -inner amenable for all $\varphi \in \Delta(\mathcal{A})$. If (m, n) is a $(\varphi, \varphi \circ T)$ -inner mean and $n(\varphi \circ T) \neq 0$, then \mathcal{B} is $\varphi \circ T$ -inner amenable.
- (2) For all $\psi \in \Delta(\mathcal{B})$, $\mathcal{A} \times_T \mathcal{B}$ is $(0, \psi)$ -inner amenable if and only if \mathcal{B} is ψ -inner amenable.
- (3) $\mathcal{A} \times_T \mathcal{B}$ is character inner amenable if and only if \mathcal{B} is character inner amenable.

PROOF. (1) Let $\varphi \in \Delta(\mathcal{A})$. Since \mathcal{A} is commutative, there exists $m \in \mathcal{A}''$ such that $m(\varphi) = 1$ and $m \square a = a \square m$ for all $a \in \mathcal{A}$. Then $(m, 0)(\varphi, \varphi \circ T) = 1$ and since \mathcal{A} is commutative, $(m, 0) \square (a, b) = (a, b) \square (m, 0)$ for all $(a, b) \in \mathcal{A} \times_T \mathcal{B}$, that is, $\mathcal{A} \times_T \mathcal{B}$ is $(\varphi, \varphi \circ T)$ -inner amenable.

Let (m, n) be a $(\varphi, \varphi \circ T)$ -inner mean and $n(\varphi \circ T) \neq 0$. Then it follows that $(n/n(\varphi \circ T))(\varphi \circ T) = 1$ and $(n/n(\varphi \circ T)) \square b = b \square (n/n(\varphi \circ T))$ for all $b \in \mathcal{B}$, that is, \mathcal{B} is $\varphi \circ T$ -inner amenable.

(2) The proof is analogous to that of (1).

(3) Assume that \mathcal{B} is character amenable. Let $(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$. First let $\varphi = 0$. Since \mathcal{B} is character amenable, there exists $n \in \mathcal{B}''$ such that $n(\psi) = 1$ and $n \square b = b \square n$ for all $b \in \mathcal{B}$. Now

$$(0, n)((0, \psi)) = n(\psi) = 1, \quad (0, n) \square (a, b) = (T''(n) \square a, n \square b)$$

and

$$(a, b) \square (0, n) = (T''(n) \square a, b \square n).$$

Since \mathcal{A} is commutative, $m \square a = a \square m$ for every $m \in \mathcal{A}''$ and $a \in \mathcal{A}$. Hence $(0, n)$ is a $(0, \psi)$ -inner mean for $\mathcal{A} \times_T \mathcal{B}$.

Second, assume that $\varphi \neq 0$. Then $(\varphi, \psi) = (\varphi, \varphi \circ T)$. Since \mathcal{A} is character inner amenable, there exists $m \in \mathcal{A}''$ such that $m(\varphi) = 1$ and $m \square a = a \square m$ for every $a \in \mathcal{A}$. Now

$$(m, 0)((\varphi, \varphi \circ T)) = m(\varphi) = 1, \\ (m, 0) \square (a, b) = (m \square a + T(b) \square m, 0) = (a, b) \square (m, 0).$$

Hence $\mathcal{A} \times_T \mathcal{B}$ is character inner amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is character inner amenable. Let $\psi \in \Delta(\mathcal{B})$. Then there exists $(m, n) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ such that $(m, n)((0, \psi)) = n(\psi) = 1$ and $(m, n) \square (a, b) = (a, b) \square (m, n)$ ($(a, b) \in \mathcal{A} \times_T \mathcal{B}$). It follows that $n(\psi) = 1$ and $n \square b = b \square n$ ($b \in \mathcal{B}$), that is, n is a ψ -inner mean for \mathcal{B} . \square

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