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Arrival Times in Quantum Theory from an Irreversible Detector Model

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We investigate a detector scheme designed to measure the arrival of a particle at $x = 0$ during a finite time interval. The detector consists of a two-state system that undergoes a transition from one state to the other when the particle crosses $x = 0$, and possesses the realistic feature that it is effectively irreversible as a result of being coupled to a large environment. The probabilities for crossing or not crossing $x = 0$ thereby derived coincide with earlier phenomenologically proposed expressions involving a complex potential. The probabilities are compared with similar previously proposed expressions involving sums over paths, and a connection with time operator approaches is also indicated.

§1. Introduction

An enduring class of questions in non-relativistic quantum theory are those that involve time in a non-trivial way.[1)-4)] Of these, the question of tunneling time is perhaps the most important.[5),6)] But some of the basic issues are most simply seen through the question. What is the probability that a particle enters a region of space for the first time during a given time interval? What makes this sort of problem interesting is that standard quantum mechanics seems to encounter some difficulty in its treatment, and there does not appear to be a unique answer; classically equivalent methods of assigning the arrival time can differ at the quantum level.

One of the key sources of difficulty with defining arrival times in quantum theory is that they involve specification of positions at two moments of time: to state that the particle enters a given spatial region at time t means that it is outside the region immediately before t *and* inside it immediately after t. Since positions at different moments of time do not commute, we do not expect to be able to associate a single hermitian operator with the arrival time. Of course, many different methods of defining arrival time then naturally suggest themselves, but, like the question of specifying phase space locations in quantum mechanics, they are not all equivalent.

To be specific, in this paper we consider the following question: Suppose we have an initial state with support only in $x > 0$. Then what is the probability that the particle enters the region $x < 0$ at any time during the time interval $[0, \tau]$?

A number of previous papers have addressed this particular problem using path integrals.[7)-9)] (These approaches are closely related to the decoherent histories approach to quantum theory.[10),11)]) The amplitudes for crossing and not crossing $x = 0$ are obtained by summing over paths which either always enter or never enter the region $x < 0$, and probabilities are then obtained by squaring amplitudes in the normal way. However, due to interference between paths, the resultant probabilities do not add up to 1, and they therefore cannot be regarded as true probabilities. A way past this difficulty is explored in Ref. 12). There, the point particle system is coupled to a thermal environment to induce decoherence of different paths in configuration space, and the correct probability sum rules were restored. Although this approach produces mathematically viable candidates for the probabilities of crossing and not crossing, they depend to some degree on the environment, and it is by no means clear how the results correspond to a particular type of measurement, even an idealized one. General theorems exist showing that decoherence of histories implies the existence at the final end of the histories of a record storing the information about the decohered histories,[10] but these have been explicitly found only in a few simple cases (see Ref. 13), for example). For these reasons, it is of interest to compare these path integral approaches with a completely different approach involving a specific model of a detector.

Let us therefore introduce a model detector which is coupled to the particle in the region $x < 0$ and such that it undergoes a transition when the coupling is switched on. Such an approach has certainly been considered before (see, e.g., Ref. 4)). The particle could, for example, be coupled to a simple two-level system that flips from one level to the other when the particle is detected. One of the difficulties of many detector models, however, is that if they are modeled by unitary quantum mechanics, the possibility of the reverse transition exists. Because quantum mechanics is fundamentally reversible, the detector could return to the undetected state under its self-dynamics, even when the particle has interacted with it.

To get around this difficulty, we appeal to the fact that realistic detectors have a very large number of degrees of freedom, and are therefore effectively *irreversible*. They are designed so that there is an overwhelming large probability for them to make a transition in one direction rather than its reverse. In this paper we introduce a simple model detector that has this property. This is achieved by coupling a two-level system detector to a large environment, which makes its evolution effectively irreversible. The description of this system is easily obtained using some standard machinery of open quantum systems, and the resulting master equation for the particle coupled to the detector actually has the Lindblad form.[14]

We are not concerned with a specific experimental arrangement, but rather, as is common in quantum measurement theory, an idealized model that has as many physically realistic features as one can reasonably incorporate. In particular, in contrast to most measurement models discussed in the literature, it has the key property of irreversibility.

The detector is described in §2. On solving the detector dynamics, an expression for the probability of entering a spacetime region is obtained. It has the appearance of a probability obtained from a wave function satisfying a Schrödinger equation with an imaginary contribution to the potential, which has previously been proposed as a phenomenological device.[1),15),16] Our detector model therefore justifies previously used phenomenological approaches.

The probabilities obtained are also readily compared with the results of the path integral approaches, and the comparison sheds some light on the shortcomings of the latter. This is described in §3. A brief mention is also made of the possible connection with time operators.

§2. The detector model

In this section we describe a detector designed to make a permanent record of whether or not a particle enters the region $x < 0$ at any time during a given finite time interval. We take the detector to be a two-level system, with levels $|1\rangle$ and $|0\rangle$, representing the states of no detection and detection, respectively. Introduce the raising and lowering operators

$$\sigma_+ = |1\rangle\langle 0|, \quad \sigma_- = |0\rangle\langle 1| \tag{2·1}$$

and let the Hamiltonian of the detector be $H_d = \frac{1}{2}\hbar\omega\sigma_z$, where

$$\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|. \tag{2·2}$$

Thus $|0\rangle$ and $|1\rangle$ are eigenstates of H_d with eigenvalues $-\frac{1}{2}\hbar\omega$ and $\frac{1}{2}\hbar\omega$ respectively. We would like to couple the detector to a free particle in such a way that the detector makes an essentially irreversible transition from $|1\rangle$ to $|0\rangle$ if the particle enters $x < 0$ and remains in $|1\rangle$ otherwise. This can be arranged by coupling the detector to a large environment of oscillators in their ground states, with a coupling proportional to $\theta(-x)$. This means that if the particle enters the region $x < 0$, the detector becomes coupled to the large environment, causing it to undergo a transition. Since the environment is in its ground state, if the detector initial state is the higher energy state $|1\rangle$, it will, with overwhelming probability, make a transition from $|1\rangle$ to the lower energy state $|0\rangle$. A possible Hamiltonian describing this process for the three-component system is

$$H = H_s + H_d + H_{\mathcal{E}} + V(x)H_{d\mathcal{E}}, \tag{2·3}$$

where the first three terms are the Hamiltonians of the particle, detector and environment, respectively, and $H_{d\mathcal{E}}$ is the interaction Hamiltonian of the detector and its environment. The simplest choice of the environment is a collection of harmonic oscillators,

$$H_{\mathcal{E}} = \sum_n \hbar\omega_n a_n^\dagger a_n, \tag{2·4}$$

and we take the coupling to the detector to be via the interaction

$$H_{d\mathcal{E}} = \sum_n \hbar \left(\kappa_n^* \sigma_- a_n^\dagger + \kappa_n \sigma_+ a_n \right). \tag{2·5}$$

An environment consisting of an electromagnetic field, for example, would give terms of this general form. $V(x)$ is a potential concentrated in $x < 0$ (and we will eventually make the simplest choice, $V(x) = \theta(-x)$, but for the moment we keep it more general). The important feature is that the interaction between the detector and its environment, causing the detector to undergo a transition, is switched on only when the particle is in $x < 0$.

A similar although more elaborate model particle detector has been studied by Schulman[17] (see also Ref. 18)). It consists of a lattice of spins in a metastable state interacting with the lattice's phonon modes, with interactions essentially the same as in Eq. (2·3). His approach has the advantage that the amplification of a microscopic event as well as the irreversibility of the measurement is modeled, but it does not appear to be possible to solve it as explicitly as the simpler model considered here. For an even more elaborate model, see Ref. 19).

We are interested in the reduced dynamics of the particle and detector with the environment traced out. Hence we seek a master equation for the reduced density operator ρ of the particle and detector. With the above choices for $H_{\mathcal{E}}$ and $H_{d\mathcal{E}}$, the derivation of the master equation is a standard calculation[20),21)] and will not be repeated here. There is the small complication of the factor of $V(x)$ in the interaction term, but this is readily accommodated. We assume a factored initial state, and we assume that the environment starts out in the ground state. In a Markovian approximation (essentially the assumption that the environment dynamics are much faster than the detector or particle dynamics), and in the approximation of weak detector-environment coupling, the master equation is

$$\dot{\rho} = -\frac{i}{\hbar}[H_s+H_d,\rho]-\frac{\gamma}{2}\left(V^2(x)\sigma_+\sigma_-\rho \ +\rho\sigma_+\sigma_-V^2(x) - 2V(x)\sigma_-\rho\sigma_+V(x)\right). \quad (2\cdot6)$$

Here, γ is a phenomenological constant determined by the distribution of oscillators in the environment and underlying coupling constants. The frequency ω in H_d is also renormalized to a new value ω'.

Equation (2·6) is the sought-after description of a particle coupled to an effectively irreversible detector in the region $x < 0$. In the dynamics of the detector-plus-environment only (i.e., with $V = 1$ and $H_s = 0$), it is readily shown that every initial state tends to the state $|0\rangle\langle 0|$ on a timescale γ^{-1}. With the particle coupled in, as in (2·6), if the initial state of the detector is chosen to be $|1\rangle\langle 1|$, it undergoes an irreversible transition to the state $|0\rangle\langle 0|$ if the particle enters $x < 0$ and remains in its initial state otherwise.

Although we have outlined the derivation of this master equation for a particular choice of the environment and detector-environment coupling, we expect that the form of the equation is more general. It is well known that, after tracing out the environment and in the approximation that the evolution is Markovian, the reduced density operator ρ of the particle and detector must evolve according to a master equation of the Lindblad form:

$$\dot{\rho} = -\frac{i}{\hbar}[H_s + H_d,\rho] + \sum_m \left(L_m\rho L_m^\dagger - \frac{1}{2}L_m^\dagger L_m\rho - \frac{1}{2}\rho L_m^\dagger L_m\right). \quad (2\cdot7)$$

This is the most general Markovian evolution equation preserving the positivity, hermiticity and trace of ρ. The operators L_m model the effects of the environment.[14)] The form (2·6) is recovered with a single Lindblad operator $L = \gamma^{\frac{1}{2}}V(x)\sigma_-$. A similar detection scheme based on a postulated master equation similar to (2·7) was previously considered in Refs. 22) and 23), although the resultant expressions for arrival time probability given below were not derived (and no microscopic origin of the equation was given).

Equation (2·6) is easily solved by writing

$$\rho = \rho_{11} \otimes |1\rangle\langle 1| + \rho_{01} \otimes |0\rangle\langle 1| + \rho_{10} \otimes |1\rangle\langle 0| + \rho_{00} \otimes |0\rangle\langle 0|, \tag{2·8}$$

where

$$\dot{\rho}_{11} = -\frac{i}{\hbar}[H_s, \rho_{11}] - \frac{\gamma}{2}\left(V(x)\rho_{11} + \rho_{11}V(x)\right), \tag{2·9}$$

$$\dot{\rho}_{01} = -\frac{i}{\hbar}[H_s, \rho_{01}] - \frac{\gamma}{2}\rho_{01}V(x) + i\frac{\hbar\omega'}{2}\rho_{01}, \tag{2·10}$$

$$\dot{\rho}_{10} = -\frac{i}{\hbar}[H_s, \rho_{10}] - \frac{\gamma}{2}V(x)\rho_{10} - i\frac{\hbar\omega'}{2}\rho_{10}, \tag{2·11}$$

$$\dot{\rho}_{00} = -\frac{i}{\hbar}[H_s, \rho_{00}] + \gamma V(x)\rho_{11}V(x). \tag{2·12}$$

(We have now set $V(x) = \theta(-x)$, so that $V^2 = V$.) We suppose that the particle starts out in an initial state $|\Psi_0\rangle$. Hence, the above equations are to be solved subject to the initial condition

$$\rho(0) = |\Psi_0\rangle\langle\Psi_0| \otimes |1\rangle\langle 1|. \tag{2·13}$$

The probability of finding the detector in the unregistered state $|1\rangle$ at time τ is

$$p_{nd} = \mathrm{Tr}\rho_{11}(\tau) = \int_{-\infty}^{\infty} dx\ \rho_{11}(x, x, \tau), \tag{2·14}$$

and the probability of finding it in the registered state $|0\rangle$ is

$$p_d = \mathrm{Tr}\rho_{00}(\tau) = \int_{-\infty}^{\infty} dx\ \rho_{00}(x, x, \tau), \tag{2·15}$$

where the trace is over the particle Hilbert space. Clearly $p_{nd}+p_d = 1$, since $\mathrm{Tr}\rho = 1$.

Note that the probability for no detection includes an integral over $x < 0$, and $\rho_{11}(x, x, \tau)$ is not necessarily zero for $x < 0$. There is therefore the possibility that the particle enters the region $x < 0$ without the detector registering the fact. This is, however, to be expected of a realistic detector; there is some probability that it will fail to do what it is supposed to do. The probability of this happening is typically small, and indeed, computation of this probability provides a useful check on the efficiency of the detector (although below we will check detector efficiency in a different way).

The formal solution to Eq. (2·9) for ρ_{11} may be written

$$\rho_{11}(t) = \exp\left(-\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right)\ \rho_{11}(0)\ \exp\left(\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right). \tag{2·16}$$

What is particularly interesting about this expression is that it can be factored into a pure state. Let $\rho_{11} = |\Psi\rangle\langle\Psi|$. Then, noting that $\rho_{11}(0) = |\Psi_0\rangle\langle\Psi_0|$, Eq. (2·16) is equivalent to

$$|\Psi(t)\rangle = \exp\left(-\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right)|\Psi_0\rangle. \tag{2·17}$$

The probability for no detection is then

$$p_{nd} = \int_{-\infty}^{\infty} dx \ |\Psi(x,\tau)|^2. \tag{2·18}$$

The pure state (2·17) evolves according to a Schrödinger equation with an imaginary contribution to the potential, $-\frac{1}{2}i\hbar\gamma V$. Complex potentials have been used in this context, as phenomenological devices, to imitate absorbing boundary conditions (see, for example Refs. 1), 15) and 16)). Here, the appearance of a complex potential is *derived* from the master equation of a particle coupled to an irreversible detector, which in turn may be derived from the unitary dynamics of the combined particle-detector-environment system.

Equation (2·12) for ρ_{00} may be formally solved to yield

$$\rho_{00}(t) = \gamma \int_0^t dt' \exp\left(-\frac{i}{\hbar}H_s(t-t')\right) V(x)\rho_{11}(t')V(x) \exp\left(\frac{i}{\hbar}H_s(t-t')\right) \tag{2·19}$$

(recalling that $\rho_{00}(0) = 0$). Inserting the solution for $\rho_{11}(t')$, the probability for detection may be written

$$\begin{aligned} p_d &= \gamma \int_0^\tau dt \int_{-\infty}^{\infty} dx \left| \left\langle x \left| \exp\left(-\frac{i}{\hbar}H_s(\tau-t)\right) V(x) \exp\left(-\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right) \right| \Psi_0 \right\rangle \right|^2 \\ &= \gamma \int_0^\tau dt \int_{-\infty}^{0} dx \left| \left\langle x \left| \exp\left(-\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right) \right| \Psi_0 \right\rangle \right|^2 \\ &= \gamma \int_0^\tau dt \int_{-\infty}^{0} dx \, |\Psi(x,t)|^2 , \end{aligned} \tag{2·20}$$

where $\Psi(x,t)$ is the wave function (2·17). The expression for the probability for detection has an appealing form: it is the integral of $|\Psi(x,t)|^2$ over the space-time region of interest. It is crucially important, however, that the wave function satisfies not the usual Schrödinger equation, but one with an imaginary contribution to the potential modeling the detector.

It is useful to write the probabilities for detection and no detection in the form

$$p_d = \mathrm{Tr}\left(\Omega \ |\Psi_0\rangle\langle\Psi_0|\right), \quad p_{nd} = \mathrm{Tr}\left(\bar{\Omega} \ |\Psi_0\rangle\langle\Psi_0|\right), \tag{2·21}$$

where

$$\Omega = \int_0^\tau dt \ \exp\left(\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right) \ \gamma V \ \exp\left(-\frac{i}{\hbar}H_s t - \frac{\gamma}{2}Vt\right), \tag{2·22}$$

$$\bar{\Omega} = \ \exp\left(\frac{i}{\hbar}H_s \tau - \frac{\gamma}{2}V\tau\right) \ \exp\left(-\frac{i}{\hbar}H_s \tau - \frac{\gamma}{2}V\tau\right). \tag{2·23}$$

The first of these expressions follows from (2·20) from the properties of the trace, and using the fact that $V^2 = V$.

Ω and $\bar{\Omega}$ are clearly not projection operators, although their properties are close to those of projectors. They are both positive operators, and $\Omega + \bar{\Omega} = 1$. The latter follows by integrating the identity

$$\frac{d\bar{\Omega}}{d\tau} = -\exp\left(\frac{i}{\hbar}H_s \tau - \frac{\gamma}{2}V\tau\right) \ \gamma V \ \exp\left(-\frac{i}{\hbar}H_s \tau - \frac{\gamma}{2}V\tau\right). \tag{2·24}$$

Moreover, these operators clearly have the desired localization properties on histories of particle positions, as is seen most clearly in the path integral expression of the next section. We do not expect to be able to associate a true projection operator with the arrival time, but here we have found a POVM, which is the next best thing.

Equations (2·21)–(2·23) are the main result of this section: expressions for the probabilities of entering or not entering a region of spacetime, derived from an irreversible detector model.

We now consider the issue of the efficiency of the detector. A simple way to do this is to introduce a second detector identical to the first and located in the region $x > 0$. Since the entire x-axis is now monitored, the probability that neither detector registers during the time interval is then a measure of the degree to which the detector will fail.

With two detectors in place, the master equation now is

$$\begin{aligned}\dot{\rho} = -\frac{i}{\hbar}[H_s, \rho] &- \frac{\gamma}{2}\left(\theta(-x)\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-\theta(-x) - 2\theta(-x)\sigma_-\rho\sigma_+\theta(-x)\right) \\ &- \frac{\gamma}{2}\left(\theta(x)\tilde{\sigma}_+\tilde{\sigma}_-\rho + \rho\tilde{\sigma}_+\tilde{\sigma}_-\theta(x) - 2\theta(x)\tilde{\sigma}_-\rho\tilde{\sigma}_+\theta(x)\right), \qquad (2{\cdot}25)\end{aligned}$$

where $\tilde{\sigma}_+$ and $\tilde{\sigma}_-$ are the raising and lowering operators for the detector in $x > 0$. This equation is solved like (2·6), by writing

$$\rho = \rho_{nd} \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| + \cdots. \qquad (2{\cdot}26)$$

We are interested only in the probability that neither detector registers, so we omit the explicit form of the other terms in (2·26). It is readily shown that

$$\dot{\rho}_{nd} = -\frac{i}{\hbar}[H_s, \rho_{nd}] - \gamma\rho_{nd}. \qquad (2{\cdot}27)$$

The probability of no detection is $\mathrm{Tr}\rho_{nd}$, and from (2·27), it clearly decays as $e^{-\gamma t}$. Hence the detector functions efficiently if the total time duration τ is much greater than γ^{-1}. (The potential inefficiency of the detector for τ not sufficiently large compared to γ^{-1} corresponds to the difficulties in defining the time operator at low momenta.[3)])

The evolution according to (2·17) for the case in which $V(x)$ is a real step function was studied in detail by Allcock,[1)] who was consequently pessimistic about the possibility of defining arrival time. This is partly because γ needs to be large for accurate detection, but in this case reflection from the potential is large, and not all of the incoming flux is absorbed. Subsequent authors have shown that his pessimism was misplaced, if additional general potentials $\tilde{V}(x)$ are permitted, which are smoother and may also need to be complex (see, for example, Refs. 15) and 24)). The considerations of this paper readily generalize to this case by replacing Eq. (2·6) with the equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_s, \rho] - \frac{\gamma}{2}\left(\tilde{V}^\dagger\tilde{V}\sigma_+\sigma_-\rho \ + \ \rho\sigma_+\sigma_-\tilde{V}^\dagger\tilde{V} \ - \ 2\tilde{V}\sigma_-\rho\sigma_+\tilde{V}^\dagger\right), \qquad (2{\cdot}28)$$

which is still of the Lindblad form, but $\tilde{V}(x)$ is generally a non-Hermitian operator. It is less clear what sort of detector-environment coupling this corresponds to in

Eq. (2·3), and this is certainly of interest to investigate. Note also that $\tilde{V}^2 \neq \tilde{V}$, but the above results are readily modified to incorporate this.

The detector described here measures whether the particle entered the region $x < 0$ at any time during $[0, \tau]$, for $\tau \gg \gamma^{-1}$. It could easily be extended to give more precise information about the time at which the particle enters $x < 0$ by using a series of similar detectors, but with a time-dependent coupling to the environment, so that the detectors can be switched on or off at a succession of times $t_1, t_2, t_3 \cdots$ (separated in time by at least γ^{-1}) in the interval $[0, \tau]$.

§3. Comparison with other approaches

The expressions we have derived for detection and no detection bear a close resemblance to previously derived path integral expressions for the probabilities of entering and not entering the region $x < 0$, so we now carry out a comparison.

For simplicity let the initial and final points lie in $x > 0$. The amplitude for remaining restricted to the region $x > 0$ is

$$g_r(x_f, \tau | x_0, 0) = \int_r \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)]\right), \qquad (3·1)$$

where the sum is over all paths in $x > 0$. The amplitude for crossing $x = 0$ at some time in the interval $[0, \tau]$ is

$$g_c(x_f, \tau | x_0, 0) = \int_c \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)]\right), \qquad (3·2)$$

where the sum is over paths that spend some time in $x < 0$ (see Refs. 25)–27) for details of the construction of these objects). Clearly $g_r + g_c = g$, where g denotes the unrestricted propagator, given by a sum over all paths. The probabilities for crossing and not crossing $x = 0$ are then obtained from these propagators, by attaching an initial state, squaring the amplitudes, and then summing over final positions in the usual way. However, as stated above, the resultant probabilities for crossing and not crossing computed in this way do not sum to 1, because of interference between the different types of paths.

Now we compare with the measurement model of the previous section. The probabilities computed here for detection and no detection in the region $x < 0$ automatically sum to 1. The probability for no detection may be computed from (2·17). The evolution operator that appears there may be written in path integral form

$$\begin{aligned} g_{nd}(x_f, \tau | x_0, 0) &= \left\langle x_f \left| \exp\left(-\frac{i}{\hbar} H_s \tau - \frac{\gamma}{2} V \tau\right) \right| x_0 \right\rangle \\ &= \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)] - \frac{\gamma}{2} \int_0^\tau dt\ V(x(t))\right). \qquad (3·3) \end{aligned}$$

The sum here is over all paths $x(t)$ connecting x_0 at time 0 to x_f at time τ. But it is clear that the potential $V(x)$ suppresses contributions from paths that enter $x < 0$.

Split the paths summed over into the two classes r and c, as above. (For simplicity, we take $x_0 > 0, x_f > 0$). Noting that $V = 0$ in $x > 0$, the path integral becomes

$$g_{nd}(x_f, \tau | x_0, 0) = \int_r \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)]\right) + \int_c \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)] - \frac{\gamma}{2} \int_0^\tau dt\ V(x(t))\right). \qquad (3\cdot4)$$

We see that (3·4) differs from (3·1) by the presence of the second term. In the second term, every path in the sum has a section lying in the region $x < 0$, and an exponential suppression factor will come into play. g_{nd} and g_r exactly coincide in the limit $\gamma \rightarrow \infty$. The resultant probabilities are, however, not very interesting.[7] Furthermore, as stated, large γ implies that most of the incoming wave packets are reflected rather than absorbed by the detector. This means that the second term in Eq. (3·4) will generally be significant and g_{nd} and g_r will not be close. From this we conclude that the purely path integral approaches to defining the arrival time, as expressed by (3·1) and (3·2), are actually rather removed from the more physically motivated expressions using a model detector derived in this paper. Including a physical mechanism for decoherence in the path integral approach[12] yields more sensible results, but they are not very closely related to the detector results. (A comparison is carried out in an earlier version of this paper.[27])

We may also write the POVM for no detection, (2·23), in another more enlightening form. Note that

$$\bar{\Omega} = U^\dagger(\tau) U(\tau), \qquad (3\cdot5)$$

where

$$U(\tau) = \exp\left(\frac{i}{\hbar} H_s \tau\right) \exp\left(-\frac{i}{\hbar} H_s \tau - \frac{\gamma}{2} V \tau\right) = T \exp\left(-\frac{\gamma}{2} \int_0^\tau dt\ V(x_t)\right). \qquad (3\cdot6)$$

Here, T denotes time ordering, and $x_t = x + pt/m$ is the position operator at time t in the Heisenberg picture. Splitting the time interval into small units δt, we see that U is a time-ordered product of operators of the form $\exp(-\gamma \delta t V/2)$. But with the choice used here, $V(x) = \theta(-x)$, we have $V^2 = V$, and hence

$$\exp\left(-\frac{\gamma}{2} \delta t V\right) = (1 - V) + V e^{-\gamma \delta t/2} = \theta(x) + \theta(-x) e^{-\gamma \delta t/2}. \qquad (3\cdot7)$$

This is therefore a projector onto the positive x-axis plus an exponentially smaller projector onto the negative x-axis. Naive expectations would lead one to guess the first term, but the addition of the second term appears to be important in making the probabilities well-defined (see Ref. 26) for related formulae).

From the above we see that the probabilities for detection and no detection depend on x and p only through the operators $\theta(x_t)$ at a series of times. This leads us to a connection with time operators. For the classical quantities x and p, we have

$$\theta\left(x+\frac{pt}{m}\right)=\theta\left(\frac{mx}{p}+t\right) \tag{3·8}$$

(for $x, p > 0$). The point is that the quantity mx/p is precisely the classical arrival time. This is the quantity that numerous authors have, not without serious difficulties, attempted to elevate to the status of a quantum operator to define quantum arrival times (see, for example, Ref. 3)). The connection between the expressions derived here and those derived using the time operator may therefore be obtained by investigating the extent to which the classical relation (3·8) persists in the quantum theory. This will be pursued elsewhere.

§4. Summary and conclusions

We have presented a detector model for the measurement of arrival time in quantum theory. It possesses the realistic feature of being effectively irreversible. The results of the scheme connect very nicely with previous approaches involving a postulated complex potential to imitate the effects of a detector.

The detector model was compared with previous approaches involving sums over paths, and the detector model exposes the limitations of the latter. Some indication of the possible connection with time operators was also given.

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