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ARITHMETIC ON SINGULAR CUBIC SURFACES

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In this note we shall give a proof of the following:

PROPOSITION 1:

(a) *Let* $V \subset \mathbb{P}^3$ *be a cubic surface, defined over the infinite perfect field* k, *and having exactly* 3 *singular points* Q_1, Q_2, Q_3. *Then* V *is* k*-birationally equivalent to a non-singular cubic surface* W *containing a* k*-rational set of* 3 *skew lines.*

(b) *Conversely, every non-singular cubic surface* $W \subset \mathbb{P}^3$ *containing a* k*-rational set of* 3 *skew lines is* k*-birationally equivalent to a cubic surface* V *with exactly* 3 *double points.*

A proof of this result was already outlined in a little known paper of B. Segre [10], but the crucial fact that W is non-singular receives no justification there. Moreover, Segre states the converse with an additional assumption, which is actually not needed, as we shall see (lemma 3).

This proposition has a number of applications to arithmetical questions; we begin by discussing a few of them:

(i) When k is a number field, it was shown by Skolem [12] that singular cubic surfaces satisfy the *Hasse principle* (i.e. if V has $\mathfrak{p}$-adic solutions for every prime $\mathfrak{p}$, then V also has a point with coordinates in k). We may clearly assume that there are exactly 3 conjugated double points (cf. Segre [9] for the other cases). Skolem remarked that the equation of V can then be written in the form

$$f(x, y, z) = N_{K_1/k}(\tau) + a\ \mathrm{Tr}_{K_1/k}(\tau) + b = 0,$$

where τ is a linear combination of x, y and z, with coefficients in the

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cubic extension $K_1 = k(Q_1)$. Let D be the discriminant of a primitive element of K_1/k. Then $K_1(\sqrt{-3D})$ is a purely cubic extension of $k^* = k(\sqrt{-3D})$, by Cardano's formula. This fact enabled Skolem to reduce the equation further to the form $N_{K^*/k^*}(\tau^*) = c^*$, where the cubic extension K^*/k^* is generated by one of the roots of the polynomial $t^3 + 3at + b$. That the Hasse principle holds for this class of cubic surfaces now follows readily from class-field theory.

By a remark of F. Châtelet [2, pp. 70–71], a cubic surface of type $N\tau^* = c^*$ is birationally equivalent to a non-singular cubic surface containing a k^*-rational set of 3 skew lines, for which the Hasse principle is also known to hold [13, theorem 7]. One could combine this remark with Skolem's argument to obtain a slightly weaker form of proposition 1(a). However, it seems more natural to use a purely geometric argument, which avoids many of the rather painful computations of [12] and yields a birational equivalence defined over the groundfield k. As a corollary we get—of course—that *it suffices to give one proof of the Hasse principle*, e.g. [13, p. 15] or [3, p. 19]; that it also holds for singular cubic surfaces follows immediately from proposition 1.

(ii) Another application of this result is to the following:

COROLLARY 1: *Let $V \subset \mathbb{P}^3$ be a singular cubic surface, defined over the field k. Let K/k be an algebraic extension of k, with degree d prime to* 3, *and suppose that $V(K) \neq \emptyset$. Then $V(k) \neq \emptyset$.*

In view of proposition 1, this is a direct consequence of [4, prop. 8.1], where we proved the analogous statement for a non-singular cubic surface with a k-rational set of 3 or 6 skew lines. The technique we used in [4, prop. 8.2] to establish this corollary was rather more complicated.

(iii) Proposition 1(b) implies the following classical result:

COROLLARY 2: *Every non-singular cubic surface $W \subset \mathbb{P}^3$, containing a k-rational set of* 3 *skew lines and at least one k-rational point, is k-birationally equivalent to a plane.*

Indeed we can map W birationally onto a cubic surface V with 3 double points Q_1, Q_2, Q_3. Let us choose a k-rational pair of points Q_4, Q_5 on V. The family of twisted cubics passing through the 5 points $Q_1, \ldots, Q_5$ determines a one-to-one correspondence between the points of V and those of a plane through Q_4 and Q_5. Indeed a twisted cubic is uniquely determined by 6 points in general position (see e.g.

[14, §11, ex. 4]). If these points form a k-rational set, the twisted cubic is therefore also defined over k (even though the construction given in [14] is irrational!). Now a general cubic through $Q_1, \ldots, Q_5$ meets V in one other point (since Q_1, Q_2 and Q_3 each count as 2 intersections); and it also meets any plane containing Q_4 and Q_5 in one other point.

The description can be made even more explicit if we note that the twisted cubics through $Q_1, \ldots, Q_5$ can be put in one-to-one correspondence with the lines going through a fixed point $T_{33}^{tet}(Q_5)$, by means of a Cremona transformation T_{33}^{tet} of $\mathbb{P}^3$ [11, p. 179 & ex. 11, p. 186], whose fundamental tetrahedron contains the points $Q_1, \ldots, Q_4$ as vertices.[1] V itself is mapped onto a quadric containing $T_{33}^{tet}(Q_5)$, and this transformation is defined over k provided both Q_4 and Q_5 are k-rational (which we may assume, since the preceding argument shows that $V(k)$ is infinite[2]).

We now proceed to the proof of proposition 1. The technique is the same as in [10], but we supply a few more details:

(a) If Q_1 has coordinates in k, projecting from Q_1 yields a k-birational equivalence of V onto a plane, which is certainly equivalent to some non-singular cubic surface W, e.g. one containing 3 k-rational skew lines. *We may therefore assume that the Q_i form a complete set of conjugates.*

Let ρ denote the plane $\langle Q_1, Q_2, Q_3\rangle$ spanned by Q_1, Q_2 and Q_3. Let l be a k-rational line in $\mathbb{P}^3$, *which meets V in three distinct points $P_1, P_2, P_3 \not\in \rho$ and is not incident with any of the finitely many lines of V passing through Q_1, Q_2 or Q_3*. Consider the 3-dimensional family $\mathfrak{M}$ of quadrics containing the base set $\Sigma = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$. If

[1] This is one of the simplest space transformations. With a suitable choice of coordinates, a possible expression is

$$T_{33}^{tet}: (x, y, z, t) \mapsto (1/x, 1/y, 1/z, 1/t).$$

A more complicated type of cubo-cubic transformation will be introduced later (see footnote [6]).

[2] Of course this is a special case of a theorem of Segre, for which a very nice geometric proof was also found by Skolem [12, pp. 308–309]. The end of the proof («*Andererseits ist klar*: . . .») is based on a continuity argument, but it can be replaced by the following remark, which applies to a field of arbitrary characteristic (we retain Skolem's notation and assume the reader is familiar with the first part of his proof): *There are only finitely many ways of obtaining each point $Q \neq P$ with Skolem's construction.* Indeed, let C_i be the intersection of the cubic surface F with the tangent plane T_i ($i = 1, 2$). We may clearly assume that C_i is irreducible (it suffices to avoid hitting one of the lines of F when choosing l). Then Q_2 belongs to the cone with vertex Q and base C_1. If C_2 meets this cone in finitely many (and therefore ≤ 9) points, our assertion is proved. Otherwise C_2 is contained in that cone; but this implies that the two double points P_1 and P_2 are collinear with the vertex Q, and this is absurd because $Q \neq P$. Since we can use infinitely many pairs (Q_1, Q_2) and no more than a finite number of them give rise to the same point Q, there must be infinitely many such points.

$\varphi_0, \ldots, \varphi_3$ are k-rational generators for this linear system, let $\Phi : V \to \mathbb{P}^3$ be the rational transformation associated with them, viz. $\Phi = (\varphi_0 : \ldots : \varphi_3)$. This is the restriction to V of a well-known Cremona transformation T_{23} (see [5, pp. 560–561]); we shall examine its effect on the surface V. We claim that the image $W = \Phi[V]$ is a non-singular cubic surface, is k-birationally equivalent to V, and contains a k-rational set of 3 skew lines. It is clear that W is defined over k, since the generators $\mathfrak{M}$ have been chosen k-rational; and Φ is a birational equivalence, since V is not contained in any exceptional set of T_{23}. In fact, let $V_0 = V - \rho - \{P_1, P_2, P_3\}$; then $\Phi | V_0$ is an embedding, in virtue of the following lemma (see e.g. [8, chap. 2, §5.5, lemma]):

LEMMA 1: *$\mathfrak{M}$ separates points and infinitely near points on V_0. It also separates the points infinitely near P_i $(i = 1, 2, 3)$.*

PROOF: Let R and S be two distinct points of V_0. We denote by $\sigma_i = \langle l, Q_i \rangle$ the plane spanned by l and Q_i; and if $D_i = \langle Q_j, Q_k \rangle$ is the line joining Q_j to Q_k $(i \neq j, k)$, we let $\tau_i = \langle D_i, R \rangle$. Of course, $R \in \sigma_i \cup \tau_i \in \mathfrak{M}$ for all $i = 1, 2, 3$; we want to show that $S \notin \sigma_i \cup \tau_i$ for some i. Now $\sigma_i \cap \sigma_j = l$ $\forall i \neq j$, and $\tau_1 \cap \tau_2 \cap \tau_3 = \{R\}$. So if $S \in \sigma_i \cup \tau_i$ $\forall i$, then $S \in \sigma_1 \cap \tau_2 \cap \tau_3$ (say). But the line $\tau_2 \cap \tau_3 = \langle R, Q_1 \rangle$ must then be contained in σ_1 (and therefore meet l); and it lies on V, since it contains the three points R, S and Q_1 (which is double). This contradicts our choice of l.

The proof for infinitely near points is practically identical: let $\vec{s} \in T_R V$ be a tangent vector at R. At least one of the planes τ_i is transversal to $\vec{s}$, say τ_1. Then $\sigma_1 \cup \tau_1$ is transversal to $\vec{s}$, unless $R \in \sigma_1$. In the latter case $R \notin \sigma_2 \cup \sigma_3$ (since $\sigma_1 \cap \sigma_j = l$ $\forall j \neq 1$), and if τ_j $(j = 2$ or $3)$ is transversal to $\vec{s}$, we may then use $\sigma_j \cup \tau_j$. If not, then $\vec{s} \in \tau_2 \cap \tau_3 = \langle R, Q_1 \rangle$ and we conclude as before, since the line $\langle R, Q_1 \rangle$ meets V with multiplicity 2 both at R and at Q_1.

Finally, given any two tangent vectors at P_i, they can be separated by means of pairs of planes of type $\rho \cup \sigma$, where σ is a suitably chosen plane that contains l (l is transversal to V at P_i, since it meets V in 3 *distinct* points). q.e.d.

The degree of W is the number of its intersections with a generic line of $\mathbb{P}^3$. The inverse image of such a line is the trace on V of the intersection of two quadrics containing the base set Σ, and therefore consists of 12 points, 3 of which are movable. (The fixed points Q_i each count as 2 intersections; to see that the 3 remaining points are indeed movable, one can consider degenerate cases, like the two quadrics

$\sigma_1 \cup \tau_1$ and $\sigma_2 \cup \tau_2$, where the τ_i are determined by an arbitrary point $R \in V_0$.) Hence W is a cubic surface.

Let $\tilde{V}$ be the variety obtained by blowing up the three non-singular points P_1, P_2, P_3. We have the following commutative diagram

$$\begin{array}{ccc} \tilde{V} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\tilde{\Phi}} & \\ V & \xrightarrow{\Phi} & W \subset \mathbb{P}^3 \end{array}$$

Let $\tilde{L}_i = \pi^{-1}(P_i)$ and $L_i = \tilde{\Phi}[\tilde{L}_i]$. One more lemma and our search for skew lines will be complete:

LEMMA 2: (i) *$\tilde{\Phi}$ is regular at every point of $\tilde{L}_i$* $(i = 1, 2, 3)$; (ii) *the L_i form a set of* 3 *skew lines*; (iii) *the points of L_i are non-singular on W.*

PROOF: All these assertions can be checked by explicit calculations or constructions; but it seems preferable to use some general properties of correspondences, from which they derive almost formally. We first recall a few facts about blowings-up (cf. [7, prop. 20.4])[3]:

Suppose that $\mathfrak{M}$ is generated by $\varphi_0, \ldots, \varphi_n$ ($n = 3$ in our case, but we now consider a somewhat more general situation) and that the *general* member φ of $\mathfrak{M}$ cuts out on the surface V a divisor (φ), on which the isolated fixed point P (which will be one of our P_i) is a k-fold point. Then it is known [8, chap. 4, §3.1, thm. 1] that the proper transform of (φ), in a neighbourhood of P, is $\pi^{-1}[\varphi] = \pi^*(\varphi) - k\tilde{L}$, where $\tilde{L} = \pi^{-1}(P)$. Hence the map $\tilde{\Phi} = (\pi^*\varphi_0 : \ldots : \pi^*\varphi_n)$ is non-regular at a point $\tilde{x}$ of $\tilde{L}$ if and only if $\tilde{x}$ belongs to all the divisors $\Delta_j = (\pi^*\varphi_j) - k\tilde{L}$ (see e.g. [8, chap. 3, §1.4, thm. 2]).

Now there is a 1-1 correspondence between the points $\tilde{x} \in \tilde{L}$ and the tangent vectors $\vec{x}$ at P. In fact, by a local isomorphism, we can even think of V as being a piece of a plane and of $\vec{x}$ as being represented by a straight line X through P. Thus we have a 1-1 correspondence between the points $\tilde{x} \in \tilde{L}$ and the 'lines' X through P (see [8, chap. 2, §4.1]). It is clear that $\tilde{x} \in \Delta_j$ if and only if, locally above P, the intersection number $(\pi^{-1}[X], \Delta_j)_{\text{loc}}$ is positive. Since this number is equal to $(\pi^*X - \tilde{L}, (\pi^*\varphi_j) - k\tilde{L})_{\text{loc}} = (\pi^*X, \pi^*(\varphi_j))_{\text{loc}} - k$, we therefore derive, using [8, chap. 4, §3.2, thm. 2], that

$$\tilde{x} \in \Delta_j \Leftrightarrow (X, (\varphi_j))_{\text{loc}} \geq k + 1$$

[3] A particularly nice exposition of this theory can be found in Šafarevič's lectures at the Tata Institute (Bombay, 1966), especially in lectures 2& 3, pp. 18–30.

Thus we have established, in particular, that $\tilde{\Phi}$ is non-regular at $\tilde{x} \in \tilde{L}$ if and only if $(X, (\varphi_j))_{\text{loc}} \geq k + 1\ \forall j$. With these preliminaries, our lemma should appear almost trivial:

(i) In our case, $k = 1$ at each P_i. By the above discussion, it follows that if $\tilde{\Phi}$ were non-regular at $\tilde{x} \in \tilde{L}_i$, there would exist a vector $\vec{x} \in T_{P_i}V$ such that $\vec{x} \in T_{P_i}\varphi_j$ for every j. But we know already that there is no such $\vec{x}$ (lemma 1).

(ii) Of course L_i has dimension 1; in fact, lemma 1 shows that $\tilde{\Phi}$ is even injective on $\tilde{L}_i$. The degree of L_i is equal to the number of its (distinct) intersections with a generic plane in $\mathbb{P}^3$; since $\tilde{\Phi}$ is regular and injective on $\tilde{L}_i$, it is therefore equal to the multiplicity k with which a general quadric of $\mathfrak{M}$ intersects V at the point P_i, namely 1.

Suppose now $y \in L_1 \cap L_2$; then y is the image by $\tilde{\Phi}$ of 2 points, $\tilde{x}_1 \in \tilde{L}_1$ and $\tilde{x}_2 \in \tilde{L}_2$. But this is absurd, because $\mathfrak{M}$ separates any two tangent vectors, $\vec{x}_1$ at P_1 and $\vec{x}_2$ at P_2. Indeed, let $\tau_i = \langle D_i, P_1 \rangle$; then P_1 is double on $\sigma_i \cup \tau_i$, which is of course transversal to $\vec{x}_2$ for some i.

(iii) By the Zariski Main Theorem, $\tilde{\Phi}|\pi^{-1}(V - \rho)$ is an isomorphism, providing W is normal at every point of each L_i. Since W is a hypersurface, this is equivalent to saying that W has only finitely many double points on L_i, i.e. that L_i is not a double line of W. And this can be seen in various ways; for instance: if L_i were double, then any plane through L_i would meet W in *one* residual line. But such a plane corresponds to a quadric with a double point at P_i, e.g. $\varphi = \sigma_1 \cup \tau_1$, where $\tau_1 = \langle D_1, P_i \rangle$. Now $\varphi \cap V$ consists of D_1 (which is blown down) and of *two* other components; a contradiction. q.e.d.

REMARK 1: The sets of 3 skew lines on a non-singular cubic surface W always occur in pairs. Indeed, given any such set, there is a unique non-singular quadric Q that contains all three lines; and $W \cap Q$ contains 3 lines from each ruling of Q. Instead of choosing the lines L_i as above, we could therefore (like Segre) have used this other set of 3 lines, which are simply the proper transforms of the curves $\sigma_i \cap V$. But our present choice shows a little more: if $V(k) \neq \emptyset$, it is possible to select the P_i k-rational, and then each of the 3 skew lines L_i is defined over k. In particular this reproves corollary 2: after mapping W onto V, we can map V onto W', where W' is a non-singular cubic surface containing 3 k-rational skew lines; and there is a very obvious birational correspondence between W' and a plane (see e.g. [7, ex. 12.6]).

It remains to prove that W is non-singular. By the foregoing lemmas, a singularity of W comes either from one of the double points of V, or from one of the lines D_i joining these points. Each of these lines is blown down to a single point $S_i \in W$. To see that S_i is non-singular, it is enough to check that a generic line through S_i meets W residually in 2 movable points. Such a line corresponds to the intersection of 2 quadrics of $\mathfrak{M}$ containing D_i. We define two special quadrics as follows: let τ_i be a plane through D_i, whose residual intersection with V is a non-degenerate conic C, and which meets l in a point $R \notin V$. A general line d of τ_i, passing through R, will then meet C in two[4] movable points T_1, T_2. If σ denotes the plane $\langle l, d \rangle$ spanned by l and d, we see that the two quadrics $\sigma_i \cup \tau_i$ and $\rho \cup \sigma$ contain T_1 and T_2, whence our assertion follows.

Finally we must examine the image of a double point Q_i and show that it does not contain any singularity of W. This part of the argument is easier to make rigorous if we use explicit equations. At this stage we can forget about the ground field[5] and assume that the coordinates (x, y, z, t) of the double points are $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$. The equation of V is then simply:

$$t(ax^2 + bxy + cxz + dyz) + x(\alpha x^2 + \beta xy + \gamma xz + \delta yz) = 0, \tag{1}$$

where $d \neq 0$, because V is irreducible. The tangent cone at the point $(0,0,0,1)$ is given by

$$ax^2 + bxy + cxz + dyz = 0. \tag{2}$$

It is degenerate if and only if $ad = bc$, in which case its equation writes $(bx + dz)(cx + dy) = 0$. Since $d \neq 0$, this represents two *distinct* planes. The condition for all three tangent cones to be degenerate is as follows:

$$ad = bc; \quad \beta d = b\delta; \quad \gamma d = c\delta. \tag{3}$$

A simple example where this situation occurs is the cubic $x^3 - yzt = 0$.

It is easy to convince oneself that the result does not depend on the particular line l chosen, as long as it satisfies the conditions imposed at the beginning of the proof. Indeed, given any two lines l and l' satisfying those conditions, there is a projective transformation that fixes Q_1, Q_2 and Q_3, and carries l into l' (cf. [1, chap. III, n° 13]). *We are*

[4] Not necessarily distinct in characteristic 2, since R might be the intersection of all the lines tangent to C! But we can always avoid this situation by selecting another plane τ_i. (If this were not possible, l would be contained in the tangent cone to V at each Q_j $(j \neq i)$, but then the line $\langle P_1, Q_j \rangle$ would lie on V.)

[5] Almost! See the remark at the end (remark 2).

therefore free to assume that l is the line $y = z = t$. The Cremona transformation and its inverse can then be written as follows:

$$(4) \quad \begin{cases} (x, y, z, t) \overset{\Phi}{\longmapsto} (x(y-t), x(y-z), z(y-t), t(y-z)) \\ (\xi, \eta, \zeta, \tau) \overset{\Phi^{-1}}{\longmapsto} (\xi\eta(\eta-\xi), \xi\eta(\tau-\zeta), \eta\zeta(\eta-\xi), \xi\tau(\eta-\xi)) \end{cases}$$

and the equation of W reads:

$$(5) \quad g(\xi, \eta, \zeta, \tau) = \tau((a\xi + c\zeta)(\eta - \xi) + (b\xi + d\zeta)(\tau - \zeta)) + \\ + \eta((\alpha\xi + \gamma\zeta)(\eta - \xi) + (\beta\xi + \delta\zeta)(\tau - \zeta)) = 0.$$

The image of the point $(0, 0, 0, 1)$ is the curve $C = \{\eta = 0; (a\xi + c\zeta)\xi - (b\xi + d\zeta)(\tau - \zeta) = 0\}$. We want to show that none of these points is singular. To this effect, we compute the derivatives of g *in the plane* $\eta = 0$ and get the following equations:

$$(6) \quad C: \quad (a\xi + c\zeta)\xi - (b\xi + d\zeta)(\tau - \zeta) = 0$$

$$(7) \quad \partial g/\partial\eta: \quad (a\xi + c\zeta)\tau - (\alpha\xi + \gamma\zeta)\xi + (\beta\xi + \delta\zeta)(\tau - \zeta) = 0$$

$$(8) \quad \partial g/\partial\zeta: \quad ((b + c)\xi + 2d\zeta - d\tau)\tau = 0$$

$$(9) \quad \partial g/\partial\tau: \quad (a\xi + c\zeta)\xi - (b\xi + d\zeta)(2\tau - \zeta) = 0$$

We distinguish two cases:

First case: $\tau = 0$

Then $\quad (6) \Rightarrow a\xi^2 + (b + c)\xi\zeta + d\zeta^2 = 0,$

and $\quad (7) \Rightarrow \alpha\xi^2 + (\beta + \gamma)\xi\zeta + \delta\zeta^2 = 0.$

But if (ξ_0, ζ_0) is a common solution of these two equations, then the line $y = z = x\,\zeta_0/\xi_0$ lies on V and is incident with l. This contradicts the choice of l.

Second case: $\tau \neq 0$

Then, in view of (6), $(9) \Rightarrow b\xi + d\zeta = 0$, and $(8) \Rightarrow d\tau = (b + c)\xi - 2b\xi = (c - b)\xi$. Therefore $(\xi, \zeta, \tau) = (d, -b, c - b)$. Now $(6) \Rightarrow a\xi + c\zeta = 0 \Rightarrow ad = bc$. Hence the tangent cone at $(0, 0, 0, 1)$ is degenerate. But at the beginning we made the assumption that the Q_i formed a complete set of conjugates. Thus if one of the tangent cones is degenerate, so are the other two. Hence (3) holds. If we insert these relations in (7), we get: $\alpha d = b\gamma = bc\delta/d = a\delta$. But then (1) splits as $(t + x\,\delta/d)(ax^2 + bxy + cxz + dyz) = 0$, in contradiction with the irreducibility of V. This completes the proof that W is non-singular.

REMARK 2: It may be of some interest to note that the proof does not carry through without the condition that the Q_i form a complete set of conjugates. For instance, the surface $tz(x+y) + x^2(x+y+z) = 0$ has 3 double points and only one degenerate tangent cone, which corresponds to a *binode* (cf. [15, chap. 1, §5]). Its image by Φ is the surface $(\tau\zeta + \eta\xi)(\eta - \xi + \tau - \zeta) + \eta\zeta(\eta - \xi) = 0$, on which the point $(1, 0, 0, 1)$ is double, although l is not in special position with respect to the lines of V.

(b) The converse is easier to prove. Segre [10] makes use of the inverse Cremona transformation T_{23}^{-1}, but he introduces the additional hypothesis that the 3 skew lines should have a common k-rational transversal in $\mathbb{P}^3$. As a matter of fact, this condition is automatically fulfilled:

LEMMA 3: *Any three skew lines L_1, L_2, L_3, forming a k-rational set, are incident with infinitely many k-rational lines $E \subset \mathbb{P}^3$.*

PROOF: The three lines are contained in a unique quadric Q, which is necessarily non-singular and defined over k. Furthermore $Q(k) \neq \emptyset$, for if K_1 is a cubic extension of k over which L_1 is defined, then $Q(K_1) \neq \emptyset$, and hence $Q(k) \neq \emptyset$. (This is a special case of proposition 2.1 of [4] and is quite easy to prove: let $S_1 \in Q(K_1)$, and consider its conjugates, S_2 and S_3; there are ∞^2 k-rational conics through these 3 points; they meet Q in 4 points, leaving a residual k-rational point.) Let S be any k-rational point of Q. The tangent plane at S contains 2 lines of Q, one of which meets L_1, L_2 and L_3. This line E, which is uniquely determined by rational conditions, is k-rational! (Alternatively: E is the residual intersection of L_1 in $\pi \cap Q$, where π denotes the plane spanned by L_1 and S. Hence it is defined over K_1. But since it lies in the tangent plane at S, it is also defined over a quadratic extension l/k. Hence it is defined over $K_1 \cap l = k$.) q.e.d.

With the help of this lemma we could complete the proof as in [10]. But it may be of more interest to introduce another birational map, which can also be described[6] as the restriction of a Cremona transformation T_{33} of $\mathbb{P}^3$:

Let Q be a k-rational point of $\mathbb{P}^3$, chosen such that (i) it does not lie on W, and (ii) the planes ρ_i spanned by L_i and Q meet W in a non-degenerate conic C_i (it is well-known [14, p. 151] that there are only 5 planes through L_i for which the conic is degenerate!). We consider the family $\mathfrak{R}$ of cubic surfaces passing through the following 15 points: $\Sigma = \{L_i \cap C_i$ (2 points for each i); $L_i \cap C_j$ (1 point for each pair $i \neq j$); $C_i \cap C_j$ (1 point for each pair $i<j$)$\}$. Σ imposes 15 independent conditions on the surfaces of $\mathfrak{R}$. (If we adjoin the point Q and one point on each conic C_i as additional constraints, the only cubic

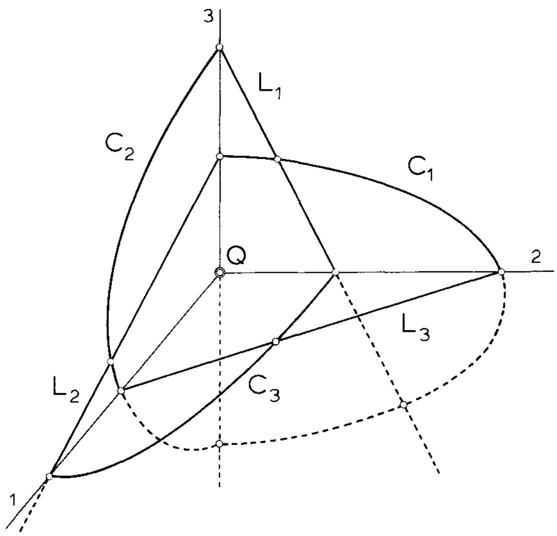

Figure 1

[6] This transformation (of type $B1$ in the classification of [6]) is similar to T_{33}^{tet}, but the fundamental tetrahedron is replaced by the configuration of the six straight lines of fig. 1. The associated linear system of cubic surfaces is the subfamily $\mathfrak{R}_Q$ of $\mathfrak{R}$ with base set $\Sigma \cup \{Q\}$. The fundamental set of the inverse transformation consists of a twisted cubic through Q_1, Q_2, Q_3 and of the 3 lines D_1, D_2, D_3 joining these points. The twisted cubic – which blows up in $\mathbb{P}^3$ to the unique quadric containing the L_i (and hence the E_i) – meets V residually in R_1, R_2 and R_3. Each line D_i of V (although it is a fixed component of the linear system) is blown down on W to the point $C_j \cap C_k$. This is because a cubic surface with two nodes Q_j, Q_k has the property that all the non-singular points on $D_i = \langle Q_j, Q_k \rangle$ have one and the same tangent plane. Hence if any element of the linear system defining Ω^{-1} (which automatically inherits a node at each Q_i) intersects V with multiplicity 2 at some non-singular point of D_i, the intersection must contain the whole line D_i with multiplicity 2. Some further details will be found in [6], but it is much simpler to use none of these global properties and to work directly on W with the map Ω. As an exercise, the reader may try to reprove proposition 1(a), using the transformation Ω^{-1} instead of Φ.

containing all these 19 points is $\rho_1 \cup \rho_2 \cup \rho_3$.) Therefore $\mathfrak{R}$ defines a rational transformation $\Omega : W \to V \subset \mathbb{P}^3$, where V is a cubic surface. Indeed the degree of V is equal to the number of its intersections with a generic line of $\mathbb{P}^3$; and such a line corresponds to the intersection of 2 cubic surfaces containing Σ. Now, one such cubic surface meets W in a sextic curve Γ_6, residual to L_1, L_2 and L_3, and having 4 points on each L_i. Hence another cubic surface meets Γ_6 in 18 points, 12 of which are on a fundamental line. The other 6 are the intersections $C_i \cap C_j$ and 3 movable points, so that the degree of V is equal to 3.

Concerning the separation of points by $\mathfrak{R}$, we first note that the 3 skew lines E_i, forming the complementary set to the L_i (see remark 1 above), are blown down to 3 points R_i. Each conic C_i is also blown down to a single point Q_i. Let now $W_0 = W - \bigcup C_i - \bigcup E_i$. We claim that Ω separates points and infinitely near points on W_0. Indeed the lines L_i are fixed for the surfaces of $\mathfrak{R}$, but – by general theory (see e.g. [16, §I.3]) – this does not affect the transformation Ω, since we can delete any fixed components of the family of divisors cut out by $\mathfrak{R}$. Now, to show that one such reduced divisor does *not* go through a point R of L_i, it suffices to show that R is *simple* on the intersection of W with a cubic surface of $\mathfrak{R}$ defining that divisor. Similarly for infinitely near points. It is now a simple matter to check that all points and infinitely near points on W_0 can be separated by means of triplets of planes containing L_1, L_2 and L_3 respectively[7].

Finally we check that $Q_1 = \Omega[C_1]$ is a double point of V: a line through Q_1 is the image of the intersection of 2 cubic surfaces containing C_1. The trace on W of one such cubic surface consists of the 3 lines L_i, the conic C_1 and a quartic curve Γ_4 going through $C_2 \cap C_3$. It is easy to see that $\Gamma_4 \cdot (L_1 + L_2 + L_3 + C_1) = 10$, so that the second cubic surface meets Γ_4 in $12 - 10 - 1 = 1$ movable point. Consequently, an arbitrary line through Q_1 meets the surface V in only one other point, and Q_1 is a double point of V, as asserted. We further note that the Q_i define a unique plane, which corresponds to the unique divisor $(\omega) = (\rho_1 \cup \rho_2 \cup \rho_3)$ that contains all the C_i. Hence they are not

[7] For completeness we give a brief sketch of the argument: let $R, S \in W_0$, with $R \neq S$. *First assume $R \notin L_i$ for any i.* Then, if $\tau_i = \langle L_i, R\rangle$, we see that $\bigcap \tau_i = \{R\}$. Otherwise, $\bigcap \tau_i$ would be a line E meeting all three L_i. But E also contains the point R; hence $E \subset W$ and E is one of the E_i; a contradiction, since $R \notin E_i$. Therefore we may assume without loss of generality that $S \notin \tau_1$. Consider $\omega = \tau_1 \cup \rho_2 \cup \rho_3$. Then $R \in \tau_1 \subset \omega$, but $S \notin \omega$, unless $S \in L_2 \cup L_3$ ($S \in \rho_2 \Rightarrow S \in L_2 \cup C_2 \Rightarrow S \in L_2$). In the latter case, however, (ω) contains S as a *simple* point, which is all we need to prove! Finally, *if $R \in L_1$ (say)*, we can use exactly the same argument, but τ_1 is replaced by the tangent plane to W at R, and we must take care not to forget multiplicities. The proof for infinitely near points is almost identical.

collinear, and this implies that the cubic surface V is normal! It follows that the points $R_i = \Omega[E_i]$ are simple, because the E_i are exceptional divisors of the first kind and we may apply theorem II.2.3 of [16]. The points $S_i = C_j \cap C_k$ are blown up to the 3 lines $D_i = \langle Q_j, Q_k \rangle$, as may be seen – for instance – by repeating the argument of lemma 2. The only difference is that, this time, the D_i are not skew, since we cannot separate the two tangent vectors $\vec{x}_i \in T_{S_i}C_k$ and $\vec{x}_j \in T_{S_j}C_k$, which both correspond to the point Q_k. The Zariski Main Theorem (used to prove the third assertion of the lemma) still applies, but of course it does not imply that the Q_i are non-singular, since the fibres above these points are infinite!

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