BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 1, January 1987

ARGUESIAN LATTICES WHICH ARE NOT LINEAR

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ABSTRACT. A *linear* lattice is one representable by commuting equivalence relations. We construct a sequence of finite lattices A_n ($n \geq 3$) with the properties: (i) A_n is not linear, (ii) every proper sublattice of A_n is linear, and (iii) any set of generators for A_n has at least n elements. In particular, A_n is then Arguesian for $n \geq 7$. This settles a question raised in 1953 by Jónsson.

1. Introduction. A lattice L is *linear* if it is representable by commuting equivalence relations. Jónsson [**6**] showed that any such lattice is Arguesian. Numerous equivalent forms of the Arguesian law are now known; it is a strong condition with important applications in coordinatization theory [**1**, **2**]. Nevertheless, the question raised by Jónsson, whether every Arguesian lattice is linear, has remained open until now.

Here we describe an infinite family $\{A_n\}$ ($n \geq 3$) of nonlinear lattices, Arguesian for $n \geq 7$ (and possibly for $n \geq 4$), settling Jónsson's question in the negative. Actually, we obtain more: a specific infinite sequence of identities strictly between Arguesian and linear, and a proof that the universal Horn theory of linear lattices is not finitely based.

2. The lattices A_n. Let $n \geq 3$. In what follows, all indices are modulo n, i.e., x_{i+1} means x_0 when $i = n-1$, etc. Let L_n be the lattice of all subspaces of a vector space v ($\dim v = 2n$) over a prime field $\mathbf{K}$ with at least 3 elements. Let $\{\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1}\}$ be a basis of v. Let

$$(1) \qquad m = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle, \qquad q_i = \langle \{\alpha_j | j \neq i\} \rangle, \qquad p_i = q_i \wedge q_{i+1},$$
$$r_i = m \vee \langle \beta_i \rangle, \qquad s_i = r_{i-1} \vee r_i,$$

where $\langle \cdots \rangle$ denotes linear span. Let

$$(2) \qquad \tilde{A}_n = [0, m] \cup [m, v] \cup \bigcup_i [p_i, r_i] \cup \bigcup_i [q_i, s_i],$$

where $[x, y] = \{z | x \leq z \leq y\}$.

$\tilde{A}_n \subset L_n$ is a sublattice; the intervals in the union (2) are its maximal complemented intervals, or *blocks*; they are the blocks of a tolerance relation on $\tilde{A}_n$ [**5**]; as such, the set S of blocks acquires a lattice structure; specifically, $0_S = [0, m]$, $1_S = [m, v]$, $a_i = [p_i, r_i]$ are atoms, $b_i = [q_i, s_i]$ are coatoms, and $a_i < b_i, b_{i+1}$ defines the order relation.

Let $\overline{m}$ ($\dim \overline{m} = n$) be another vector space, with basis $\{\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1}\}$. Define $\overline{p}_i$, $\overline{q}_i$ by analogy with (1). Let $F = \bigcup_i [p_i, v]$; $F \subset \tilde{A}_n$ is an order filter. Within F, $\bigcup_i [p_i, m]$ is an order ideal. Set up a "twisting" isomorphism

Received by the editors March 25, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 06C05.

τ of $\bigcup_i [p_i, m]$ with the order filter $\bigcup_i [\overline{p}_i, \overline{m}] \subset [0, \overline{m}]$ as follows: for each i, the atoms of $[\overline{p}_i, \overline{m}]$ are of the form $\langle r\overline{\alpha}_i + s\overline{\alpha}_{i+1}, \overline{\alpha}_{i+2}, \dots, \overline{\alpha}_{i-1}\rangle$ where $(r : s)$ is a ratio of elements of $\mathbf{K}$. Put $\tau(\overline{p}_i) = p_i$, $\tau(\overline{m}) = m$, and $\tau(\langle r\overline{\alpha}_i + s\overline{\alpha}_{i+1}, \overline{\alpha}_{i+2}, \dots, \overline{\alpha}_{i-1}\rangle) = \langle r\alpha_i + s\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_{i-1}\rangle$ *except, when* $i = 0$, *put*

$$\tau(\langle r\overline{\alpha}_0 + s\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_{n-1}\rangle) = \langle -r\alpha_0 + s\alpha_1, \alpha_2, \dots, \alpha_{n-1}\rangle.$$

This definition is consistent on $\overline{q}_i$ and makes $\tau(\overline{q}_i) = q_i$.

Let

$$A_n = F \cup [0, \overline{m}]/(x = \tau(x))_{x \in \bigcup_i [\overline{p}_i, \overline{m}]}.$$

A_n is a modular lattice and has the same block decomposition (2) as $\tilde{A}_n$, hence the same skeleton lattice S. Composing τ with the automorphism of $[0, \overline{m}]$ induced by the linear transformation $\overline{\alpha}_1 \mapsto -\overline{\alpha}_1, \dots, \overline{\alpha}_k \mapsto -\overline{\alpha}_k$, other $\overline{\alpha}_i$ fixed, shows that the exceptional interval $[\overline{p}_0, \overline{m}]$ in the definition of τ could as well have been $[\overline{p}_k, \overline{m}]$, up to an isomorphism of A_n respecting the p_i, q_i, r_i, s_i.

3. Properties of A_n.

THEOREM. *A_n is not a linear lattice.*

PROOF. In [**3**], the author introduced "higher Arguesian identities"

$$D_n\colon\quad a_0 \wedge \Big(a_0' \vee \bigwedge_{i=1}^{n-1} [a_i \vee a_i'] \Big)$$
$$\leq a_1 \vee \Big((a_0' \vee a_1') \wedge \bigvee_{i=1}^{n-1} [(a_i \vee a_{i+1}) \wedge (a_i' \vee a_{i+1}')] \Big)$$

which hold in all linear lattices. D_3 is the Arguesian law [**4**]. If we take $a_i = p_i + \langle \beta_i \rangle$ for all i, $a_i' = p_i + \langle \beta_i + \alpha_i + \alpha_{i+1}\rangle$ for $i \neq 0$, and $a_0' = p_0 + \langle \beta_0 - \alpha_0 + \alpha_1\rangle$, D_n fails in A_n. In particular, A_3 is not Arguesian. This minimally non-Arguesian lattice was discovered by Pickering [**8**].

THEOREM. *Every proper sublattice of A_n is linear.*

PROOF. $\bigcup_i [p_i, r_i]$ generates A_n, so a proper sublattice $N \subset A_n$ will have $N \cap [p_i, r_i] \subset [p_i, r_i]$ strictly for some i. We can assume $[p_i, m]$ is the exceptional interval in the definition of τ. We show $[p_i, r_i]$ (which is a projective plane over $\mathbf{K}$) possesses an automorphism fixing $N \cap [q_i, r_i]$ and $N \cap [q_{i+1}, r_i]$ and acting as τ on $N \cap [p_i, m]$. This is proved by classifying maximal proper sublattices of $[p_i, r_i]$ and their possible orientations relative to m, q_i, q_{i+1}, which leads to 13 cases to check, some trivial, none difficult.

It follows that $\tilde{A}_n$ has a sublattice isomorphic to N, so N is linear.

THEOREM. *If $X \subseteq A_n$ generates A_n, then $|X| \geq n$.*

PROOF. For each j, $0_S \cup 1_S \cup \bigcup_{i \neq j} a_i \cup \bigcup_{i \neq j} b_i$ is a sublattice of A_n because $\{0_S, 1_S\} \cup \{a_i, b_i \mid i \neq j\}$ is a sublattice of S. For each j, therefore, some $x_j \in X$ is an element of $a_j \cup b_j$ and not an element of any other block. This requires n distinct elements of X.

4. Conclusions. The results of §3 imply that no finite set of identities, or even universal Horn sentences, can completely characterize linearity; in particular, the Arguesian law is insufficient, since it holds in A_n for $n \geq 7$. It is known, however, how to characterize linear lattices by an *infinite* set of universal Horn sentences [**3**, **7**].

If, as appears likely, the identity D_{n-1} holds in A_n $(n \geq 4)$, we would have that D_{n-1} does not imply D_n, showing that $\{D_n\}$ forms a hierarchy of progressively strictly stronger linear lattice identities. We remark that generator-counting will not suffice for this, since A_n has a set of generators X with $|X| = n + 3$. We conjecture $n + 3$ is minimal, which would imply A_4 is Arguesian.

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