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# ARITHMETICAL FUNCTIONS INVOLVING EXPONENTIAL DIVISORS: NOTE ON TWO PAPERS BY L. TÓTH

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**Abstract.** Asymptotic estimates of L. Tóth [5, 6] on the summatory functions of three arithmetical functions involving exponential divisors are improved. For two of them the improvement is on the upper bound of the size of the remainder term ($O$-estimate), and is reached by appealing to lattice points estimates using exponent pairs due to Krätzel [1], and by having as well a closer look at the first terms of the generating Dirichlet series. For the third one, a lower bound on the size of the remainder term ($\Omega$-estimate) is replaced by two-sided oscillation ($\Omega_{\pm}$-estimate), by appealing to a method of Pétermann and Wu [2].

## 1. Notation and definitions

An *exponential divisor* ($e$-divisor) $d = p_1^{b_1} \cdots p_r^{b_r}$ of $n = p_1^{a_1} \cdots p_r^{a_r}$, satisfies by definition $b_i \mid a_i \quad (i = 1, \ldots, r)$. The integer $n$ is thus called *exponentially squarefree* ($e$-squarefree) if all the $a_i$ are squarefree. These two notions were introduced by M.V. Subbarao [4]. Other authors further extended the analogies with notions related to usual divisors. For instance, if $n$ and $m$ have the same prime divisors, we call $\kappa(n)(= \kappa(m)) := p_1 \cdots p_r$ their *kernel*, and then their *greatest common exponential divisor* ($e$-gcd) is defined as $(n,m)_{(e)} := \prod_{1\le i\le r} p_i^{(a_i,b_i)}$. And if $(n,m)_{(e)} = \kappa(n) = \kappa(m)$ we say that $n$ and $m$ are *exponentially-coprime* ($e$-coprime).

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Several functions related to exponential divisors, as the number $\tau^{(e)}(n)$ and the sum $\sigma^{(e)}(n)$ of $e$-divisors of $n$ to begin with, were studied by Subbarao and then by several other authors: see [5] for references.

In [5] and [6], L. Tóth studied some such functions, three of which are the subjects of this note. These are: (i) the number $t^{(e)}(n)$ of $e$-squarefree $e$-divisors of $n$, (ii) the number $\phi^{(e)}(n)$ of divisors $d$ of $n$ which are $e$-coprime with $n$ (the $e$-analogue of the Euler function $\phi$), and (iii) $\tilde{P}(n) := \sum_{1\le j\le n, \kappa(j)=\kappa(n)} (j,n)_{(e)}$ (the $e$-analogue of the Pillai function $P(n) := \sum_{1\le j\le n} (j,n)$ [3]).

Let $\zeta$ denote the Riemann zeta function. Let $\phi$ and $\mu$ be the Euler and Möbius functions. For a positive integer $n$ put as usual $\omega(n)$ for the number of distinct prime divisors of $n$. For a positive integer $k$ let $\mathbf{1}_k$ be the characteristic function of the integers $n$ of the form $n = m^k$ (where $m$ is an integer), and similarly let $\mu_k(n) = \mu(m)$ if $n = m^k$ and $\mu_k(n) = 0$ otherwise.

## 2. Results

Tóth proved the following estimates for the summatory functions of $t^{(e)}(n)$, $\phi^{(e)}(n)$ and $\tilde{P}(n)$.

**Theorem A.** *We have*

$$E_t(x) := \sum_{n\le x} t^{(e)}(n) - C_1 x - C_2 x^{1/2} = O(x^{1/4+\epsilon})$$

*for every $\epsilon > 0$, where $C_1$ and $C_2$ are constants given by*

$$C_1 = \prod_p \left(1 + \frac{1}{p^2} + \sum_{a\ge 6} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right),$$

$$C_2 = \zeta\left(\frac{1}{2}\right) \prod_p \left(1 + \sum_{a\ge 4} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{a/2}}\right).$$

**Theorem B.** *We have*

$$E_\phi(x) := \sum_{n\le x} \phi^{(e)}(n) - C_3 x - C_4 x^{1/3} = O(x^{1/5+\epsilon})$$

*for every $\epsilon > 0$, where $C_3$ and $C_4$ are constants given by*

$$C_3 = \prod_p \left(1 + \sum_{a \geq 3} \frac{\phi(a) - \phi(a-1)}{p^a}\right),$$

$$C_4 = \zeta\left(\frac{1}{3}\right) \prod_p \left(1 + \sum_{a \geq 5} \frac{\phi(a) - \phi(a-1) - \phi(a-3) + \phi(a-4)}{p^{a/3}}\right).$$

**Theorem C.** *We have*

$$E_p(x) := \sum_{n \leq x} \tilde{P}(n) - C_5 x^2 = \begin{cases} O(x(\log x)^{5/3}), \\ \Omega(x \log\log x), \end{cases}$$

*where the constant $C_5$ is given by*

$$C_5 := \frac{1}{2} \prod_p \left(1 + \sum_{a \geq 2} \frac{\tilde{P}(p^a) - p\tilde{P}(p^{a-1})}{p^{2a}}\right).$$

**Notes.**

(1) Theorem A is Theorem 4 in [6], which however contains two misprints: the term $1/p^2$ is missing in the factor defining $C_1$, and the rightmost exponent in the factors defining $C_2$ is incorrect ($\omega(a-4)$ instead of $\omega(a-3)$; the same mistake is repeated in the proof on p.164). Theorem B is Theorem 1 in [5], which also contains misprints: the rightmost term in the factor defining $C_4$ is incorrect ($-\phi(a-4)$ instead of $+\phi(a-4)$), and the product symbol $\prod$ is missing. The $O$-estimate in Theorem C is Theorem 3 in [5], and the $\Omega$-estimate is a direct consequence of Theorem 4 in [5], which states that $\limsup_{n\to\infty} \tilde{P}(n)/(n \log\log n) = 6e^\gamma/\pi^2$.

(2) The proofs of Theorems A and B make use of estimates due to Krätzel for

$$\Delta(a, b; x) := \sum_{n_1^a n_2^b \leq x} 1 - \zeta(b/a)x^{1/a} - \zeta(a/b)x^{1/b}$$

in the case where $a$ and $b$ are integers with $1 \leq a < b$. The elementary Theorem 5.3 in [1] yields $\Delta(a, b; x) = O(x^{1/(2a+b)})$, and is applied to the case $a = 1,\ b = 2$ for the proof of Theorem A, and to the case $a = 1,\ b = 3$ for the proof of Theorem B.

But, from more elaborate arguments involving exponent pairs in this same Chapter 5 of [1], we see that $\Delta(1,2;x) = O(x^\tau)$ with $\tau < 1/4$ and $\Delta(1,3;x) = O(x^\varphi)$ with $\varphi < 1/5$. This will be used in the Proof of Theorem 1 below.

[For the best known values of $\tau$ and $\varphi$: Theorem 5.11 p.223 yields $\Delta(1,2;x) = O(x^{37/167+\epsilon})$ $(x \to \infty)$ for every $\epsilon > 0$ (see the Note on Section 5.3 on p.230), and Theorem 5.12 p.227 yields $\Delta(1,3;x) = O(x^{0.175}(\log x)^2)$ $(x \to \infty)$ with the exponent pair (1/14, 11/14) (as indicated in the small table at the bottom of page 227)].

There are two objects to this note. The first one is to refine the argument yielding Theorems A and B, and to prove

**Theorem 1.** *We have* $E_t(x) = O(x^{1/4})$ *and* $E_\phi(x) = O(x^{1/5}\log x)$.

The other object is to replace the $\Omega$-estimate in Theorem C by an oscillation estimate.

**Theorem 2.** *We have*

$$E_P(x) = \Omega_\pm(x \log\log x).$$

## 3. Proofs

**Proof of Theorem 1.** We begin with $E_t$. The proof of Theorem A in [6] exploits the expression

$$T(s) := \sum_{n\ge1} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s) \quad (\sigma > 1),$$

where, for $v(p^a) := 2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}$ $(a \ge 4)$ and $v(p^a) = 0$ $(1 \le a \le 3)$, the series

$$V(s) := \sum_{n\ge1} \frac{v(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^{4s}} + \sum_{a\ge5} \frac{v(p^a)}{p^{as}}\right)$$

is absolutely convergent for $\sigma > 1/4$.

A closer look thus easily shows that $V(s) = H(s)/\zeta(4s)$, with

$$H(s) := \sum_{n\ge1} \frac{h(n)}{n^s} = \prod_p \left(1 + \frac{2}{p^{6s}} + \sum_{a\ge7} \frac{h(p^a)}{p^{as}}\right).$$

Since $|h(p^a)| = |(\mathbf{1}_4 * v)(p^a)| \le \sum_{i\le a} 2^{\omega(i)} = O(a^2)$, we see that $H(s)$ converges absolutely for $\sigma > 1/6$.

Now if $H_0(s) := \zeta(s)\zeta(2s)/\zeta(4s) =: \sum_{n\ge1} h_0(n)n^{-s}$, we have $h_0 = \mathbf{1} * \mathbf{1}_2 * \mu_4$, whence by using the fact that $\Delta(1,2;x) = O(x^\tau)$ for some $\tau < 1/4$ (see Note (2) above) we have

$$\sum_{n\le x} h_0(n) = \sum_{n=n_1n_2^2N^4\le x} \mu(N) =$$

$$= \sum_{N\le x^{1/4}} \mu(N) \left(\zeta(2)\frac{x}{N^4} + \zeta\left(\frac{1}{2}\right)\frac{x^{1/2}}{N^2} + O\left(\frac{x^\tau}{N^{4\tau}}\right)\right).$$

From the prime number theorem under the form $\sum_{n\ge y} \mu(n)/n = o(1)$ $(y \to \infty)$ it follows that, if $\ell > 1$, $\sum_{n<y} \mu(n)/n^\ell = 1/\zeta(\ell) + o(y^{1-\ell})$, whence

$$\sum_{n\le x} h_0(n) = \frac{\zeta(2)}{\zeta(4)}x + \frac{\zeta(1/2)}{\zeta(2)}x^{1/2} + O(x^{1/4}).$$

Finally, with $t^{(e)} = h * h_0$, we see that

$$\sum_{n\le x} t^{(e)}(n) = \frac{\zeta(2)}{\zeta(4)}H(1)x + \frac{\zeta(1/2)}{\zeta(2)}H(1/2)x^{1/2} + O(x^{1/4}).$$

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The proof of $E_\phi(x) = O(x^{1/5})$ is similar. Instead of considering as in [5] the expression

$$\Phi(s) := \sum_{n\ge1} \frac{\phi^{(e)}(n)}{n^s} = \zeta(s)\zeta(3s)U(s) =$$

$$= \zeta(s)\zeta(3s) \prod_p \left(1 + \frac{2}{p^{5s}} + \sum_{a\ge6} \frac{u(p^a)}{p^{as}}\right) \quad (\sigma > 1),$$

where the Dirichlet series for $U(s)$ converges absolutely for $\sigma > 1/5$, we note that

$$U(s) = (\zeta(5s))^2 J(s) = (\zeta(5s))^2 \prod_p \left(1 - \frac{3}{p^{6s}} + \sum_{a\geq 7} \frac{j(p^a)}{p^{as}}\right),$$

where the Dirichlet series for $J(s)$ converges absolutely for $\sigma > 1/6$. Indeed $j = \mu_5 * \mu_5 * u$ where $u(p^a) = \phi(a) - \phi(a-1) - \phi(a-3) + \phi(a-4)$ $(a \geq 5)$ and $u(p^a) = 0$ $(1 \leq a \leq 4)$, whence $j(p^a) = O(a)$ (more precisely, $|j(p^a)| \leq 8a$). Thus by using the fact that $\Delta(1, 3; x) = O(x^{\varphi})$ for some $\varphi < 1/5$ we obtain, similarly as before,

$$\sum_{n\leq x} \phi^{(e)}(n) = \zeta(3)(\zeta(5))^2 J(1)x + \zeta(1/3)(\zeta(5/3))^2 J(1/3)x^{1/3} + O(x^{1/5}\log x).$$

**Proof of Theorem 2.** We leave this proof to the reader, whom we refer to the proof of Theorem 3 in [2], since the argument there may be very closely followed with only minor adaptations. Indeed the latter theorem establishes that $\sum_{n\leq x} \sigma^{(e)}(n) = Dx^2 + \Omega_{\pm}(x \log\log x)$ for some constant $D$ by exploiting the expression $\sum_{n\geq 1} \sigma^{(e)}(n)n^{-s} = \zeta(s-1)\zeta(2s-1)(\zeta(3s-2)^{-1}K(s)$, where the Dirichlet series for $K(s)$ absolutely converges for $\sigma > 3/4$; and similarly we have (see Lemma 3 of [5]) $\sum_{n\geq 1} \tilde{P}(n)n^{-s} = \zeta(s-1)\zeta(2s-1)(\zeta(3s-2)^{-1}W(s)$, where the Dirichlet series for $W(s)$ absolutely converges for $\sigma > 3/4$.

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