JOHN CORCORAN

ARISTOTLE'S NATURAL DEDUCTION SYSTEM

> Here and elsewhere we shall not obtain the best insight into things until we actually see them growing from the beginning.
>
> *Aristotle*

In the present article we attempt to show that Aristotle's syllogistic is an *underlying logic* which includes a natural deductive system and that it is not an axiomatic theory as had previously been thought. We construct a mathematical model which reflects certain structural aspects of Aristotle's logic and we examine both the mathematical properties of the model and the relation of the model to the system of logic envisaged in certain scattered parts of *Prior* and *Posterior Analytics*.

Our interpretation restores Aristotle's reputation as a logician of consummate imagination and skill. Several attributions of shortcomings and logical errors to Aristotle are shown to be without merit. Aristotle's logic is found to be self-sufficient in several senses. In the first place, his theory of deduction is logically sound in every detail. (His indirect deductions have been criticized, but incorrectly on our account.) In the second place, Aristotle's logic presupposes no other logical concepts, not even those of propositional logic. In the third place, the Aristotelian system is seen to be complete in the sense that every valid argument expressible in his system admits of a deduction within his deductive system; i.e., every semantically valid argument is deducible.

There are six sections in this article. The first section includes methodological remarks, a preliminary survey of the present interpretation and a discussion of the differences between our interpretation and that of Łukasiewicz. The next three sections develop the three parts of the mathematical model. The fifth section deals with general properties of the model and its relation to the Aristotelian system. The final section contains conclusions.

J. Corcoran (ed.), Ancient Logic and Its Modern Interpretations, 85–131.

1. Preliminaries

1.1. *Mathematical Logics*

Logicians are beginning to view mathematical logic as a branch of applied mathematics which constructs and studies mathematical models in order to gain understanding of logical phenomena. From this standpoint mathematical logics are comparable to the mathematical models of solar systems, vibrating strings, or atoms in mathematical physics and to the mathematical models of computers in automata theory[1] (cf. Kreisel, p. 204). Thus one thinks of mathematical logics as mathematical models of real or idealized logical systems.

In the most common case a mathematical logic can be thought of as a mathematical model composed of three interrelated parts: a 'language', a 'deductive system' and a 'semantics'. The language is a syntactical system often designed to reflect what has been called the logical form of propositions (cf. Church, pp. 2, 3). The elements of the language are called sentences. The deductive system, another syntactical system, contains elements sometimes called formal proofs or formal deductions. These elements usually involve sequences of sentences constructed in accord with syntactical rules themselves designed to reflect actual or idealized principles of reasoning (cf. Church, pp. 49–54). Finally, the semantics is usually a set-theoretic structure intended to model certain aspects of meaning (cf. Church, pp. 54ff), e.g., how denotations attach to noun phrases and how truth-values attach to sentences.[2]

Many theories of logic involve a theory of propositional forms, a theory of deductive reasoning and a theory of meaning (cf. Church, pp. 1, 3, 23). Such theories are intended to account for logical phenomena relating to a natural language or to an ideal language perhaps alleged to underlie natural language, or even to an artificial language proposed as a substitute for natural language. In any case, it is often possible to construct a mathematical model which reflects many of the structural aspects of 'the system' envisaged in the theory. Once a mathematical logic has been constructed, it is possible to ask definite, well-defined questions concerning how well, or to what degree and in what respects, the model reflects the structure of 'the system' envisaged by the theory. Such activity usually contributes toward the clarification of the theory in question. Indeed any attempt to construct such a model necessarily involves an organized and detailed study of the theory and often raises questions not considered by the author of the theory.

1.1.1. *Underlying logics.* Because some articulations of the above viewpoint admit of certain misunderstandings, a few further comments may be in order. Consider a deductive science such as geometry. We may imagine that geometry presupposes its own subject matter which gives rise to its own laws, some of which are taken without deductive justification. In addition, geometry presupposes a geometrical language. The activity of deductively justifying some laws on the basis of others further presupposes a system of demonstrative discourses (the deductions). The activity of establishing by means of reinterpretations of the language of geometry that certain geometrical statements are independent of others further presupposes a system of reinterpretations of the language. The last three presupposed systems taken together from the *underlying logic* (cf. Church, p. 58, 317; Tarski, p. 297) of geometry.

Although the underlying logic is not a science it can be the subject matter of a scientific investigation. Of course, there is much more to be said about this approach to the study of deductive sciences, but what has been said should be sufficient to enable the reader to see that there is a clear distinction to be made between *logic* as a scientific study of underlying logics on one hand, and the *underlying logic* of a science on the other. It is roughly the difference between zoology and fishes. A science has an underlying logic which is treated scientifically by the subject called logic. Logic, then, is a science (in our sense, not Aristotle's), but an underlying logic of a science (Aristotle's sense) is not a science; rather it is a complex, abstract system presupposed by a science. Some of the possibility for confusion could be eliminated by using the term 'science' in Aristotle's sense and the term 'metascience' to indicate activities such as logic. Then we could say that a science presupposes an underlying logic which is then studied in a metascience, viz. logic.

It is unfortunate that in a previous article (Corcoran, 'Theories') I spoke of the 'science of logic' for what I should have termed 'the metascience, logic' or 'the science of logics'. That unfortunate usage, among other things, brought about Mary Mulhern's justified criticism (cf. her paper below) to the effect that I am myself guilty of blurring a distinction which I take to be crucial to understanding Aristotle's logic (metascience).

Readers of Mulhern's article should be advised that the present paragraphs were added as a result of Mulhern's remarks, which are still important and interesting but, hopefully, no longer applicable to me.

1.2. *The Data*

In the present paper we consider only Aristotle's theory of non-modal logic, which has been called 'the theory of the assertoric syllogism' and 'Aristotle's syllogistic'. Aristotle presents the theory almost completely in Chapters 1, 2, 4, 5 and 6 of the first book of *Prior Analytics*, although it presupposes certain developments in previous works – especially the following two: first, a theory of form and meaning of propositions having an essential component in *Categories* (Chapter 5, esp. 2a34–2b7); second, a doctrine of opposition (contradiction) more fully explained in *Interpretations* (Chapter 7, and cf. Ross, p. 3). Bocheński has called this theory 'Aristotle's second logic' because it was evidently developed after the relatively immature logic of *Topics* and *Sophistical Refutations*, but before the theory of modal logic appearing mainly in Chapters 3 and 8–22 of *Prior Analytics* I. On the basis of our own investigations we have come to accept the essential correctness of Bocheński's chronology and classification of the *Organon* (Bocheński, p. 43; Łukasiewicz, p. 133; Tredennick, p. 185).

Although the theory is rather succinctly stated and developed (in five short chapters), the system of logic envisaged by it is discussed at some length and detail throughout the first book of *Prior Analytics* (esp. Chapters 7, 23–30, 42 and 45) and it is presupposed (or applied) in the first book of *Posterior Analytics*. Book II of *Prior Analytics* is not relevant to this study.

1.3. *Theories of Deduction Distinguished From Axiomatic Sciences*

We agree with Ross (p. 6), Scholz (p. 3) and many others that the theory of the categorical syllogisms is a logical theory concerned in part with deductive reasoning (as this term is normally understood). Because a recent challenge to this view has gained wide popularity (Łukasiewicz, Preface to 2nd ed.) a short discussion of the differences between a theory of deduction (whether natural or axiomatic) and an axiomatic science is necessary.

A theory of deduction puts forth a number of principles (logical axioms and rules of inferences) which describe deductions of conclusions from premises. All principles of a theory of deduction are necessarily metalinguistic – they concern constructions involving object language sentences and, as was said above, a theory of deduction is one part of a theory of logic (which deals with grammar and meaning as well). Theories of deduction (and, of course, deductive systems) have been classified as 'natural' or 'axiomatic' by means of a loose criterion based on the prominence of logical axioms as opposed to rules – the more rules the more natural, the more axioms the more axiomatic. On one extreme we find the so-called Jaskowski-type systems which have no logical axioms and which are therefore most properly called 'natural'. On the other extreme there are the so-called Hilbert-type systems which employ infinitely many axioms though only one rule and which are most properly called 'axiomatic'. The reason for the choice of the term 'natural' may be attributed to the fact that our normal reasoning seems better represented by a system in which rules predominate, whereas axiomatic systems of deduction seem contrived in comparison (cf. Corcoran, 'Theories', pp. 162–171).

A science, on the other hand, deals not with reasoning (actual or idealized) but with a certain universe or domain of objects insofar as certain properties and relations are involved. For example, arithmetic deals with the universe of numbers in regard to certain properties (odd, even, prime, perfect, etc.) and relations (less than, greater than, divides, etc.). Aristotle was clear about this (*Posterior Analytics* I, 10, 28) and modern efforts have not obscured his insights (Church, pp. 57, 317–341). The laws of a science are all stated in the object language whose non-logical constants are interpreted as indicating the required properties and relations and whose variables are interpreted as referring to objects in the universe of discourse. From the axioms of a science other laws of the science are deduced by *logical reasoning*. Thus an axiomatic science, though not itself a logical system, presupposes a logical system for its deductions (cf. Church, pp. 57, 317). The logic which is presupposed by a given science is called the *underlying logic* of the science (cf. Church, p. 58 and Tarski, p. 297).

It has been traditional procedure in the presentation of an axiomatic science to leave the underlying logic implicit. For example, neither in Euclid's geometry nor in Hilbert's does one find any codification of the logical rules used in the deduction of the theorems from the axioms and definitions. It is also worth noting that even Peano's axiomatization of arithmetic and Zermelo's axiomatization of set theory were both presented originally without explicit description of the underlying logic (cf. Church, p. 57). The need to be explicit concerning the underlying logic developed late in modern logic.

1.4. *Preliminary Discussion of the Present Interpretation*

We hold that in the above-mentioned chapters of *Prior Analytics*, Aristotle developed a logical theory which included a theory of deduction for deducing categorical conclusions from categorical premises. We further hold that Aristotle treated the logic thus developed as the underlying logic of the axiomatic sciences discussed in the first book of *Posterior Analytics*. The relation of the relevant parts of *Prior Analytics* to the first book of *Posterior Analytics* is largely the same as the relation of Church's Chapter 4, where first order logic is developed, to the part of Chapter 5 where the axiomatic science of arithmetic is developed with the preceding as its underlying logic. This interpretation properly includes the traditional view (cf. Ross, p. 6 and Scholz, p. 3) which is supported by reference to the *Analytics* as a whole as well as to crucial passages in the *Prior Analytics* where Aristotle tells what he is doing (*Prior Analytics* I, 1; and cf. Ross, p. 2). In these passages Aristotle gives very general definitions – in fact, definitions which *seem* to have more generality than he ever uses (cf. Ross, p. 35).

In this article the term *syllogism* is not restricted to arguments having only two premises. Indeed, were this the case, either here or throughout the Aristotelian corpus, the whole discussion would amount to an elaborate triviality. Barnes (*q.v.*) has argued that *at least* two premises are required. Additional reasons are available. That Aristotle did *not* so restrict his usage *throughout* is suggested by the form of his definition of syllogism (24b19–21), by his statement that every demonstration is a syllogism (25b27–31; cf. 71b17, 72b28, 85b23), by the content of Chapter 23 of *Prior Analytics* I and by several other circumstances to be mentioned below. Unmistakable evidence that Aristotle applied the term in cases of more than two premises is found in *Prior Analytics* I, 23 (esp. 41a17) and in *Prior Analytics* II, 17, 18 and 19 (esp. 65b17, 66a18 and 66b2). However, it is equally clear that in many places Aristotle does restrict the term to the two-premise case. It may be possible to explain

Aristotle's emphasis on two-premise syllogisms by reference to his discovery (*Prior Analytics* I, 23) that *if* all two-premise syllogisms are deducible in his system *then* all syllogisms without restriction are so deducible. As mentioned above, in this article the term has the more general sense. Thus 'sorites' are syllogisms (but, of course, enthymemes are not).

The *Analytics* as a whole forms a treatise on scientific knowledge (24a, 25b28–31). On Aristotle's view every item of scientific knowledge is either known in itself by experience (or some other non-deductive method) or else deduced from items known in themselves (*Posterior Analytics, passim*, esp. II, 19). The *Posterior Analytics* deals with the acquisition and deductive organization of scientific knowledge. It is the earliest general treatise on the axiomatic method[3] in sciences. The *Prior Analytics*, on the other hand, develops the underlying logic used in the inference of deductively known scientific propositions from those known in themselves; but the logic of the *Prior Analytics* is not designed solely for such use (cf., e.g., 53b4–11; Kneale and Kneale, p. 24).

According to Aristotle's view, once the first principles have been discovered, all subsequent knowledge is gained by means of 'demonstrative syllogisms', syllogisms having antecedently known premises, and it is only demonstrative syllogisms which lead to 'new' knowledge (*Posterior Analytics* I, 2). Of course, the knowledge thus gained is in a sense not 'new' because it is already implicit in the premises (*Posterior Analytics* I,1).

According to more recent terminology (cf. Mates, *Elementary Logic*, p. 3) a *premise-conclusion argument* (*P-c* argument) is simply a set of sentences called the *premises* together with a single sentence called the *conclusion*. Of course the conclusion need not follow from the premises, if it does then the argument is said to be *valid*. If the conclusion does not follow, the argument is *invalid*. It is obvious that even a valid argument with known premises does not *prove anything* – one is not expected to come to know the conclusion by reading the argument because there is no reasoning expressed in a *P-c* argument. For example, take the premises to be the axioms and definitions in geometry and take the conclusion to be any complicated theorem which actually follows. Such a valid argument, far from demonstrating anything, is the very kind of thing which needs 'demonstrating'. *In 'demonstrating' the validity of an argument one adds more sentences until one has constructed a chain of reasoning pro-*

ceeding from the premises and ending with the conclusion. The result of such a construction is called a *deductive argument* (premises, conclusion, plus a chain of reasoning) or, more briefly, a *deduction.* If the reasoning in a deduction actually shows that the conclusion follows from the premises the deduction is said to be *sound*; otherwise *unsound.* Given this terminology we can say that by *perfect syllogism* Aristotle meant precisely what we mean by *sound deduction* and that Aristotle understood the term *syllogism* to include both valid *P-c* arguments and sound deductions[4] (cf. 24b19–32). For Aristotle an invalid premise-conclusion argument is *not* a syllogism at all (cf. Rose, pp. 27–28). In an imperfect syllogism the conclusion follows, but it is not evident that it does. An imperfect syllogism is 'potentially perfect' (27a2, 28a16, 41b33, and Patzig, p. 46) and it is made perfect by adding more propositions which express a chain of reasoning from the premises to the conclusion (24b22–25, 28a1–10, 29a15, *passim*). Thus a demonstrative syllogism for Aristotle is a sound deduction with antecedently known premises (71b9–24, 72a5, *passim*).

That 'a demonstrative syllogism', for Aristotle, is not simply a valid *P-c* argument with appropriately known premises is already obvious from his view that such syllogisms are productive of knowledge and conviction (73a21; Ross, pp. 508, 517; also cf. Church, p. 53). *A fortiori*, a syllogism cannot be a single sentence of a certain kind, as other interpreters have suggested (see below; cf. Corcoran, 'Aristotelian Syllogisms' and cf. Smiley).

Aristotle is quite clear throughout that treatment of scientific knowledge presupposes a treatment of syllogisms (in particular, of perfect syllogisms). In order to be able to produce demonstrative syllogisms one must be able to reason deductively, i.e., to produce perfect syllogisms. Demonstration is a kind of syllogism but not vice versa (25b26–31, 71b22–24). According to our view outlined above, Aristotle's syllogistic includes a theory of deduction which, in his terminology, is nothing more than a theory of perfecting syllogisms. More specifically and in more modern parlance, Aristotle's syllogistic includes a *natural deduction system* by means of which categorical conclusions are deduced from categorical premises. The system countenances two types of deductions (direct and indirect) and, except for 'conversions', each application of a rule of inference is (literally) a first figure syllogism. Moreover, as will be clear below, *Aristotle's theory of deduction* is fundamental in the sense that it *pre-*

supposes no other logic, not even propositional logic.[5] It also turns out that the Aristotelian system (cf. Section 5 below) is complete in the sense that every valid *P-c* argument composed of categorical sentences can be 'demonstrated' to be valid by means of a formal deduction in the system. In Aristotelian terminology this means that every imperfect syllogism can be perfected by Aristotelian methods.

As will become clear below in Section 4, our interpretation is able to account for the correctness of certain Aristotelian doctrines which previous scholars have had to adjudge incorrect. For example, both Łukasiewicz (p. 57) and Patzig (p. 133) agree that Aristotle believed that all deductive reasoning is carried out by means of syllogisms, i.e., that imperfect syllogisms are perfected by means of perfect syllogisms, but they also hold that Aristotle was wrong in this belief (Łukasiewicz, p. 44; Patzig, pp. 135). Rose (p. 55) has wondered how one syllogism can be used to prove another but he did not make the mistake of disagreeing with Aristotle's view. Indeed, in the light of our own research one can see that Rose was very close (p. 53) to answering his own question. We quote in part:

> We have seen how Aristotle establishes the validity of... imperfect [syllogisms]... This amounts to presenting an extended argument with the premises of the imperfect [syllogism]... as... premises... using several intermediate steps, ... finally reaching as the ultimate conclusion the conclusion of the imperfect [syllogism]... being established. A natural reaction... is to think of the first figure [syllogisms]... as axioms and the imperfect [syllogisms]... as theorems and to ask to what extent Aristotle is dealing with a formal deductive system.

This would be natural indeed to someone not concerned with formal 'natural' deductive systems. To someone concerned with the latter, it would be natural to consider the first figure syllogisms as 'applications' of rules of inference, to consider the imperfect syllogisms as derived arguments, and then to scrutinize Chapters 2 and 4 (*Prior Analytics* I) in search of parts needed to complete the specification of a natural deductive system. What Rose calls 'an extended argument' is simply a deduction or, in Aristotle's terms, a discourse got by perfecting an imperfect syllogism. Rose had already seen the relevance of pointing out (p. 10) that the term 'syllogism' had been in common use in the sense of 'mathematical computation'. One would not normally apply the term 'computation' to mere data-and-answer reported in the form of an equation, e.g. $(330+1955= =2285)$. The *sine qua non* of a computation would seem to be the intermediate steps, and one might be inclined to call the mere data-plus-answer complex an 'imperfect computation' or a 'potential computation'. A 'perfect' or 'completed' computation would then be the entire complex of data, answer and intermediate steps. At one point Patzig seems to have been closer to our view than Rose. We quote from Patzig (p. 135), who sometimes uses 'argument' for 'syllogism'.

> ... the odd locution 'a potential argument' (synonymous with 'imperfect argument' ...) which, as was shown, properly means 'a potentially *perfect* argument'... has no clear sense unless we assume that Aristotle intended to state a procedure by which 'actual' syllogisms could be produced from these 'potential' ones, i.e., actually evident syllogisms produced from potentially evident ones.

Although Rose seems to have missed our view by failing to consider the possibility of a natural deduction system in Aristotle, Patzig was diverted in less subtle ways, as well. In the first place Patzig uncritically accepted the false conclusion of previous interpreters that all perfect syllogisms are in the first figure and thus arrives at the strange view that imperfect syllogisms are "as it were disguised first figure syllogisms" (*loc. cit.*). Secondly, and surprisingly, Patzig (p. 136) seems to be unaware of the distinction between a valid *P-c* argument and a sound deduction having the same premises and conclusion.

1.5. *The Łukasiewicz View and Its Inadequacies*

In order to contrast our view with the Łukasiewicz view it is useful to represent categorical statements with a notion which is mnemonic for readers of twentieth century English.

Amd	*A*ll *m* are *d*.
Smd	*S*ome *m* is *d*.
Nmd	*N*o *m* is *d*.
$md	*S*ome *m* is *not d*.

Łukasiewicz holds that Aristotle's theory of syllogistic is an axiomatic science which presupposes 'a theory of deduction' unknown to Aristotle (p. 14, 15, 49). The *universe* of the Łukasiewicz science is the class of secondary substances (man, dog, animal, etc.) and the relevant *relations* are those indicated above by *A*, *N*, *S*, and $, i.e., the relations of inclusion, disjointness, partial inclusion and partial non-inclusion respectively (pp. 14–15). Accordingly, he understands Aristotle's schematic letters (*alpha,* *beta, gamma, mu, nu, xi, pi, rho* and *sigma*) as variables ranging over the class of secondary substances and he takes A, N, S and \$ as non-logical constants (*ibid.*). Some of the axioms of the Łukasiewicz science correspond to Aristotelian syllogisms. But his axioms are single sentences (not arguments) and they are generalized with respect to the schematic letters (see Mates, *op. cit.*, p. 178). For example, the argument scheme

All Z are Y.
All X are Z.
So All X are Y.

corresponds to the following sort of axiom in the Łukasiewicz system

$$\forall xyz((Azy \;\&\; Axz) \supset Axy).$$

It should be noted, however, that Łukasiewicz does not *use* quantifiers in his reconstruction of Aristotle's syllogistic (p. 83). Universal quantification is nevertheless expressed in the theorems of the Łukasiewicz reconstruction – it is expressed by means of 'free variables', as can be verified by noticing the 'Rule of Substitution' that Łukasiewicz uses (p. 88). Indeed, the deductive system of the underlying logic presupposed by Aristotle (according to Łukasiewicz) is more than a propositional logic – it is what today would be called a free variable logic, a logic which involves truth-functions *and* universal quantification (expressed by free variables). Łukasiewicz refers to the deductive system of the underlying logic as 'the theory of deduction' and he sometimes seems to ignore the fact that a free variable logic is more than simply a propositional logic. [Using propositional logic alone one cannot derive Ayy from Axx (i.e., $\forall yAyy$ from $\forall xAxx$) but in a free variable logic it is done in one step.]

The Łukasiewicz view is ingenious and his book contains a wealth of useful scholarship. Indeed it is worth emphasizing that without his book the present work could not have been done in even twice the time. Despite the value of the book, its viewpoint must be incorrect for the following reasons. In the first place, as mentioned above, Łukasiewicz (p. 44) does not take seriously Aristotle's own claims that imperfect syllogisms are "proved by means of syllogisms". He even says that Aristotle was wrong in this claim. In the second place, he completely overlooks the many passages in which Aristotle speaks of perfecting imperfect syllogisms (e.g., *Prior Analytics*, 27a17, 29a30, 29b1–25). Łukasiewicz (p. 43) understands 'perfect syllogism' to indicate only the [valid] syllogisms in the first figure. This leads him to neglect the crucial fact that Chapters 4, 5 and 6 of *Prior Analytics* deal with Aristotle's theory of deduction. Thirdly, Aristotle is clear in *Posterior Analytics* (I, 10) about the nature of axiomatic sciences and he nowhere mentions syllogistic as a science (Ross, p. 24), but Łukasiewicz still wants to regard the syllogistic as such. (Łukasiewicz does seem uneasy (p. 44) about the fact that Aristotle does not call his basic syllogisms 'axioms'.) Indeed, as Scholz has already noticed (p. 6), Aristotle could not have regarded the syllogistic as a science because to do so he would have had to take the syllogistic as its own underlying logic. Again, were the Łukasiewicz system to be a science in Aristotle's terms then its universe of discourse would have to form a genus (e.g., *Posterior Analytics* I, 28) – but Aristotle nowhere mentions the class of secondary substances as a genus. Indeed, on reading the tenth chapter of the *Posterior Analytics* one would expect that *if* the syllogistic were a science *then* its genus would be mentioned on the first page of *Prior Analytics*. Not only does Aristotle fail to indicate the subject matter required by the Łukasiewicz view, he even indicates a different one – viz. demonstration – but not as a genus (*Prior Analytics*, first sentence).[6] In the fourth place, if the syllogistic were an axiomatic science and *A*, *N*, *S* and $ were relational terms, as Łukasiewicz must have it, then awkward questions ensue: (a) Why are these not mentioned in *Categories*, Chapter 7, where relations are discussed? (b) Why did Aristotle not seek for axioms the simplest and most obvious of the propositions involving these relations, i.e., 'Everything is predicated of all of itself' and 'Everything is predicated of some of itself'? In fact Aristotle may have deliberately avoided 'self-predication', although he surely knew of several reflexive *relations* (identity, equality, congruence). Łukasiewicz counts this as an oversight and adds the first of the above self-predications as a 'new' axiom. In connection with the above questions we may also note that the relations needed in the Łukasiewicz science are of a different 'logical type' than those considered by Aristotle in *Categories* – the former relate secondary substances whereas the latter relate primary substances, Fifth, if indeed Aristotle is axiomatizing a system of true relational sentences on a par with the system of true relational sentences which characterize the ordering of the numbers, as Łukasiewicz must and does claim (pp. 14, 15, 73), then again awkward questions ensue: (a) Why is there no discussion anywhere in the second logic of the general topic of relational sentences? (b) Why does Aristotle axiomatize only one such system? The 'theory of congruence' (equivalence relations) and the 'theory of the ordering of numbers' (linear order) are obvious, similar systems and nowhere does Aristotle even hint at the analogies. Sixth, as Łukasiewicz himself implicitly recognizes in a section called 'Theory of Deduction' (pp. 79–82), if the theory of syllogisms is understood as an axiomatic science then, as indicated above, it would presuppose an underlying logic (which Łukasiewicz supplies). But all indications in the Aristotelian corpus suggest not only that Aristotle regarded the theory of syllogistic as the most fundamental sort of reasoning (Kneale and Kneale, p. 44, and even Łukasiewicz, p. 57) but also that he regarded its logic as *the* underlying logic of all axiomatic sciences.[7] Łukasiewicz himself says, "It seems that Aristotle did not suspect the existence of a system of logic besides his theory of the syllogism" (p. 49). Seventh, the view that syllogisms are sentences of a certain kind and not extended discourses is incompatible with Aristotle's occasional but essential reference to *ostensive* syllogisms and to *per impossibile* syllogisms (41a30–40, 45a23, 65b16, e.g.). These references imply that some syllogisms have *internal structure even over and above 'premises' and 'conclusion'*. Finally, although Łukasiewicz gives a mathematically precise system which obtains and rejects 'laws' corresponding to those which Aristotle obtains and rejects, the Łukasiewicz system neither justifies nor accounts for the methods that Aristotle used. Our point is that the method is what Aristotle regarded as most important. In this connection, Aristotle obtained metamathematical results using methods which are clearly accounted for by the present interpretation but which must remain a mystery on the Łukasiewicz interpretation.[8]

It will be seen that Aristotle's theory of deduction contains a self-sufficient natural deduction system which presupposes no other logic. Perhaps the reason that Aristotle's theory of deduction has been overlooked is that it differs radically from many of the 'standard' modern systems. It has no axioms, it involves no truth-functional combinations and it lacks both the explicit and implicit quantifiers (in the modern sense).

1.6. *The Importance of the Issue*

Universally absent from discussions of this issue is reference to why it is important. My opinion is this: *if* the Łukasiewicz view is correct *then* Aristotle cannot be regarded as the founder of logic. Aristotle would merit this title no more than Euclid, Peano, or Zermelo insofar as these men are regarded as founders, respectively, of axiomatic geometry, axiomatic arithmetic and axiomatic set theory. (Aristotle would be merely the founder of 'the axiomatic theory of universals'.) Each of the former three men set down an axiomatization of a body of information *without* explicitly developing the underlying logic. That is, each of these men put down axioms and regarded as theorems of the system the sentences obtainable from the axioms by logical deductions *but* without bothering to say what a logical deduction is. Łukasiewicz is claiming that this is what Aristotle did. In my view, logic must begin with observations explicitly related to questions concerning the nature of an underlying logic. In short, logic must be explicitly concerned with deductive reasoning.

If Łukasiewicz is correct *then* the Stoics were the genuine founders of logic. Of course, my view is that in the *Prior Analytics* Aristotle developed the underlying logic for the axiomatically organized sciences that he discussed in the *Posterior Analytics* and that he, therefore, is the founder of logic.

2. The language L

In formulating a logic which is to serve as the underlying logic for several axiomatic sciences it is standard to define a 'master language' which involves: (1) punctuation, (2) finitely many logical constants, (3) infinitely many variables and (4) infinitely many non-logical constants or content words (cf. Church, p. 169). Any given axiomatic science will involve all of the logical constants and all of the variables, but only finitely many content words. The full infinite set of content words plays a role only in abstract theoretical considerations. In Aristotle there is no evidence of *explicit* consideration of a master language, although theoretical considerations involving infinitely many content words do occur in *Posterior Analytics* (I, 19, 20, 21). It is worth noticing that there is no need to postulate *object language* variables for Aristotle's system.

The vocabulary of the master language (L) involved in the present development of Aristotle's logic consists in the four logical constants (A, N, S and \$) and an infinite set U of non-logical constants (u_1, u_2, u_3, ...). The latter play the roles of 'categorical terms'. The rule of formation which defines 'sentence of *L*' is simply the following: a *sentence of L* is the result of attaching a logical constant to a string of two distinct non-logical constants. Thus each sentence of *L* is one of the following where *x* and *y* are distinct content words: *Axy*, *Nxy*, *Sxy*, *$xy*.

It is to be emphasized that no sentence of *L* has two occurrences of the same content word (or non-logical constant). This means, in the above terminology, that the system eschews self-predication. Self-predication is here avoided because Aristotle avoids it in the system of the *Prior Analytics* (so our model needs to do so for faithfulness) and also because, as J. Mulhern (pp. 111–115) has argued, Aristotle had theoretical reasons for such avoidance. Thus, contrary to the Łukasiewicz interpretation (p. 45), Aristotle's 'omission of the laws of identity' (All *X* are *X*; Some *X* are *X*) need not be construed as an oversight. The textual situation is the following: In the whole of the passages which contain the 'second logic' there is no appearance of self-predication. The only appearance of self-predication in *Analytics* is in the second book of *Prior Analytics* (63b40–64b25), which was written later. In this passage the sentences 'No knowledge is knowledge' and 'Some knowledge is not knowledge' appear as conclusions of syllogisms with contradictory premises and there are ample grounds for urging the extrasystematic character of the examples. In any case, no affirmative self-predications occur at all. Indeed, it may be possible to explain the absence of a doctrine of logical truth in Aristotle as being a practical 'consequence' of the fact that there are no logically true sentences in his abstract language.

It is readily admitted, however, that the reader's subjective feelings of 'naturalness' will color his judgment concerning which of the choices is an interpolation. If self-predications are thought to be 'naturally present' then our decision to exclude them will seem an interpolation. On the other hand, if they are thought to be 'naturally absent' then the Łukasiewicz inclusion will seem an interpolation. The facts that they do not occur in the second logic and that the system works out without them may tip the scales slightly in favor of the present view. Perhaps further slight evidence that Aristotle *needed* to exclude them can be got by noticing that the mood *Barbara* with a necessary major and necessary conclusion (regarded as valid by Aristotle) is absurdly invalid when the predicate and middle are identical.

Some may also question our omission of the 'indefinite propositions' like 'Men are greedy' which lack 'quantification' (cf. M. Mulhern, p. 51). Although these are mentioned by Aristotle, he seems to treat them as extra-systematic insofar as his system of scientific reasoning is concerned. In the first book of *Prior Analytics* (43a24–44) Aristotle also seems to exclude both adjectives and proper names from scientific languages. Łukasiewicz (p. 7) seems correct in saying that both the latter were banned because neither can be used both in subject and in predicate positions (also see Kneale and Kneale, p. 67 and Patzig, p. 6). It must also be noted that our model makes no room for relatives (and neither does the Łukasiewicz interpretation).

Even if subsequent research shows that these opinions are incorrect, our model need not be changed. However, its significance will change. Inclusion of proper nouns, adjectives, relatives and/or indefinite propositions would imply only *additions* to our model; no other changes would be required. Our language seems to be a sublanguage, at least, of any faithful analogue of the abstract language of Aristotle's system.[9]

The language L (just defined) is an abstract mathematical object designed in analogy with what might be called the ideal language envisaged in Aristotle's theory of scientifically meaningful statements. In effect each sentence in L should be thought of as representing a specific categorical proposition. The structure of a sentence in L is supposed to reflect the structure of the specific categorical proposition it represents. For example, if u and v represent the universals 'man and 'animal' then the structure of Auv should reflect the structure of the proposition 'All men are animals'. It is to be emphasized that a sentence in L is supposed to represent a particular proposition (as envisaged by Aristotle's theory) and *not* a propositional form, propositional function, proposition scheme or anything of the sort. There is no need within Aristotle's theory, nor within our model, of postulating the existence of propositional functions, propositional schemes or even object language variables. Our view is that Aristotle used metalinguistic variables, but that he neither used nor had a doctrine concerning object language variables.[10]

2.1. *Topical Sublanguages*

As was said above, Aristotle developed his logic largely (but not solely) as the underlying logic of the various sciences. In the first book of *Posterior Analytics*, Aristotle develops his view of the organization of sciences and at several places therein he makes it clear that each science has its own genus and its own peculiar terms (*Posterior Analytics* I; 7, 9, 10, 12, 28). A given science can have only finitely many terms (88b6–7; cf. Barnes, p. 123; Ross, p. 603) and it is somehow wrong (impossible?) to mix terms from different sciences.[11] Aristotle even goes so far as to claim that a proposition which seems common to two sciences is really two analogous propositions (76a37–b2).

We conclude that each science has its own *finite* language. We call such a special language a 'topical sublanguage' of the 'master' language. The notion of 'base' in Lewis and Langford (p. 348) corresponds to the finite vocabulary of terms of a topical sublanguage. It is very likely that Aristotle would have regarded his master language not as literally infinite but rather as indefinitely large or perhaps as potentially infinite.

2.2. *Grammatical Concepts*

Once the language has been defined, we can define some useful concepts which depend only on the language, i.e., which are independent of semantic and/or deductive notions. As above, a *premise-conclusion argument* (*P-c* argument) is a set *P* of sentences together with a single sentence *c*; *P* is called the *premises* and *c* is called the *conclusion.* Four things are to be noted at this point. First, Aristotle seems to have no term equivalent in meaning to '*P-c* argument'; each time he refers by means of a common noun to a *P-c* argument it is always by means of the term 'syllogism' which carries the connotation of validity (cf. Rose, p. 27). Second, Aristotle never refers to *P-c* arguments having the empty set of premises (which is not surprising, if only because none are valid). Third, although the 'laws of conversion' involve arguments having only a single premise, Aristotle did not recognize that fact, insisting repeatedly that every syllogism must have at least two premises (e.g., *Prior Analytics*, 42a8, 53b19; *Posterior Analytics* 73a9). Fourth, there is no question that Aristotle treated, in detail, syllogisms with more than two premises (e.g., *Prior Analytics* I, 23, 25, 42; *Posterior Analytics* I, 25, also see above). In fact, *Posterior Analytics* implicitly considers syllogisms whose premises are all of the axioms of a science (*Posterior Analytics* I, 10) and it explicitly considers the possibility of syllogisms with infinitely many premises (*Posterior Analytics* I, 19, 20, 21).

Underlying much of Aristotle's thought (but never explicitly formulated) is the notion of *form of argument*, but only in the relational sense in which one argument can be said *to be in the same form as* another. This notion is purely syntactic and can be defined given the language alone. In particular, let (P, c) and (P', c') be two arguments. (P, c) *is in the same form as* (P', c') if and only if there is a one-one correspondence between their respective sets of content words so that substitution according to the correspondence converts one argument into the other. In order to exhibit examples let us agree to represent an argument by listing the premises and conclusion – indicating the conclusion by a question mark.

Example 1: The following two arguments are in the same form by means of the one-one correspondence on the right:

Aab	*Acd*	*a*	*c*
Sbc	*Sda*	*b*	*d*
$ab	*$cd*	*c*	*a*
?*Ncd*	?*Nae*	*d*	*e*

Example 2: In the following pairs the respective arguments are not in the same form:

Aab	*Aab*	*Aab*	*Aab*	*Aab*	*Aab*
Sbc	*Sbc*	*Sac*	*$ac*	?*Nac*	*$ac*
?*Nac*	?*Nca*	?*$ac*	?*$ac*		?*Nac*

It follows from the definition that in order for two arguments to be in the same form, it is necessary that they have (1) the same number of premises, (2) the same number of distinct content words and (3) the same number of sentences of any of the four kinds.

It is obvious that one need know absolutely nothing about how the sentences in *L* are to be interpreted or how one 'reasons' about their logical interrelations in order to be able to decide whether two arguments are in the same form. Relative to this system, the notion of form is purely grammatical (cf. Church. pp. 2–3).

Define $P+s$ as the result of adjoining the sentence *s* with the set *P*.

Finally we define *Nxy* and *Axy* to be *contradictories* respectively of *Sxy* and *$xy* (and vice versa) and we define the function *C* which when applied to a sentence in *L* produces its contradictory. The table of the function is given below.

	C
Axy	$\$xy$
Nxy	Sxy
Sxy	Nxy
$\$xy$	Axy

3. THE SEMANTIC SYSTEM S

Aristotle regarded the truth-values of the non-modal categorical propositions as determined extensionally (*Prior Analytics*, 24a26 ff.).[12] Thus, for Aristotle: (1) 'All X is Y' is true if the extension of X is included in that of Y; (2) 'No X is Y' is true if the extension of X is disjoint with that of Y; (3) 'Some X is Y' is true if an object is in both extensions and (4) 'Some X is not Y' is true if some object in the extension of X is outside of the extension of Y. Thus, given the meanings of the logical constants, the truth-values of the categorical sentences are determined by the extensions of the universals involved in the manner just indicated. Now imagine that the content words (characters in U) are correlated with the secondary substances (sortal universals) and consider the following *interpretation* i of L. The interpretation ix of the content word x is the extension of the secondary substance correlated with x. Given i we can easily define a function V^i which assigns the correct truth-value to each sentence in L as follows:

(1) $V^i(Axy) = t$ if ix is included in iy,
$V^i(Axy) = f$ if ix is not included in iy.
(2) $V^i(Nxy) = t$ if ix is disjoint with iy,
$V^i(Nxy) = f$ if ix is not disjoint with iy.
(3) $V^i(Sxy) = t$ if ix is not disjoint with iy,
$V^i(Sxy) = f$ if ix is disjoint with iy.
(4) $V^i(\$xy) = t$ if ix is not included in iy,
$V^i(\$xy) = f$ if ix is included in iy.

The function i defined above may be regarded as the *intended interpretation* of L. In order to complete the construction of the semantics for L we must specify, in addition, the non-intended or 'possible' interpretations of L. The non-intended interpretations of a language are structures which share all 'purely logical' features with the intended interpretation. What is essential to the intended interpretation is that it assigns to each content word a set of primary substances (individuals) which 'could be' the extension of a secondary substance. Since Aristotle held that every secondary substance must subsume at least one primary substance (*Categories*, 2a34–2b7), we give the following general definition of an interpretation of L: j is an *interpretation* of L if and only if j is a function which assigns a non-empty set[13] to each member of U. The general definition of truth-values of sentences of L under an arbitrary interpretation j is exactly the same as that for the intended interpretation.

The absence of the notion of universe of discourse warrants special comment if only because it is prominent, not only in modern semantics but also in Aristotles treatment of axiomatic science (see above). In the first place, this concept plays no role in the system of the *Prior Analytics*, which is what we are building a model for. So we deliberately leave it out, although from a modern point of view it is unnatural to do so. [Of course, in an underlying logic based on a topical sublanguage, universes of discourse *are* needed (each science has its genus). To supply them we would require that, for each j, each jx is a subclass of some set, say Dj, given in advance. Its omission has no mathematical consequences.] In the second place there may be a tradition (cf. Jaskowski, p. 161; Patzig, p. 7) which holds that Aristotle prohibited his content words from having the universe as extension. (So both the null set *and* the universe would be excluded. Since the universe of sets is not itself a set, our definitions respect the tradition without special attention – and perhaps without special significance.[14])

It must be admitted that Aristotle nowhere makes specific reference to alternative interpretations nor does he anywhere perform operations which suggest that he had envisaged alternative interpretations. Rather it seems that at every point he thought of his ideal language as interpreted in what we would call its intended interpretation. Moreover, it is doubtful that Aristotle ever conceived of a language apart from its intended interpretation. In other words, it seems that Aristotle did not separate logical syntax from semantics (but cf. *De. Int.*, chapter 1 and *Soph. Ref.*, chapter 1).

3.1. *Semantic Concepts*

In terms of the semantics of L just given, we define some additional useful notions as follows. A sentence s is said to be *true* [*false*] *in an interpre-*

tation j if $V^j(s) = t$ [$V^j(s) = f$]. If s true in j then j is called *a true interpretation of s*. If P is a set of sentences all of which are true in j then j is called a *true interpretation of P* and if every true interpretation of P is a true interpretation of c then P is said to (logically) *imply* c (written $P \vDash c$). If P implies c then the argument (P, c) is *valid*, otherwise (P, c) is *invalid*. A *counter interpretation* of an argument (P, c) is a true interpretation of the premises, P, in which the conclusion, c, is false. When (P, c) is valid, c is said to be a *logical consequence*[15] of P.

By reference to the definitions just given one can show the following important semantic principle – which is suggested by Aristotle's 'contrasting instances' method of establishing invalidity of arguments (below and cf. Ross, pp. 28, 292–313 and Rose, pp. 37–52).

(3.0) *Principle of counter interpretations. A premise-conclusion argument is invalid if and only if it has a counter interpretation.*

The import of this principle is that whenever an argument is invalid it is possible to reinterpret its content words in such a way as to make the premises true and the conclusion false. It is worth remembering that the independence of the Parallel Postulate from the other 'axioms' of geometry was established by construction of a counter interpretation, a reinterpretation of the language of geometry in which the other axioms were true and the Parallel Postulate false (cf. Cohen and Hersh, and also, Frege, pp. 107–110).[16]

Perhaps the most important semantic principle underlying Aristotle's logical work is the following, also deducible from the above definitions.

(3.1.) *Principle of Form: An argument is valid if and only if every argument in the same form is also valid.*

Aristotle tacitly employed this principle[17] throughout the *Prior Analytics* in two ways. First, to establish the validity of all arguments in the same form as a given argument, he establishes the validity of an arbitrary argument in the same form as the argument in question (i.e. he establishes the validity of an argument leaving its content words unspecified). Second, to establish the invalidity of all arguments in the same form as a given argument, he produces a specific argument in the required form for which the intended interpetation is a counter interpretation.[18] The latter, of course, is the method of 'contrasting instances'. In neither of these operations, which are applied repeatedly by Aristotle, is it necessary to postulate either alternative interpretations or argument forms (over and above individual arguments; cf. Sections 3.2 and 3.3 below).

The final semantic consideration is the semantic basis of what will turn out to be Aristotle's theory of deduction. The clauses of the following principle are easily established on the basis of the above definitions.

(3.2.) *Semantic Basis of Aristotle's Theory of Deduction: let x, y, and z be different members of U. Let P be a set of sentences and let d and s be sentences.*

Law of Contradictions:

(C) For all j, $V^j(s) \neq V^j(C(s))$,
[i.e., in every interpretation, contradictions have different truth values].

Conversion Laws:

(C1) $Nxy \vDash Nyx.$
(C2) $Axy \vDash Syx.$
(C3) $Sxy \vDash Syx.$

Laws of Perfect Syllogisms:

(PS1) $\{Azy, Axz\} \vDash Axz.$
(PS2) $\{Nzy, Axz\} \vDash Nxy.$
(PS3) $\{Azy, Sxz\} \vDash Sxy.$
(PS4) $\{Nzy, Sxz\} \vDash \$xy.$

Reductio Law:

(R) $P \vDash d$ *if* $P + C(d) \vDash s$ and $P + C(d) \vDash C(s).$

The law of contradictions, the conversion laws, and the laws of perfect syllogisms are familiar and obvious. The *reductio* law says that for d to follow from P it is sufficient that P and the contradiction of d together imply both a sentence s and its contradictory $C(s)$. Although Aristotle regarded all of the above clauses as obviously true, he does not completely neglect metalogical questions[19] concerning them.

As far as I can tell Aristotle did not raise the metalogical question concerning *reductio* reasoning in the *Analytics*. In Chapter 2 of the first book of the *Prior Analytics* he puts down the conversion laws and then offers what seem to be answers to the metalogical questions concerning their validity. Specifically, he establishes (C1) by a kind of metasystematic *reductio* proof which presupposes (1) non-emptiness of term-extensions, (2) contradictory opposition between *Nxy* and *Sxy*, and (3) that existence of an object having properties *x* and *y* precludes the truth of *Nyx*. Then, taking (C1) as established, he establishes (C2) and (C3) by *reductio* reasoning. Two chapters later he gives obviously semantic justification for the four laws of perfect syllogisms.

3.2. *An Alternative Semantic System*

Instead of having a class of interpretations some logicians prefer to 'do as much semantics as possible' in terms of the following two notions: (1) truth-valuation in the intended interpretation and (2) form (cf. Quine, *Philosophy*, p. 49 and Corcoran, 'Review'). Such logicians would have a semantic system containing exactly one interpretation, the intended interpretation, and they would *define* an argument to be valid if every argument in the same form with true premises (relative to the intended interpretation) has a true conclusion (relative to the intended interpretation). Ockham's razor would favor the new 'one-world' semantics over the above 'possible-worlds' semantics (Quine, *op. cit.*, p. 55). Within a framework of a one-world semantics *invalidity* would be established in the same way as above (and as in Aristotle).

It does not seem possible to establish by reference to the Aristotelian corpus whether one semantic system agrees better with Aristotle's theory than the other. The main objection to the one-world semantics is that it makes logical issues depend on 'material reality' rather than on 'logical possibilities'. For example, *if* the intended interpretation is so structured that for every pair of content words the extension of one is identical to the extension of the other or else disjoint with it *then Axy* 'logically implies' *Ayx*. Thus in order to get the usual valid arguments in a one-world semantics it is necessary to make additional assumptions about the intended interpretation (cf. Quine, *op. cit.*, p. 53). Proponents of the one-world semantics prefer additional assumptions concerning 'the real world' to additional assumptions about 'possible worlds'. Since the mathematics involved with the semantics of the previous section involves fewer arbitrary decisions than does the semantics of this section we have chosen to make the former the semantic system of our model of Aristotle's system. It is very likely that proponents of the one-world view could honestly weight the available evidence so that attribution of the one-world semantics to Aristotle is more probable. If the current dialogue between proponents of the two views continues the above may well become an important historical issue.

3.3. *Forms of Arguments*

Above we used the term *form* only in relational contexts: (P, c) is in the same form as (P^*, c^*). During previous readings of this paper, auditors insisted on knowing what logical forms 'really are' and whether Aristotle used them as theoretical entities. Perhaps the best way of getting clear about the first problem is to first see an 'explication' of the notion. The following explication is a deliberate imitation of Russell's explication of *number* in terms of the relation 'has the same number of members as'.

Consider the class of all arguments and imagine that it is partitioned into non-empty subsets so that all and only formally similar arguments are grouped together. Define *Forms* to be these subsets. If we use this notion of *Form*, then many of the traditional uses of the substantive *form* (not the relative) are preserved. Taking *in* in the sense of membership, we can say that (P, c) is in the same form as (P^*, c^*) if and only if (P, c) is a member of the same Form that (P^*, c^*) is a member of.

A Form is simply a set of formally similar arguments. Unfortunately, this clear notion of form is *not* the one that has been traditionally invoked. The traditional 'argument form' is supposed to be like a (real) argument except that it doesn't have (concrete) terms. Putting variables for the terms will not help because new variables can be substituted without changing the 'form'. Proponents of 'forms' fall back on saying that an 'argument form'is that which all formally similar arguments have in common, but (seriously) what can this be except membership in a class of formally similar arguments? In any case there are no textual grounds for imputing to Aristotle a belief in argument Forms (or, for that matter, in 'argument forms', assuming that sense can be made of that notion).

4. The deductive system *D*

We have already implied above that a theory of deduction is intended to specify what steps of deductive reasoning may be performed in order to come to know that a certain proposition c follows logically from a certain set *P* of propositions. Aristotle's theory of deduction is his theory of perfecting syllogisms. As stated above, our view is that a perfect syllogism is a discourse which expresses correct reasoning from premises to conclusion. In case the conclusion is immediate, nothing need be added to make the implication clear (24a22). In case the conclusion does not follow immediately, then additional sentences must be added (24b23, 27a18, 28a5, 29a15, 29a30, 42a34, etc.). A valid argument by itself is only potentially perfect (27a2, 28a16, 41b33): it is 'made perfect' (29a33, 29b5, 29b20, 40b19, etc.) by, so to speak, filling its interstices.

According to Aristotle's theory, there are only two general methods[20] for perfecting an imperfect syllogism – either directly (ostensively) or indirectly (*per impossibile*) (e.g., 29a30–29bl, 40a30, 45b5–10, 62b29–40, *passim*). In constructing a direct deduction of a conclusion from premises one interpolates new sentences by applying conversions and first figure syllogisms to previous sentences until one arrives at the conclusion. Of course, it is permissible to repeat an already obtained line. In constructing an indirect deduction of a conclusion from premises one adds to the premises, as an additional hypothesis, the contradictory of the conclusion; then one interpolates new sentences as above until both of a pair of contradictory sentences have been reached.

Our deductive system *D*, to be defined presently, is a syntactical mathematical model of the system of deductions found in Aristotle's theory of perfecting syllogisms.

Definition of D. First restate the laws of conversion and perfect syllogisms as rules of inference.[21] Use the terms 'a *D*-conversion of a sentence' to indicate the result of applying one of the three conversion rules to it. Use the terms '*D*-inference from two sentences' to indicate the result of applying one of the perfect syllogism rules to the two sentences.

A *direct deduction in D of c from P* is a finite list of sentences ending with *c*, beginning with all or some of the sentences in *P*, and such that each subsequent line (after those in *P*) is either (a) a repetition of a previous line, (b) a *D*-conversion of a previous line or (c) a *D*-inference from two previous lines.

An *indirect deduction in D of c from P* is a finite list of sentences ending in a contradictory pair, beginning with a list of all or some of the sentences in *P* followed by the contradictory of *c*, and such that each subsequent additional line (after the contradictory of *c*) is either (a) a repetition of a previous line, (b) a *D*-conversion of a previous line or (c) a *D*-inference from two previous lines.

All examples of deductions will be annotated according to the following scheme: (1) Premises will be prefixed by '+' so that '+*Axy*' can be read 'assume *Axy* as a premise'. (2) After the premises are put down we interject the conclusion prefixed by '?' so that '?*Axy*' can be read 'we want to show why *Axy* follows'. (3) The hypothesis of an indirect (*reductio*) deduction is prefixed by '*h*' so that '*hAxy*' can be read 'suppose *Axy* for purposes of reasoning'. (4) A line entered by repetition is prefixed by '*a*' so that '*aAxy*' can be read 'we have already accepted *Axy*'. (5) Lines entered by conversion and syllogistic inference are prefixed by '*c*' and '*s*', respectively. (6) Finally, the last line of an indirect deduction has '*B*' prefixed to its other annotation so that '*BaAxy*' can be read 'but we have already accepted *Axy*', etc. We define an *annotated deduction in D* to be a deduction in *D* annotated according to the above scheme. In accordance with now standard practice we say that *c is deducible from P in D* to mean that there is a deduction of *c* from *P* in the system *D*. It is also sometimes convenient to use the locution 'the argument (*P*, *c*) is deducible in *D*'.

The following is a consequence of the above definitions (cf. Frege, pp. 107–11).

(4.1) *Deductive Principle of Form: An argument is deducible in D if and only if every argument in the same form is also deducible.*

The significance of *D* is as follows. We claim that *D* is a faithful mathematical model of Aristotle's theory of perfecting syllogisms in the sense that every perfect syllogism (in Aristotle's sense) corresponds in a direct and obvious way to a deduction in *D*. Thus what can be added to an imperfect syllogism to render it perfect corresponds to what can be 'added' to a valid argument to produce a deduction in *D*. In the case of a direct deduction the 'space' between the premises and conclusion is filled up in accordance with the given rules.

In order to establish these claims as well as they can be established (taking account of the vague nature of the data), the reader may go through the deductions presented by Aristotle and convince himself that each may be faithfully represented in *D*. We give four examples below; three direct deductions and one indirect deduction. The others raise no problems.

We reproduce two of Aristotle's deductions (27a5–15; Rose, p. 34), each followed by the corresponding annotated deductions in *D*.

(1)	Let *M* be predicated of no *N*	+ *Nnm*
	and of All *X*	+ *Axm*
	(*conclusion omitted in text*).	(?*Nxn*)
	Then since the negative premise converts	
	N belongs to no *M*.	*cNmn*
	But it was supposed that *M* belongs to all *X*.	*aAxm*
	Therefore *N* will belong to no *X*.	*sNxn*
(2)	Again, if *M* belongs to all *N*	+ *Anm*
	and to no *X*,	+ *Nxm*
	X will belong to no *N*.	?*Nnx*
	For if *M* belongs to no *X*,	*aNxm*
	X belongs to no *M*.	*cNmx*
	But *M* belonged to all *N*.	*aAnm*
	Therefore *X* will belong to no *N*.	*sNnx*

Below we reproduce Aristotle's words (28b8-12) followed by the corresponding annotated deduction in *D*.

(3)	For if *R* belongs to all *S*,	+ *Asr*
	P belongs to some *S*,	+ *Ssp*
	P must belong to some *R*.	?*Srp*
	Since the affirmative statement is convertible	
	S will belong to some *P*,	*cSps*
	consequently since *R* belongs to all *S*,	*aAsr*
	and *S* to some *P*,	*aSps*
	R must also belong to some *P*:	*sSpr*
	therefore *P* must belong to some *R*.	*cSrp*

To exemplify an indirect deduction we do the same for 28 b17–20.

(4)	For if *R* belongs to all *S*,	+ *Asr*
	but *P* does not belong to some *S*,	+ *$sp*
	it is necessary that *P* does not belong to some *R*.	?*$rp*
	For if *P* belongs to all *R*,	*hArp*
	and *R* belongs to all *S*,	*aAsr*
	then *P* will belong to all *S*:	*sAsp*
	but we assumed that it did not.	*Ba$sp*

Readers can verify the following (by 'translating' Aristotle's proofs of the syllogisms he proved, using ingenuity in the other cases).

(4.2) *All valid arguments in any of the four traditional figures*[22] *are deducible in D.*

4.1. *Deductive Concepts*

As is to be expected given the above developments, a deductive concept is one which can be defined in terms of concepts employed in the deductive system without reference to semantics. In many cases one relies on semantic insights for the motivation to delimit one concept rather than another. This is irrelevant to the criterion for distinguishing deductive from semantic concepts; just as reliance on mechanical insight for motivation to define mathematical concepts is irrelevant to distinguishing physical and mathematical concepts.

Already several deductive notions have been used – 'direct deduction', 'indirect deduction', 'rule of inference', 'deducible from', 'contradictory' (as used here), etc. Relative to *D* the notion of consistency is defined as follows. A set *P* of sentences is *consistent* if no two deductions from *P* have contradictory conclusions. If there are two deductions from *P* one of which yields the contradictory of the conclusion of the other then, of course, *P* is *inconsistent*.

Aristotle did not have occasion to define the notion of inconsistency but he showed a degree of sophistication lacking in some current thinkers by discussing valid arguments having inconsistent premise sets[23] (63b40-64b25).

4.2. *Some Metamathematical Results in Aristotle*

Generally speaking, a metamathematical result is a mathematical result concerning a logical or mathematical system. Such results can also be called metasystematic. The point of the terminology is to distinguish the results codified by the system from results concerning the system itself. The latter would necessarily be stated in the metalanguage and codified in a metasystem. It is also convenient (but sometimes artificial) to distinguish intrasystematic and intersystematic results. The former would concern mathematical relations among parts of the given system whereas the latter would concern mathematical relations between the given system and another system. The artificiality arises when the 'other' system is acactually a part of the given system.

It is worth noting that the theorem/metatheorem confusion cannot arise in discussion of Aristotle's syllogistic for the reason that there are no theorems. This observation is important but it is not deep. It is simply a reflection of two facts: first, that within the passages treating the second logic Aristotle did not consider the possibility of 'logical truths' (object language sentences true in virtue of logic alone); second, and more importantly, that Aristotle regarded logic as a 'canon of inference' rather than as a codification of 'the most general laws of nature'.

Given the three-part structure of a logic one can anticipate four kinds of metasystematic results: 'grammatical' results which concern the language alone; 'semantic' results which concern the language and the semantic system; 'proof-theoretic' results which concern the language and the deductive system; and 'bridge' results which bridge or interrelate the semantic system with the deductive system. Since the Aristotelian grammar is so trivial, there is nothing of interest to be expected there. The semantics, however, is complex enough to admit of analogues to modern semantic results. For example, the analogue to the Löwenheim–Skolem theorem is that any satisfiable set of sentences of *L* involving no more than n content words is satisfiable in a universe of not more than 2^n objects (for proof see Corcoran, 'Completeness'). Unfortunately there are no semantic results (in this sense) in Aristotle's 'second logic'. As mentioned above, Aristotle may not have addressed himself to broader questions concerning the semantic system of his logic. As is explained in detail below, most of Aristotle's metasystematic results are proof-theoretic: they concern the relationship between the deductive system *D* and various subsystems of it. There is, however, one bridge result, viz., the completeness of the deductive system relative to the semantics. Unfortunately, Aristotle's apparent inattention to semantics may have prevented him from developing a rigorous proof of completeness.

There are several metasystematic results in the 'second logic', none of which have been given adequate explanation previously. We regard an explanation of an Aristotelian metasystematic result to be adequate only when it accounts for the way in which Aristotle obtained the result.

4.2.1. *Aristotle's Second Deductive System D2.* As already indicated above, the first five chapters of the 'second logic' (*Prior Analytics* I, 1, 2, 4, 5, 6) include a general introductory chapter, two chapters presenting the system and dealing with the first figure and two chapters which present deductions for the valid arguments in the second and third figures.[24] The next chapter (Chapter 7) is perhaps the first substantial metasystematic discussion in the history of logic.

The first interesting metasystematic passage begins at 29a30 and merely summarizes the work of the preceding three chapters. It reads as follows

It is clear too that all the imperfect syllogisms are made perfect by means of the first figure. All are brought to conclusion either ostensively or *per impossibile.*

From the context it is obvious that by 'all' Aristotle means 'all second and third figure'. Shortly thereafter begins a long passage (29b1–25) which states and proves a substantial metasystematic result. We quote (29b1–2)

It is possible also to reduce all syllogisms to the universal syllogisms in the first figure.

Again 'all' is used as above; 'reduce to' *here* means 'deduce by means of' and 'universal syllogism' means 'one having an *N* or *A* conclusion'. What Aristotle has claimed is that all of the syllogisms previously proved can be established by means of deductions which do not involve the 'particular' perfect syllogistic rules (PS3 and PS4). Aristotle goes on to explain in concise, general, but mathematically precise terms exactly how one can construct the twelve particular deductions which would substantiate the claim. Anyone can follow Aristotle's directions and thereby construct the twelve formal deductions in our system *D*.

In regard to the validity of the present interpretation these facts are significant. Not only have we accounted for the content of Aristotle's discovery but we have also been able to reproduce exactly the methods that he used to obtain them. Nothing of this sort has been attempted in previous interpretations (cf. Łukasiewicz, p. 45).

Let *D2* indicate the deductive system obtained by deleting PS3 and PS4 from *D*. Aristotle's metaproof shows that the syllogisms formerly deduced in *D* can also be deduced in *D2*. On the basis of the next chapter (*Prior Analytics* I, 23) of the 'second logic' (cf. Bocheński, p. 43; Łukasiewicz, p. 133; Tredennick, p. 185) it becomes clear that Aristotle thinks that he has shown that *every* syllogism deducible in *D* can also be deduced in *D2*. On reading the relevant passages (29b1-25) it is obvious that Aristotle has not proved the result. However, it is now known that the result is correct; it follows immediately from the main theorem of Corcoran 'Completeness' (*q.v.*). But regardless of the correctness of Aristotle's proof one must credit him with conception of the first significant hypothesis in proof theory.

4.2.2. *Redundancy of Direct Deductions.* Among indirect deductions it is interesting to distinguish two subclasses on the basis of the role of the added hypothesis. Let us call an indirect deduction *normal* if a rule of inference is applied to the added hypothesis and *abnormal* otherwise. In many of the abnormal cases, one reasons from the premises ignoring the added hypothesis until the desired conclusion is reached and then one notes 'but we have assumed the contradictory'.

Aristotle begins Chapter 29 (*Prior Analytics* I) by stating that whatever can be proved directly can also be proved indirectly. He then gives two examples of normal indirect deductions for syllogisms he has already deduced directly. Shortly thereafter (45b1–5) he says,

> Again if it has been proved by an ostensive syllogism that *A* belongs to no *E*, assume that *A* belongs to some *E* and it will be proved *per impossibile* to belong to no *E*. Similarly with the rest.

The first sentence means that by interpolating the added hypothesis *Sea* into a direct deduction of *Nea* one transforms it to an indirect deduction of the same conclusion. See the diagram below.

```
+————                               +————
+————                               +————
+————                               +————
?Nea         Transforming to:       ?Nea
                                    hSea
 ————                                ————
 ————                                ————
 ————                                ————
 ————                                ————
 Nea                                 Nea
                                     BaSea
```

The second quoted sentence is meant to indicate that the same result holds regardless of the form of the conclusion. In other words, Aristotle has made clear the fact that whatever can be deduced by a direct deduction can also be deduced by an abnormal indirect deduction, i.e., that direct deductions are redundant from the point of view of the system as a whole.[25]

We feel that this is additional evidence that Aristotle was self-consciously studying interrelations among deductions – exactly as is done in Hilbert's 'proof theory' (e.g., cf. van Heijenoort, p. 137).

4.3. *Indirect Deductions or a Reductio Rule?*

To the best of my knowledge Aristotle considered indirect reasoning to be a certain style of deduction. After the premises are set down one adds the contradictory of what is to be proved and then proceeds by 'direct reasoning' to each of a pair of contradictory sentences. Imagine, however, the following situation: one begins an indirect deduction as usual and immediately gets bogged down. Then one sees that there is a pair of contradictories, say s and $C(s)$, such that (1) s can be got from what is already assumed by indirect reasoning and (2) that $C(s)$ can be got from s together with what is already assumed by direct reasoning.

In a normal context of mathematics there would be no problem – the outlined strategy would be carried out without a second thought. In fact the situation is precisely what is involved in a common proof of 'Russell's Theorem' (no set contains exactly the sets which do not contain themselves). It involves using *reductio* reasoning as a structural rule of inference (cf., e.g., Corcoran, 'Theories', pp. 162ff). The trouble is that the strategy requires the addition of a *second* hypothesis and this is not countenanced by the Aristotelian system (41a33–36).

The salient differences between a system with indirect deductions and a system with a *reductio* rule are the following. In the case of indirect deductions, one can add but one additional hypothesis (viz. the contradictory of the conclusion to be reached) and one cannot in general use an indirectly obtained conclusion later on in a deduction. Once the indirectly obtained conclusion is reached the indirect deduction is, by definition, finished. An indirectly obtained conclusion is never written as such in the deduction. In the case of the *reductio* rule one can add as many additional hypotheses as desired; once an indirectly obtained conclusion is reached it is written as an intermediate conclusion usable in subsequent reasoning.

The deductive system of Jeffrey (*q.v.*) consists solely of indirect deductions whereas the system of Anderson and Johnstone (*q.v.*) has a *reductio* rule.

Metamathematically, one important difference is the following. Where one has a *reductio* rule it is generally easy to prove the metamathematical result that $C(d)$ is (indirectly) deducible from P whenever each of a pair of contradictions is separately deducible from $P + d$. This result can be difficult in the case where one does not have a *reductio* rule – especially when each of the pair of contradictions was reached indirectly.

In order to modify the system (or systems) to allow such 'iterated or nested *reductio* strategies' one would abandon the distinction between direct and indirect deductions; in the place of the indirect deductions one would have (simply) deductions which employ one or more applications of a *reductio* rule. Statements of such *reductio* rules are in general easily obtained *but* they involve several ideas which would unnecessarily complicate this article. Let us assume that *D2* has been modified[26] to permit iterated or nested *reductio* deductions and let us call the new system *D3*.

Now we have two final points to make. In the first place, in one clear sense, nothing is gained by adding the *reductio* rule because, since *D2* is known to be complete and *D3* is sound, every argument deducible in *D3* is already deducible in *D2*. In the second place, Aristotle may well have been thinking of *reductio* as a rule of inference but either lacked the motivation to state it as such or else actually stated it as such only to have his statements deleted or modified by copyists. It may even be the case that further scholarship will turn up convincing evidence for a *reductio* rule in the extant corpus. This is left as an open problem in Aristotle scholarship.[27]

4.4. *Extended Deductions*

In the course of a development of an axiomatic science it would be silly, to say the least, to insist on starting each new deduction from scratch. We quite naturally use as premises in each subsequent deduction not only the axioms of the science but also any or all previously proved theorems. Thus at any point in a development of an axiomatic science the last theorem proved is proved not by a deduction from the axioms but rather by a deduction from the axioms *and* previously proved theorems. In effect, we can think of the entire sequence of deductions, beginning with that of the first theorem and ending with that of the last proved theorem as an 'extended deduction' with several conclusions. If the basic deductive system is D (above) then the 'extended deductions' can be defined recursively as follows. (In D we define '*deduction of c from P*' where c is an individual sentence. Now we defined '*extended deduction of C from P*' where C is a set of sentences.)

Definition of Deductive System DE.

(a) All direct and indirect deductions in D of c from P are extended deductions in DE of $\{c\}$ from P.

(b) *If* F' is an extended deduction in DE of C from P and F is a deduction in D of d from $P + C$ *then* the result of adjoining F to the end of F' is an extended deduction in DE of $C+d$ from P.

Thus an extended deduction in DE of $\{c_1, c_2, \ldots, c_n\}$ from P could be (the concatenation of) a sequence of *component deductions* (all in D) the $i+1$st of which is a deduction of c_{i+1} from one or more members of $P + \{c_1, c_2, \ldots, c_i\}$. Soundness of the system of extended deductions is almost immediate given the following principle which holds in the 'possible-worlds' semantics of Section 3 above.

(4.0) *Semantic Principle of Extended Deduction:*

$$P \models d \text{ if } P + C \models d \text{ and, for all } s \text{ in } C, P \models s.$$

The significance of the system of extended deductions is as follows. In the first place, it is natural (if not inevitable) to consider such a system in the course of a study of axiomatic sciences. Thus, we must consider the possibility that the underlying logic of the axiomatic sciences discussed in *Posterior Analytics* had as its deductive system a system similar to the system of extended deductions. Secondly, this system loosens to some extent the constraint of not being able to *use* indirectly obtained results in deductions in D. (Although the constraint there resulted from an absence of a *reductio* rule, strictly speaking, there is still no *reductio* rule in DE.)[28]

It may be relevant to point out here that, since an Aristotelian science has only a finite number of principles (axioms *and* theorems), for formal purposes each science can be identified with a single extended deduction.

Here we wish to consider briefly the possibility that the underlying logic presupposed in *Posterior Analytics* is a system of extended deductions. At the outset, we should say that there are no grounds whatsoever for thinking that Aristotle restricted the use of the term 'demonstration' to the two-premise cases. Next we note that *if Posterior Analytics* requires a system of extended deductions *then* there are grounds for limiting the component deductions (direct and indirect) to ones having at most two premises. Thus we are considering the possibility that every 'demonstration' is an extended deduction whose components are all deductions having one or two premises. If this possibility were established, it could provide an alternative account for the passages where 'syllogism' is clearly used in the restricted sense, given that there are passages which refer to demonstrations *as chains of syllogisms.* The latter, however, do not seem to exist in *Analytics* (cf. 25b27, 71b17, 72b28, 85b23), but there is one tempting passage in *Topics* (100a27). In any case, we have been unsuccessful in our attempt to construct persuasive support for this possibility. (cf. Smiley.)

5. The Mathematical Logic *I*

In the previous three sections we considered the components of several mathematical logics any one of which could be taken as a reasonably faithful model of the system (or systems) of logic envisaged in Aristotle's theory (or theories) of syllogistic. The model (hereafter called *I*) which we take to be especially important has *L* as language, *S* as semantics and *D* as deductive system. It is our view that *I* is the system most closely corresponding to Aristotle's explicit theory.[29]

Concerning any mathematical logic there are two kinds of questions. In the first place, there are *internal* questions concerning the mathematical properties of the system itself. For example, we have compared the deductive system *D* with the semantics *S* by asking whether every deducible argument is valid (problem of soundness) and conversely whether every valid argument is deducible (problem of completeness). Both of these questions and all other internal questions are perfectly definite mathematical questions concerning the logic *as a mathematical object.* And if they are answered, then they are answered by the same means used to answer any mathematical question – viz. by logical reasoning from the definitions of the systems together with the relevant mathematical laws. In the second place, there are *external* questions concerning the relationship of the model to things outside of itself. In our case the most interesting question is a fairly vague one – viz. how well does our model represent 'the system' treated in Aristotle's theory of the syllogism?

As the various components of the model were developed, we considered the external questions in some detail and concluded that the model can be used to account for many important aspects of the development of Aristotle's theory, as recorded in the indicated parts of *Analytics*. Moreover, the logic *I* adds nothing to what Aristotle wrote except for giving an explicit reference to 'possible worlds' and formulating a systematic definition of formal deductions. It is especially important to notice that the deductive system involves nothing different in kind from what Aristotle explicitly used – no 'new axioms' were needed and no more basic sort of reasoning was presupposed.

As far as internal questions are concerned it is obvious that *I* is sound, i.e., that all arguments deducible in *D* are valid. This is clear from Section 3 above. The completeness of *I* has been proved[30] – i.e., we have been able to demonstrate as a mathematical fact concerning the logic *I* that every argument valid according to the semantics *S* can be obtained by means of a formal deduction in *D*. Thus not only is Aristotle's logic self-sufficient in the sense of not presupposing any more basic logic but it is also self-sufficient in the sense that no further sound rules can be added without reduncancy.

5.1. *The Possibility of a Completeness Proof in Prior Analytics*

According to Bocheński's view (p. 43), in which we concur, Chapter 23 follows Chapter 7 in *Prior Analytics*, Book I. As already indicated Chapter 7 shows that all syllogisms in the three figures are "perfected by means of the universal syllogisms in the first figure". Chapter 23 (40b17–23) begins with the following words.

> It is clear from what has been said that the syllogisms in these figures are made perfect by means of universal syllogisms in the first figure and are reduced to them. That every syllogism without qualification can be so treated will be clear presently, when it has been proved that every syllogism is formed through one or the other of these figures.

The same chapter (41b3–5) ends thus.

> But when this has been shown it is clear that every syllogism is perfected by means of the first figure and is reducible to the universal syllogisms in this figure.

From these passages *alone* we might suppose that the intermediate material contained the main part of a completeness proof for *D2*, which depended on a 'small' unproved lemma. We might further suppose that the imagined completeness proof had the following three main parts. First, it would define a new deductive system which had the syllogisms in all three figures as rules. Second, it would prove the completeness of the new system. Third, it would show that every deduction in the new system can be transformed into a deduction in *D2* having the same premises and conclusion.

Unfortunately, the text will not support this interpretation. Before considering a more adequate interpretation one can make a few historical observations. In the first place, even raising a problem of completeness seems to be a very difficult intellectual achievement. Indeed, neither Boole nor Frege nor Russell asked such questions.[31] Apparently no one stated a completeness problem[32] before it emerged naturally in connection with the underlying logic of modern Euclidean geometry in the 1920's (Corcoran, 'Classical Logic', pp. 41, 42), and it is probably the case that no completeness result (in this *exact* sense) was printed before 1951 (cf. Corcoran, 'Theories', p. 177 for related results), although the necessary mathematical tools were available in the 1920's. In the second place, Aristotle does not seem to be clear enough about his own semantics to understand the problem. If he had been, then he could have solved the problem definitively for any finite 'topical sublogic' by the same methods employed in *Prior Analytics* (I, 4, 5, 6). In fact, in these chapters he 'solves' the problem for a 'topical sublogic' having only three content words.

In the intervening passages of Chapter 23 Aristotle seems to argue, not that *every* syllogism is deducible in *D2*, but rather that *any syllogism deducible at all* is deducible in *D2*. And, as indicated in his final sentence, he does not believe he has completed his argument. He reasons as follows. In the first place he asserts without proof that any syllogism deducible by means of syllogisms in the three figures is deducible in *D2* (but here he is overlooking the problem of iterated *reductio* mentioned in Section 4.3 above). In any case, granting him that hypothesis, he then argues that any syllogism deducible at all is deducible by means of the syllogisms in the three figures, thus: Every deduction is either direct or hypothetical – the latter including both indirect deductions and those involving *ecthesis* (see above). He considers the direct case first. Here he argues that every direct deduction must have at least two premises as in the three figures and that in the two-premise case the conclusion has already been proved. Then he simply asserts that it is "the same if several middle terms should be necessary" (41a18). In considering the hypothetical deductions he takes up indirect deductions first and observes that after the contradictory of the conclusion is also assumed one proceeds as in the direct case – concluding that the reduction to *D2* is evident in this case also (41a35ff). Finally, he simply asserts that it is the same with the other hypothetical deductions. But this he has immediate misgivings about (41b1). He leaves the proof unfinished to the extent that the non-indirect hypothetical deductions have not been completely dealt with.

6. Conclusion

As a kind of summary of our research we present a review of what we take to be the fundamental achievements of Aristotle's logical theory. In the first place, he clearly distinguished the role of deduction from the role of experience (or intuition) in the development of scienctific theories. This is revealed by his distinction between the axioms of a science and the logical apparatus used in deducing the theorems. Today this would imply a distinction between logical and nonlogical axioms; but Aristotle had no idea of logical axioms (but cf. 77a22–25). Indeed, he gave no systematic discussion of logical truth (*Axx* is not even mentioned once). In the second place, Aristotle developed a natural deduction system which he exemplified and discussed at great length. Moreover, he formulated fairly intricate metamathematical results relating his central system to a simpler one. It is also important to notice that Aristotle's system is sound and strongly complete. In the third place, Aristotle was clear enough about logical consequence so that he was able to discover the method of counter instances for establishing invalidity. This method is the cornerstone of all independence (or invalidity) results, though it probably had to be rediscovered in modern times cf. Cohen and Hersh). In the fourth place, his distinction between perfect and imperfect syllogisms suggests a clear understanding of the difference between deducibility and implication – a distinction which modern logicians believe to be their own (cf. Church, p. 323, fn. 529). In the fifth place, Aristotle used principles concerning form repeatedly and accurately, although it is not possible to establish that he was able to state them nor is even clear that he was consciously aware of them as logical principles.

The above are all highly theoretical points – but Aristotle did not merely theorize; he carried out his ideas and programs in amazing detail despite the handicap of inadequate notation. In the course of pursuing details Aristotle originated many important discoveries and devices. He described indirect proof. He used syntactical variables (alpha, beta, etc.) to stand for content words – a device whose importance in modern logic has not been underestimated. He formulated several rules of inference and discussed their interrelations.

Philosophers sometimes say that Aristotle is the best introduction to philosophy. This is perhaps an exaggeration. One of the Polish logicians once said that the *Analytics* is the best introduction to logic. My own reaction to this remark was unambiguously negative – the severe difficulties in reading the *Analytics* form one obstacle and I felt then that the meager results did not warrant so much study. After carrying out the above research I can compromise to the following extent. I now believe that Aristotle's logic is rich enough, detailed enough, and sufficiently representative of modern logics that a useful set of introductory lectures on mathematical logic could be organized around what I have called the main Aristotelian system.

From a modern point of view, there is only one mistake which can sensibly be charged to Aristotle: his theory of propositional forms is very seriously inadequate. It is remarkable that he did not come to discover this for himself, especially since he mentions specific proofs from arithmetic and geometry. If he had tried to reduce these to his system he may have seen the problem (cf. Mueller, pp. 174–177). But, once the theory of propositional forms is taken for granted, there are no important inadequacies attributable to Aristotle, given the historical context. Indeed, his work is comparable in completeness and accuracy to that of Boole and seems incomparably more comprehensive than the Stoic or medieval efforts. It is tempting to speculate that it was the oversimplified theory of propositional forms that made possible the otherwise comprehensive system. A more adequate theory of propositional forms would have required a much more complicated theory of deduction – indeed, one which was not developed until the present era.

ACKNOWLEDGMENTS

Previous versions of sections of this paper have been presented at various places including: Buffalo Logic Colloquium (March 1971), Pennsylvania Logic Colloquium (April 1971), Hammond Society (Johns Hopkins, April 1971), Laval University (June 1971), University of Montreal (October, 1971), SUNY/Albany (October, 1971), Association for Symbolic Logic (December, 1971), University of Quebec (March, 1972), Cambridge University (June, 1972). Several persons at these talks, or afterward, made useful suggestions, especially Charles Kahn, John Glanville, Lynn Rose, and John Kearns. Throughout the Winter and Spring semesters of 1971–72, I consulted extensively with John Mulhern and Mary Mulhern. My research assistants, Keith Ickes, David Levin, John Herring and Terry Nutter, deserve thanks for conscientious editing and proof-reading of many drafts.

Other papers of mine complementing and overlapping this one appear in *Journal of Symbolic Logic, Mind* and *Archiv für Geschichte der Philosophie*. The first of these three contains the completeness proof. The second treats the question of whether syllogisms are arguments or conditional sentences (as Łukasiewicz holds). The third paper is the result of deleting parts from a paper to which additions were made in formulating the present article.

In February 1972, Smiley's article (*q.v.*) which is in remarkable agreement with the above was brought to my attention. It should be noted that Smiley has considered some questions which have not been treated here. Many ideas expressed in this paper have been colored by consultation with Smiley and by Smiley's article. In particular, my estimate of the value of the Łukasiewicz work has been revised downward as a result of discussion with Smiley.

State University of New York at Buffalo

NOTES

[1] It should be realized that the notion of a 'model' used here is the ordinary one used in discussion of, e.g., wooden models of airplanes, plastic models of boats, etc. Here the adjective 'mathematical' indicates the kind of material employed in the model. I.e., here we are talking about models 'constructed from' mathematical objects. Familiar

mathematical objects are numbers, (mathematical) points, lines, planes, (syntactic) characters, sets, functions, etc. Here we *need* as basic elements only syntactic characters, but the development below also presupposes sets *ab initio*. It should also be realized that a mathematical model is *not* a distinctive sort of mathematical entity – it is simply a mathematical entity conceived of as analogous to something else.

[In order to avoid excessive notes bracketed expressions are used to refer by author (and/or by abbreviated title) and location to items in the list of references at the end of this article. Unless otherwise stated, translations are taken from the Oxford translation (see 'Aristotle').]

[2] These ideas are scattered throughout Church's introductory chapter, but in Schoenfield (*q.v.*) Sections 2.4, 2.5 and 2.6 treat, respectively, languages, semantic systems and deductive systems.

[3] From the best evidence of the respective dates of the *Analytics* (Ross, p. 23) and Euclid's *Elements* (Heath, pp. 1, 2), one can infer that the former was written in the neighborhood of fifty years before the latter. The lives of the two authors probably overlapped; Aristotle is known to have been teaching in Athens from 334 until 322 (Edel, pp. 40, 41) and it is probable both that Euclid received his mathematical training from Aristotle's contemporaries and that he flourished c. 300 (Heath, p. 2). In any case, from internal evidence Ross (p. 56) has inferred that Euclid was probably influenced by the *Analytics*. Indeed, some scholarship on the *Elements* makes important use of Aristotle's theory of the axiomatic organization of science (cf. Heath, pp. 117–124). However, it should be admitted that Hilbert's geometry (*q. v.*) is much more in accord with Aristotle's principles than is Euclid's. For example, Hilbert leaves some terms 'undefined' and he states his universe of discourse at the outset, whereas Euclid fails on both of these points, which were already clear Aristotelian requirements.

[4] Aristotle may have included deductive arguments which would be sound were certain intermediate steps added; cf. Section 5.1 below.

[5] This will account somewhat for the otherwise inexplicable fact already noted by Łukasiewicz (p. 49) and others that there are few passages in the Aristotelian corpus which could be construed as indicating an awareness of propositional logic.

[6] In a doubly remarkable passage (p. 13) Łukasiewicz claims that Aristotle did not reveal the object of his logical theory. It is not difficult to see that Łukasiewicz is correct in saying that Aristotle nowhere admits to the purpose which Łukasiewicz imputes to him. However, other scholars have had no difficulty in discovering passages which do reveal Aristotle's true purpose (cf. Ross, pp. 2, 24, 288; Kneale and Kneale, p. 24).

[7] This point has already been made by Kneale and Kneale (pp. 80–81), who point out further difficulties with Łukasiewicz's interpretation. For yet further sensitive criticism see Austin's review and also Iverson, pp. 35–36.

[8] Although we have no interest in giving an account of how Łukasiewicz may have arrived at his view, it may be of interest to some readers to note the possibility that Łukasiewicz was guided in his research by certain attitudes and preferences not shared by Aristotle. The Łukasiewicz book seems to indicate the following: (1) Łukasiewicz preferred to consider logic as concerned more with truth than with either logical consequence or deduction (e.g., pp. 20, 81). (2) He understands 'inference' in such a way that correctness of inference depends on starting with true premises (e.g., p. 55). (3) He feels that propositional logic is somehow objectively more fundamental than quantificational or syllogistic logic (e.g., pp. 47, 79). (4) He tends to concentrate his attention on axiomatic deductive systems to the neglect of natural systems. (5) He

tends to underemphasize the differences between axiomatic deductive systems and axiomatic sciences. (6) He places the theory of the syllogism on a par with a certain branch of pure mathematics (pp. 14, 15, 73) and he believes that logic has no special relation to thought (pp. 14, 15). Indeed, he seems to fear that talk of logic as a study of reasoning necessarily involves some sort of psychologistic view of logic. (7) He believes that content words or non-logical constants cannot be introduced into logic (pp. 72, 96). The Łukasiewicz attitudes are shared by several other logicians, notably, in this context, by Bocheński (*q.v.*). It may not be possible to argue in an objective way that the above attitudes are incorrect but one can say with certainty that they were not shared by Aristotle.

[9] Exclusion of proper names, relatives, adjectives and indefinite propositions is based more on a reading of the second logic as a whole than on specific passages (but cf. 43a25–40). M. Mulhern, in substantial agreement with this view, has shown my previous attempts to base it on specific passages to be inconclusive as a result of reliance on faulty translation. Her criticisms together with related ones by Charles Kahn (University of Pennsylvania) and Dale Gottlieb (Johns Hopkins) have led to the present version of the last two paragraphs.

[10] Rose (p. 39) has criticized the Łukasiewicz view that no syllogisms with content words are found in the Aristotelian corpus. Our view goes further in holding that *all* Aristotelian syllogisms have content words, i.e., that Aristotle nowhere refers to argument forms or propositional functions. All apparent exceptions are best understood as metalinguistic reference to 'concrete syllogisms'. This view is in substantial agreement with the view implied by Rose at least in one place (p. 25).

[11] In many of the locations cited above Aristotle seems remarkably close to a recognition of 'category mistakes' – a view that nonsense of some sort results from mixing terms from different sciences in the same proposition (e.g., 'the sum of two triangles is a prime number').

[12] It must be recognized that other interpretations are possible – cf. Kneale and Kneale, pp. 55–67. However, in several places (e.g., 85a31–32) Aristotle seems to imply that a secondary substance is nothing but its extension.

[13] This would explain the so-called existential import of *A* and *N* sentences. Notice that, according to this view, existential import is a result of the semantics of the *terms* and has no connection whatever with the meaning of 'All'. In particular, the traditional concern with the meaning of 'All' was misplaced – the issue is properly one of the meaning of categorical terms. As far as we have been able to determine this is the first clear theoretical account of existential import based on textual material.

[14] Jaskowski (*loc. cit.*) gives no textual grounds. There are, however, some passages (e.g., 998b22) which imply that the class of all existent individuals is not a genus. In subsequent developments of 'Aristotelian logic' which include 'negative terms', exclusion of the universe must be maintained to save exclusion of the null set.

[15] This is the mathematical analogue of the classical notion of logical consequence which is clearly presupposed in traditional work on so-called 'postulate theory'. It is important to notice that we have offered only a mathematical analogue of the concept and not a definition of the concept itself. The basic idea is this: Each interpretation represents a 'possible world'. To say that it is logically impossible for the premises to be true and the conclusion false is to say that there is no possible world in which the premises actually are true and the conclusion actually is false. The analogue, therefore, is that no true interpretation of the premises makes the conclusion false. Church (p. 325) attributes this mathematical analogue of logical consequence to Tarski

(pp. 409–420), but Tarski's notion of true interpretation (model) seems too narrow (at best too vague) in that no mention of alternative universes of discourse is made or implied. In fact the limited Tarskian notion seems to have been already known even before 1932 by Lewis and Langford (p. 342), to whom, incidentally, I am indebted for the terms 'interpretation' and 'true interpretation' which seem heuristically superior to the Tarskian terms 'sequence' and 'model', the latter of which has engendered category mistakes – a 'model of set of sentences' in the Tarskian sense is by no means a model, in any ordinary sense, of a set of sentences.

[16] The method of 'contrasting instances' is a fundamental discovery in logic which may not yet be fully appreciated in its historical context. Because Łukasiewicz (p. 71) misconstrued the Aristotelian framework, he said that modern logic does not employ this method. It is obvious, however, that all modern independence (invalidity) results from Hilbert (pp. 30–36) to Cohen (see Cohen and Hersh) are based on developments of this method. Indeed, there were essentially no systematic investigations of questions of invalidity from the time of Aristotle until Beltrami's famous demonstration of the invalidity of the argument whose premises are the axioms of geometry less the Parallel Postulate and whose conclusion is the Parallel Postulate itself (Heath, p. 219). Although there is not a single invalidity result in the *Port Royal Logic* or in Boole's work, for example, modern logic is almost characterizable by its wealth of such results – all harking back to Aristotle's method of contrasting instances.

[17] The Principle of Form is generally accepted in current logic (cf. Church, p. 55). Recognition of its general acceptance is sometimes obscured by two kinds of apparent challenges – each correct in its own way but not to the point at issue. (1) Ryle wants to say (e.g.) that 'All animals are brown' implies 'All horses are brown' and, so, that implication is not a matter of form alone (Ryle, pp. 115–116). It is easy to regard the objection as verbal because, obviously, Ryle is understanding an argument to be 'valid' if addition of certain truths as premises will produce an argument valid in the above sense. (2) Oliver makes a more subtle point (p. 463). He attacks a variant of the Principle of Form by producing examples of the following sort.

If Axy then Nxy	If Sxy then Axy
Nxy	Axy
?Axy	?Sxy

According to Oliver's usage these two arguments are in the same form and yet the one on the left is obviously invalid (suppose x indicates 'men' and y 'horses') while the one on the right is obviously valid (in fact the conclusion follows immediately from the second premise). The resolution is that Oliver's notion of 'being in same form' is not the traditional one; rather it is a different but equally useful notion. Oliver takes two arguments to be in the same form if there is a scheme which subsumes both. Since both are subsumed under the scheme '(if P then Q, Q/P)' they are in the same form. It so happens that the scheme is not a valid scheme; it subsumes both valid and invalid arguments. He does allow the correctness of the above principle as stated (Oliver, p. 465).

[18] Rose (p. 39) emphasizes the fact that Aristotle would establish the invalidity of several arguments at once by judicious choice of interrelated counter interpretations.

[19] A logical question concerning the validity of an argument is settled by using presupposed procedures to deduce the conclusion from the premises. A metalogical question concerns the validity of the presupposed procedures and is usually 'answered' in terms of a theory of meaning (or a semantic system).

[20] One is impressed with the sheer number of times that Aristotle alludes to the fact that there are but two methods of perfecting syllogisms – and this makes it all the more remarkable that an apparent third method occurs, the so-called method of *ecthesis*. There are two ways of explaining the discrepancy. In the first place, *ecthesis* is not a method of proof on a par with the direct and indirect methods; rather it consists in a class of rules of inference on a par with the class of conversion rules and the class of perfect syllogism rules (see below). In the second place, and more importantly, *ecthesis* is clearly extrasystematic relative to Aristotle's logical system (or systems). It is only used three times (Łukasiewicz, p. 59), once in a clearly metalogical passage (25a17) and twice redundantly (28a23, 28b14).

[21] Specifically, for example with regard to the first conversion rule (C1), define the set-theoretic relation [RCl] on L such that for all s and s' in L, s[RCl]s' iff for some x and y in U, $s=Nxy$ and $s'=Nyx$. Thus the *rule* [RCl] is, in effect, the set of all 'its applications'. Generally speaking, an n-placed rule of inference is an $n+1$ – placed relation on sentences. But, of course, not necessarily vice versa (cf. Corcoran, 'Theories', pp. 171–175).

[22] Quine has conveniently listed all such arguments in pp. 76–79 of his *Methods of Logic*. Incidentally, the reader should regard the notion of 'valid argument' in principle 4.2 as convenient parlance for referring to Quine's list – so that no semantic notions have been used in this section in any essential way.

[23] There seems to be a vague feeling in some current circles that an argument with inconsistent premises should not be regarded as an argument at all and that an 'authentic' deduction cannot begin with an inconsistent premise set. However, the only way of determining that a premise set is inconsistent is by deducing contradictory conclusions from it. Thus it would seem that those who wish to withhold 'authenticity' from deductions with inconsistent premise sets must accept the 'authenticity' of those very deductions in order to ascertain their 'non-authenticity'. One must admit, however, that the issue does seem to involve convention (*nomos*) more than nature (*physis*). On the other hand, how does one determine the natural joints of the fowl except by noting where the neatest cuts are made? (cf. *Phaedrus*, 265e).

[24] For an interesting solution to 'the mystery of the fourth figure' (the problem of explaining why Aristotle seemed to stop at the third figure) see Rose, *Aristotle's Syllogistic*, pp. 57–79.

[25] It is in the interest of accuracy that we reluctantly admit that Aristotle also *seems* to claim the converse. It is germane also to observe that, although the above claim is substantiated not only by examples but also by a general formula, the converse is false.

It is also relevant to point out that the existence of this metaproof provides a negative answer to a question raised by William Parry concerning the nature of indirect deductions in Aristotle. Parry wondered whether Aristotle required that the contradiction explicitly involve one of the premises. An affirmative answer would rule out abnormal indirect deductions which, as indicated above, form the basis of Aristotle's metaproof.

[26] For example, the whole revised system $D3$ can be obtained from the system of Corcoran and Weaver (p. 373) by the following changes in the latter. (1) Change the language to L. (2) Replace negations by contradictions. (3) Replace the rules of conditionals and modal operators by the conversion and syllogism rules.

[27] As an indication that Aristotle's clarity concerning *reductio* is significant one may note with Iverson (p. 36) that Łukasiewicz (p. 55) misunderstood indirect proof.

[28] The consideration of extended deductions emerged from a suggestion by Howard Wasserman (Linguistics Department, University of Pennsylvania).

[29] Of course one should not overlook the historical importance of II (the logic having components *L*, *S* and *D2*) nor should the possible importance of *IE* (the logic having components *L*, *S* and *DE*) be minimized. In this connection we have been asked whether there are deductive systems other than *D*, *DE*, *D2* and *D3* implicit in the second logic. This question is confidently answered negatively, even though Patzig (p. 47) alleges to have found other systems in *Prior Analytics* I, 45. It is clear that this chapter merely investigates certain interrelationships among the three figures without raising any issues concerning alternative deductive systems. Although Aristotle speaks of 'reducing' first figure syllogisms to the other figures there is no mention of 'perfecting' first figure syllogisms (or any others for that matter) by means of syllogisms in the other figures. Indeed, because of Aristotle's belief that syllogisms can be *perfected* only through the first figure, one should not expect to find any deductive systems besides those based on first figure syllogistic rules. In addition, one may note that Bocheński (p. 79) alleges to have found other deductive systems outside of the second logic in *Prior Analytics* II, 10. But this chapter is the last of a group of three which together are largely repetitious of the material in *Prior Analytics* I, 45 which we just discussed.

[30] See Corcoran, 'Completeness' and/or 'Natural Deduction'.

[31] Mates (*Stoic Logic*, pp. 4, 81, 82, 111, 112) has argued that the Stoics believed their deductive system to be complete. But had the Aristotelian passage (from 40b23 up to but not including 41b1) been lost Mates would have equivalent grounds for saying that Aristotle believed his system complete. There are no grounds for thinking that the problem was raised in either case.

[32] Unfortunately, the Łukasiewicz formulation makes it possible to confuse these problems with the so-called decision problems. The two types of problems are distinct but interrelated to the extent that decidable logics are generally (but not necessarily) complete. It is hardly necessary to mention the fact that ordinary first order predicate logic is complete but not decidable (Jeffrey, pp. 195ff; Kneale and Kneale, pp. 733–734).

BIBLIOGRAPHY

Anderson, J. and Johnstone, H., *Natural Deduction*, Belmont, Calif. 1963.

Aristotle, *The Works of Aristotle Translated into English* (ed. by W. D. Ross), Vol. 1, Oxford 1928.

Austin, J. L., 'Review of Łukasiewicz's *Aristotle's Syllogistic*', *Mind* **61** (1952), 395–404.

Barnes, J., 'Aristotle's Theory of Demonstration', *Phronesis* **14** (1969), 123–152.

Bocheński, I. M., *History of Formal Logic*, (transl. by I. Thomas), Notre Dame, Indiana, 1961.

Church, A., *Introduction to Mathematical Logic*, Princeton 1956.

Cohen, P. J. and Hersh, R., 'Non-Cantorian Set Theory', *Scientific American*, December 1967, pp. 104–116.

Corcoran, J., 'Three Logical Theories', *Philosophy of Science* **36** (1969), 153–177.

Corcoran, J., 'Review of Quine's *Philosophy of Logic*', *Philosophy of Science* **39** (1972), 88–90.

Corcoran, J., 'Conceptual Structure of Classical Logic', *Philosophy and Phenomenological Research* **33** (1972), 25–47.

Corcoran, J., 'Aristotle's Natural Deduction System', presentation at December 1971 meeting of Association for Symbolic Logic, abstract in *Journal of Symbolic Logic* **37** (1972), 437.

Corcoran, J., 'Completeness of an Ancient Logic', presented at Laval University, June 1971 and at the Buffalo Logic Colloquium, September 1971; *Journal of Symbolic Logic* **37** (1972), 696–702.

Corcoran, J., 'Aristotelian Syllogisms: Valid Arguments or True Generalized Conditionals', forthcoming in *Mind.*

Corcoran, J., 'A Mathematical Model of Aristotle's Syllogistic', *Archiv für Geschichte der Philosophie* **55** (1973), 191–219.

Corcoran, J., and Weaver, G., 'Logical Consequence in Modal Logic', *Notre Dame Journal of Formal Logic* **19** (1969), 370–384.

Edel, A., *Aristotle*, New York 1967.

Frege, G., 'On the Foundations of Geometry', *Jahresbericht* **15** (1906); reprinted in Kluge (ed. and transl.), *Gottlob Frege On the Foundations of Geometry* etc., New Haven and London 1971.

Heath, T., *Euclid's Elements*, Vol. 1 (2nd ed.), New York 1956.

Hilbert, D., *Foundations of Geometry*, (transl. by E. J. Townsend), LaSalle, Illinois, 1965.

Iverson, S. L., *Reduction of the Aristotelian Syllogism*, M. A. Thesis in Philosophy, State University of New York at Buffalo, May, 1964.

Jaskowski, St., 'On the Interpretations of Aristotelian Categorical Propositions in the Predicate Calculus', *Studia Logica* **24** (1969), 161–174.

Jeffrey, R., *Formal Logic: Its Scope and Limits*, New York 1967.

Kneale, W. and Kneale, M., *The Development of Logic*, Oxford 1962.

Kreisel, G., 'Mathematical Logic: What Has it Done for the Philosophy of Mathematics?', in *Bertrand Russell Philosopher of the Century* (ed. by R. Schoenman), London 1967.

Lewis, C. I. and Langford, C. H., *Symbolic Logic*, New York 1959.

Łukasiewicz, J., *Aristotle's Syllogistic* (2nd. ed.), Oxford 1957.

Mates, B., *Stoic Logic*, Berkeley and Los Angeles 1961.

Mates, B., *Elementary Logic*, New York 1965.

Mueller, Ian, 'Stoic and Peripatetic Logic', *Archiv für Geschichte der Philosophie* **51** (1969), 173–187.

Mulhern, J. J., *Problems of the Theory of Predication in Plato's Parmenides, Theatetus, and Sophista*, Ph. D. dissertation in Philosophy, State University of New York at Buffalo, February, 1970.

Mulhern, M. M., *Aristotle's Theory of Predication: The Categoriae Account*, Ph. D. Dissertation in Philosophy, State University of New York at Buffalo, September, 1970.

Oliver, J. W., 'Formal Fallacies and Other Invalid Arguments', *Mind* **76** (1967), 463–78.

Patzig, G., *Aristotle's Theory of the Syllogism* (transl. by J. Barnes), Dordrecht 1968.

Quine, W., *Methods of Logic* (revised ed.), New York 1959.

Quine, W., *Philosophy of Logic*, Englewood Cliffs, N. J., 1970.

Rose, L., *Aristotle's Syllogistic*, Springfield Illinois, 1968.

Ross, W. D., *Aristotle's Prior and Posterior Analytics*, Oxford 1965.

Ryle, G., *Dilemmas* (paperback ed.), London 1960.

Schoenfield, J., *Mathematical Logic*, Reading, Mass., 1967.

Scholz, H., *Concise History of Logic* (transl. by K. Leidecker), New York 1961.

Smiley, T., 'What is a Syllogism?', *Journal of Philosophical Logic* **2** (1973), 136–154.
Tarski, A., *Logic, Semantics and Metamathematics* (transl. by J. Woodger), Oxford 1956.
Tredennick, H., 'Introduction', in *Aristotle, The Organon*, Vol. 1, Cambridge, Mass. 1949, pp. 182–195.
van Heijenoort, J., *From Frege to Godel*, Cambridge, Mass. 1967.
Weaver, G. (see Corcoran and Weaver).