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ARRANGEMENTS OF HYPERPLANES AND VECTOR BUNDLES ON P^n

I. DOLGACHEV AND M. KAPRANOV

Introduction. Let X be a smooth algebraic variety and D be a divisor with normal crossing on X. The pair (X, D) gives rise to a natural sheaf $\Omega_X^1(\log D)$ of differential 1-forms on X with logarithmic poles on D. For each point $x \in X$ the space of sections of this sheaf in a small neighborhood of x is generated over $\mathcal{O}_{X,x}$ by regular 1-forms and by forms $d \log f_i$ where $f_i = 0$ is a local equation of an irreducible component of D containing X. This sheaf (and its exterior powers) was originally introduced by Deligne [De] to define a mixed Hodge structure on the open variety $X - D$. An important feature of the sheaf $\Omega_X^1(\log D)$ is that it is locally free, i.e., can be regarded as a vector bundle on X.

In this paper we concentrate on a very special case when $X = P^n$ is a projective space and $D = H_1 \cup \cdots \cup H_m$ is a union of hyperplanes in general position. It turns out that the corresponding vector bundles are quite interesting from the geometric point of view. It was shown in an earlier paper [K] of the second author that in this case $\Omega_{P^n}^1(\log D)$ defines an embedding of P^n into the Grassmann variety $G(n, m-1)$ whose image becomes, after the Plücker embedding, a Veronese variety V_n^{m-3}, i.e., a variety projectively isomorphic to the image of P^n under the map given by the linear system of all hypersurfaces of degree $m - 3$. In the case when the hyperplanes osculate a rational normal curve in P^n, the bundle $\Omega_{P^n}^1(\log D)$ coincides with the secant bundle E_n^m of Schwarzenberger [Schw1–2]. The corresponding Veronese variety consists in this case of chordal $(n-1)$-dimensional subspaces to a rational normal curve in P^{m-2}.

The main result of this paper (Theorem 7.2) asserts that in the case $m \geqslant 2n + 3$ the arrangement of m hyperplanes $\mathscr{H} = \{H_1, \ldots, H_m\}$ can be uniquely reconstructed from the bundle $E(\mathscr{H}) = \Omega_{P^n}^1(\log \bigcup H_i)$ unless all of its hyperplanes osculate the same rational normal curve of degree n. To prove this we study the variety $C(\mathscr{H})$ of jumping lines for $E(\mathscr{H})$. The consideration of this variety is traditional in the theory of vector bundles on P^n (see [Bar, Hu]). In our case this variety is of some geometric interest. For example, if $n = 2$, i.e., we deal with m lines in P^2, then (in the case of odd m) $C(\mathscr{H})$ is a curve in the dual P^2 containing the points corresponding to lines from $\mathscr{H}$. The whole construction therefore gives a canonical way to draw an algebraic curve through a collection of points in (the dual) P^2.

For 5 points $p_1, \ldots, p_5$ in P^2 this construction gives the unique conic through p_i.

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For 7 points $p_1, \dots, p_7$ the construction gives a plane sextic curve for which p_i are double points. A nonsingular model of this curve is isomorphic to the plane quartic curve of genus 3 which is classically associated to 7 points via Del Pezzo surfaces of degree 2 (see [C2] [DO]).

For an even number of lines in P^2 the set of jumping lines is typically finite. In this case, more interesting is the curve of *jumping lines of second kind* introduced by Hulek [Hu]. The study of this curve will be carried out in the subsequent paper [DK].

Let us only formulate the answer for 6 lines (considered as points $p_1, \dots, p_6$ in the dual plane). In this case Hulek's curve will be a sextic of genus 4 of which p_i are nodes. It is described as follows. Blow up the points p_i. The result is isomorphic to a cubic surface S in P^3. The inverse images of the points p_i and the strict preimages of quadrics through various 5-tuples of p_i form a Schläfli double sixer of lines on the cubic surface. To each such double sixer there is classically associated a quadric Q in P^3 called the *Schur quadric* [Schur], [R] (see also [B], p. 162). It is uniquely characterized by the property that the corresponding pairs of lines of the double sixer are orthogonal with respect to Q. Our sextic curve in P^2 lifted to the surface S becomes the intersection $S \cap Q$.

In fact, much of Hulek's general theory of stable bundles on P^2 with odd first Chern class can be neatly reformulated in terms of (suitably generalized) Schur quadrics. This will be done in [DK].

Thus our approach gives a unified treatment of many classical constructions associating a curve to a configuration of points in a projective space. It appears that a systematic study of logarithmic bundles in other situations (like surfaces other than P^2) will provide a rich supply of concrete examples and give additional insight into the geometry related to vector bundles.

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1. Arrangements of hyperplanes.

1.1. Let V be a complex vector space of dimension $n + 1$ and $P^n = P(V)$ be the projective space of lines in V. Let $\mathscr{H} = (H_1, \dots, H_m)$ be a set (arrangement) of hyperplanes in P^n. Dually it defines a set (a configuration) of m points in the dual vector space $\check{P}^n = P(V^*)$. We say that $\mathscr{H}$ is in (linearly) general position if the intersection of any $k \leqslant n + 1$ hyperplanes from $\mathscr{H}$ is of codimension exactly k. Throughout this paper we shall mostly deal with arrangements in general position.

1.2. We choose a linear equation $f_i \in V^*$ for each hyperplane H_i of $\mathscr{H}$. This system of choices defines a linear map

$$\alpha_{\mathscr{H}}\colon \mathbf{C}^m \to V^*; \qquad (\lambda_1, \dots, \lambda_m) \mapsto \sum \lambda_i f_i.$$

The kernel of this map will be denoted by $I_{\mathscr{H}}$. It consists of linear relations between the linear forms f_i. By transposing, we obtain a linear map

$$\alpha_{\mathscr{H}}^*: (\mathbf{C}^m)^* \to I_{\mathscr{H}}^*. \tag{1.1}$$

Assume that $m \geqslant n + 2$ so that $I_{\mathscr{H}} \neq 0$. After making a natural identification between the space $\mathbf{C}^m$ and its dual space $(\mathbf{C}^m)^*$ (defined by the bilinear form $\sum x_i y_i$), we obtain from the map (1.1) m linear forms on the space $I_{\mathscr{H}}$, i.e., an arrangement of hyperplanes in $P(I_{\mathscr{H}})$. We denote the arrangement thus obtained by $\mathscr{H}^{as}$ and refer to it as the *associated* arrangement. It is clear that $\mathscr{H}$ is in general position if and only if the restriction of the map $\alpha_{\mathscr{H}}$ to any coordinate subspace $\mathbf{C}^k$ in $\mathbf{C}^m$ with $k \leqslant n + 1$ is of maximal rank. This implies that $\mathscr{H}^{as}$ is in general position if and only if $\mathscr{H}$ is. In the latter case, the dimension of $P(I_{\mathscr{H}})$ is equal to $m - n - 2$. From now on whenever we speak about the association we assume that the arrangements are in general position. Note that to define the map $\alpha_{\mathscr{H}}$ we need a choice of order on the set of hyperplanes from $\mathscr{H}$. Making this choice we automatically make a choice of order on the set $\mathscr{H}^{as}$.

The notion of association was introduced by A. Coble [C1]. For modern treatment see [DO]. This notion was rediscovered, under the names "duality" or "orthogonality" several times later, notably in the context of combinatorial geometries (see [CR], §11) and hypergeometric functions (see [GG]).

Obviously the association is a self-dual operation, so $(\mathscr{H}^{as})^{as} = \mathscr{H}$, where we make the canonical identification between the spaces V and V^{**}.

Let V and I be two vector spaces of dimensions $n + 1$ and $m - n - 1$ respectively, and let $\mathscr{H}$ and $\mathscr{H}'$ be two arrangements of m hyperplanes in $P(V)$ and $P(I)$ respectively. We shall say that $\mathscr{H}$ and $\mathscr{H}'$ are associated if there is a projective isomorphism $P(I) \to P(I_{\mathscr{H}})$ taking $\mathscr{H}'$ to the associated configuration $\mathscr{H}^{as}$ (matching the ordering, if it was made). In particular, when $m = n + 2$, we can speak about *self-associated* configurations.

By duality we can speak about associated configurations of points in projective spaces. The following proposition (equivalent to a result by A. Coble) gives a criterion of being associated (resp. self-associated) in terms of the Segre (resp. Veronese) embedding. We state it in terms of configurations of points.

1.3. PROPOSITION. (a) *Let V, I be vector spaces of dimensions $n + 1$ and $m - n - 1$ respectively. Let $p_i \in P(V)$, $q_i \in P(I)$, $i = 1, \dots, m$, be two configurations of m points. Let $s(p_i, q_i) \in P(V \otimes I)$ be the image of the pair (p_i, q_i) with respect to the Segre embedding s: $P(V) \times P(I) \to P(V \otimes I)$. The configurations of points $(p_1, \dots, p_m)$ and $(q_1, \dots, q_m)$ are associated to each other if and only if the points $s(p_i, q_i)$ are projectively dependent but any proper subset of them is projectively independent.*

(b) *Let V be a vector space of dimension $n + 1$ and $p_i \in P(V)$, $i = 1, \dots, 2n + 2$ be a configuration of points. Let $v(p_i) \in P(S^2 V)$ be the image of p_i under the Veronese embedding v: $P(V) \to P(S^2 V)$. The configuration $(p_1, \dots, p_{2n+2})$ is self-associated if and only if the points $v(p_i)$ are projectively dependent but any proper subset of them is projectively independent.*

Proof. (b) follows from (a). For the proof of (a), see, e.g., [K].

1.4. Let $\mathscr{H}$ be an arrangement of m hyperplanes in $P(V)$ and $\mathscr{H}^{as}$ be the associated arrangement in the space $P(I_{\mathscr{H}})$. Let

$$W = \left\{(\lambda_1, \ldots, \lambda_m) \in \mathbf{C}^m \colon \sum \lambda_i = 0\right\}.$$

Define a linear map

$$t_{\mathscr{H}} \colon I_{\mathscr{H}} \otimes V \to W$$

by the formula

$$t_{\mathscr{H}}((a_1, \ldots, a_m), v) = (a_1 f_1(v), \ldots, a_m f_m(v)).$$

This map considered as an element of the tensor product $I_{\mathscr{H}}^* \otimes V^* \otimes W$ will be of considerable importance in the sequel. We shall refer to it as the *fundamental tensor* of the configuration $\mathscr{H}$.

It is clear that the fundamental tensor $t_{\mathscr{H}^{as}} \in (I_{\mathscr{H}^{as}})^* \otimes I_{\mathscr{H}}^* \otimes W = V^* \otimes I_{\mathscr{H}}^* \otimes W$ of the associated configuration $\mathscr{H}^{as}$ is obtained from the fundamental tensor $T_{\mathscr{H}} \in I_{\mathscr{H}}^* \otimes V^* \otimes W$ by interchanging of factors in the tensor product.

In coordinates, fixing a basis $e_1, \ldots, e_{n+1}$ in V and its dual basis in V^*, let $A = \|a_{ij}\|_{1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant m}$ be the matrix whose columns are the coordinates of the linear functions f_i and $B = \|b_{ij}\|_{1 \leqslant i \leqslant m-n-1, 1 \leqslant j \leqslant m}$ be a similar matrix for the associated arrangement. We can choose B in such a way that $B \circ A = 0$. Then the coordinates of the tensor $t_{\mathscr{H}}$ are given by the formula

$$(t_{\mathscr{H}})_{ijk} = b_{ik} a_{kj}.$$

1.5. Proposition. *Suppose that $\mathscr{H}$ is in general position. Then for any nonzero vector $v \in V$, the linear operator $t_{\mathscr{H}}(v) \colon I_{\mathscr{H}} \to W$ defined by the fundamental tensor $t_{\mathscr{H}}$, is injective.*

Proof. If $(a_1, \ldots, a_m) \in \operatorname{Ker}(t_{\mathscr{H}}(v))$, then $a_i f_i(v) = 0$ for all $i = 1, \ldots, m$. Let $J = \{i \colon f_i(v) = 0\}$. Then for any $i \notin J$ we have $a_i = 0$. Since $\mathscr{H}$ is in general position, $|J| \leqslant n$. Hence $\sum a_i f_i = 0$ is a nontrivial linear relation between $\leqslant n$ linear functions among f_i. This contradicts the assumption of general position for $\mathscr{H}$.

2. Logarithmic bundles.

2.1. Let $\mathscr{H} = (H_1, \ldots, H_m)$ be an arrangement of m hyperplanes in $P^n = P(V)$ in general position. We shall define the divisor $\bigcup H_i$ also by $\mathscr{H}$. This divisor has normal crossing. This means that for any point $x \in P^n$, its local equation can be given by $t_1 \ldots t_k = 0$ where $t_1, \ldots, t_k$ is a part of a system of local parameters at x.

In this situation one can define the sheaf $\Omega^1_{P^n}(\log \mathscr{H})$ of differential 1-forms with logarithmic poles along $\mathscr{H}$ (see [De]). It is a subsheaf of the sheaf $j_*\Omega^1_U$ where $U = P^n - \mathscr{H}$ and $j: U \hookrightarrow P^n$ is the embedding. If $x \in P^n$ and $t_1 \dots t_k = 0$ is a local equation of the divisor $\mathscr{H}$ near x, as above, then the section of $\Omega^1_{P^n}(\log \mathscr{H})$ near x are meromorphic differential forms which can be expressed as $\omega + \sum u_i d \log t_i$ where ω is a 1-form and u_i are functions, all regular near x. It is not difficult to see that the sheaf $\Omega^1_{P^n}(\log \mathscr{H})$ is locally free of rank n (see [De]).

We shall denote the sheaf $\Omega^1_{P^n}(\log \mathscr{H})$ by $E(\mathscr{H})$ and call it the *logarithmic bundle* associated to $\mathscr{H}$. It will be the main object of study in this paper. We will not make a distinction between vector bundles and locally free coherent sheaves of their sections.

2.2. The sheaf $E(\mathscr{H})^*$ dual to $E(\mathscr{H})$ has a nice interpretation in terms of vector fields. We say that a regular vector field ∂ defined in some open subset $U \subset P^n$ is tangent to $\mathscr{H}$ if for any $x \in U$, the vector $\partial(x)$ lies in the intersection of the tangent hyperplanes at x to all H_i containing x. (In particular, $\partial(x) = 0$ if x is a point of n-tuple intersection.) Such fields form a coherent subsheaf $T_{P^n}(\log \mathscr{H})$ in the tangent sheaf T_{P^n}. It is easy to see by local calculations that this sheaf is isomorphic to the dual sheaf $E(\mathscr{H})^*$.

2.3. PROPOSITION. *Let ε_i: $H_i \hookrightarrow P^n$ be the embedding map. We have the canonical exact sequence of sheaves on P^n*

$$0 \to \Omega^1_{P^n} \to E(\mathscr{H}) \xrightarrow{\text{res}} \bigoplus_{i=1}^{m} \varepsilon_{i*}\mathcal{O}_{H_i} \to 0 \tag{2.1}$$

where res is the Poincaré residue morphism defined locally by the formula

$$a_1 d \log t_1 + \cdots + a_k d \log t_k + b_{k+1}\, dt_{k+1} + \cdots + b_n\, dt_n$$

$$\mapsto (a_1(x), \dots, a_k(x), 0, \dots, 0)$$

where $(t_1, \dots, t_n)$ is a system of local coordinates at x such that $t_1 \dots t_k = 0$ is a local equation of the divisor $\mathscr{H}$ at x.

Proof. See [De].

The next two propositions follow simply from the above exact sequence.

2.4. PROPOSITION. *The Chern polynomial $c(E(\mathscr{H})) = \sum c_i(E(\mathscr{H}))t^i$ of the bundle $E(\mathscr{H})$ is given by*

$$c(E(\mathscr{H})) = (1 - ht)^{-m+n+1}$$

where h is the class of a hyperplane in P^n. In particular, the determinant $\bigwedge^n E(\mathscr{H})$ is isomorphic to the line bundle $\mathcal{O}(m - n - 1)$ on P^n.

2.5. PROPOSITION. (a) *The space $H^0(P^n, E(\mathscr{H}))$ has dimension $m-1$ and consists of forms*

$$\sum_{i=1}^{m} \alpha_i d\log f_i = d\log\left(\prod_{i=1}^{m} f_i^{\alpha_i}\right), \qquad \alpha_i \in \mathbf{C}, \sum \alpha_i = 0.$$

(b) *More generally,* $\dim H^0(E(\mathscr{H})(k)) = (n+1)\binom{k+n-1}{n} - \binom{k+n}{n} + m\binom{k+n-1}{n-1}$.

(c) $H^i(E(\mathscr{H})(k)) = 0$ *for* $1 \leqslant i \leqslant n-2$ *and any* $k \in \mathbf{Z}$.

Note that we can now identify the space $H^0(P^n, E(\mathscr{H}))$ with the space W introduced in 1.4.

2.6. The logarithmic bundles can be obtained from the bundle $\Omega^1_{P^n}$ by applying elementary transformations of vector bundles. These transformations were introduced first in the case of vector bundles over curves by A. Tyurin [T], and their general definition is due to Maruyama [M1–2]. Let us recall this concept.

Let E be a rank r vector bundle over a smooth algebraic variety X and $Z \subset X$ a hypersurface. Denote by $i\colon Z \to X$ the embedding. Suppose that we have chosen some quotient bundle F of the restriction i^*E. Then we have a surjective map of sheaves $E \to i_*F$ on X. We define the coherent sheaf $\mathrm{Elm}^-_{Z,F}$ as the kernel of this surjection. It is easy to see that it is locally free of rank r, i.e., can also be regarded as a vector bundle. This bundle is called the elementary transformation of E along (Z, F).

Note than when E is a line bundle and $F = i^*E$, then $\mathrm{Elm}^-_{Z,F}$ is just the twisted sheaf $E(-Z)$.

2.7. The bundle E can be reconstructed from its elementary transformation by applying the "inverse" elementary transformation $\mathrm{Elm}^+_{Z,F}$. In the situation of 2.6, the definition of Elm^+ is as follows. Let $E(Z)$ be the sheaf whose sections are sections of E with simple poles along Z. Then $\mathrm{Elm}^+_{Z,F}(E)$ is a subsheaf of $E(Z)$ whose sections after multiplying by the local equation of Z belong to $\mathrm{Elm}^-_{Z,F}(E) = \mathrm{Ker}\{E \to i^*F\}$. It is easy to see that Elm^+ and Elm^- are mutually inverse operations.

2.8. For any $1 \leqslant i \leqslant m$ let $\mathscr{H}_{\leqslant i}$ be the truncated arrangement $(H_1, \ldots, H_i)$. By definition, $\mathscr{H}_{\leqslant 0} = \varnothing$ and $E(\mathscr{H}_{\leqslant 0}) = \Omega^1_{P^n}$. The residue exact sequence from 2.3 induces the exact sequence

$$0 \to E(\mathscr{H}_{\leqslant i-1}) \to E(\mathscr{H}_{\leqslant i}) \to \varepsilon_{i*}\mathcal{O}_{H_i} \to 0.$$

Passing to the dual exact sequence and using the adjunction formula we find the exact sequence

$$0 \to E(\mathscr{H}_{\leqslant i})^* \to E(\mathscr{H}_{\leqslant i-1})^* \to \varepsilon_{i*}\mathcal{O}_H(1) \to 0. \tag{2.2}$$

Thus, by definition, we obtain the following.

2.9. PROPOSITION. *For each $i \leqslant m$ we have isomorphisms*

$$E(\mathscr{H}_{\leqslant i-1}) \cong \mathrm{Elm}^{-}_{H_i, \varepsilon_{i*}\mathcal{O}_{H_i}}(E(\mathscr{H}_{\leqslant i})),$$

$$E(\mathscr{H}_{\leqslant i}) \cong \mathrm{Elm}^{-}_{H_i, \varepsilon_{i*}\mathcal{O}_{H_i}(1)}(E(\mathscr{H}_{\leqslant i-1})^*).$$

It is the second isomorphism which will be useful for us in §5 later.

2.10. PROPOSITION. *Assume $1 \leqslant m \leqslant n+1$. Then*

$$E(\mathscr{H}) \cong (\mathcal{O}_{P^n})^{\oplus(m-1)} \oplus \mathcal{O}_{P^n}(-1)^{n+1-m}.$$

Proof. Since $m \leqslant n+1$, we can choose homogeneous coordinates $x_1, \ldots, x_{n+1}$ in $P^n = P(V)$ such that the hyperplane H_i, $1 \leqslant i \leqslant m$, is given by the equality $x_i = 0$. By Serre's theorem [H] coherent sheaves on P^n correspond to graded $\mathbf{C}[x_1, \ldots, x_{n+1}]$-modules (modulo finite-dimensional ones), the correspondence being given by $\mathscr{F} \mapsto \bigoplus H^0(P^n, \mathscr{F}(i))$. We shall describe the module corresponding to $E(\mathscr{H})$. Denote this module by M'. The ring $\mathbf{C}[x_1, \ldots, x_{n+1}]$ will be denoted shortly by A.

Denote by $\xi = \sum x_i \partial/\partial x_i$ the Euler vector field on V. By Lie_ξ and i_ξ we shall denote the Lie derivative along ξ and the contraction of 1-forms with ξ.

Let $\tilde{\mathscr{H}} \subset V$ be the configuration of coordinate hyperplanes $\{x_i = 0\}$, $i = 1, \ldots, m$. Let M be the space of all global sections of the sheaf $\Omega^1_V(\log \tilde{\mathscr{H}})$ on V. It is a graded A-module; the graded component M_r consists of forms ω such that $\mathrm{Lie}_\xi\, \omega = r \cdot \omega$.

It is clear that the space of section $H^0(P^n, E(\mathscr{H})(p))$ can be identified with the subspace in M_r consisting of forms ω such that $i_\xi \omega = 0$. Hence our module M' corresponding to $E(\mathscr{H})$ is the kernel of the homomorphism $M \to A$ given by i_ξ.

The graded A-module M is free: it is isomorphic to $A^m \oplus A^{n-m}(-1)$ with the basis $d \log x_1, \ldots, d \log x_m, dx_{m+1}, \ldots, dx_{n+1}$, the first m elements being in degree 0, the remaining ones in degree 1. Since $i_\xi(d \log x_i) = 1$, $i_\xi(dx_i) = x_i$, we find that M' is the kernel of the homomorphism

$$A^m \oplus A^{n-m}(-1) \to A, \qquad (a_1, \ldots, a_n) \mapsto \sum_{j=1}^{m} a_j + \sum_{j=m+1}^{n+1} a_j x_j. \tag{2.3}$$

However, an element $(a_1, \ldots, a_n)$ from the kernel of (2.3) is uniquely determined by the components $(a_2, \ldots, a_n)$ which may be arbitrary: we just define a_1 to be equal to $-(\sum_{j=2}^{m} a_j + \sum_{j=m+1}^{n+1} a_j x_j)$. This means that M is isomorphic to $A^{m-1} \oplus A(-1)^{n-m+1}$ as a graded A-module. Hence, by Serre's theorem, $E(\mathscr{H}) \cong (\mathcal{O}_{P^n})^{\oplus(m-1)} \oplus \mathcal{O}_{P^n}(-1)^{n+1-m}$.

2.11. The logarithmic bundles can be used to define a map from the projective space to a Grassmannian with the image isomorphic to a Veronese variety. Let us explain this in more detail.

By a Veronese variety, we mean a subvariety in a projective space $P^{\binom{n+d}{n}-1}$ which is projectively isomorphic to the image of the Veronese mapping

$$v_{n,d}: P^n = P(V) \to P^{\binom{n+d}{n}-1} = P(S^d V). \tag{2.4}$$

Let E be a vector space of dimension $n + d$ and $G(n, E)$ the Grassmannian of n-dimensional linear subspaces in E. We shall often identify it with the Grassmannian $G(d, E^*)$ of d-dimensional subspaces in the dual space E^*. Consider its Plücker embedding

$$G(n, E) \hookrightarrow P\left(\bigwedge^n E\right) = P^{\binom{n+d}{n}-1}. \tag{2.5}$$

Note that the dimensions of the ambient spaces for the Plücker embedding and the Veronese embedding coincide. Therefore it makes sense to speak about n-dimensional Veronese varieties in the Grassmannian $G(n, E)$. The following result, proven in [K], shows that the logarithmic bundle $E(\mathscr{H})$ defines an embedding of P^n into a Grassmannian whose image is a Veronese variety.

2.12. THEOREM. *Let $\mathscr{H}$ be an arrangement of $m \geqslant n + 2$ hyperplanes in P^n in general position. Denote by W the space $H^0(P^n, E(\mathscr{H})) \cong \mathbf{C}^{m-1}$. For any point $x \in P^n$ consider the subspace of W consisting of all sections vanishing at x, and let $\phi_{\mathscr{H}}(x)$ be the dual subspace of W^*. Then*:

(a) *the dimension of $\phi_{\mathscr{H}}(x)$ equals n for all $x \in P^n$;*
(b) *the correspondence $x \mapsto \phi_{\mathscr{H}}(x)$ is a regular embedding $\phi_{\mathscr{H}}: P^n \hookrightarrow G(n, W^*)$;*
(c) *the image $\phi_{\mathscr{H}}(P^n)$ in $G(n, W^*)$ becomes, after the Plücker embedding $G(m-n-1, W) \subset P(\bigwedge^n W^*)$, a Veronese variety.*

In particular, $E(\mathscr{H})$ is the inverse image of the bundle $\mathscr{S}^*$ on $G(n, W^*)$ where $\mathscr{S}$ is the tautological subbundle over $G(n, W^*)$.

2.13. COROLLARY. *Assume that $m = n + 2$. Then $E(\mathscr{H}) \cong T_{P^n}(-1)$ where T_{P^n} is the tangent bundle of P^n.*

Proof. Since $\dim(W) = n + 1$, the map $\phi_{\mathscr{H}}$ defines an isomorphism $P^n = P(V) \to G(n, W^*) = G(1, W) = P(W)$. In this case the tautological subbundle $\mathscr{S}$ on $G(n, W^*)$ is isomorphic to $\Omega^1_{P(W)}(1)$. Hence $E(\mathscr{H})$ is isomorphic to $T_{P^n}(-1)$.

2.14. Let us call a rank-n vector bundle E on P^n *normalized* if $c_1(E) \in \{0, -1, \ldots, -n+1\}$. If E is any rank-n vector bundle E on P^n and $c_1(E) = na + b$ where $a \in \mathbf{Z}$, $b \in \{0, -1, \ldots, -n+1\}$ then we denote by E_{norm} the normalized bundle $E(-a)$.

In our case $c_1(E(\mathscr{H})) = m - n - 1$ so the normalized bundle $E_{norm}(\mathscr{H})$ has the form $E(\mathscr{H})(-d+1)$, where $m = 1 + nd + r$, $0 \leqslant r \leqslant n-1$. Its first Chern class equals $r - n$. The case when the first Chern class of the normalized bundle is zero, i.e., when $m = nd + 1$, will play a special role for us since many results below rely on a good theory of jumping lines for bundles with $c_1 = 0$ (see [Bar]).

3. Steiner bundles.

Vector bundles of logarithmic forms turn out to belong to a more general class of bundles remarkable for the existence of a very simple resolution.

3.1. DEFINITION. *A vector bundle E on $P^n = P(V)$ is called a Steiner bundle if E admits a resolution of the form*

$$0 \to I \otimes \mathcal{O}_{P^n}(-1) \xrightarrow{\tau} W \otimes \mathcal{O}_{P^n} \to E \to 0 \tag{3.1}$$

where I and W are vector spaces identified with the corresponding trivial vector bundles.

The bundles of this type were considered earlier by several people (see [E] [BS]). The name "Steiner bundles" will be explained later in this section.

Note that applying the exact sequence of cohomology, we immediately obtain

$$W \cong H^0(P^n, E), \qquad I \cong H^0(P^n, E \otimes \Omega^1_{P^n}(1)). \tag{3.2}$$

3.2. PROPOSITION. *A vector bundle E is a Steiner bundle if and only if the cohomology groups $H^q(P^n, E \otimes \Omega^p(p))$ vanish for all $q > 0$ and also for $q = 0, p > 1$. The resolution (3.1) is defined functorially in E. More precisely, the map $\tau: I \otimes \mathcal{O}_{P^n}(-1) \to W \otimes \mathcal{O}_{P^n}$ in this resolution is the only nontrivial differential $d_1^{-1,0}: E_1^{-1,0} \to E_1^{0,0}$ of the Beilinson spectral sequence with the first term*

$$E_1^{pq} = H^q(P^n, F \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(p)$$

converging to E in degree 0 and to 0 in degrees $\neq 0$.

The proposition follows easily from considering the Beilinson spectral sequence (see [E], Proposition 2.2).

3.3. COROLLARY. *The property of being a Steiner bundle is an open property.*

3.4. A map τ between sheaves $I \otimes \mathcal{O}_{P^n}(-1)$ and $W \otimes \mathcal{O}_{P^n}$, as in (3.1), is uniquely determined by a tensor

$$t \in \operatorname{Hom}(V, \operatorname{Hom}(I, W)) = V^* \otimes I^* \otimes W. \tag{3.3}$$

This tensor should be such that the map τ is fiberwise injective.

Thus we see that the fundamental tensor $t_{\mathscr{H}}$ of an arrangement of hyperplanes in P^n (see 1.4) allows one to define a coherent sheaf as the cokernel of the map

$$\tau_{\mathscr{H}}: I_{\mathscr{H}} \otimes \mathcal{O}_{P^n}(-1) \to W \otimes \mathcal{O}_{P^n}. \tag{3.4}$$

Here the spaces $I = I_{\mathscr{H}}$ and W are defined in 1.2 and 1.4 respectively. It turns out that this sheaf is isomorphic to our logarithmic bundle $E(\mathscr{H})$.

3.5. THEOREM. *Let $\mathscr{H}$ be an arrangement of m hyperplanes in general position in $P(V)$. Suppose that $m \geqslant n+2$. Then the logarithmic bundle $E(\mathscr{H})$ is a Steiner bundle. The corresponding tensor is the fundamental tensor $t_{\mathscr{H}}$ of the configuration $\mathscr{H}$.*

Proof. Let $v \in V$ be a nonzero vector. The fiber of the map (3.4) over the point $\mathbf{C}v \in P(V)$ has, in the notation of §1, the form

$$t_{\mathscr{H}}(v)\colon I_{\mathscr{H}} \to W, \qquad (a_1, \dots, a_m) \mapsto (a_1 f_1(v), \dots, a_m f_m(v)). \tag{3.5}$$

To prove our theorem, we shall construct, for any v, an explicit isomorphism between Coker $t_{\mathscr{H}}(v)$ and the fiber at $\mathbf{C}v$ of the bundle $E(\mathscr{H})$. Consider the map of vector spaces

$$\pi_v\colon W \to E(\mathscr{H})_{\mathbf{C}v}, \qquad (a_1, \dots, a_m) \mapsto \sum a_i (d \log f_i)|_{\mathbf{C}v}, \tag{3.6}$$

where $E(\mathscr{H})_{\mathbf{C}v}$ is the fiber of $E(\mathscr{H})$ at $\mathbf{C}v$. It follows from Theorem 2.12 (a) that π_v is a surjection. Thus our theorem is a consequence of the following lemma.

3.6. LEMMA. *We have* Ker π_v = Im $t_{\mathscr{H}}(v)$. *In other words, a section $\sum \lambda_i d \log f_i$, $\sum \lambda_i = 0$ of the bundle $E(\mathscr{H})$ vanishes at $\mathbf{C}v$ if and only if $\lambda_i = a_i f_i(v)$ for some $(a_1, \dots, a_m) \in I_{\mathscr{H}}$.*

Proof. It suffices to show that Im $t_{\mathscr{H}}(v) \subset$ Ker π_v since the spaces in question have the same dimension.

Let $J = \{i\colon \mathbf{C}v \in H_i\}$ and $H_J = \bigcap_{i \in J} H_i$. A section ω of $E(\mathscr{H})$ vanishes at $\mathbf{C}v$ if and only if ω is regular near $\mathbf{C}v$ as a 1-form and, moreover, vanishes on the tangent subspace to H_J.

Suppose that $\lambda_i = a_i f_i(v)$ where $(a_1, \dots, a_m) \in I_{\mathscr{H}}$. Then for $i \in J$ we have $\lambda_i = 0$ since $f_i = 0$ on H_i. Hence the form $\omega = \sum \lambda_i d \log f_i$ is regular at $\mathbf{C}v$. Let $\xi \in V$ be such that $f_i(\xi) = 0$ for $i \in J$, i.e., ξ represents a vector tangent to H_J at $\mathbf{C}v$. Then the value of ω on this tangent vector equals

$$\sum_{i \notin J} \lambda_i \frac{f_i(\xi)}{f_i(v)} = \sum_{i \notin J} a_i \frac{f_i(v) f_i(\xi)}{f_i(v)} = \sum_{i \notin J} a_i f_i(\xi) = \sum_{i=1}^{m} a_i f_i(\xi) = 0.$$

This proves the lemma and hence Theorem 3.5.

Let us mention that is possible to give an alternative proof of Theorem 3.5 by using Proposition 3.2.

3.7. Let E be a rank-r Steiner bundle on P^n given by the resolution (3.1). We have noticed already that $W = H^0(P^n, E)$. It is obvious that E is generated by its global sections. Hence we obtain a regular map $\gamma\colon P^n \to G(r, W^*)$ that takes a point $x \in P^n$ into the dual of the subspace of sections vanishing at x. This map can be defined "synthetically" by means of the following "Grassmannian Steiner construction" [K].

Let $m = \dim(W) + 1$ so that $\dim(I) = m - 1 - r$. Take $m - r - 1$ projective subspaces $L_1, \ldots, L_{m-r-1}$ in the projective space $P(W^*)$, each of codimension $n + 1$. Denote by $]L_i[$ the "star" of L_i, i.e., the projective space of dimension n formed by hyperplanes in $P(W^*)$ containing L_i. Identify all the stars $]L_i[$ with each other by choosing projective isomorphisms $\phi_i\colon P^n \to]L_i[$. Suppose that for any $x \in P^n$ the corresponding hyperplanes $\phi_i(x)$ are independent. Consider the locus of subspaces in $P(W^*)$ of codimension $m - r - 1$ (i.e., of dimension $r - 1$) which are intersections of the corresponding hyperplanes from stars $]L_i[$, i.e., the subspaces of the form

$$\phi_1(x) \cap \cdots \cap \phi_{m-r-1}(x), \qquad x \in P^n. \tag{3.7}$$

This locus lies in $G(n, W^*)$. It is called the Grassmannian Steiner construction. This is a straightforward generalization of the classical Steiner construction of rational normal curves; see [GH], Ch. 4, §3. The following proposition shows that this construction is equivalent to that of Steiner bundle. This explains the name.

3.8. Proposition. *Let X be a projective space of dimension n embedded in some way into the Grassmannian $G^n(W)$ of codimension-n subspaces in W,* $\dim(W) = m - 1$. *Let Q be the rank-n bundle on $G^n(W)$, whose fiber over a subspace $L \subset W$ is W/L. Let E be the restriction of Q to X. The possibility of representing X by the Grassmannian Steiner construction is equivalent to the fact that E is a Steiner bundle.*

Proof. The choice of $m - n - 1$ parametrized star ($]L_i[, \phi_i\colon P^n = P(V) \to]L_i[$) is equivalent to the choice of $m - n - 1$ surjective linear operators $a_i\colon W^* \to V^*$. Namely, given such a_i, we associate to any hyperplane in V^*, i.e., to any point of $P(V)$ its inverse image under a_i. Thus a point $\pi \in G^n(W)$ corresponding to $x \in P(V)$ is $\bigcap \operatorname{Ker}(a_i(x))$. Define a linear map $A\colon \mathbf{C}^{m-n-1} \to \operatorname{Hom}(W^*, V^*)$ by setting $a_i = A(e_i)$ where $e_1, \ldots, e_{m-n-1}$ is the standard basis of $\mathbf{C}^{m-n-1}$. Denote the space $\mathbf{C}^{m-n-1}$ by I. This defines a tensor $t \in I^* \otimes W \otimes V^*$ which, in its turn, defines a morphism of sheaves $I \otimes \mathcal{O}_{P(V)}(-1) \to W \otimes \mathcal{O}_{P(V)}$. Our bundle E must be the cokernel of this morphism. Indeed, dualizing, we have to show that E^* is the kernel of the dual map $W^* \otimes \mathcal{O}_{P(V)} \to I^* \otimes \mathcal{O}_{P(V)}(1)$. This is defined by a linear map $A^\dagger\colon V \to \operatorname{Hom}(W^*, I^*)$ associated to t. The fiber of this bundle over a point $\mathbf{C}v$ of $P(V)$ is equal to the kernel of the linear map $A^\dagger(v)\colon L^* \to I^*$. The latter is dual to the point of $X \subset G^n(W)$ corresponding to x. This identifies the fibers. The converse reasoning is obvious.

3.9. Proposition [E]. *The rank of a nontrivial Steiner bundle on P^n is greater than or equal to n.*

Proof. Let r be the rank. A Steiner bundle is given by a linear map $V \to \operatorname{Hom}(I, W)$ where $\dim(I) = \dim(W) - r$. Let D be the subvariety of $\operatorname{Hom}(I, W)$ consisting of linear maps of not maximal rank. It is well known that its codimension equals $r + 1$ (see [ACGH], p. 67). Therefore if V is of dimension $> r + 1$, every linear map $t\colon V \to \operatorname{Hom}(I, W)$ will map some nonzero vector $v \in V$ to a matrix of not maximal rank. This does not occur for Steiner bundles.

Thus logarithmic bundles provide examples of Steiner bundles of maximal possible rank. In the rest of this section we shall consider only rank-n Steiner bundles on P^n.

3.10. Recall that a vector bundle E is called stable if for any torsion-free coherent subsheaf $F \subset E$ we have

$$\deg(F)/\mathrm{rk}(F) < \deg(E)/\mathrm{rk}(E).$$

It is well known that the property of stability is preserved under tensoring with invertible sheaves.

The following fact is a particular case of results of Bohnhorst and Spindler ([BH], Theorem 2.7).

3.11. THEOREM. *Any nontrivial Steiner bundle on P^n is stable.*

3.12. PROPOSITION. *Let E be a nontrivial rank-3 Steiner bundle on P^3 with $c_1(E) = 3k$ (i.e., $\dim(W) = 3k + 3$). Then the normalized bundle $E_{norm} = E(-k)$ is an instanton bundle on P^3; i.e., $c_1(E_{norm}) = 0$ and $H^1(P^3, E_{norm}(-2)) = 0$.*

Proof. The proof follows at once from the resolution (3.1).

3.15. COROLLARY. *Let $\mathscr{H}$ be a configuration of m hyperplanes in P^n in general position with $m > n + 2$. Then:*

(a) *the logarithmic bundle $E(\mathscr{H})$ is stable;*
(b) *if $n = 3$ and $m = 3d + 1$, then the normalized bundle $E_{norm}(\mathscr{H}) = E(\mathscr{H})(-d + 1)$ is an instanton bundle on P^3.*

Denote by $M_{P^2}(a, b)$ the moduli space of stable rank-2 vector bundles on P^2 with $c_1 = a$, $c_2 = b$. It is known to be an irreducible algebraic variety of dimension $4b - a^2 - 3$ (see [OSS], Ch. 2, §4). Note that $M_{P^2}(a, b)$ is isomorphic to the moduli space of normalized bundles, namely to $M_{P^2}(0, (4b - a^2)/4)$ for a even and to $M_{P^2}(-1, (4b - a^2 + 1)/4)$ if a is odd.

3.16. COROLLARY. *If $a = m - 3$, $b = \binom{m-2}{2}$ for some m, then the moduli space $M_{P^2}(a, b)$ contains a dense Zariski open subset consisting of Steiner bundles.*

In other words, a generic stable bundle with these Chern classes is a Steiner bundle.

Proof. The property of being a Steiner bundle is open (Corollary 3.3). The said moduli space indeed contains Steiner bundles—the logarithmic bundles corresponding to configurations of m lines: they are stable by Theorem 3.11 and have the required Chern classes by Proposition 2.2. Since the moduli space is irreducible, we are done.

Let us reformulate the above corollary in terms of normalized bundles.

3.17. COROLLARY. *For any $d > 0$ each of the moduli spaces $M_{P^2}(0, d(d-1))$, $M_{P^2}(-1, (d-1)^2)$ has an open dense subset consisting of twisted Steiner bundles.*

3.18. Notice that $\dim M_{P^2}(3, 6) = \dim M_{P^2}(-1, 4) = 12$. On the other hand, arrangements of 6 lines in P^2 also depend on 12 parameters. We will show later that the map $\mathscr{H} \mapsto E(\mathscr{H})$ from the space of arrangements of 6 lines to the moduli space $\dim M_{P^2}(3, 6)$ is generically injective. This will show that a generic bundle from $M_{P^2}(3, 6)$ is a logarithmic bundle associated to an arrangement of 6 lines in P^2.

3.19. The operation of association discussed in §1 can be extended to Steiner bundles. Namely, we can view the defining tensor (3.3) as a linear map $V^* \otimes I^* \to W$ and consider the corresponding map

$$\tau' \colon V \otimes \mathcal{O}_{P(I)}(-1) \to W \otimes \mathcal{O}_{P(I)}$$

of vector bundles on the projective space $P(I)$.

3.20. PROPOSITION-DEFINITION. *The map τ' is injective on all the fibers if and only if τ is. In this case the Steiner bundle* $\operatorname{Coker}(\tau')$ *is said to be associated to the Steiner bundle $E =$* $\operatorname{Coker}(\tau)$ *and denoted by E^{as}.*

Proof. The condition that τ is not fiberwise injective means that there are nonzero $v \in V$, $i \in I$ such that $t(v \otimes i) = 0$. The same condition is equivalent to the fact that τ' is not fiberwise injective. This proves the "proposition" part.

The next proposition follows immediately from definitions of 1.4 and its proof is left to the reader.

3.21. PROPOSITION. *Let $\mathscr{H}$ be an arrangement of hyperplanes in $P(V)$ in general position and $\mathscr{H}^{as}$ be its associated arrangement in $P(I)$. Then there is a natural isomorphism of vector bundles*

$$E(\mathscr{H}^{as}) \cong (E(\mathscr{H}))^{as}.$$

4. Monoids, codependence and monoidal complexes.

In this section we describe some constructions of projective geometry which will be used in the study of logarithmic bundles, more precisely, in the description of jumping lines for such bundles.

4.1. We shall work in projective space P^n with homogeneous coordinates $x_0, \ldots, x_n$. Projective subspaces in P^n will be shortly called *flats*. For a subset $S \subset P^n$ let $\langle S \rangle$ denote the flat (projective subspace) spanned by S. In particular, for two points $p \neq q \in P^n$ the notation $\langle p, q \rangle$ means the line through p and q.

Let X be a hypersurface in a smooth algebraic variety Y and $x \in X$ be a point. We say that x is a k-tuple point of Y if the whole $(k-1)$st infinitesimal neighborhood $x^{(k-1)} \subset Y$ is contained in X.

As usual, if $\mathscr{L}$ is a line bundle on a projective variety X, we shall denote by $|\mathscr{L}|$ the complete linear system of divisors on X formed by zero loci of sections of $\mathscr{L}$; i.e., $|\mathscr{L}| = P(H^0(X, \mathscr{L}))$.

4.2. Let $Z \subset P^n$ be an irreducible variety. A hypersurface $X \subset P^n$ of degree d is called a Z-monoid if each point of Z is a $(d-1)$-tuple point of X. For example, a Z-monoid of degree 2 is just a quadric containing Z.

We denote by $M_d(Z)$ the projective subspace of $|\mathcal{O}_{P^n}(d)|$ formed by all Z-monoids of degree d.

We shall be mostly interested in the case when $Z \subset P^n$ is a flat. In this case, denoting $c = \operatorname{codim} Z$, we find by easy calculation that

$$\dim M_d(Z) = \binom{c+d-2}{d-1}(n-c-1) + \binom{c+d-1}{d} - 1. \tag{4.1}$$

In particular, if $\operatorname{codim} Z = 2$, then $\dim M_d(Z) = nd$.

4.3. Proposition. *Let $Z \subset P^n$ be a flat of dimension $k \leqslant n-2$. Any Z-monoid is a rational variety ruled in P^k's.*

Proof. Projecting X from the subspace Z we find a rational map to P^{n-k-1} whose fibers are flats of dimension k. Indeed, take any $(k+1)$-dimensional flat L containing Z. Then Z is a hyperplane in L. The intersection $L \cap X$ is a hypersurface of degree d in L containing $d-1$ times the hyperplane Z. This means that $L \cap X = (d-1)Z + H(L)$ where $H(L)$ is some hyperplane in L. So $H(L)$ is the fiber of the said rational map over L, as claimed.

4.4. For any flat $Z \subset P^n$ we denote by $]Z[$ the star of Z, i.e., the projective space of hyperplanes containing Z (cf. 3.8). Obviously $\dim\,]Z[\, = \operatorname{codim} Z - 1$.

Assume that $\operatorname{codim} Z = 2$. There is a simple way to construct irreducible Z-monoids of degree d by means of the classical Steiner construction. Take any point $x \in P^n - Z$ and any regular map

$$\psi\colon\ ]Z[\ \cong P^1 \to\ ]x[\ \cong P^{n-1}$$

of degree $d-1$; i.e., a map given by a linear subsystem of $|\mathcal{O}_{P^1}(d-1)|$. Denote by H the unique hyperplane containing Z and x and assume that $\psi(H) \neq H$. Consider the variety $X(Z, x, \psi)$ which is the union of codimension-2 flats $L \cap \psi(L)$. $L \in\,]Z[$. We claim that this is a Z-monoid of degree d containing the point x.

In fact, take a line l which has no common points with $Z \cup \{x\}$. Then $]Z[$ is identified with l by the correspondence taking $H \in\,]Z[$ to the intersection point $x_H = H \cap l$. The map ψ defines a degree-$(d-1)$ map $f\colon l \to l$ defined as follows: $x_H \mapsto \psi(H) \cap l$. The graph of this map $\Gamma_f \subset l \times l \cong P^1 \times P^1$ intersects the diagonal in d points. Therefore l intersects $X(Z, x, \psi)$ at d points so $\deg X(Z, x, \psi) = d$. To see that $X(Z, x, \psi)$ is a Z-monoid, we take any general hyperplane P containing Z. The intersection $X(Z, x, \psi) \cap P$ is a hypersurface Y in P of degree d which settheoretically is the union of Z of another hyperplane in P, the latter entering with multiplicity 1. This implies that Z enters with multiplicity $(d-1)$ into Y so Z consists of $(d-1)$-tuple points of $X(Z, x, \psi)$.

Conversely, every Z-monoid containing a point x outside Z is equal to $X(Z, x, \psi)$ for some regular map ψ of degree d from $]Z[$ to $]x[$. This follows from the proof of Proposition 4.3. And so we have proven the following fact.

4.5. PROPOSITION. *Let $Z \subset P^n$ be a codimension-2 flat and $x \in P^n - Z$. Then there is a bijection between the set of irreducible Z-monoids of degree d containing x and the set of regular maps $\psi:]Z[\to]x[$ of degree $d-1$ with the property that $\psi(\langle Z, x\rangle) \neq \langle Z, x\rangle$.*

4.6. We are going to relate monoids to a property of point sets in projective spaces which we call *codependence.*

Let $\mathbf{p} = (p_1, \ldots, p_r)$ be an ordered r-tuple of points in P^{n-1} and $\mathbf{q} = (q_1, \ldots, q_r)$ be an ordered r-tuple of points in P^1. We say that $\mathbf{p}$ and $\mathbf{q}$ are *d-codependent* if there is a hypersurface $Y \subset P^{n-1} \times P^1$ of bi-degree (1, d) which contains the points (x_i, y_i). We say that $\mathbf{p}$ and $\mathbf{q}$ are *strongly d-codependent* if there is an irreducible such hypersurface.

4.7. PROPOSITION. *Let $(\Lambda_1, \ldots, \Lambda_r)$ be an ordered r-tuple of hyperplanes in P^{n-1} and $(q_1, \ldots, q_r)$ be an ordered r-tuple of points in P^1. Let us regard each Λ_i as a point p_i in the dual projective space $\check{P}^{n-1}$. Then $\mathbf{p} = (p_1, \ldots, p_r)$ and $\mathbf{q} = (q_1, \ldots, q_r)$ are d-codependent (resp. strongly d-codependent) if and only if there is a regular map $\psi: P^1 \to P^{n-1}$ of degree $\leqslant d$ (resp. of degree exactly d) such that $\psi(q_i) \in \Lambda_i$.*

Proof. Suppose that $\mathbf{p}$ and $\mathbf{q}$ are strongly d-codependent. Let Y be an irreducible hypersurface of bidegree $(1, d)$ in $\check{P}^{n-1} \times P^1$ containing all (p_i, q_i). Denote by $F(u, v) = F(u_0, \ldots, u_{n-1}; v_0, v_1)$ the bihomogeneous equation of Y. We obtain, for any $v = (v_0, v_1)$ a linear form $u \mapsto F(u, v)$ on P^{n-1}. Since Y is irreducible, this form is nonzero and so its kernel is a hyperplane, denoted $\psi(v)$, in $\check{P}^{n-1}$, i.e., a point in the initial P^{n-1}. This gives the desired map ψ from P^1 to P^{n-1}. The rest of the proof is obvious.

4.8. COROLLARY. *Let $Z \subset P^n$ be a codimension-2 flat and $q, p_1, \ldots, p_r$ be points in $P^n - Z$. The following conditions are equivalent:*

(i) *There exists a Z-monoid of degree d (resp. an irreducible Z-monoid of degree d) containing $q, p_1, \ldots, p_r$.*
(ii) *There exists a regular map $\psi:]Z[\to]q[$ of degree $\leqslant d-1$ (resp. of degree exactly $d-1$) such that $\psi(\langle Z, p_i\rangle)$ contains the line $\langle q, p_i\rangle$.*
(iii) *The collection of points $\langle q, p_i\rangle \in]q[\cong P^{n-1}$ and $\langle Z, p_i\rangle \in]Z[\cong P^1$ are $(d-1)$-codependent (resp. strongly $(d-1)$-codependent).*

The proof is immediate.

4.9. PROPOSITION. *For $r = nd$ all $(d-1)$-codependent pairs of nd-tuples $(x_1, \ldots, x_{nd}, y_1, \ldots, y_{nd})$ form a hypersurface Ξ in $(P^{n-1})^{nd} \times (P^1)^{nd}$. Let $x_{i0}, \ldots, x_{i,n-1}$*

be the homogeneous coordinates of the point $x_i \in P^{n-1}$ and let y_{i0}, y_{i1} be the homogeneous coordinates of the point $y_i \in P^1$. The equation of Ξ is the determinant of size $nd \times nd$ whose ith row is the following vector of length nd.

$$(x_{i0}y_{i0}^{d-1}, x_{i0}y_{i0}^{d-2}y_{i1}, \ldots, x_{i0}y_{i1}^{d-1}, x_{i1}y_{i0}^{d-1}, x_{i1}y_{i0}^{d-2}y_{i1}, \ldots, x_{i_1}y_{i1}^{d-1}; \ldots; x_{i,n-1}y_{i0}^{d-1},$$

$$x_{i,n-1}y_{i0}^{d-2}y_{i1}, \ldots, x_{i,n-1}y_{i1}^{d-1}).$$

Proof. Let V be the space of polynomials in $(x_0, \ldots, x_{n-1}, y_1, y_2)$ homogeneous of degree 1 in x_i and of degree $d-1$ in y_j. Then $\dim(V) = nd$. The entries of the determinant in question are the values of nd monomials forming a basis of V on our nd points. So the vanishing of the determinant is equivalent to the linear dependence of the vectors given by these values, i.e., to the $(d-1)$-codependence of (x_i) and (y_i).

4.10. Let $p_1, \ldots, p_{nd+1}$ be $nd+1$ points in P^n in general position. The *monoidal complex* $C(p_1, \ldots, p_{nd+1})$ is, by definition, the locus of all the codimension-2 flats $Z \subset P^n$ for which there exists a Z-monoid of degree d containing $p_1, \ldots, p_{nd+1}$.

According to the classical terminology of Plücker, by complexes one meant 3-parametric families of lines in P^3, i.e., a hypersurface in the Grassmannian $G(2, 4)$. As we shall see, our $C(p_1, \ldots, p_{nd+1})$ is a hypersurface in the Grassmannian $G(n-1, n+1)$. This explains the word "complex".

Monoidal complexes will be used in the next section to describe jumping lines of logarithmic vector bundles.

4.11. THEOREM. *Let $G = G(n-1, n+1)$ be the Grassmannian of codimension-2 flats in P^n. Then*:

(a) *for any points $p_1, \ldots, p_{nd+1} \in P^n$ in general position, the variety $C(p_1, \ldots, p_{nd+1})$ is either the whole G or a hypersurface in G. In the latter case this hypersurface comes with a distinguished equation in Plücker coordinates, defined uniquely up to Plücker relations. The degree of this equation (which may be reducible and may contain multiple factors) equals $nd(d-1)/2$;*

(b) *any codimension-2 flat Z containing one of the points p_i belongs to $C(p_1, \ldots, p_{nd+1})$ and, moreover, is a $(d-1)$-tuple point of it (with respect to the scheme structure defined by the above equation).*

The proof of this theorem will be organized as follows. Our first step will be to analyze the equation of $C(p_1, \ldots, p_{nd+1})$ and find its degree. The second step will be to prove part (b) of the theorem, again by using the equation. These two steps will be done in 4.12 and 4.13 respectively.

4.12. We shall define codimension-2 flats by pairs of linear forms whose coefficients are put into rows of a $2 \times (n+1)$ matrix

$$A = \begin{pmatrix} a_{10} & a_{11} & \ldots & a_{1n} \\ a_{20} & a_{21} & \ldots & a_{2n} \end{pmatrix}.$$

The flat corresponding to A will be denoted by $Z(A)$. Its Plücker coordinates are just 2 by 2 minors of A. A representation of a flat Z as $Z(A)$ gives a parametrization of the pencil $]Z[$ of hyperplanes through Z, i.e., an explicit identification $]Z[= P^1$. Explicitly, to a point $(t_1 : t_2) \in P^1$ we associate the hyperplane given by the equation

$$\sum_{j=0}^{n} (t_1 a_{1j} + t_2 a_{2j}) x_j = 0.$$

Denote the last point p_{nd+1} by q. We can choose a coordinate system in P^n in such a way that q has coordinates $(1:0:\ldots:0)$. The projective space $]p_{nd+1}[$ of lines through q is identified with P^{n-1}. Explicitly, if $p = (b_0:\ldots:b_n) \in P^n$ is another point, then the line $\langle q, p\rangle$ has homogeneous coordinates $(b_1:\ldots:b_n)$.

Let b_{ij}, $j = 0, \ldots, n$, be the homogeneous coordinates of the point $p_i \in P^n$; $i = 1, \ldots, nd$. The hyperplane $\langle Z(A), p_i\rangle \in]Z(A)[$ has, under the above identification $]Z(A)[= P^1$, the homogeneous coordinates $(\sum_j a_{1j} b_{ij}, \sum_j a_{2j} b_{ij})$.

Applying Corollary 4.8, we find that a flat $Z(A)$ belongs to the variety $C(p_1, \ldots, p_{nd+1})$ if and only if the two nd-tuples of points

$$((b_{i1}, \ldots, b_{in}) \in P^{n-1}, i = , \ldots, nd) \quad \text{and} \quad \left(\left(\sum_j a_{1j} b_{ij}, \sum_j a_{2j} b_{ij}\right) \in P^1, i = 1, \ldots, nd\right)$$

are $(d-1)$-codependent. Substituting them into the determinant of Proposition 4.9, we find an equation of matrix elements a_{ij} whose degree in these elements equals $nd(d-1)$ (since each entry of the determinant will have degree $d-1$ in a_{ij}). The Plücker coordinates, being 2 by 2 minors of A, have degree 2 in a_{ij}. Hence the degree of the equation in Plücker coordinates equals $nd(d-1)/2$ as required.

4.13. Let us prove part (b) of Theorem 4.11. By symmetry it suffices to prove that if $p_{nd+1} \in Z$, then Z is a $(d-1)$-tuple point of $C(p_1, \ldots, p_{nd+1})$. Let us keep the notations and conventions introduced in the proof of part (a); in particular, assume that $p_{nd+1} = (1:0;\ldots:0)$. A flat $Z(A)$ contains p_{nd+1} if and only if $a_{10} = a_{20} = 0$.

First, let us prove that such $Z(A)$ lies in $C(p_1, \ldots, p_{nd+1})$. By Corollary 4.8, this means that the collections of points

$$(b_{i1}:\ldots:b_{in}) \in P^{n-1} \quad \text{and} \quad (t_i, s_i) = \left(\sum_{j=1}^{n} a_{1j} b_{ij}, \sum_{j=1}^{n} a_{2j} b_{ij}\right) \in P^1, i = 1, \ldots, nd$$

are always $(d-1)$-codependent. To show this, we construct a polynomial $F(b_1, \ldots, b_n, t, s)$ homogeneous of degree 1 in $b_1, \ldots, b_n$ and of degree $(d-1)$ in t, s such that for all i we have $F(b_{i1}, \ldots, b_{in}, t_i, s_i) = 0$. In fact, we can construct at least $d-1$ linearly independent such polynomials, namely

$$F_m(b_1, \ldots, b_n, t, s) = \left(\sum_{j=1}^{n} a_{2j} b_j\right) t^m s^{d-1-m} - \left(\sum_{j=1}^{n} a_{1j} b_j\right) t^{m-1} s^{d-m}, \qquad m = , 2, \ldots, d-1.$$

The possibility of finding $d-1$ such polynomials means that the kernel of the $nd \times nd$-matrix whose determinant, by Proposition 4.9, defines $C(p_1, \ldots, p_{nd+1})$, has dimension $\geqslant d-1$. Since matrices with such properties are $(d-1)$-tuple points of the variety of degenerate matrices, we find that $Z(A)$ is a $(d-1)$-tuple point of $C(p_1, \ldots, p_{nd+1})$.

So we have proven all the assertions of Theorem 4.11.

Unfortunately, we do not know whether the case $C(p_1, \ldots, p_{nd+1}) = G$ occurs. It does not occur if $n = 2$ (see Corollary 5.4). Another case when this does not occur is as follows.

4.14. PROPOSITION. *If $d \leqslant 3$ and n is arbitrary, then for any $nd+1$ points $p_1, \ldots, p_{nd+1} \in P^n$ in linearly general position the monoidal complex $C(p_1, \ldots, p_{nd+1})$ does not coincide with the whole Grassmannian G.*

Proof. Suppose the contrary, i.e., that for any codimension 2 flat Z there is a Z-monoid X of degree d through $p_1, \ldots, p_{nd+1}$. If we take Z to lie in the hyperplane $H = \langle p_1, \ldots, p_n \rangle$, then we find that any such monoid X should be the union of H and some Z-monoid of degree $d-1$ through $p_{n+1}, \ldots, p_{nd+1}$. Let $H' = \langle p_{n+1}, \ldots, p_{2n} \rangle$. We take $Z = H \cap H'$. By the above, the coorresponding monoid should be the union of H, H' and a Z-monoid of degree $d-2$ through the remaining points $p_{2n+1}, \ldots, p_{nd+1}$. If $d = 2$, this means that $2n+1$ generic points $p_1, \ldots, p_{2n+1}$ lie on the union of two hyperplanes $H \cup H'$, which is impossible. If $d = 3$, the above means that $n+1$ points $p_{2n+1}, \ldots, p_{3n+1}$ lie on a monoid of degree 1, i.e., on a hyperplane, which is also impossible.

4.15. Examples in P^2. Consider first the case $n = 2$. Then each $2d+1$ points in P^2 in general position define the monoidal complex $C(p_1, \ldots, p_{2d+1})$. It is a curve of degree $d(d-1)$ with $(d-1)$-tuple points at each p_i. Let us consider some particular cases.

(a) Let $d = 2$. Then the curve $C(p_1, \ldots, p_5)$ is just the unique conic through the points p_i.

(b) Let $d = 3$. Then $C(p_1, \ldots, p_7)$ is a curve of degree 6 and genus 3 with double points at p_i. By definition, it is the locus of all possible singular points of cubics through $p_1, \ldots, p_7$. This curve has the following (classical, see [DO]) description.

Let $S \xrightarrow{\sigma} P^2$ be the blow-up of P^2 in $p_1, \ldots, p_7$. This is a Del Pezzo surface. Its anticanonical linear system has dimension 2 and defines a double cover $S \xrightarrow{\tau} \tilde{P}^2$ (this $\tilde{P}^2$ is different from the first one) ramified along a plane quartic curve $C' \subset \tilde{P}^2$. We claim that C' is birationally isomorphic to $C(p_1, \ldots, p_7)$.

Indeed, the anticanonical linear system of P^2 consists of cubic curves. The curves of the anticanonical linear system of the blown-up surface S can be viewed, after projection $S \xrightarrow{\sigma} P^2$, as plane cubics through $p_1, \ldots, p_7$. Denote this linear system by $\mathscr{L} \cong P^2$. The second projective plane $\tilde{P}^2$ is the space of lines in $\mathscr{L}$. The projection τ associates to a point $p \in S$ with projection $z = \sigma(p) \in P^2$ the set of all plane cubics through $p_1, \ldots, p_7$ which also meet p. (So this set is a line in $\mathscr{L}$, i.e., a pencil of cubics.) All the cubics from this pencil also contain some ninth point p' which is conjugate to p with respect to the double cover τ. The map τ ramifies at p when $p' = p$, i.e., the cubics from $\mathscr{L}$ have a node at $z = \tau(p)$.

(c) Let $d = 4$. Then $C(p_1, \ldots, p_9)$ is a curve of degree 12 with triple points at $p_1, \ldots, p_9$. Its genus equals 28. We do not know any special geometric significance of this curve.

4.16. Examples in P^3. Let us consider the case $n = 3$. The monoidal complex $C(p_1, \ldots, p_{3d+1})$ is a line complex of degree $3d(d-1)/2$.

(a) For $d = 2$, i.e., for 7 points in P^2, we get the so-called *Montesano complex*. It consists of lines in P^3 which lie on a quadric passing through points $p_1, \ldots, p_7$. It is not difficult to see that this complex (which is a threefold in $G(2, 4)$) is isomorphic to a P^1-bundle over a Del Pezzo surface of degree 2. This latter surface is obtained by blowing up the seven points of P^2 corresponding to $p_1, \ldots, p_7 \in P^3$ by association (see 1.2 above). We refer to [Mo] for more details about the geometry of this complex.

(b) Let $d = 3$. The complex $C(p_1, \ldots, p_9)$ is a complex of degree 9 consisting of lines which appear as double lines of cubic surfaces through $p_1, \ldots, p_9$.

4.17. Examples in P^n. We consider only the case $d = 2$, i.e., of $2n + 1$ points in P^n. This case gives a complex of codimension-2 flats which can be called the *generalized Montesano complex*. We consider all quadrics in P^n through $p_1, \ldots, p_{2n+1}$ and pick those among them which contain a P^{n-2}, i.e., quadrics of rank $\leqslant 4$. The collection of all $(n-2)$-flats on all the quadrics of rank $\leqslant 4$ through p_i gives a hypersurface in the Grassmannian $G(n-1, n+1)$ whose degree equals n. This is our complex.

Note the case when all p_i lie on a rational normal curve C in P^n of degree n. In this case the generalized Montesano complex will be the locus of all $(n-2)$-flats intersecting the curve C. Its equation will be the Chow form of C, i.e., the resultant of two indeterminate polynomials of degree n. This is a consequence of the following easy fact.

4.18. LEMMA. *Let $C \subset P^n$ be a rational normal curve and Z a codimension-2 flat in P^n. Then the two conditions are equivalent:*

(i) *there exists a quadric through C and Z;*
(ii) $C \cap Z \neq \varnothing$.

Proof. (ii) $\Rightarrow$ (i): Let $x \in C \cap Z$. Let $|\mathcal{O}(2)|$ be the linear system of all quadrics in P^n. We consider three projective subspaces $L, M, N \subset |\mathcal{O}(2)|$ consisting respectively of quadrics containing C, Z, and x. Then $L, M \subset N$. The codimension of L in the whole $|\mathcal{O}(2)|$ is $2n + 1$, and hence its codimension in N is $2n$. The space M has dimension $2n$. Hence $L \cap M \neq \varnothing$, so there exists a quadric with required properties.

(i) $\Rightarrow$ (ii): Let Q be a quadric containing $C \cup Z$. Then C and Z are subvarieties in Q of complementary dimensions. In the case $n > 3$ (as well as in the case when $n = 3$ and Q is singular) this alone implies that the intersection is nonempty. If $n = 3$ and Q is smooth the nonemptiness follows from the fact that C regarded as a curve on $Q = P^1 \times P^1$ has bidegree $(1, 2)$ or $(2, 1)$. The case $n = 2$ is trivial.

5. Monoidal complexes and splitting of logarithmic bundles.

5.1. One of the main tools for the study of vector bundles on P^n is the restriction of bundles to projective subspaces to P^n, especially to lines. By Grothendieck's theorem any vector bundle on P^1 splits into a direct sum of line bundles $\bigoplus \mathcal{O}_{P^1}(a_i)$.

In this section we use this approach for logarithmic bundles $E(\mathscr{H})$ where $\mathscr{H} = (H_1, \ldots, H_m)$ is an arrangement of m hyperplanes in P^n in general position. Let us write the number m in the form $m = nd + 1 + r$ where d, r are integers and $0 \leqslant r \leqslant n - 1$. Call a line l in P^n a *jumping line* for $E(\mathscr{H})$ (or for $\mathscr{H}$, if no confusion arises) if the restriction $E(\mathscr{H})|_l$ is not isomorphic to $\mathcal{O}_l(d)^{\oplus r} \oplus \mathcal{O}_l(d-1)^{\oplus(n-r)}$.

Of special interest for us will be the case $m = nd + 1$. In this case the normalized bundle $E(\mathscr{H})_{norm} = E(\mathscr{H})(-d+1)$ has first Chern class 0. A line l will be in this case a jumping line for $\mathscr{H}$ if the restriction $E(\mathscr{H})_{norm}|_l$ is nontrivial, i.e., not isomorphic to $\mathcal{O}_l^{\oplus n}$.

The main result of this section is as follows.

5.2. THEOREM. *Suppose $m = nd + 1$. Let $p_1, \ldots, p_{nd+1}$ be points in $\check{P}^n$ corresponding to hyperplanes $H_1, \ldots, H_{nd+1} \in H$. For any line $l \subset P^n$ let $]l[$ be the corresponding codimension-2 flat in $\check{P}^n$. Then a line $l \subset P^n$ is a jumping line for the bundle $E(\mathscr{H})$ if and only if the flat $]l[$ belongs to the monoidal complex $C(p_1, \ldots, p_{nd+1})$. In particular, the locus of jumping lines of $E(\mathscr{H})$ is either the whole $G(2, n+1)$, or the support of a divisor in the Grassmannian $G(2, n+1)$ of degree $nd(d-1)/2$.*

5.3. COROLLARY. *Assume $n = 2$. Then the monoidal complex $C(p_1, \ldots, p_{2d+1})$ (which is in this case a subvariety in the dual plane $\check{P}^2$), does not coincide with the whole $\check{P}^2$.*

Proof. This is a consequence of Theorems 5.2, 3.11, and of the Grauert-Mülich theorem [OSS] which implies that the locus of jumping lines of a stable rank-2 bundle on P^2 is in fact a curve.

5.4. COROLLARY. *Assume that $d \leqslant 3$. Then for any configuration $\mathscr{H}$ of $nd + 1$ hyperplanes $H_1, \ldots, H_{nd+1} \subset P^n$ in general position the locus of jumping lines of the bundle $E_{norm}(\mathscr{H})$ does not coincide with the whole Grassmannian.*

Proof. This follows from Proposition 4.14.

Applying Corollary 4.8, we can give an equivalent, more geometric description of the property of a line to be jumping.

5.5. COROLLARY. *Suppose $m = nd + 1$. Let $l \subset P^n$ be a line intersecting the H_i in distinct points. Then l is a jumping line for $\mathscr{H}$ if and only if there is a regular map $\psi: l \to H_{nd+1}$ of degree $\leqslant d - 1$ such that $\psi(l \cap H_i) \in H_{nd+1} \cap H_i$ for $i = 1, \ldots, nd$.*

This reformulation is asymmetric: one of the hyperplanes, namely H_{nd+1}, acts as a "screen". Of course, any other H_i can be chosen for this role.

5.6. Let E be a vector bundle on P^n. We say that E is projectively trivial if $E \cong \mathcal{O}_{P^n}(a)^{\oplus b}$ for some $a \in \mathbf{Z}, b \in \mathbf{Z}_+$. In this case the projective bundle $P(E)$ is trivial and, moreover, canonically trivialized. To get the trivialization we note that $P(E) = P(E(-a))$. If W is the space of sections of $E(-a)$, then $E(-a)$ is canonically identified with $W \otimes \mathcal{O}_{P^n}$. Hence $P(E)$ is canonicaly identified with $P^n \times P(W)$. For any two points $x, x' \in P^n$ we get the identification of fibers

$$\Psi_{E,x,x'}\colon P(E_x) \to P(W) \to P(E_{x'}).$$

We shall call this system of identifications the *canonical projective connection* of the projectively trivial bundle E.

In our situation of logarithmic bundles it follows that whenever $m = nd + 1$ and l is not a jumping line for $\mathscr{H}$, we get a canonical projective connection in the restricted bundle $E(\mathscr{H})|_l$. We are going to describe this connection explicitly.

Note that the fiber of the bundle $E(\mathscr{H})^* = T_{P^n}(\log \mathscr{H})$ at any point $x \in P^n$ not lying on any H_i is identified with the tangent space $T_x P^n$. Therefore the fiber $P(E(\mathscr{H}^*)_x)$ is canonically identified with the projective space P_x^{n-1} of all lines through x. This means that for any nonjumping line l the projective connection gives us isomorphisms which we denote

$$\Phi_{\mathscr{H},l,x,x'}\colon P_x^{n-1} \to P_{x'}^{n-1}, \qquad x, x' \in l - \mathscr{H}.$$

Our next result describes this identification.

5.7. PROPOSITION. *Let $m = nd + 1$ and let l be a nonjumping line for $\mathscr{H}$. Let $x \in l - \mathscr{H}$ be any point and $\lambda \in P_x^{n-1}$ be a line in P^n through x. Then there is a unique regular map $\psi = \psi_{x,\lambda}\colon l \to H_{nd+1}$ of degree d such that $\psi(l \cap H_i) \subset H_{nd+1} \cap H_i$ for each $i = 1, \ldots, nd$ and $\psi(x) = \lambda \cap H_{nd+1}$. For any other point $x' \in l - \mathscr{H}$ the value at λ of the projective connection map $\Phi_{\mathscr{H},l,x,x'}\colon P_x^{n-1} \to P_{x'}^{n-1}$ equals the line $\langle x', \psi_{x,\lambda}(x')\rangle \in P_{x'}^{n-1}$.*

Now we start to prove our results. We shall begin with Theorem 5.2. We need two lemmas.

5.8. LEMMA. *A vector bundle E^* on P^1 of rank n and first Chern class $(-n(d-1))$ does not have the form $\mathcal{O}(-d+1)^{\oplus n}$ if and only if $H^0(P^1, E^*(d-2)) \neq 0$.*

Proof. As any bundle on P^1, our E^* has the form $\bigoplus_{i=0}^n \mathcal{O}_l(a_i)$ where $\sum a_i = -n(d-1)$. The condition $(a_1, \ldots, {}_n) \neq (-(d-1), \ldots, -(d-1))$ is equivalent, under the above constraint on the sum, to the condition "$\exists i\colon a_i \geqslant -d+2$" which is tantamount to $H^0(P^1, E^*(d-2)) \neq 0$.

The next lemma concerns the case when $\mathscr{H}$ consists of just one hyperplane H. In this case, as we have seen in Proposition 2.10, the logarithmic bundle $E(\mathscr{H})$ is itself projectively trivial. So the canonical projective connection on $E(\mathscr{H})$ gives identifications

$$\Phi_{H,x,x'}\colon P_x^{n-1} \to P_{x'}^{n-1}, \qquad x, x' \in P^n - H.$$

5.9. LEMMA. *The identification $\Phi_{H,x,x'}$ takes a line λ through x to the line $\langle \lambda \cap H, x' \rangle$ through x'.*

Proof. Let $\mathbf{H} \subset \mathbf{C}^{n+1}$ be the linear hyperplane corresponding to the projective hyperplane $H \subset P^n$. By Proposition 2.10, we have an isomorphism $E(\mathscr{H}) \cong O_{P^n}(-1)^{\oplus n}$. We can make this statement more precise by showing the existence of a natural isomorphism

$$E(\mathscr{H}) \cong \mathbf{H}^* \otimes \mathcal{O}_{P^n}(-1).$$

Denote the space $\mathbf{C}^{n+1}$ shortly by V. Let x be any point of $P^n = P(V)$ and let $\mathbf{x}$ be the 1-dimensional subspace in V representing x. The tangent space $T_x P^n$ is canonically identified with $\mathbf{x} \otimes V/\mathbf{x}$. Denote by U the open set $P^n - H$. If $x \in U$, then the map $\mathbf{H} \to V \to V/\mathbf{x}$ is an isomorphism, so we get the identification $T_x P^n = \mathbf{x}^* \otimes \mathbf{H}$. Correspondingly, the fiber at x of $\Omega^1_{P^n}$ becomes identified with $\mathbf{x} \otimes \mathbf{H}^*$, i.e., with the fiber at x of $\mathbf{H}^* \otimes \mathcal{O}_{P^n}(-1)$. We get an isomorphism of restricted bundles ϕ: $E(\mathscr{H})|_U \cong \mathbf{H}^* \otimes \mathcal{O}_{P^n}(-1)$. Using the fact that $E(\mathscr{H})$ is isomorphic to $\mathcal{O}_{P^n}(-1)^{\oplus n}$, we can extend the isomorphism ϕ to the whole P^n. In this model for $E(\mathscr{H})$ the fiber P^{n-1}_x of $E(\mathscr{H})^*$ at x is canonically identified with H by assigning to the line λ through x the point $\lambda \cap H$. Our lemma follows from this immediately.

5.10. Now we are ready to prove Theorem 5.2. Let us consider the bundle $T_{P^n}(\log \mathscr{H})$ as the result of successive elementary transformations starting with the bundle $T_{P^n}(\log H_{nd+1})$, as in Proposition 2.9. The latter bundle is projectively trivial. Consider a line $l \subset P^n$. We can assume that $l \cap H_i$ are distinct points of l. Then the restriction to l of the bundle $T_{P^n}(\log \mathscr{H})$ is the elementary transformation of $T_{P^n}(\log H_{nd+1}) = \mathcal{O}_l(1)^{\oplus n}$ with respect to points $y_i = l \cap H_i$ and subspaces $T_{y_i} H_i \subset T_{y_i} P^n$.

5.11. LEMMA. *Consider on the projective line P^1 the vector bundle $\mathcal{O}_{P^1}^{\oplus n} = \mathcal{O}_{P^1} \otimes E$ where E is an n-dimensional vector space. Let $y_1, \ldots, y_{nd}$ be distinct points of P^1 and $\Lambda_1, \ldots, \Lambda_{nd}$ be hyperplanes in E. We regard Λ_i as a hyperplane in the fiber of our bundle over y_i. Then the following conditions are equivalent:*

(i) *The elementary transformation* $\mathrm{Elm}_{\{y_1,\ldots,y_{nd}\},\{\Lambda_1,\ldots,\Lambda_{nd}\}}(E \otimes \mathcal{O}_{P^1}(1))$ *is not isomorphic to* $\mathcal{O}_{P^1}(-d+1)^{\oplus n}$.
(ii) *The two nd-tuples* $(\Lambda_1, \ldots, \Lambda_{nd}) \in P(E)^{nd}$ *and* $(y_1, \ldots, y_{nd}) \in (P^1)^{nd}$ *are d-codependent in the sense of 4.6.*

Proof. Denote the elementary transformation in condition (i) by Elm. Then, By Lemma 5.8, condition (i) is equivalent to nonvanishing of $H^0(\mathrm{Elm}(E \otimes \mathcal{O}(1))(d-2))$. Let x_0, x_1 be homogeneous coordinates in P^1. A section of $\mathrm{Elm}(E \otimes \mathcal{O}(-1))(d-2)$ is, by definition, a homogeneous polynomial $s(x) = (x_0, x_1)$ of degree $d-1$ with values in E such that $(y_i) \subset \Lambda_i$. This is exactly the characterization of $(d-1)$-codependence given in Proposition 4.7.

5.12. COROLLARY. *Let $\mathscr{H} = (H_1, \ldots, H_{nd+1})$ be a configuration of hyperplanes in P^n in general position. A line $l \subset P^n$ not lying in any H_i is a jumping line for $\mathscr{H}$ if and only if the nd-tuples $(H_1 \cap H_{nd+1}, \ldots, H_{nd} \cap H_{nd+1}) \in (\check{H}_{nd+1})^{nd}$ and $(l \cap H_1, \ldots, l \cap H_{nd+1}) \in l^{nd}$ are $(d-1)$-codependent.*

Proof. Let l be given and suppose that l does not lie in H_{nd+1}. Let $\mathbf{H}_{nd+1}$ be the linear hyperplane in $\mathbf{C}^{n+1}$ corresponding to H_{nd+1}.

By Proposition 5.7 the restriction of $T_{P^n}(\log H_{nd+1})$ to l is isomorphic to $\mathbf{H}_{nd+1} \otimes \mathcal{O}_l(1)$. The restriction of the bundle $E^*(\mathscr{H})$ to l is the elementary transformation of this projectively trivial bundle with respect to points $y_i = l \cap H_i$ and hyperplanes $\Lambda_i = T_{y_i}H_i$. So we can apply Lemma 5.11. An explicit projective trivialization of the bundle $T_{P^n}(\log H_{nd+1})$ given in Proposition 5.7 identifies the projectivizations of fibers at every point $y \in l - H_{nd+1}$ with H_{nd+1}. Under this identification our hyperplanes Λ_i correspond to hyperplanes $H_{nd+1} \cap H_i \subset H_{nd+1}$. So the assertion follows from Lemma 5.8.

5.13. To finish the proof of Theorem 5.2, let us reformulate Corollary 5.12 in terms of the dual space $\check{P}^n$. Hyperplanes H_i correspond to points p_i of $\check{P}^n$; the line l corresponds to a flat Z of codimension 2. The projective space $\check{H}_{nd+1}$ of hyperplanes in H_{nd+1} becomes the space $P^{n-1}_{p_{nd+1}}$ of lines through p_{nd+1}, and l itself becomes identified with the pencil $]Z[$ of lines through Z. Under these identifications the hyperplane $H_{nd+1} \cap H_i$ in H_{nd+1} corresponds to the line $\langle p_{nd+1}, p_i \rangle$. Now Theorem 5.2 follows from the definition of the monoidal complex and Corollary 4.8.

5.14. Proposition 5.7 now becomes just a reformulation of the fact that the projective connection in the projectively trivial bundle $E = \mathcal{O}_{P^1}(a)^{\oplus b}$ is induced by global sections of $E(-a)$. So it is proven.

5.15. For any stable rank-r bundle E on P^n, there exists a Zariski open set $U \subset G(2, n+1)$ such that for $l \in U$ the splitting type $(a_1 \geqslant \cdots \geqslant a_r)$ of $E|_U$ is constant. A splitting type $(a_1, \ldots, a_r)$ is called *generic* (or rigid) if $a_1 - a_r \leqslant 1$. By the Grauert-Mülich theorem it is always generic if $r = 2$. We conjecture that the splitting type of $E(\mathscr{H})$ is always generic.

6. Schwarzenberger bundles.

In this secion we shall show that our logarithmic bundles generalize the construction of Schwarzenberger [Schw1–2] of vector bundles of rank n on P^n.

6.1. Note that the following choices are equivalent:

(i) an isomorphism $P^n \cong |\mathcal{O}_{P^1}(n)|$ of P^n with the n-fold symmetric product of P^1;
(ii) a dual isomorphism $\check{P}^n \cong |\mathcal{O}_{P^1}(n)|^*$;
(iii) a map $v\colon P^1 \to \check{P}^n = |\mathcal{O}_{P^1}(n)|^*$ given by a complete linear system;
(iv) a rational normal curve of degree n (Veronese curve, for short) $\check{C}$ in $\check{P}^n$, the image of the map in (iii);
(v) a map $\check{v}\colon P^1 \to P^n \cong |\mathcal{O}_{P^1}(n)|$ given by a complete linear system;
(vi) a Veronese curve C in P^n, the image of this map.

Fix any of them. Then every point $x \in P^n$ is identified with a positive divisor D_x of degree n on P^1. Choose some $m \geqslant n + 2$. Let $V(x)$ be the subspace of sections $s \in H^0(P^1, \mathcal{O}(m-2))$ whose divisor of zeros $\operatorname{div}(s)$ satisfies the condition $\operatorname{div}(s) \geqslant D_x$. Denote by $V(x)^\perp \subset H^0(P^1, \mathcal{O}(m-2))^*$ the orthogonal subspace. Its dimension is equal to n. In this way we obtain a map

$$P^n \to G(n, H^0(P^1, \mathcal{O}(m-2))^*) = G(n, m-1), \qquad x \mapsto V(x)^\perp. \tag{6.1}$$

Let S be the tautological bundle on $G(n, m-1)$ whose fiber over a point represented by an n-dimensional linear subspace is this subspace. The pull-back, with respect to (6.1), of S is a rank-n vector bundle on P^n. It is defined by any of the above six choices, in particular, by a choice of the Veronese curve $C \subset P^n$. We denote the dual bundle by $E(C, m)$ and call it the *Schwarzenberger bundle* of degree m associated to C. Thus fibers of $E(C, m)$ have the form

$$E(C, m)_x = \frac{H^0(P^1, \mathcal{O}(m-2))}{\{s \in H^0(P^1, \mathcal{O}(m-2)) \colon \operatorname{div}(s) \geqslant D_x\}}. \tag{6.2}$$

If we fix another isomorphism $P^{m-2} \cong |\mathcal{O}_{P^1}(m-2)|$, this time by means of another Veronese curve R of degree $m-2$ in $\check{P}^{m-2}$, then we can view each point $x \in P^n$ as a positive divisor D_x on R and the space $P(V(x)^\perp)$ as the projective subspace $\langle D_x \rangle$ spanned by D_x, i.e., as an $(n-1)$-secant flat of R. For this reason the projective bundle $P(E(C, m))$ is called the *n-secant bundle* of R (see [Schw2]).

One can easily show that $E(C, m)$ is generated by its space of global sections which is canonically isomorphic to $H^0(P^1, \mathcal{O}(m-2)) = \mathbf{C}^{m-1}$.

6.2. The bundle $E(C, m)$ is in fact a Steiner bundle. This fact is well known (see e.g., [BS], Example 2.2). Let us give a precise statement.

Fix an isomorphism $P^n = P(V)$, $\dim V = n + 1$. Then the choice of a Veronese curve C is given by an isomorphism $V \cong S^n A$, where A is a 2-dimensional vector space and the points of the curve C are represented by the n-th powers l^n, $l \in A$. Consider the multiplication map

$$t \colon V \otimes S^{m-n-2} A = S^n A \otimes S^{m-n-2} A \to S^{m-2} A. \tag{6.3}$$

6.3. Proposition. *The Schwarzenberger bundle $E(C, m)$ is a Steiner bundle on $P^n = P(S^n A)$ defined by vector spaces $I = S^{m-n-2} A$, $W = S^{m-2} A$ and the tensor $t \colon V \otimes I \to W$ given by (6.3).*

Proof. This follows from formula (6.2) and the fact that for two sections $f \in H^0(P^1, \mathcal{O}(a))$, $g \in H^0(P^1, \mathcal{O}(b))$ we have $\operatorname{div}(f) \geqslant \operatorname{div}(g)$ if and only if f is divisible by g.

6.4. Theorem. *Let $\mathcal{H} = (H_1, \ldots, H_m)$ be an arrangement of m hyperplanes in P^n in general position. Suppose that all H_i considered as points of $\check{P}^n$ lie on a Veronese curve $\check{C}$. (Equivalently, all H_i osculate the dual Veronese curve $C \subset P^n$.) Then there is* *an isomorphism*

$$E(\mathscr{H}) \cong E(C, m).$$

Proof. This is equivalent to Theorem 3.8.5 from [K] which describes the Veronese variety in the Grassmannian corresponding to $E(\mathscr{H})$. We prefer to give a direct proof here.

Let $I_{\mathscr{H}}$ be the space defined in 1.4 and $W \subset \mathbf{C}^m$ be the space of vectors with sum of coordinates zero (see formula (1.2)). We shall construct explicit isomorphisms

$$\alpha\colon S^{m-n-2}(A) \to I_{\mathscr{H}}, \qquad \beta\colon S^{m-2}A \to W,$$

which take the multiplication tensor (6.3) into the fundamental tensor $t_{\mathscr{H}}$ which defines $E(\mathscr{H})$ as a Steiner bundle.

Let $f_i = 0$ be the equation of the hyperplane H_i from H. The condition that H_i osculates C means that f_i considered as an element of $V^* = S^nA^*$ can be written in the form u_i^n where $u_i \in A^*$. Let $p_i \in P^1 = P(A)$ be the point corresponding to u_i. Let us identify the space $S^{m-2}A = H^0(P^1, \mathcal{O}(m-2))$ with the space $H^0(P^1, \Omega^1_{P^1}(p_1 + \cdots + p_m))$ of forms with simple poles at $(p_1, \ldots, p_m)$. After that the map β is defined by the formula

$$\beta(\omega) = (\operatorname{res}_{p_1}(\omega), \ldots, \operatorname{res}_{p_m}(\omega)), \qquad \omega \in H^0(P^1, \Omega^1_{P^1}(p_1 + \cdots + p_m)). \tag{6.4}$$

By the residue theorem the sum of components of $\beta(\omega)$ equals 0, i.e., $\beta(\omega) \in W$. Now, if we fix a point q different from the p_i's, then we can identify the space $H^0(P^1, \mathcal{O}(n)) = S^nA$ with the space of rational functions on P^1 with poles of order $\leqslant n$ at q. Let us denote this latter space by $L(nq)$. Let us also regard the space $S^{m-n-2}A$ as the space $H^0(P^1, \Omega^1(p_1 + \cdots + p_m - nq))$ of forms with at most simple poles at p_i and with a zero of order $\leqslant n$ at q. This is a subspace of $H^0(P^1, \Omega^1_{P^1}(p_1 + \cdots + p_m)$, and we define α to be the restriction of β to this subspace.

Let us see that this is correct, i.e., the image of α indeed lies in the space $I_{\mathscr{H}}$. By definition (see 1.4), $I_{\mathscr{H}} \subset \mathbf{C}^m$ consists of $(\lambda_1, \ldots, \lambda_m)$ such that $\sum \lambda_i f_i(v) = 0$ for every $v \in V$. In our case $\lambda_i = \operatorname{resp}_{p_i}(\omega)$, where $\omega \in H^0(P^1, \Omega^1(p_1 + \cdots + p_m - nq))$. Since we have identified $V = S^nA = L(nq)$, we have, for any $v \in V$,

$$\sum \lambda_i f_i(v) = \sum \lambda_i(u_i^n, v) = \sum \operatorname{resp}_{p_i}(\omega) \cdot (u_i^n, v) = \sum \operatorname{res}_{p_i}(\omega \cdot v) = 0$$

since the form $\omega \cdot v$ has poles only at p_i.

This shows the correctness of the definition of maps α and β. After the identification given by these maps it is obvious that the multiplication tensor becomes identified with the fundamental tensor $t_{\mathscr{H}}$.

6.5. Corollary. *Suppose that $\mathscr{H}$, $\mathscr{H}'$ are two arrangements of m hyperplanes in general position in P^n such that all the hyperplanes from $\mathscr{H}$ and $\mathscr{H}'$ osculate the same fixed Veronese curve $C \subset P^n$. Then the logarithmic bundles $E(\mathscr{H})$ and $E(\mathscr{H}')$ are isomorphic.*

6.6. PROPOSITION. *Let C, C' be two Veronese curves in P^n such that for some $m \geqslant n+2$ the Schwarzenberger bundles $E(C, m)$ and $E(C', m)$ are isomorphic. Then $C = C'$.*

Proof. We shall show that C can be recovered intrinsically from $E(C, m)$. Since $E(C, m)$ is a Steiner bundle, its defining tensor $t: V \otimes I \to W$ is determined by $E(C, m)$ itself (see Proposition 3.2). We know that there are isomorphisms $V \cong S^n A$, $I \cong S^{m-n-2}A$, $W \cong S^{m-2}A$, with $\dim A = 2$ which take the tensor t into the multiplication tensor (6.3). We shall see that as soon as such isomorphisms exist, the Veronese curves in $P(V)$, $P(I)$, $P(W)$ consisting of perfect powers of elements of A are defined by t alone. Indeed, t gives a morphism $T: P(V) \times P(I) \to P(W)$. The Veronese curve R in $P(W)$ is recovered as the locus of $w \in P(W)$ such that $T^{-1}(w)$ consists of just one point, say (v, i). The loci of v (resp. i) corresponding to various $w \in R$ constitute the Veronese curves of perfect powers in $P(V)$ and $P(I)$ respectively. Proposition 6.6 is proven.

6.7. Consider as an example the case when $m = n + 3$. It is well known that any $n + 3$ points in general position in P^n lie on a unique Veronese curve (see [GH], p. 530). So Corollary 6.5 and Proposition 6.6 lead to the following conclusion.

For two arrangements $\mathscr{H}$ and $\mathscr{H}'$ of $n + 3$ hyperplanes in general position in P^n, the logarithmic bundles $E(\mathscr{H})$ and $E(\mathscr{H}')$ are isomorphic if and only if the Veronese curves osculated by $\mathscr{H}$ and $\mathscr{H}'$ coincide.

Moreover, any deformation of a bundle $E(\mathscr{H})$ is again of this type, as the following proposition shows.

6.8. PROPOSITION. *Any Steiner bundle E on P^n of rank n with* $\dim I = 2$, $\dim W = n + 2$ *is a Schwarzenberger bundle $E(C, n + 3)$ for some Veronese curve $C \subset P^n$.*

Proof. Let E^{as} be the associated Steiner bundle on P^1 (see 3.20). By Proposition 3.21, it suffices to show that $E^{as} = E(\mathscr{H})$ for some arrangement of points on P^1. But this is obvious since E^{as} is of rank 1 and hence is determined by its first Chern class, which is equal to $n + 1$ in our case. Thus taking any $n + 3$ points on P^1, we realize E^{as} as a logarithmic bundle and hence realize E as a Schwarzenberger bundle.

7. A Torelli theorem for logarithmic bundles.

7.1. Let $\mathscr{A}_{gen}(m, n)$ be the variety of all arrangements of m *unordered* hyperplanes in P^n in general position. So $\mathscr{A}_{gen}(m, n)$ is an open subset in the symmetric product $\mathrm{Sym}^m(\check{P}^n)$. (Note that we do not factorize modulo projective transformations.) The correspondence $\mathscr{H} \mapsto E(\mathscr{H})$ defines a map

$$\mathscr{A}_{gen}(m, n) \mapsto M(n, (1 - ht)^{n+1-m}), \tag{7.1}$$

where $M(n, (1 - ht)^{n+1-m})$ is the moduli space of stable rank-n bundles on P^n with Chern polynomial $(1 - ht)^{n+1-m}$. We are interested in the question of whether this map is an embedding. The statements that some moduli space is embedded into another are traditionally called "Torelli theorems" after the classical Torelli theorem about the embedding of the moduli space of curves into the moduli space of Abelian varieties. The following theorem, which is the main result of this section, shows that Ψ is very close to an embedding, at least for large m.

7.2. THEOREM. *Let $m \geqslant 2n + 3$ and let $\mathscr{H}$, $\mathscr{H}'$ be two arrangements of m hyperplanes in P^n in general position. Suppose that the corresponding logarithmic bundles $E(\mathscr{H})$ and $E(\mathscr{H}')$ are isomorphic. Then one of the two possibilities holds:*

(1) *$\mathscr{H} = \mathscr{H}'$ (possibly after reordering the hyperplanes).*
(2) *There exists a Veronese curve $C \subset P^n$ such that all hyperplanes from $\mathscr{H}$ and $\mathscr{H}'$ osculate this curve. In this case $E(\mathscr{H})$ and $E(\mathscr{H}')$ are isomorphic to the Schwarzenberger bundle $E(C, m)$.*

7.3. To prove Theorem 7.2 we have to recover (as far as possible) the configuration $\mathscr{H}$ from the bundle $E(\mathscr{H})$. The key idea is that the lines lying in each $\mathscr{H}_i$ are special jumping lines.

More precisely, we shall call a line $l \subset P^n$ a *superjumping line* for $\mathscr{H}$ if the restriction $E(\mathscr{H})|_l$ contains as a direct summand a sheaf $\mathcal{O}_l(a)$ with $a \leqslant 0$. As before, let us write $m = nd + 1 + r$ with $0 \leqslant r < n$. Clearly the line l is superjumping if and only if the restriction $E_{norm}(\mathscr{H})|_l$ of the normalized bundle contains $\mathcal{O}_l(b)$ with $b \leqslant -d$.

7.4. PROPOSITION. *Any line lying in one of the hyperplanes H_i of the arrangement $\mathscr{H}$ is a superjumping line for $\mathscr{H}$.*

Proof. We can assume that $l \subset H_1$. Let F be a vector bundle on P^1. The property that F contains $\mathcal{O}(a)$, with $a \leqslant 0$ as a direct summand, is equivalent to the fact that $H^1(P^1, F(-2)) \neq 0$.

We shall therefore prove that for $F = E(\mathscr{H})|_l$ the above cohomology does not vanish. Since the dimension of the cohomology groups varies semicontinuously with l, it is enough for our purpose to assume that $l \subset H_1$ is not contained in any other H_i, $i \neq 1$. The residue exact sequence (2.1) gives a surjection

$$F \to \mathcal{O}_l \oplus \bigoplus_{i=2}^{m} \mathbf{C}_{l \cap H_i} \to 0.$$

Hence $F(-2)$ maps surjectively onto $\mathcal{O}_l(-2)$. Since for coherent sheaves on P^1 the functor H^1 is right exact, we get a surjection $H^1(F(-2)) \to H^1(\mathcal{O}_l(-2)) = \mathbf{C}$. The proposition is proven.

We want now to reformulate the condition of being a superjumping line in terms of the dual projective space $\check{P}^n$.

7.5. PROPOSITION. *Let $\mathscr{H} = (H_1, \ldots, H_m)$ be as before and let $l \subset P^n$ be a line not lying in any H_i. Let p_i be the point of the dual projective space $\check{P}^n$ corresponding to H_i. Let also $Z \subset \check{P}^n$ be the codimension-2 flat corresponding to the line l. Then the following two conditions are equivalent*:

(i) *l is a superjumping line for $\mathscr{H}$.*
(ii) *There exists a quadric $Q \subset P^n$ (of rank $\leqslant 4$) containing all the points $p_1, \ldots, p_m$ and the flat Z.*

Proof. Consider the dual bundle E^* to $E = E(\mathscr{H})$. In other words, E^* is the bundle $T_{P^n}(\log \mathscr{H})$. A line l is superjumping for $\mathscr{H}$ if and only if $H^0(l, E^*) \neq 0$. By reasoning analogous to that in the proof of Corollary 4.8 and Lemma 5.11, the existence of a section of $E^*|_l$ is equivalent to the existence of a regular map $\psi\colon l \to H_m$ of degree 1 such that $\psi(l \cap H_i) \subset H_m \cap H_i$ for $i = 1, \ldots, m-1$. A map ψ is just an identification of l with some line l' in H_m. Let Λ, Λ' be codimension-2 flats in $\check{P}^n$ corresponding to l, l'. The map ψ gives an identificaton Ψ of the projective lines (pencils) $]\Lambda[$, $]\Lambda'[$ formed by hyperplanes through Λ and Λ' respectively. Such an identification defines, by Steiner's construction [GH], a quadric Q of rank $\leqslant 4$. Explicitly, Q is the union of codimension-2 subspaces of the form $\Pi \cap \Psi(\Pi)$ where $\Pi \in]\Lambda[$ is a hyperplane through Λ. This proves Proposition 7.5.

7.6. We would like now to characterize the hyperplanes H_i of $\mathscr{H}$ as those of which every line is superjumping for $\mathscr{H}$. To do this, it is again convenient to use the dual projective space $\check{P}^n$ and the points $p_i \in \check{P}^n$ corresponding to H_i. Let us call a point $q \in \check{P}^n$ *adjoint* to $p_1, \ldots, p_m$ if q does not coincide with any of p_i, and for any codimension-2 flat $Z \subset \check{P}^n$ containing q there is a quadric containing $Z, p_1, \ldots, p_m$. For a fixed point $q \in \check{P}^n$ let $H \subset P^n$ be the corresponding hyperplane. Proposition 7.5 shows that q is adjoint to $p_1, \ldots, p_m$ if and only if any line in H is superjumping for $\mathscr{H} = (H_1, \ldots, H_m)$. Thus Theorem 7.2 is equivalent to the following fact (in which we write P^n instead of $\check{P}^n$).

7.7. THEOREM. *Let $p_1, \ldots, p_m$ be points in P^n in linearly general position and $m \geqslant 2n+3$. Then*:

(a) *unless all p_i lie on one Veronese curve, there are no points adjoint to $p_1, \ldots, p_m$;*
(b) *if all p_i do lie on one Veronese curve C, then points of C, and only they, are adjoint to $p_1, \ldots, p_m$.*

Part (b) is a consequence of Lemma 4.21. We shall concentrate on the proof of part (b). The proof will be based on the classical Castelnuovo lemma [GH].

7.8. LEMMA. *Let $m \geqslant 2n+3$ and $p_1, \ldots, p_m$ be points in P^n in linearly general position which impose $\leqslant 2n+1$ conditions on quadrics. Then p_i lie on a Veronese curve.*

We shall prove the following fact which, together with Castelnuovo lemma, will imply Theorem 7.7.

7.9. PROPOSITION. *If $m \geqslant 2n$ and q is adjoint to $p_1, \dots, p_m$, then $q, p_1, \dots, p_m$ impose exactly $2n + 1$ conditions on quadrics.*

Proof. Let L be the linear system of all quadrics through the first $2n$ points $p_1, \dots, p_{2n}$. It is well known that L has codimension $2n$ in the linear system $|\mathcal{O}(2)|$ of all quadrics and quadrics from L cut out precisely $p_1, \dots, p_{2n}$. For any codimension-2 flat $Z \subset P^n$ let $M(Z)$ be the linear system of all quadrics through Z. It has dimension $2n$ (see 4.2). To establish Proposition 7.9 it suffices to prove the following fact.

7.10. PROPOSITION. *Let q be any point different from $p_1, \dots, p_{2n}$. Let $L_1 \subset L$ be the linear system of all quadrics through $p_1, \dots, p_{2n}, q$. Then*:

(a) *for a generic codimension-2 flat Z containing q the intersection $L \cap M(Z)$ consists of one point;*
(b) *the points $L \cap M(Z)$ for generic Z through q as above span L_1 as a projective subspace.*

Proof of 7.9 from 7.10. Suppose we know Proposition 7.10. Note that L_1 has codimension $2n + 1$ in $|\mathcal{O}(2)|$. This follows from the fact that quadrics from L (in fact, just rank-2 quadrics from L) cut out $p_1, \dots, p_{2n}$ and nothing else.

Let $L_2 \subset L_1$ be the linear system of quadrics through all p_i, $i = 1, \dots, m$ and q. We shall show that $L_2 = L_1$. Indeed, suppose that L_2 is a proper subspace in L_1. Since q is adjoint to $p_1, \dots, p_m$, the intersection $L_2 \cap M(Z)$ is nonempty for any codimension-2 flat Z through q. But for generic such Z the intersection $L \cap Z = L_1 \cap Z$ consists of just one point and these points span L_1. Hence L_2 should miss some of these points, giving a contradiction.

Proof of Proposition 7.10(a). Suppose that the statement is wrong. Then for every Z through q the linear system $L \cap M(Z)$ contains a pencil. This means that for any additional point $r \in P^n$ and any Z through q there will be a quadric containing $p_1, \dots, p_{2n}, r$ and Z. Now we shall move r and Z in a special way to get a contradiction.

Let us number the points $p_1, \dots, p_{2n}$ so as to ensure that the hyperplane $\langle p_1, \dots, p_n \rangle$ does not contain q. Let $H = \langle q, p_1, \dots, p_{n-1} \rangle$. Take 1-parameter families $Z = Z(t)$, $r = r(t)$, $t \in \mathbf{C}$ with the following properties:

(1) $Z(t)$, $r(t)$ lie in H for any t;
(2) $Z(0) = \langle q, p_1, \dots, p_{n-2} \rangle$, the point $r(0)$ is a generic point inside $Z(0)$;
(3) For $t \neq 0$ the flat $Z(t)$ intersects $Z(0)$ transversely inside H and the curve $r(t)$ intersects $Z(0)$ transversely at $t = 0$.

Let $Q(t)$ be a quadric containing $p_1, \dots, p_{2n}, r(t), Z(t)$ (which exists by our assumption). We can choose $Q(t)$ for $t \neq 0$ to depend algebraically on t. Let $Q = \lim_{t \to 0} Q(t)$. Then Q contains the flat $\langle q, p_1, \dots, p_{n-2} \rangle$ and, in addition, the embedded tangent space to Q at points $p_1, \dots, p_{n-2}$ and $r(0)$ coincides with the hyperplane H. This means that the lines $\langle p_i, p_{n-1} \rangle$ and $\langle r(0), p_{n-1} \rangle$ will lie on Q. This implies that the whole hyperplane H will be part of Q. Hence the remaining $n + 1$

points $p_n, \ldots, p_{2n}$ should lie on another hyperplane, which is impossible. Part (a) of Proposition 7.10 is proven.

7.11. Now we shall concentrate on the proof of part (b) of Proposition 7.10. Note that we can reformulate it as follows.

7.12. PROPOSITION. *The linear system L_1 of quadrics through $p_1, \ldots, p_{2n}$ and q is spanned by quadrics of rank $\leqslant 4$ contained in this system.*

Indeed, the union of $L_1 \cap M(Z)$ for all codimension-2 flats Z through q coincide with the part of L_1 consisting of quadrics of rank $\leqslant 4$.

So we shall prove Proposition 7.12. Note that is a consequence of the following fact.

7.13. LEMMA. *The linear system L of quadrics through $p_1, \ldots, p_{2n}$ is spanned by $(1/2)\binom{2n}{n}$ quadrics of rank 2 contained in L.*

Indeed, suppose we know Lemma 7.13. Let Q be any quadric in L_1. By the lemma, Q is a linear combination of rank-2 quadrics $Q_1, \ldots, Q_N$ which lie in L but not necessarily in L_1. Since L_1 is a hyperplane in L, any pencil $\langle Q_i, Q_j \rangle$ intersects L_1 and Q lies in the projective span of their intersection points (for all i, j). But any quadric from any pencil $\langle Q_i, Q_j \rangle$ has rank $\leqslant 4$ since the Q_i have rank 2. This reduces our statement to Lemma 7.13.

To prove Lemma 7.13, let $D \subset |\mathcal{O}(2)|$ be the locus of quadrics in P^n of rank $\leqslant 4$.

7.14. LEMMA. *The dimension of D equals $2n$, the degree of D equals $(1/2)\binom{2n}{n}$, and D spans the projective space $|\mathcal{O}(2)|$.*

Proof of Lemma 7.14. The space D is the image of the map $f\colon \check{P}^n \times \check{P}^n \to |\mathcal{O}(2)|$ which takes $(H, H') \mapsto H \cup H'$. The map f is generically two-to-one. The inverse image under f of the standard bundle $\mathcal{O}(1)$ on the projective space $|\mathcal{O}(2)|$ is $\mathcal{O}(1, 1)$, and its degree (self-intersection index) is $\binom{2n}{n}$. This proves the statements about the dimension and the degree. The fact that D spans $|\mathcal{O}(2)|$ follows since D contains the Veronese variety of double planes which by itself spans $|\mathcal{O}(2)|$.

End of the proof of Lemma 7.13. Note that L intersects D in finitely many points whose number is equal to the degree of D. These points are, moreover, smooth points of D (being quadrics of rank exactly 2). Hence the intersection is transversal at any of the points.

Suppose that the intersection points do not span L and are contained in some hyperplane $M \subset L$.

Take any quadric $X \in D$ which does not belong to L and consider the codimension-$2n$ subspace W in L spanned by M and X. Then W intersects D in more points than the degree of D. This means that the intersection will contain a curve (denote it $C(X)$) passing through one of the $(1/2)\binom{2n}{n}$ rank-2 quadrics which belong to M. Clearly there will be one of these quadrics, say, Q, which will be contained in any $C(X)$.

The embedded tangent space $T_Q C(X)$ is contained in the tangent space to D at Q. Since L intersects D transversely, $T_Q C(X)$ does not intersect M in any point other than Q. On the other hand, since D spans $|\mathcal{O}(2)|$, for a generic $X \in D$ the intersection of the projective span $\langle M, X\rangle$ with $T_Q D$ will consist of Q alone. Since $T_Q C(X) \subset T_Q D \cap \langle M, X\rangle$, we get a contradiction.

This finishes the chain of reductions proving Proposition 7.10(b). The proof of Theorem 7.7 is finished.

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Dolgachev: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109, USA; igor.dolgachev@um.cc.umich.edu

Kapranov: Department of Mathematics, Northwestern University, Evanston, Illinois 60208, USA; kapranov@chow.math.nwu.edu