Arithmetic Differential Equations

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ABSTRACT. This research monograph develops an arithmetic analogue of the theory of ordinary differential equations in which derivations are replaced by Fermat quotient operators. The theory is then applied to the construction and study of certain quotient spaces of algebraic curves with respect to correspondences having infinite orbits.

Preface

The main purpose of this research monograph is to develop an arithmetic analogue of the theory of ordinary differential equations. In our arithmetic theory the "time variable" t is replaced by a fixed prime integer p. Smooth real functions, $t \mapsto x(t)$, are replaced by integer numbers $a \in \mathbf{Z}$ or, more generally, by integers in various (completions of) number fields. The derivative operator on functions,

$$x(t) \mapsto \frac{dx}{dt}(t),$$

is replaced by a "Fermat quotient operator" δ which, on integer numbers, acts as $\delta : \mathbf{Z} \to \mathbf{Z}$,

$$a \mapsto \delta a := \frac{a - a^p}{p}.$$

Smooth manifolds (configuration spaces) are replaced by algebraic varieties defined over number fields. Jet spaces (higher order phase spaces) of manifolds are replaced by what can be called "arithmetic jet spaces" which we construct using δ in place of d/dt. Usual differential equations (viewed as functions on usual jet spaces) are replaced by "arithmetic differential equations" (defined as functions on our "arithmetic jet spaces"). Differential equations (Lagrangians) that are invariant under certain group actions on the configuration space are replaced by "arithmetic differential equations" that are invariant under the action of various correspondences on our varieties.

As our main application we will use the above invariant "arithmetic differential equations" to construct new quotient spaces that "do not exist" in algebraic geometry. To explain this we start with the remark that (categorical) quotients of algebraic curves by correspondences that possess infinite orbits reduce to a point in algebraic geometry. In order to address the above basic pathology we propose to "enlarge" algebraic geometry by replacing its algebraic equations with our more general arithmetic differential equations. The resulting new geometry is referred to as $\delta-$*geometry*. It then turns out that certain quotients that reduce to a point in algebraic geometry become interesting objects in $\delta-$geometry; this is because there are more invariant "arithmetic differential equations" than invariant algebraic equations. Here are 3 classes of examples for which this strategy works:

1) *Spherical case*. Quotients of the projective line $\mathbf{P}^1$ by actions of certain finitely generated groups (such as $SL_2(\mathbf{Z})$);

2) *Flat case*. Quotients of $\mathbf{P}^1$ by actions of postcritically finite maps $\mathbf{P}^1 \to \mathbf{P}^1$ with (orbifold) Euler characteristic zero;

3) *Hyperbolic case*. Quotients of modular or Shimura curves (e.g. of $\mathbf{P}^1$) by actions of Hecke correspondences.

Our results will suggest a general conjecture according to which the quotient of a curve (defined over a number field) by a correspondence is non-trivial in $\delta-$geometry for almost all primes p if and only if the correspondence has an "analytic uniformization" over the complex numbers. Then the 3 classes of examples above correspond to *spherical, flat, and hyperbolic* uniformization respectively.

Material included. The present book follows, in the initial stages of its analysis, a series of papers written by the author [**17**]-[**28**]. A substantial part of this book consists, however, of material that has never been published before; this includes our Main Theorems stated at the end of Chapter 2 and proved in the remaining Chapters of the book. The realization that the series of papers [**17**]-[**28**] consists of pieces of one and the same puzzle came relatively late in the story and the unity of the various parts of the theory is not easily grasped from reading the papers themselves; this book is an attempt at providing, among other things, a linear, unitary account of this work. Discussed are also some of the contributions to the theory due to C. Hurlburt [**71**], M. Barcau [**2**], and K. Zimmerman [**29**].

Material omitted. A problem that was left untouched in this book is that of putting together, in an adelic picture, the various $\delta-$geometric pictures, as p varies. This was addressed in our paper [**26**] where such an adelic theory was developed and then applied to providing an arithmetic differential framework for functions of the form $(p, a) \mapsto c(p, a)$, p prime, $a \in \mathbf{Z}$, where $L(a, s) = \sum_n c(n, a)n^{-s}$ are various families of $L-$functions parameterized by a. Another problem not discussed in this book is that of generalizing the theory to higher dimensions. A glimpse into what the theory might look like for higher dimensional varieties can be found in [**17**] and [**3**]. Finally, we have left aside, in this book, some of the Diophantine applications of our theory such as the new proof in [**18**] of the Manin-Mumford conjecture about torsion points on curves and the results in [**21**], [**2**] on congruences between classical modular forms.

Prerequisites. For most of the book, the only prerequisites are the basic facts of algebraic geometry (as found, for instance, in R. Hartshorne's textbook [**66**]) and algebraic number theory (as found, for instance, in Part I of S. Lang's textbook [**87**]). In later Chapters more background will be assumed and appropriate references will be given. In particular the last Chapter will assume some familiarity with the $p-$adic theory of modular and Shimura curves. From a technical point of view the book mainly addresses graduate students and researchers with an interest in algebraic geometry and / or number theory. However, the general theme of the book, its strategy, and its conclusions should appeal to a general mathematical audience.

Plan of the book. We will organize our presentation around the motivating "quotient space" theme. So quotient spaces will take center stage while "arithmetic jet spaces" and the corresponding analogies with the theory of ordinary differential equations will appear as mere tools in our proofs of $\delta-$geometric theorems. Accordingly, the Introduction starts with a general discussion of strategies to construct quotient spaces and continues with a brief outline of our $\delta-$geometric theory. We also include, in our Introduction, a discussion of links, analogies, and / or discrepancies between our theory and a number of other theories such as: differential equations on smooth manifolds [**114**], the Ritt-Kolchin differential algebra [**117**], [**84**], [**32**], [**13**], the difference algebraic work of Hrushovski and Chatzidakis [**69**], the theory of dynamical systems [**109**], Connes' non-commutative geometry [**36**], the theory of Drinfeld modules [**52**], Dwork's theory [**53**], Mochizuchi's $p-$adic Teichmüller theory [**111**], Ihara's theory of congruence relations [**73**] [**74**], and the work of Kurokawa, Soulé, Deninger, Manin, and others on the "field with one element" [**86**], [**128**], [**98**], [**46**]. In Chapter 1 we discuss some algebro-geometric preliminaries; in particular we discuss analytic uniformization of correspondences on algebraic curves. In Chapter 2 we discuss our $\delta-$geometric strategy in detail and we state our main conjectures and a sample of our main results. In Chapters 3, 4, 5 we develop the general theory of arithmetic jet spaces. The corresponding 3 Chapters deal with the global, local, and birational theory respectively. In Chapters 6, 7, 8 we are concerned with our applications of $\delta-$geometry to quotient spaces: the corresponding 3 Chapters are concerned with correspondences admitting a spherical, flat, or hyperbolic analytic uniformization respectively. Details as to the contents of the individual Chapters are given at the beginning of each Chapter. All the definitions of new concepts introduced in the book are numbered and an index of them is included after the bibliography. A list of references to the main results is given at the end of the book. Internal references of the form Theorem $x.y$, Equation $x.y$, etc. refer to Theorems, Equations, etc. belonging to Chapter x (if $x \neq 0$) or the Introduction (if $x = 0$). Theorems, Propositions, Lemmas, Corollaries, Definitions, and Examples are numbered in the same sequence; Equations are numbered in a separate sequence. Here are a few words about the dependence between the various Chapters. The impatient reader can merely skim through Chapter 1; he/she will need to read at least the (numbered) "Definitions" (some of which are not standard). Chapters 2-5 should be read in a sequence. Chapters 6-8 are largely (although not entirely) independent of one another but they depend upon Chapters 2-5.

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