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Arithmetic Fuzzy Models

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***Abstract*—It is well known that a fuzzy rule base can be interpreted in different ways. From a logical point of view, the conjunctive interpretation is preferred, while from a practical point of view, the disjunctive interpretation has been dominantly present. Each of these interpretations results in a specific fuzzy relation modelling the fuzzy rule base. Basic interpolation requirements naturally suggest a corresponding inference mechanism: the direct image for the conjunctive interpretation, and the subdirect image for the disjunctive interpretation. Interpolation then corresponds to solvability of some system of fuzzy relational equations. In this paper, we show that other types of fuzzy relations, closely related to Takagi–Sugeno models, are of major interest as well. These fuzzy relations are based on addition and multiplication only, whence the name arithmetic fuzzy models. Under some mild requirements, these fuzzy relations turn out to be solutions of the same systems of fuzzy relational equations. The impact of these results is both theoretical and practical: there exist simple solutions to systems of fuzzy relational equations, other than the extremal solutions that have received all the attention so far, that are moreover easy to implement.**

***Index Terms*—Direct image, fuzzy relational equation, fuzzy rule base, interpolation, subdirect image.**

I. INTRODUCTION

On many occasions, *fuzzy rule-based systems* have been demonstrated to be powerful tools in modelling, decision making and automatic control. In essence, such a system consists of two main components: a fuzzy rule and an inference mechanism. The choice of an appropriate fuzzy relation modelling the fuzzy rule base and of a compatible inference mechanism are crucial for the proper functioning of the whole system.

Consider two arbitrary universes X and Y. The classes of fuzzy sets in X and Y are denoted as $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. The information present in a given fuzzy rule base is contained in pairs of input-output fuzzy sets $(\mathbf{A}_1, \mathbf{B}_1)$, …, $(\mathbf{A}_n, \mathbf{B}_n)$, expressing that fuzzy set $\mathbf{B}_i \in \mathcal{F}(Y)$ is assigned to fuzzy set $\mathbf{A}_i \in \mathcal{F}(X)$ [1].

There exist two standard approaches to modelling a given fuzzy rule base by an appropriate fuzzy relation $R \in \mathcal{F}(X \times Y)$. Consider a left-continuous t-norm $*$ and its residual operation $\rightarrow_*$ (residual implication) defined by $a \rightarrow_* b = \sup\{c \in [0,1] \mid a * c \le b\}$ [2]. The first approach consists in constructing the fuzzy relation $\hat{\mathbf{R}}_* \in \mathcal{F}(X \times Y)$ defined by

$$\hat{\mathbf{R}}_*(x,y) = \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_* \mathbf{B}_i(y)) \ . \quad (1)$$

As stated by Dubois et al. [3]: "In the above view, each piece of information (fuzzy rule) is viewed as a constraint. This view naturally leads to a conjunctive way of merging the individual pieces of information since the more information, the more constraints and the less possible values to satisfy them." This fact together with the fact that the minimum operation as well as other t-norms are appropriate interpretations of conjunction (the logical connective AND) and residual operations are appropriate interpretations of implication [4]–[8], the above statement leads to the conclusion that the fuzzy relation $\hat{\mathbf{R}}_*$ defined by (1) is a proper model of the following set of fuzzy rules

$$\begin{array}{ccc} \textbf{IF } x \text{ is } \mathcal{A}_1 & \textbf{THEN} & y \text{ is } \mathcal{B}_1 \\ & \dots & \\ & \textbf{AND} & \\ & \dots & \\ \textbf{IF } x \text{ is } \mathcal{A}_n & \textbf{THEN} & y \text{ is } \mathcal{B}_n \end{array} \quad (2)$$

where $\mathcal{A}_i$ and $\mathcal{B}_i$ are membership predicates represented by fuzzy sets $\mathbf{A}_i \in \mathcal{F}(X)$ and $\mathbf{B}_i \in \mathcal{F}(Y)$.

The second approach to modelling a given fuzzy rule base, initiated by a successful experimental application by Mamdani and Assilian [9], consists in constructing the fuzzy relation $\check{\mathbf{R}}_* \in \mathcal{F}(X \times Y)$ defined by

$$\check{\mathbf{R}}_*(x,y) = \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)) \ . \quad (3)$$

Obviously, the fuzzy relation $\check{\mathbf{R}}_*$ can hardly be considered as a model of fuzzy rule base (2). As mentioned above, a t-norm is an appropriate interpretation of conjunction, not of implication; moreover, the maximum operation disjunctively aggregating all rules has nothing in common with the logical connective AND.

We again recall the work of Dubois et al. [3]: "It seems that fuzzy rules modelled by (3) are not viewed as constraints but are considered as pieces of data. Then the maximum in (3) expresses accumulation of data". This fact together with the known fact that the maximum operation as well as other t-conorms are appropriate interpretations of disjunction (the logical connective OR) [6], [7] leads to the conclusion that the fuzzy relation $\check{\mathbf{R}}_*$ defined by (3) is a proper model of the

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following set of fuzzy rules

$$\begin{array}{ccc} x \text{ is } \mathcal{A}_1 & \textbf{AND} & y \text{ is } \mathcal{B}_1 \\ & \dots & \\ & \textbf{OR} & \\ & \dots & \\ x \text{ is } \mathcal{A}_n & \textbf{AND} & y \text{ is } \mathcal{B}_n \end{array} \tag{4}$$

It is worth mentioning that distinguishing between the conditional (IF–THEN) form of fuzzy rules (2) and the Cartesian product (AND) form of fuzzy rules (4) at the syntactical level is not commonly done, but it can be found e.g. in [10]–[12]. Usually only the form (2) is considered because of several, mainly historical, reasons and the differences are taken into account only at the semantical level. But the differences can play a crucial role in further implementations and, therefore, they should be kept in mind.

Remark 1.1: For a detailed discussion of both forms of fuzzy rules, we refer to [6] in which the topic is elaborated from the point of view of mathematical logic. We also refer to [7] in which the authors investigate the problem from the point of view of *fuzzy logic in narrow sense* (algebraic and logical background) as well as from the point of view of *fuzzy logic in broader sense* (extensions serving the modeling of vagueness). The implicative and conjunctive approaches to fuzzy rules are also jointly addressed in [11]–[13].

II. Systems of fuzzy relational equations

Each fuzzy rule-based system adopts an inference mechanism. It is a deduction rule determining an output $\mathbf{B} \in \mathcal{F}(Y)$ for a given input $\mathbf{A} \in \mathcal{F}(X)$. In particular, this output is defined as an image of $\mathbf{A}$ under the fuzzy relation $\mathbf{R} \in \mathcal{F}(X \times Y)$ modelling the given fuzzy rule base. In most cases, one uses the *direct image* (sup-$*$ composition),

$$\mathbf{B} = \mathbf{A} \circ_* \mathbf{R}\,, \tag{5}$$

which stems from the *compositional rule of inference* introduced by Zadeh [14]. It is defined by

$$(\mathbf{A} \circ_* \mathbf{R})(y) = \bigvee_{x \in X} (\mathbf{A}(x) * \mathbf{R}(x,y))\,, \tag{6}$$

and it is worth mentioning that its logical background coincides with the *generalized modus ponens* [6].

A fuzzy rule base may be viewed as a partial function from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$, i.e. as a mapping that assigns $\mathbf{B}_i \in \mathcal{F}(Y)$ to $\mathbf{A}_i \in \mathcal{F}(X)$, for $i = 1, \dots, n$. The purpose of building a fuzzy inference module on the basis of the fuzzy rule base is to extend this partial function to a total function. It means that, in some 'reasonable manner', we have to associate with any $\mathbf{A} \in \mathcal{F}(X)$ some $\mathbf{B} \in \mathcal{F}(Y)$. This should be done in such a way that any input $\mathbf{A}_i$ is exactly mapped to $\mathbf{B}_i$, for $i = 1, \dots, n$. Otherwise, the total function would not be an extension of the partial one. This requirement leads to the following *system of direct image equations*

$$\mathbf{A}_i \circ_* \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1, \dots, n\,. \tag{7}$$

A fuzzy relation $\mathbf{R} \in \mathcal{F}(X \times Y)$ which satisfies (7) is called a *solution* of the system of direct image equations.

We recall some basic results concerning systems of direct image equations (see e.g. [15]–[17]).

Theorem 2.1: System (7) is solvable if and only if $\hat{\mathbf{R}}_*$ is a solution of this system. In case of solvability, $\hat{\mathbf{R}}_*$ is the greatest solution of (7).

Theorem 2.1 is a crucial theorem in the study of direct image equations. Beside the fact that it provides a necessary and sufficient condition for the solvability of system (7), it determines a particular solution which turns out to be the greatest solution. This means that if there exists a solution $\mathbf{R}$ to the given system, then necessarily $\mathbf{R}(x,y) \leq \hat{\mathbf{R}}_*(x,y)$, for any $(x,y) \in X \times Y$.

Its particular importance is as follows. Whenever we deal with fuzzy rule base (2) modelled by $\hat{\mathbf{R}}_*$, the direct image is the first choice for an inference mechanism, since $\hat{\mathbf{R}}_*$ holds a unique position in the set of all possible solutions of the corresponding system of direct image equations. First, there are no other solutions when $\hat{\mathbf{R}}_*$ is not a solution. Second, if $\hat{\mathbf{R}}_*$ is a solution, it is the greatest one.

Let us recall a theorem specifying conditions under which even $\check{\mathbf{R}}_*$ is a solution of system (7) [16], [18]. Hence, it can also be used as a solvability criterion (sufficient condition) for this system. Note that a fuzzy set is called normal if it has at least one element with membership degree equal to one. The following theorem uses the biresidual operation $\leftrightarrow_*$ corresponding to $*$, defined by $a \leftrightarrow_* b = (a \rightarrow_* b) \wedge (b \rightarrow_* a)$.

Theorem 2.2: Let all $\mathbf{A}_i$, $i = 1, \dots, n$, be normal. Then $\check{\mathbf{R}}_*$ is a solution of (7) if and only if the condition

$$\bigvee_{x \in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \leq \bigwedge_{y \in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y)) \tag{8}$$

holds for any $i, j \in \{1, \dots, n\}$.

Theorem 2.2 specifies a condition under which $\check{\mathbf{R}}_*$, connected to the direct image inference mechanism, is an appropriate model of a fuzzy rule base. On the other hand, whenever $\check{\mathbf{R}}_*$ is an appropriate model, the fuzzy relation $\hat{\mathbf{R}}_*$ is an appropriate model too. So, $\check{\mathbf{R}}_*$ does not hold a unique position among all possible solutions as does $\hat{\mathbf{R}}_*$. However, it is not a common approach to fix an inference mechanism first and then search for an appropriate model of a given fuzzy rule base which would be a solution of a corresponding system of fuzzy relational equations. On the contrary, usually a fuzzy relation is selected first based on certain arguments. It is therefore natural to adopt another inference mechanism which yields a system of fuzzy relational equations for which $\check{\mathbf{R}}_*$ holds an analogous position among all possible solutions.

Let us recall the subdirect image (inf-$\rightarrow_*$ composition)

$$\mathbf{B} = \mathbf{A} \lhd_* \mathbf{R} \tag{9}$$

which is related to the *triangular subcomposition* introduced by Bandler and Kohout [19], [20]. It is defined by

$$(\mathbf{A} \lhd_* \mathbf{R})(y) = \bigwedge_{x \in X} (\mathbf{A}(x) \rightarrow_* \mathbf{R}(x,y))\,. \tag{10}$$

The subdirect image, in contrast to the direct image, has no connection to the generalized modus ponens deduction rule and its motivation was quite different [19]. On the other hand, as mentioned in [21], the inference mechanism need not necessarily be logical, but simply a mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ fulfilling certain properties.

Next, we recall some basic facts [15], [17] about *systems of subdirect image equations* formulated as follows

$$\mathbf{A}_i \lhd_* \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1, \ldots, n\,. \tag{11}$$

They should justify our further use of the subdirect image as an inference mechanism.

Theorem 2.3: System (11) is solvable if and only if $\check{\mathbf{R}}_*$ is a solution of this system. In case of solvability, $\check{\mathbf{R}}_*$ is the smallest solution of (11).

Let us recall the following theorem [22].

Theorem 2.4: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal. Then $\hat{\mathbf{R}}_*$ is a solution of (11) if and only if the condition

$$\bigvee_{x \in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \leq \bigwedge_{y \in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y))$$

holds for any $i, j \in \{1, \ldots, n\}$.

We observe that $\check{\mathbf{R}}_*$ holds precisely the same position among all solutions of system (11) as the fuzzy relation $\hat{\mathbf{R}}_*$ holds in the case of the system of direct image equations. So, if we adopt the idea of [21] that the inference mechanism can be understood as some mapping, without deeper connection to a logical deduction rule, nothing prevents us from treating the subdirect image as a special kind of inference mechanism.

Based on the facts and theorems above, we claim that fuzzy relation $\check{\mathbf{R}}_*$ should be treated together with the subdirect image, no matter that Theorem 2.4 determines conditions under which even $\hat{\mathbf{R}}_*$ "would work" together with the subdirect image (from the interpolation point of view). The reason is that, in practice, we first determine an interpretation of the fuzzy rule base and then select an inference mechanism. Consequently, we claim that the set of fuzzy rules (2) predetermines the direct image inference mechanism, while the set of fuzzy rules (4) predetermines the subdirect image inference mechanism.

On the other hand, it should be stressed that considering the subdirect image inference mechanism together with $\hat{\mathbf{R}}_*$ or the direct image inference mechanism together with $\check{\mathbf{R}}_*$ might lead to significant computational savings. However, in these cases, the interpolation property is not ensured, which might lead to unsatisfactory results. This issue is not related to the present investigation and we refer the reader to [23], [24].

Remark 2.5: It is worth mentioning that condition (8) appearing in Theorems 2.2 and 2.4 is from the practical point of view not very convenient. On the other hand, if the antecedent fuzzy sets form a so-called $*$-semi-partition [25], condition (8) is fulfilled automatically [26].

III. Additive fuzzy models

A lot of work has been done in the field of fuzzy relational equations [15], [27]-[31], mainly aiming at the identification of the greatest and smallest solutions, maximal and minimal solutions, solvability conditions, etc. Unfortunately, this work has rarely attracted the attention of practitioners. One of the reasons is the popularity of neuro-fuzzy models [32] and Takagi–Sugeno models [33].

Recall that Takagi and Sugeno proposed fuzzy rules of the following form

$$\textbf{IF } x \text{ is } \mathcal{A}_i \quad \textbf{THEN} \quad y \text{ is } f_i(x), \qquad i = 1, \ldots, n\,, \tag{12}$$

where the conditional relationship is determined by a so-called fuzzy implication [33]. However, they model this implication by the weighted arithmetic mean

$$y = \frac{\sum_{i=1}^{n} \mathbf{A}_i(x) f_i(x)}{\sum_{i=1}^{n} \mathbf{A}_i(x)}\,, \tag{13}$$

which obviously has nothing in common with logical implication and so, does not correspond to fuzzy rules in the conditional form (2). However, because of their powerful approximation capabilities, Takagi–Sugeno models became very popular in the fuzzy community. The consequent parts $f_i(x)$ of the rules (12) are usually polynomial functions, so that we can talk about k-th order Takagi–Sugeno models, where $k \in \mathbb{N}$ denotes the degree of the consequent polynomial functions.

Remark 3.1: Takagi–Sugeno models are basically data-driven, i.e. determined on the basis of a finite input-output data set. Beside this standard approach to the identification of a fuzzy model, there exists an integral (continuous) version of the 0-th order Takagi–Sugeno model (minimizing a modified criterion) and developed based on the theory of fuzzy transforms [34].

Often the so-called *Ruspini* condition [35] is imposed, requiring that

$$\sum_{i=1}^{n} \mathbf{A}_i(x) = 1\,, \qquad \text{for all } x \in X\,, \tag{14}$$

and therefore, the interpretation of a 0-th order Takagi–Sugeno model is given by

$$y = \sum_{i=1}^{n} \mathbf{A}_i(x) b_i\,, \tag{15}$$

where $b_i \in \mathbb{R}$ are the right-hand sides of the Takagi–Sugeno rules.

Considering the crisp values b_i as singletons $\mathbf{B}_i$, i.e. special fuzzy sets in Y, and having in mind that the product is a particular t-norm [2] and the fact that we impose the Ruspini condition, leads to the following natural fuzzy relation modelling Takagi–Sugeno rules with fuzzy consequents

$$\mathbf{R}_*^{\oplus}(x, y) = \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y))\,, \tag{16}$$

where $\oplus$ is the Łukasiewicz t-conorm and $*$ is an arbitrary t-norm. Moreover, it coincides with standard fuzzy relations appearing in the neuro-fuzzy literature [32].

The fuzzy relation $\mathbf{R}_*^{\oplus}$ is related to the disjunctive fuzzy model $\check{\mathbf{R}}_*$, where the disjunction is now modelled by the Łukasiewicz t-conorm. Therefore, $\mathbf{R}_*^{\oplus}$ given by (16) can be considered as a model of fuzzy rule base (4). For obvious reasons, this model is called *additive* [36].

Remark 3.2: Modelling fuzzy rule base (2) by the fuzzy relation $\hat{\mathbf{R}}_*$ can be viewed in the light of "conjunctive normal forms" (CNF), while modelling fuzzy rule base (4) by $\check{\mathbf{R}}_*$

can be viewed in the "disjunctive normal forms" (DNF) [37]. These normal forms were proposed to investigate fuzzy models from an approximation point of view. The additive fuzzy models $\mathbf{R}_*^{\oplus}$ correspond to the "additive normal forms" (ANF), which were motivated by one particular additive normal form in [37] and then further studied in [38], [39]. The relationship between additive normal forms and the fuzzy transform (and hence also Takagi–Sugeno rules) has been discussed in [38].

IV. Additive fuzzy models and systems of fuzzy relational equations

We start by stating the definition of generalized orthogonality [37], which has proven crucial in the study of additive normal forms [37], [39].

Definition 4.1: We say that a collection of fuzzy sets $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \dots, n$, fulfills the orthogonality condition if

$$\bigoplus_{\substack{i=1\\ i\neq j}}^{n} \mathbf{A}_i(x) = 1 - \mathbf{A}_j(x) \tag{17}$$

for any $j \in \{1, \dots, n\}$.

As shown above, the Ruspini condition seems to be pivotal for the additive fuzzy models given by (16). The following lemma shows that both conditions are equivalent.

Lemma 4.2: A collection of fuzzy sets $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \dots, n$, fulfills the orthogonality condition if and only if (14) holds.

Proof: Suppose that the orthogonality condition is fulfilled. Consider arbitrary $j \in \{1, \dots, n\}$. First, let $x \in X$ be such that $\mathbf{A}_j(x) \in]0, 1]$, then

$$\begin{aligned} 1 >& 1 - \mathbf{A}_j(x) = \bigoplus_{\substack{i=1\\ i\neq j}}^{n} \mathbf{A}_i(x) \\ 1 >& \min\left(\sum_{\substack{i=1\\ i\neq j}}^{n} \mathbf{A}_i(x), 1\right) \\ 1 >& \sum_{\substack{i=1\\ i\neq j}}^{n} \mathbf{A}_i(x) \end{aligned}$$

and therefore (14) is fulfilled.

Second, let $x \in X$ be such that $\mathbf{A}_j(x) = 0$. Then (17) implies that there necessarily exists an index $k \neq j$ such that either $\mathbf{A}_k(x) \in]0, 1[$ or $\mathbf{A}_k(x) = 1$. The rest goes as above.

The converse part of the proof, showing that (14) implies (17) is trivial. ■

A. Subdirect image equations

Since the additive fuzzy model (16) can be considered as a model of fuzzy rule base (4), it is expected to be related to the subdirect image inference mechanism. This subsection investigates this relationship via subdirect image equations.

Theorem 4.3: Let all $\mathbf{A}_i$, $i = 1, \dots, n$, be normal and fulfill the Ruspini condition. Then system (11) is solvable and $\mathbf{R}_*^{\oplus}$ is a solution.

Proof: Consider arbitrary $j \in \{1, \dots, n\}$ and

$$\mathbf{B}(y) = \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y))\right) .$$

Since the supremum is the smallest t-conorm and $\to_*$ is increasing in its second argument, it holds that

$$\begin{aligned} \mathbf{B}(y) &\geq \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y))\right) \\ &\geq \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* (\mathbf{A}_j(x) * \mathbf{B}_j(y))\right) . \end{aligned}$$

Since $(a \to_* (a * b)) \geq b$, it follows that $\mathbf{B} \supseteq \mathbf{B}_j$.

On the other hand, since $\to_*$ is increasing in its second argument, it holds that

$$\begin{aligned} &\mathbf{B}(y) \\ &= \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* (\bigoplus_{\substack{i=1\\ i\neq j}}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y))) \oplus (\mathbf{A}_j(x) * \mathbf{B}_j(y))\right) \\ &\leq \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* (\bigoplus_{\substack{i=1\\ i\neq j}}^{n} (\mathbf{A}_i(x) * 1)) \oplus (1 * \mathbf{B}_j(y))\right) \\ &= \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* \left(\bigoplus_{\substack{i=1\\ i\neq j}}^{n} (\mathbf{A}_i(x)) \oplus \mathbf{B}_j(y)\right)\right) . \end{aligned}$$

The Ruspini condition yields

$$\mathbf{B}(y) \leq \bigwedge_{x\in X} (\mathbf{A}_j(x) \to_* ((1 - \mathbf{A}_j(x)) \oplus \mathbf{B}_j(y))) .$$

Let $x' \in X$ be such that $\mathbf{A}_j(x') = 1$. Since $1 \to_* b = b$, it holds that

$$\begin{aligned} \mathbf{B}(y) &\leq (\mathbf{A}_j(x') \to_* ((1 - \mathbf{A}_j(x')) \oplus \mathbf{B}_j(y))) \\ &= (1 \to_* \mathbf{B}_j(y)) = \mathbf{B}_j(y) , \end{aligned}$$

which yields $\mathbf{B} \subseteq \mathbf{B}_j$. ■

Due to Theorem 2.3 we can state the following corollary of Theorem 4.3.

Corollary 4.4: Let all $\mathbf{A}_i$, $i = 1, \dots, n$, be normal and fulfill the Ruspini condition. Then $\check{\mathbf{R}}_*$ is a solution of system (11).

Moreover, it can be demonstrated on the basis of the proof of Theorem 4.3, that $\mathbf{R}_*^{\oplus}$ is not the only additive fuzzy model which is a solution of system (11).

Proposition 4.5: Let all $\mathbf{A}_i$, $i = 1, \dots, n$, be normal and fulfill the Ruspini condition. Furthermore, let $\blacktriangle$ be a t-norm such that $* \leq \blacktriangle$. Then the fuzzy relation $\mathbf{R}_{\blacktriangle}^{\oplus}$ is a solution of system (11).

Proof: Let $\mathbf{B}$ be defined as in the proof of Theorem 4.3. Consider arbitrary $j \in \{1, \dots, n\}$ and

$$\mathbf{B}_{\blacktriangle}(y) = \bigwedge_{x\in X} \left(\mathbf{A}_j(x) \to_* \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) \blacktriangle \mathbf{B}_i(y))\right) .$$

Since $\rightarrow_*$ is increasing in its second argument, it holds that $\mathbf{B}_\blacktriangle(y) \geq \mathbf{B}(y) \geq \mathbf{B}_j(y)$. The proof of the opposite inequality is identical to that in the proof of Theorem 4.3 as ▲ has neutral element 1 as well. ■

Let us briefly summarize results of this subsection. Theorem 4.3 yields an easy-to-check condition guaranteeing the proper performance of an additive fuzzy model connected to the subdirect image inference mechanism. Moreover, the assumptions refer to the antecedent fuzzy sets only, and its fulfillment can be ensured prior to an identification process. Since no solvability is assumed, this theorem also has a theoretical impact, as it specifies a sufficient solvability condition, which then implies that $\check{\mathbf{R}}_*$ is a solution as well. Finally, due to Proposition 4.5, a wide variety of t-norms can be used in the additive fuzzy models. Indeed, we can use an additive fuzzy model based on any t-norm stronger than the t-norm $*$ used in the direct image inference mechanism.

B. Direct image equations

This subsection focuses on systems of direct image equations. Theorem 2.2 states that, under certain conditions, the disjunctive fuzzy model $\check{\mathbf{R}}_*$ is a solution of system (7). Here, we examine the suitability of the additive fuzzy model for this system.

Theorem 4.6: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition, and let $* \leq \otimes$, where $\otimes$ is the Łukasiewicz t-norm. Then system (7) is solvable and $\mathbf{R}_*^\oplus$ is a solution.

Proof: Consider arbitrary $j \in \{1, \ldots, n\}$ and

$$\mathbf{B}(y) = \bigvee_{x \in X} \left(\mathbf{A}_j(x) * \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)) \right) .$$

Let $x \in X$ be such that $\mathbf{A}_j(x') = 1$. Since the supremum is the smallest t-conorm and $1 \rightarrow_* b = b$, it holds that

$$\begin{aligned} \mathbf{B}(y) &\geq \bigvee_{x \in X} \left(\mathbf{A}_j(x) * \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)) \right) \\ &\geq (\mathbf{A}_j(x') * (\mathbf{A}_j(x') * \mathbf{B}_j(y))) \\ &= 1 * (1 * \mathbf{B}_j(y)) = \mathbf{B}_j(y) , \end{aligned}$$

which yields $\mathbf{B} \supseteq \mathbf{B}_j$.

On the other hand, similarly as in the proof of Theorem 4.3, it holds that

$$\mathbf{B}(y) \leq \bigvee_{x \in X} \left(\mathbf{A}_j(x) * \left(\bigoplus_{\substack{i=1 \\ i \neq j}}^{n} (\mathbf{A}_i(x)) \oplus \mathbf{B}_j(y) \right) \right) .$$

The Ruspini condition yields

$$\mathbf{B}(y) \leq \bigvee_{x \in X} (\mathbf{A}_j(x) * ((1 - \mathbf{A}_j(x)) \oplus \mathbf{B}_j(y))) .$$

Since $(1 - a) \oplus b = a \rightarrow_\otimes b$ and $* \leq \otimes$, it follows that

$$\mathbf{B}(y) \leq \bigvee_{x \in X} (\mathbf{A}_j(x) \otimes (\mathbf{A}_j(x) \rightarrow_\otimes \mathbf{B}_j(y))) .$$

Finally, since $a \otimes (a \rightarrow_\otimes b) \leq b$, we obtain $\mathbf{B} \subseteq \mathbf{B}_j$. ■

Due to Theorem 2.1 we can state the following corollary of Theorem 4.6.

Corollary 4.7: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition, and let $* \leq \otimes$. Then $\hat{\mathbf{R}}_*$ is a solution of system (7).

Theorem 4.6 requires to use a t-norm that is even weaker than the Łukasiewicz t-norm, which already is a very weak t-norm. Hence, for practical applications, perhaps only the case $* = \otimes$ is of interest. In this case, the Łukasiewicz t-norm is used both in the sup-$\otimes$ composition as inference mechanism, as well as for connecting antecedent and consequent fuzzy sets in the corresponding fuzzy model $\mathbf{R}_\otimes^\oplus$.

This result can again be strengthened in the following proposition.

Proposition 4.8: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition. Furthermore, let ▲ be an arbitrary t-norm and let $* \leq \otimes$. Then the fuzzy relation $\mathbf{R}_\blacktriangle^\oplus$ is a solution of (7).

Proof: The proof is a variation on the proof of Theorem 4.6 and is therefore omitted. ■

The above proposition forces us to use a t-norm weaker than the Łukasiewicz t-norm in the inference mechanism, while the interpretation of the fuzzy rule base can be built w.r.t. an arbitrary t-norm ▲. Similarly as for the subdirect image equations, only normality of the antecedents and the Ruspini condition are assumed.

V. Multiplicative fuzzy models

The additive fuzzy models with as t-norm the product, which we primarily focus on because of our initial motivation, could be called *arithmetic fuzzy models* since they use arithmetic operations only. They were motivated by a variety of fuzzy methods such as neuro-fuzzy systems or Takagi-Sugeno rules using weighted average approaches rather than logical operations.

As we have shown in Section IV, the investigation of arithmetic fuzzy models does not only yield results concerning the proper usage of such models, but also points out new solutions to well-known systems of fuzzy relational equations. An important observation is the fact that the fulfillment of two, in practice very often required, conditions leads to the solvability of the corresponding system of fuzzy relational equations. These results are relevant for practice as these conditions only relate to the antecedent fuzzy sets. This enables us to identify a fuzzy rule base in such a way that it ensures the solvability of a corresponding system of fuzzy relational equations, even with arbitrary consequent fuzzy sets, e.g. identified from data using some algorithm.

The impact of these results is obviously more significant in the case of systems of subdirect image equations, because in that case we assumed a weak t-norm in the inference mechanism.

In view of the two different inference mechanisms, it seems quite natural to introduce also *multiplicative fuzzy models*:

$$\mathbf{R}_*^\otimes(x, y) = \bigotimes_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_* \mathbf{B}_i(y)) . \quad (18)$$

The multiplicative fuzzy model $\mathbf{R}_*^{\otimes}$ is related to the conjunctive fuzzy model $\hat{\mathbf{R}}_*$, where the conjunction is now modelled by the Łukasiewicz t-norm. When using as t-norm the product, we obtain a second arithmetic fuzzy model.

VI. Multiplicative fuzzy models and systems of fuzzy relational equations

This section investigates under which conditions the multiplicative fuzzy model is a solution of a system of direct image equations or a system of subdirect image equations.

A. Direct image iquations

Theorem 6.1: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition. Then system (7) is solvable and $\mathbf{R}_*^{\otimes}$ is a solution.

Proof: Consider arbitrary $j \in \{1, \ldots, n\}$ and

$$\mathbf{B}(y) = \bigvee_{x \in X} \left(\mathbf{A}_j(x) * \bigotimes_{i=1}^{n} (\mathbf{A}_i(x) \to_* \mathbf{B}_i(y)) \right) .$$

Since the minimum operation is the greatest t-norm, it holds that

$$\begin{aligned}\mathbf{B}(y) &\leq \bigvee_{x \in X} \left(\mathbf{A}_j(x) * \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \to_* \mathbf{B}_i(y)) \right) \\ &\leq \bigvee_{x \in X} (\mathbf{A}_j(x) * (\mathbf{A}_j(x) \to_* \mathbf{B}_j(y))) .\end{aligned}$$

Since $a * (a \to_* b) \leq b$, it follows that $\mathbf{B} \subseteq \mathbf{B}_j$.

On the other hand, let $x' \in X$ be such that $\mathbf{A}_j(x') = 1$, then the Ruspini condition implies $\mathbf{A}_i(x') = 0$ for all $i \neq j$. It then holds that

$$\begin{aligned}\mathbf{B}(y) &\geq \left(\mathbf{A}_j(x') * \bigotimes_{i=1}^{n} (\mathbf{A}_i(x') \to_* \mathbf{B}_j(y)) \right) \\ &= \bigotimes_{i=1}^{n} (\mathbf{A}_i(x') \to_* \mathbf{B}_j(y)) \\ &= (1 \to_* \mathbf{B}_j(y)) \otimes \left(\bigotimes_{\substack{i=1 \\ i \neq j}}^{n} (0 \to_* \mathbf{B}_j(y)) \right)\end{aligned}$$

Since $1 \to_* b = b$ and $0 \to_* b = 1$, it follows that $\mathbf{B} \supseteq \mathbf{B}_j$. ■

Due to Theorem 2.1 we can state the following corollary of Theorem 6.1.

Corollary 6.2: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition. Then $\hat{\mathbf{R}}_*$ is a solution of system (7).

Similar to the case of additive fuzzy models and systems of subdirect image equations, we can use a wide variety of residuation operations in the multiplicative fuzzy models.

Proposition 6.3: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition. Furthermore, let $\blacktriangle$ be a t-norm such that $* \leq \blacktriangle$. Then $\mathbf{R}_\blacktriangle^{\otimes}$ is a solution of system (7).

Proof: Similar to that of Theorem 6.1. ■

Propositions 4.5 and 6.3 both relate to the direct image inference mechanism and ensure that we may use a wide variety of models based on any t-norm stronger than the t-norm $*$ used in the inference mechanism. While Proposition 4.5 applies to additive fuzzy models, Proposition 6.3 does so for multiplicative fuzzy models.

B. Subdirect image equations

Theorem 6.4: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition, and let $* \leq \otimes$, where $\otimes$ is the Łukasiewicz t-norm. Then system (11) is solvable and $\mathbf{R}_*^{\otimes}$ is a solution.

Proof: Consider arbitrary $j \in \{1, \ldots, n\}$ and

$$\mathbf{B}(y) = \bigwedge_{x \in X} (\mathbf{A}_j(x) \to_* \bigotimes_{i=1}^{n} (\mathbf{A}_i(x) \to_* \mathbf{B}_i(y))) .$$

Let $x \in X$ be such that $\mathbf{A}_j(x') = 1$. Since $\to_*$ is increasing in its second argument and $1 \to_* b = b$, it holds that

$$\begin{aligned}\mathbf{B}(y) &\leq \bigwedge_{x \in X} (\mathbf{A}_j(x) \to_* (\mathbf{A}_j(x) \to_* \mathbf{B}_j(y))) \\ &\leq \mathbf{A}_j(x') \to_* (\mathbf{A}_j(x') \to_* \mathbf{B}_j(y)) \\ &= 1 \to_* (1 \to_* \mathbf{B}_j(y)) = \mathbf{B}_j(y) ,\end{aligned}$$

which proves that $\mathbf{B} \subseteq \mathbf{B}_j$.

Let us define $\mathbf{B}_x \in \mathcal{F}(Y)$ as follows

$$\mathbf{B}_x(y) = \mathbf{A}_j(x) \to_* \bigotimes_{i=1}^{n} (\mathbf{A}_i(x) \to_* \mathbf{B}_i(y)) .$$

Obviously, $\mathbf{B}(y) = \bigwedge_{x \in X} \mathbf{B}_x(y)$.

To prove the converse inclusion $\mathbf{B} \supseteq \mathbf{B}_j$, it suffices to show that $\mathbf{B}_x \supseteq \mathbf{B}_j$ for any $x \in X$. Let us consider the following three cases.

(a) First, let $x \in X$ be such that $\mathbf{A}_j(x) = 0$. Since $0 \to_* b = 1$, it holds that

$$\mathbf{B}_x(y) = 0 \to_* \bigotimes_{i=1}^{n} (\mathbf{A}_i(x) \to_* \mathbf{B}_i(y)) = 1 ,$$

which implies that $\mathbf{B}_x(y) = 1$ for any $y \in Y$, and thus $\mathbf{B}_x \supseteq \mathbf{B}_j$.

(b) Second, let $x \in X$ be such that $\mathbf{A}_j(x) = 1$, then the Ruspini condition implies that $\mathbf{A}_i(x) = 0$ for all $i \neq j$. Therefore $0 \to_* b = 1$ and $1 \to_* b = b$ lead to

$$\begin{aligned}\mathbf{B}_x(y) &= 1 \to_* \left(\bigotimes_{\substack{i=1 \\ i \neq j}}^{n} (0 \to_* \mathbf{B}_i(y)) \otimes (1 \to_* \mathbf{B}_j(y)) \right) \\ &= (\bigotimes_{\substack{i=1 \\ i \neq j}}^{n} 1) \otimes \mathbf{B}_j(y) = \mathbf{B}_j(y) .\end{aligned}$$

(c) Finally, let $x \in X$ be such that $\mathbf{A}_j(x) \in]0, 1[$. From

$* \le \otimes$ it follows that $\rightarrow_* \ge \rightarrow_\otimes$, and thus

$$\mathbf{B}_x(y) \ge \mathbf{A}_j(x) \rightarrow_* \left(\bigotimes_{\substack{i=1\\ i\neq j}}^{n} (\mathbf{A}_i(x) \rightarrow_\otimes \mathbf{B}_i(y)) \otimes (\mathbf{A}_j(x) \rightarrow_\otimes \mathbf{B}_j(y)) \right) .$$

Since $a \rightarrow_\oplus b \ge 1 - a$, it follows that

$$\mathbf{B}_x(y) \ge \mathbf{A}_j(x) \rightarrow_* \left(\bigotimes_{\substack{i=1\\ i\neq j}}^{n} (1 - \mathbf{A}_i(x)) \otimes (\mathbf{A}_j(x) \rightarrow_\otimes \mathbf{B}_j(y)) \right) .$$

Since $a \rightarrow_\otimes b = (1-a) \oplus b$, we

$$\mathbf{B}_x(y) \ge \mathbf{A}_j(x) \rightarrow_* \left(\bigotimes_{\substack{i=1\\ i\neq j}}^{n} (1 - \mathbf{A}_i(x)) \otimes ((1 - \mathbf{A}_j(x)) \oplus \mathbf{B}_j(y)) \right) .$$

Using the expression for the n-ary version of the Łukasiewicz t-norm, we obtain

$$\bigotimes_{\substack{i=1\\ i\neq j}}^{n} (1 - \mathbf{A}_i(x)) = 0 \vee \left(\left(\sum_{\substack{i=1\\ i\neq j}}^{n} 1 - \mathbf{A}_i(x) \right) - (n-2) \right) = 0 \vee \left(1 - \sum_{\substack{i=1\\ i\neq j}}^{n} \mathbf{A}_i(x) \right) .$$

Since $\mathbf{A}_j(x) \in]0,1[$, the Ruspini condition yields

$$\bigotimes_{\substack{i=1\\ i\neq j}}^{n} (1 - \mathbf{A}_i(x)) = 0 \vee \mathbf{A}_j(x) = \mathbf{A}_j(x) .$$

Hence, we can proceed as follows

$$\mathbf{B}_x(y) \ge \mathbf{A}_j(x) \rightarrow_* (\mathbf{A}_j(x) \otimes ((1 - \mathbf{A}_j(x)) \oplus \mathbf{B}_j(y))) .$$

(i) Consider $y \in Y$ such that $(1 - \mathbf{A}_j(x)) + \mathbf{B}_j(y) \ge 1$. Since $0 \rightarrow_* a = 1$, it holds that

$$\mathbf{B}_x(y) \ge \mathbf{A}_j(x) \rightarrow_* (\mathbf{A}_j(x) \otimes 1) = 1 ,$$

which implies that $\mathbf{B}_x(y) = 1$ for any $y \in Y$, and thus $\mathbf{B}_x \supseteq \mathbf{B}_j$.

(ii) Consider $y \in Y$ such that $(1 - \mathbf{A}_j(x)) + \mathbf{B}_j(y) < 1$. Then

$$\mathbf{B}_x(y) \ge \mathbf{A}_j(x) \rightarrow_* \mathbf{B}_j(y) .$$

Since $a \rightarrow_* b \ge b$, it follows that $\mathbf{B}_x(y) \ge \mathbf{B}(y)$, and thus $\mathbf{B}_x \supseteq \mathbf{B}_j$.

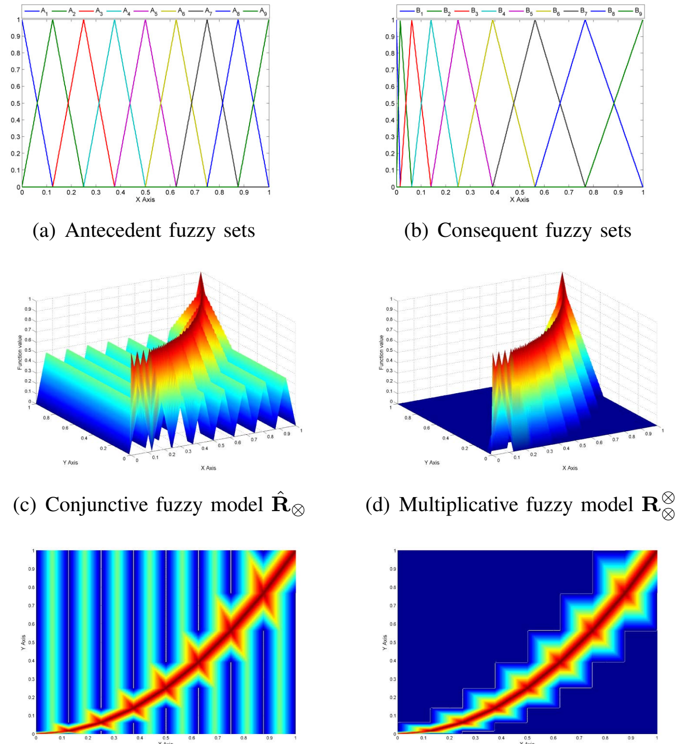
■

Similar to the case of additive fuzzy models and systems of direct image equations, the Łukasiewicz t-norm is too weak in the case of multiplicative fuzzy models and systems of subdirect image equations. Hence, only the case $\mathbf{R}^{\otimes}_{\otimes}$ is of practical importance. Again, Theorem 6.4 can be strengthened.

Proposition 6.5: Let all $\mathbf{A}_i$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition. Furthermore, let $\blacktriangle$ be a t-norm such that $\blacktriangle \le \otimes$. Then the fuzzy relation $\mathbf{R}^{\otimes}_{\blacktriangle}$ is a solution of (11).

Proposition 6.5 forces us to use a residual operation $\rightarrow_\blacktriangle$ adjoint to a t-norm $\blacktriangle$ weaker than the Łukasiewicz t-norm in the fuzzy relation interpreting a given fuzzy rule base, while the corresponding inference mechanism can be based on an arbitrary residual operation $\rightarrow_*$.

Example 6.6: To visually demonstrate the difference between the standard conjunctive fuzzy model and the multiplicative one, let us consider the following example. Consider the pairs of input-output fuzzy sets $(\mathbf{A}_i, \mathbf{B}_i)$, $i = 1, \ldots, 9$, in $\mathcal{F}([0,1]) \times \mathcal{F}([0,1])$ approximating the quadratic function $y = x^2$. The fuzzy sets $\mathbf{A}_i$ are triangular and form a uniform fuzzy partition of $X = [0,1]$, and thus fulfill the Ruspini condition. Also the fuzzy sets $\mathbf{B}_i$ are triangular, with kernel points equal to x^2 and x the kernel point of the corresponding $\mathbf{A}_i$. In Figure 1, the conjunctive and multiplicative fuzzy models are displayed, both using the Łukasiewicz t-norm (or its residual operation) between antecedent and consequent fuzzy sets.



(a) Antecedent fuzzy sets (b) Consequent fuzzy sets

(c) Conjunctive fuzzy model $\hat{\mathbf{R}}_\otimes$ (d) Multiplicative fuzzy model $\mathbf{R}^{\otimes}_{\otimes}$

(e) Conjunctive fuzzy model $\hat{\mathbf{R}}_\otimes$ - view from above (f) Multiplicative fuzzy model $\mathbf{R}^{\otimes}_{\otimes}$ - view from above

Fig. 1. Comparison of $\hat{\mathbf{R}}_\otimes$ and $\mathbf{R}^{\otimes}_{\otimes}$.

VII. Demonstration

A. Additive fuzzy models

Let us recapitulate the results of Section IV. Let all $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \ldots, n$, be normal and fulfill the Ruspini condition. Consider a fuzzy rule base (4) where the antecedents are represented by the given fuzzy sets $\mathbf{A}_i$ and the consequents are represented by arbitrary fuzzy sets $\mathbf{B}_i \in \mathcal{F}(Y)$. Due to Theorem 4.3, the fuzzy relation

$$\mathbf{R}_{\otimes}^{\oplus}(x,y) = \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) \otimes \mathbf{B}_i(y))$$

is a solution of the system of fuzzy relational equations

$$\mathbf{A}_i \lhd_{\otimes} \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1,\dots,n\,.$$

Moreover, due to Corollary 4.4 and Proposition 4.5, the fuzzy relations

$$\check{\mathbf{R}}_{\otimes}(x,y) \;=\; \bigvee_{i=1}^{n}(\mathbf{A}_i(x) \otimes \mathbf{B}_i(y))$$

$$\mathbf{R}_{\odot}^{\oplus}(x,y) \;=\; \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) \odot \mathbf{B}_i(y)) = \sum_{i=1}^{n}\mathbf{A}_i(x)\mathbf{B}_i(y)\,,$$

with $\odot$ the product t-norm, are solutions of this system as well. Furthermore, due to Theorem 4.6, the fuzzy relation $\mathbf{R}_{\otimes}^{\oplus}$ is also a solution of the following system of fuzzy relational equations

$$\mathbf{A}_i \circ_{\otimes} \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1,\dots,n\,,$$

and due to Proposition 4.8, another solution of the latter system is the fuzzy relation $\mathbf{R}_{\odot}^{\oplus}$.

This means that the fuzzy models $\check{\mathbf{R}}_{\otimes}$, $\mathbf{R}_{\otimes}^{\oplus}$ and $\mathbf{R}_{\odot}^{\oplus}$ are safe models [40] of fuzzy rule base (4) from the fuzzy interpolation point of view, when adopting the subdirect image with the Łukasiewicz t-norm as the corresponding inference mechanism. The two latter arithmetic fuzzy models $\mathbf{R}_{\otimes}^{\oplus}$ and $\mathbf{R}_{\odot}^{\oplus}$ are even safe in the case of the direct image with the Łukasiewicz t-norm as the corresponding inference mechanism.

B. Multiplicative fuzzy models

Similarly, we discuss the results of Section VI. Let all $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1,\dots,n$ be normal and fulfill the Ruspini condition. Consider a fuzzy rule base (2) where the antecedents are represented by the given fuzzy sets $\mathbf{A}_i$ and the consequents are represented by arbitrary fuzzy sets $\mathbf{B}_i \in \mathcal{F}(Y)$. Due to Theorem 6.1, the fuzzy relation

$$\mathbf{R}_{\otimes}^{\otimes}(x,y) = \bigotimes_{i=1}^{n}(\mathbf{A}_i(x) \to_{\otimes} \mathbf{B}_i(y))$$

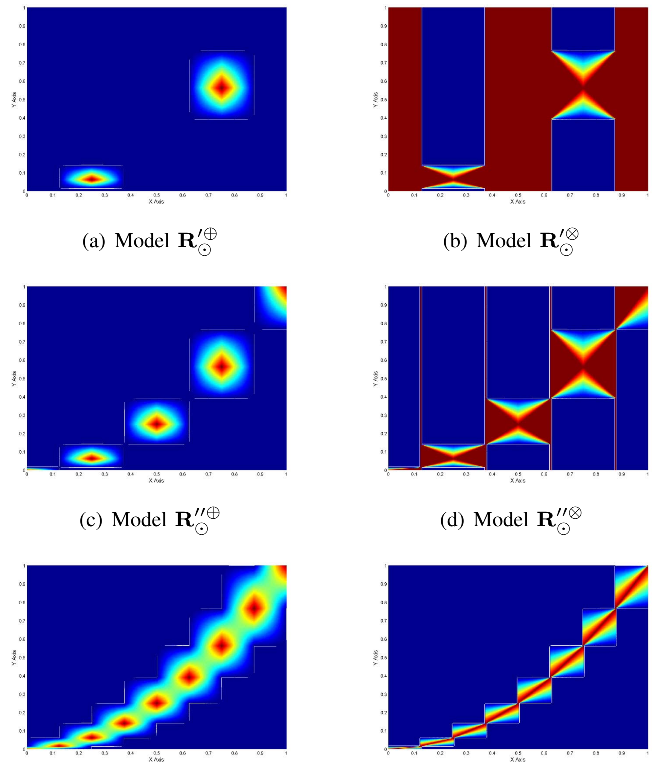
is a solution of the system of fuzzy relational equations

$$\mathbf{A}_i \circ_{\otimes} \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1,\dots,n\,.$$

Moreover, due to Corollary 6.2 and Proposition 6.3, the fuzzy relations

$$\hat{\mathbf{R}}_{\otimes}(x,y) \;=\; \bigwedge_{i=1}^{n}(\mathbf{A}_i(x) \to_{\otimes} \mathbf{B}_i(y))$$

$$\mathbf{R}_{\odot}^{\otimes}(x,y) \;=\; \bigotimes_{i=1}^{n}(\mathbf{A}_i(x) \to_{\odot} \mathbf{B}_i(y))\,,$$



(a) Model $\mathbf{R}'^{\oplus}_{\odot}$ (b) Model $\mathbf{R}'^{\otimes}_{\odot}$

(c) Model $\mathbf{R}''^{\oplus}_{\odot}$ (d) Model $\mathbf{R}''^{\otimes}_{\odot}$

(e) Model $\mathbf{R}^{\oplus}_{\odot}$ (f) Model $\mathbf{R}^{\otimes}_{\odot}$

Fig. 2. Examples of arithmetic fuzzy models.

with $\odot$ the product t-norm, are solutions of this system as well. Furthermore, due to Theorem 6.4, the fuzzy relation $\mathbf{R}_{\otimes}^{\otimes}$ is also a solution of the following system of fuzzy relational equations

$$\mathbf{A}_i \lhd_{\otimes} \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1,\dots,n,$$

and due to Proposition 6.5, even of the system

$$\mathbf{A}_i \lhd_{\odot} \mathbf{R} = \mathbf{B}_i\,, \qquad i = 1,\dots,n\,.$$

This means that the fuzzy models $\hat{\mathbf{R}}_{\otimes}$, $\mathbf{R}_{\odot}^{\otimes}$ and $\mathbf{R}_{\otimes}^{\otimes}$ are safe models [40] of the fuzzy rule base (2) from the fuzzy interpolation point of view, when adopting the direct image with the Łukasiewicz t-norm as the corresponding inference mechanism. The latter arithmetic fuzzy model $\mathbf{R}_{\otimes}^{\otimes}$ is even safe in the case of the subdirect image with the Łukasiewicz t-norm or the subdirect image with the product t-norm as the corresponding inference mechanism.

C. Example

In this subsection, we present several figures of the proposed arithmetic fuzzy models.

Consider again the 9 pairs of input-output fuzzy sets from Example 6.6. We consider the additive and multiplicative fuzzy models, both with respect to the product t-norm, taking into account two, five or all of these pairs. The fuzzy models $\mathbf{R}'^{\oplus}_{\odot}$ and $\mathbf{R}'^{\otimes}_{\odot}$ are based on the pairs $(\mathbf{A}_3, \mathbf{B}_3)$ and $(\mathbf{A}_7, \mathbf{B}_7)$, the fuzzy models $\mathbf{R}''^{\oplus}_{\odot}$ and $\mathbf{R}''^{\otimes}_{\odot}$ are additionally based on $(\mathbf{A}_1, \mathbf{B}_1)$, $(\mathbf{A}_5, \mathbf{B}_5)$ and $(\mathbf{A}_9, \mathbf{B}_9)$. Finally, the fuzzy models

$\mathbf{R}_{\odot}^{\oplus}$ and $\mathbf{R}_{\odot}^{\otimes}$ are based on all pairs. These fuzzy models are depicted in Figure 2.

In principle, fuzzy rule base (4) is understood as a collection of data points, and is predetermined to be modelled by a disjunctive fuzzy model (in general, so even by an additive one). Similarly, fuzzy rule base (2) is meant as a set of conditional constraints valid at once, and is predetermined to be modelled by a conjunctive fuzzy model (in general, so even by a multiplicative one). Figure 2 shows how these originally absolutely incompatible fuzzy models converge to each other when the corresponding antecedent fuzzy sets approach a Ruspini partition.

VIII. Approximation abilities

So far, we have provided an in-depth study of the interpolation properties of additive and multiplicative fuzzy models. However, approximation abilities of these fuzzy models might be of interest as well. The universal approximation abilities of disjunctive and conjunctive fuzzy models are well known. In this section, we investigate whether this crucial property also holds for arithmetic fuzzy models.

A. Additive fuzzy models

First of all, let us focus on additive fuzzy models and their approximation abilities.

Theorem 8.1: Let X, Y be two closed real intervals and let $f : X \to Y$ be an arbitrary continuous function. Then for arbitrary $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $\mathbf{A}_i \in \mathcal{F}(X)$, $\mathbf{B}_i \in \mathcal{F}(Y)$, $i = 1, \ldots, n$, such that

$$\text{if } \mathbf{R}_*^{\oplus}(x, y) > 0 \text{ , then } |y - f(x)| < \varepsilon \,,$$

where $x \in X$, $y \in Y$ and $\mathbf{R}_*^{\oplus}$ is the additive fuzzy model given by (16).

Proof: Let $X = [a, b]$. The continuity of f on X implies that f is uniformly continuous, i.e., for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, x' \in X$ it holds that

$$\text{if } |x - x'| < \delta \text{ , then } |f(x) - f(x')| < \varepsilon/2 \,. \quad (19)$$

Let us fix an arbitrary $\varepsilon > 0$ and a related $\delta > 0$ according to (19). Let us choose $n > 2$ such that

$$h = \frac{|b - a|}{n - 1} < \frac{\delta}{2} \,.$$

Denote $x_1 = a$ and $x_i = x_{i-1} + h$, for $i = 2, \ldots, n$. Let $U_i = [x_{i-1}, x_{i+1}]$, then $f(U_i) = [a_i, b_i]$. Denote the midpoint of the interval $[a_i, b_i]$ by y_i and construct the interval $V_i \subseteq Y$ as follows

$$V_i =]y_i - \varepsilon/2, y_i + \varepsilon/2[\,.$$

Since $|x_{i-1} - x_{i+1}| < \delta$ and because of the continuity of f, it holds that $|b_i - a_i| < \varepsilon/2$ and thus $f(U_i) \subset V_i$.

Let $\mathbf{A}_i \in \mathcal{F}(X)$, $i = 1, \ldots, n$, be such that $\mathbf{A}_i(x_i) = 1$ and $\mathbf{A}_i(x) > 0$ if and only if $x \in]x_{i-1}, x_{i+1}[$ where $x_0 = x_1$, $x_{n+1} = x_n$. Let $\mathbf{B}_i \in \mathcal{F}(Y)$, $i = 1, \ldots, n$, be such that $\mathbf{B}_i(y) > 0$ if and only if $y \in V_i$. Take an arbitrary $x' \in X$ such that $x' \notin \{x_i \mid i = 1, \ldots, n\}$. Then there exist U_i, U_{i+1} such that $x' \in U_i$ and $x' \in U_{i+1}$. In this case $\mathbf{A}_i(x') > 0$ as well as $\mathbf{A}_{i+1}(x') > 0$, while $\mathbf{A}_j(x') = 0$ for $j \notin \{i, i+1\}$. Furthermore, $f(x') \in V_i$ as well as $f(x') \in V_{i+1}$.

Take an arbitrary $y \in Y$ such that $\mathbf{R}_*^{\oplus}(x', y) > 0$. Then

$$\mathbf{R}_*^{\oplus}(x', y) = ((\mathbf{A}_i(x') * \mathbf{B}_i(y)) \oplus (\mathbf{A}_{i+1}(x') * \mathbf{B}_{i+1}(y)))$$

and hence, $\mathbf{R}_*^{\oplus}(x', y) > 0$ occurs if and only if $\mathbf{B}_i(y) > 0$ or $\mathbf{B}_{i+1}(y) > 0$. Without loss of generality, let $\mathbf{B}_i(y) > 0$ then $y \in V_i$. Finally,

$$|y - f(x')| = |y - y_i + y_i - f(x')| \leq |y - y_i| + |y_i - f(x')| \leq \varepsilon.$$

In case $x' \in \{x_i \mid i = 1, \ldots, n\}$, the proof uses the same technique and is therefore omitted. ■

As mentioned before, additive fuzzy models are closely related to the disjunctive ones. Hence, the *Center Of Gravity* (COG) defuzzification method standardly used in connection with the disjunctive fuzzy models seems to be appropriate for the additive fuzzy models as well. Let us recall that the COG defuzzification of a fuzzy set $B \in \mathcal{F}(Y)$ is given as follows

$$\mathrm{COG}(B) = \frac{\int_Y y \cdot B(y) \; dy}{\int_Y B(y) \; dy} \quad (20)$$

assuming that (20) is a well-defined formula. Since the center of gravity formula averages non-zero values, Theorem 8.1 yields the following direct corollary.

Corollary 8.2: Let X, Y be two closed real intervals and let $f : X \to Y$ be an arbitrary continuous function. Then for arbitrary $\varepsilon > 0$, there exists an additive fuzzy model $\mathbf{R}_*^{\oplus}$ given by (16) such that for every $x \in X$

$$|\,\mathrm{COG}(\mathbf{R}_*^{\oplus}(x, \cdot)) - f(x)| < \varepsilon \,.$$

Corollary 8.2 states the universal approximation property of additive fuzzy models connected to an appropriate defuzzification method, the standard COG. Obviously, even other defuzzification methods would fit into the whole concept based on non-zero values of the model in an ε-neighborhood of the approximated function.

B. Multiplicative fuzzy models

Similarly as for the additive fuzzy models, we discuss the approximation abilities of the multiplicative ones.

Theorem 8.3: Let X, Y be two closed real intervals and let $f : X \to Y$ be an arbitrary continuous function. Then for arbitrary $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $\mathbf{A}_i \in \mathcal{F}(X)$, $\mathbf{B}_i \in \mathcal{F}(Y)$, $i = 1, \ldots, n$, such that

$$\text{if } \mathbf{R}_*^{\otimes}(x, y) = 1 \text{ , then } |y - f(x)| < \varepsilon \,,$$

where $x \in X$, $y \in Y$ and $\mathbf{R}_*^{\otimes}$ is the multiplicative fuzzy model given by (18).

Proof: The first part of the proof only repeats the proof of Theorem 8.1, i.e., we assume the continuity of f on $X = [a, b]$ and similarly we fix ε, δ and choose $n > 2$ to keep $h < \frac{\delta}{2}$. Let also x_i be given as above as well as U_i, V_i and y_i.

Let $\mathbf{A}_i \in \mathcal{F}(X)$, for $i = 1, \ldots, n$, be such that $\mathbf{A}_i(x_i) = 1$ if and only if $x \in]x_{i-1}, x_{i+1}[$. Let $\mathbf{B}_i \in \mathcal{F}(Y)$, for $i = 1, \ldots, n$, be such that $\mathbf{B}_i(y) = 1$ if and only if $y \in V_i$. Take an arbitrary $x' \in X$ such that $x' \notin \{x_i \mid i = 1, \ldots, n\}$. Then there exist U_i, U_{i+1} such that $x' \in U_i$ and $x' \in U_{i+1}$.

In this case $\mathbf{A}_i(x') = 1$ as well as $\mathbf{A}_{i+1}(x') = 1$, while $\mathbf{A}_j(x') < 1$ for $j \notin \{i, i+1\}$. Furthermore, $f(x') \in V_i$ as well as $f(x') \in V_{i+1}$.

Take an arbitrary $y \in Y$ such that $\mathbf{R}_*^{\otimes}(x', y) = 1$. Then

$$\mathbf{R}_*^{\otimes}(x', y) = ((\mathbf{A}_i(x') * \mathbf{B}_i(y)) \otimes (\mathbf{A}_{i+1}(x') * \mathbf{B}_{i+1}(y)))$$

and hence, $\mathbf{R}_*^{\otimes}(x', y) = 1$ occurs if and only if $\mathbf{B}_i(y) = 1$ and $\mathbf{B}_{i+1}(y) = 1$. Then $y \in V_i$ and also $y \in V_{i+1}$. So, we can write

$$|y - f(x')| = |y - y_i + y_i - f(x')| \leq |y - y_i| + |y_i - f(x')| \leq \varepsilon\,.$$

In case $x' \in \{x_i \mid i = 1, \ldots, n\}$, the proof uses the same technique and is therefore omitted. ■

Theorem 8.1 assumes $\mathbf{R}_*^{\oplus}(x, y) > 0$, while Theorem 8.3 assumes $\mathbf{R}_*^{\otimes}(x, y) = 1$. However, this distinction fully fits into the different points of view on both types of fuzzy rules as discussed in Section I. Also the approximation ability of the defuzzified output causes no problem at all, since the multiplicative fuzzy models are closely related to the conjunctive ones. Hence, COG defuzzification is not appropriate, while the *Mean Of Maxima* (MOM) defuzzification method is preferred [41]. Let us recall that the MOM defuzzification of a fuzzy set $B \in \mathcal{F}(Y)$ is given as follows

$$\mathrm{MOM}(B) = \begin{cases} \frac{\sum_{y \in \mathrm{Ceil}(B)} y}{|\,\mathrm{Ceil}(B)|} & , \text{ if } |\,\mathrm{Ceil}(B)| < \infty, \\ \frac{\int_{\mathrm{Ceil}(B)} y \ dy}{\int_{\mathrm{Ceil}(B)} 1 \ dy} & , \text{ if } \int_{\mathrm{Ceil}(B)} 1 \ dy \neq 0, \end{cases} \quad (21)$$

where $\mathrm{Ceil}(B) = \{y \mid B(y) = \mathrm{Height}(B)\}$ and $\mathrm{Height}(B) = \sup\{B(y) \mid y \in Y\}$.

From formula (21) it is easy to see that MOM takes into account only Ceil points. In case of *coherence* (non-emptiness of the Core of the output fuzzy set for any crisp input $x' \in X$ [3]), Ceil and Core are identical. Hence, the MOM defuzzification focuses only on Core points, which fully explains why Theorem 8.3 assumed $\mathbf{R}_*^{\otimes}(x, y) = 1$.

Remark 8.4: Let us remark that the coherence is not only a technical assumption, but it is a very powerful and highly desirable property certifying the non-existence of conflicting rules in a fuzzy rule base [3]. Furthermore, this condition may be very useful for further investigations [41].

Due to the properties of the MOM formula, Theorem 8.3 yields the following corollary.

Corollary 8.5: Let X, Y be two closed real intervals and let $f : X \to Y$ be an arbitrary continuous function. Then for arbitrary $\varepsilon > 0$, there exists a multiplicative fuzzy model $\mathbf{R}_*^{\otimes}$ given by (18) such that for every $x \in X$

$$|\,\mathrm{MOM}(\mathbf{R}_*^{\otimes}(x, \cdot)) - f(x)| < \varepsilon\,.$$

Although Corollary 8.5 states the universal approximation property of multiplicative fuzzy models connected to the MOM method, even other defuzzification methods focusing on Core points (First of Maxima, Last of Maxima) would also guarantee the universal approximation property.

C. Fuzzy control benchmark

Due to the universal approximation ability of both types of arithmetic fuzzy models, they have a very wide application potential, just as for the more established disjunctive and implicative fuzzy models. Since fuzzy control is among the most frequent fields of application, we have chosen a fuzzy control benchmark for our demonstration. The chosen benchmark consists of controlling a simplified *inverted pendulum*-like process described by the second order differential equation

$$y'' - 10\sin(y) = u(t)\,, \quad (22)$$

where y is the output function, t denotes the time variable and u denotes a control action. The inverted pendulum is a very frequently chosen benchmark, see e.g. [42], [43].

TABLE I
CONTROL FUZZY RULES IN A LOOK-UP TABLE. ABBREVIATIONS B, M, S, Z DENOTE LABELS "BIG", "MEDIUM", "SMALL", "ZERO"; THE PREFIX N STANDS FOR "NEGATIVE", WHILE P STANDS FOR "POSITIVE".

Rules	NM	NS	Z	PS	PM
NM	NB	NB	NM	NS	Z
NS	NB	NM	NS	Z	PS
Z	NM	NS	Z	PS	PM
PS	NS	Z	PS	PM	PB
PM	Z	PS	PM	PB	PB

In every field of application including fuzzy control, the overall quality of a given model is highly dependent on many aspects, including the fuzzy rule base, the defuzzification method, the sampling period, etc. To avoid a change in scope of this paper, we do not focus on all these aspects and select a simple fuzzy rule base, simple (triangular) shapes of antecedent and consequent fuzzy sets, a standard defuzzification method (COG) and we build disjunctive as well as additive fuzzy models, which are even in such a simplified setting supposed to be able to control the given system.

The process control was simulated using the LFLCsim software, a dedicated software tool for the simulation of ODE process control in a closed loop. It directly uses fuzzy inference schemes and defuzzification methods implemented in the LFLC2000 software package [44] which also allows users to edit fuzzy rule bases.

A fuzzy PD controller has been designed, i.e. the error E (the distance to the stable position $y = 0$) and its change dE in a sampling period have been used as antecedent variables, while the control action U has been used as consequent variable. Triangular antecedent fuzzy sets were uniformly distributed on both antecedent axes in order to fulfill the Ruspini condition. Consequent fuzzy sets meet the same properties on the output axis. The sampling period has been set to 0.020 seconds.

A fuzzy rule base consisting of 25 fuzzy rules was determined, see Table I. It is a simple version of a fuzzy PD controller stemming from a similar one published in [45]. Both a disjunctive and an additive fuzzy model using the Łukasiwicz t-norm (i.e. $\check{\mathbf{R}}_{\otimes}$ and $\mathbf{R}_{\otimes}^{\oplus}$) were constructed. Both models were combined with the COG defuzzification method, which is appropriate for these models.

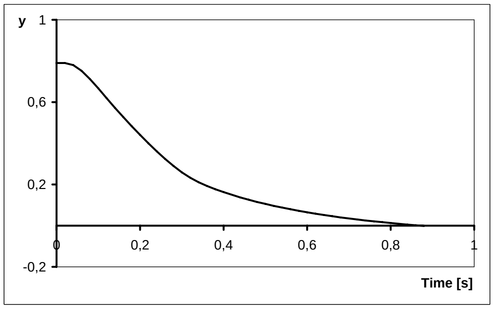


Fig. 3. Additive fuzzy model process control - system output y versus time.

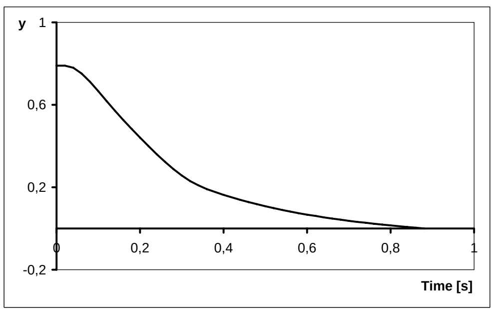


Fig. 4. Disjunctive fuzzy model process control - system output y versus time.

The process stable point was set to 0, i.e., zero corresponds to the upright direction of the inverted pendulum. The initial position was set to 0.79 (approx. $\pi/4$).

The additive fuzzy model of the fuzzy rule base in Table I together with the COG defuzzification method reached the stable point in 44 control actions, i.e. in 0.88 seconds, see Figure 3. The disjunctive fuzzy model of the same fuzzy rule base with the COG defuzzification method also reached the stable point in 44 control actions, see Figure 4.

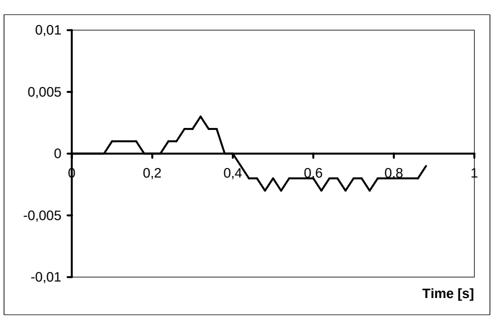


Fig. 5. Difference of both process control graphs.

Figure 5 displays the difference between both control graphs. The disjunctive fuzzy model $\check{\mathbf{R}}_{\otimes}$ converged faster to the stable point during the first part of the process control (positive difference), while in the second part the additive fuzzy model $\mathbf{R}_{\otimes}^{\oplus}$ took the advantage (negative difference). Overall, the additive fuzzy model performed slightly better since the system position y was 0.001 closer to the desired stable position on average. Since the benchmark differences are negligible, it can be stated that the newly suggested arithmetic fuzzy models perform at least as good as the more established standard fuzzy models.

From the applicability point of view, we may conclude that the arithmetic fuzzy models behave similarly to the standard ones and may serve as alternative — especially in cases where the standard fuzzy models do not meet the fundamental fuzzy interpolation property. In such cases, it is highly desirable to replace the standard fuzzy models by other fuzzy models that do meet this condition and that have the same application potential.

IX. Conclusions

We have discussed two interpretations of a fuzzy rule base together with the corresponding inference mechanisms. From the fuzzy interpolation point of view, we have explained that the disjunctive fuzzy model should be treated together with the subdirect image, while the conjunctive fuzzy model should be preferably treated together with the direct image.

Motivated by widely known practical methods, we have introduced additive fuzzy models and have shown that they are closely related to the disjunctive ones. Analogously, we have proposed multiplicative fuzzy models being closely related to the conjunctive fuzzy models. We have investigated both newly defined *arithmetic fuzzy models* from the fuzzy interpolation point of view w.r.t. both images, i.e. as possible solutions of the corresponding systems of fuzzy relational equations.

Similarly as for disjunctive and conjunctive fuzzy models, stronger results are obtained when additive fuzzy models are combined with the subdirect image, and when multiplicative fuzzy models are combined with the direct image.

The crucial role of the Ruspini condition for the four types of fuzzy models has been underlined and visually demonstrated. It is shown that this standard property together with the normality of antecedent fuzzy sets assures the solvability of related systems of fuzzy relation equations. In other words, by an appropriate setting of antecedent fuzzy sets (partitioning the input space), the solvability may be guaranteed beforehand with no further restrictions on consequent fuzzy sets.

Finally, the universal approximation property of the newly proposed fuzzy models has been discussed. It has been shown that both types of arithmetic fuzzy models, combined with appropriate standard defuzzification methods, preserve this crucial property. Hence, they are suitable models of fuzzy rule bases, not only from the fuzzy interpolation point of view, but also from the approximation point of view. The applicability of the arithmetic fuzzy models has been justified on a fuzzy control benchmark as well.

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Unfortunately, Lenka's life-path has been untimely finished before she achieved her 26th birthday. On March 4th, Lenka tragically deceased after a car accident. Before this grievous incident, she significantly participated on the research published in this paper. This article is a tribute to her scientific work.