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ASCENDING BID AUCTIONS WITH BEHAVIORALLY CONSISTENT BIDDERS

by

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1. Introduction

The mounting experimental evidence to the effect that decision makers tend to violate expected utility theory in a systematic manner when presented with choices among risky prospects led to the development of new models of decision making under risk and under uncertainty.[1] These models depart from expected utility theory and, in particular, from the independence axiom (or the sure thing principle) that lends this theory its linear structure. Without the independence axiom, however, the optimal strategy of a player in a game involving random outcomes and a sequence of moves is, in general, dynamically inconsistent. In other words, the continuation of an optimal strategy formulated at the outset, as of a subsequent decision node, may differ from the optimal continuation strategy as of this node. The issue here is similar to the problem of intertemporal consumption decisions with time dependent discount rate analyzed by Strotz (1956). In the spirit of Strotz's suggestion on how to deal with this problem we proposed in a recent paper (Karni and Safra (1987)) to model the behavior of a player whose preferences are nonlinear in the probabilities as follows: We regard the same player at different decision nodes as different agents. The preferences of agents representing a given player are represented by the same utility functional and each agent acts in his own self-interest. Finally, the behavior of the player in a game is a sequential equilibrium of a game among his representing agents.[2] A player whose behavior may be represented in this manner is said to be **behaviorally consistent**.

In Karni and Safra (1987) we applied the notion of behavioral consistency to the analysis of an ascending bid auction. In this paper we analyze a different ascending bid auction using the same characterization of individual behavior. The interest in the study of the auction presented in section 2 below is twofold: First, while involving only a slight modification of the rules, the ascending bid auction considered in this paper leads to a significant change in the conclusions. In particular, the use of Bayesian equilibrium replaces the sequential equilibrium concept that seemed natural for the analysis of the other auction and, more importantly, it is possible to characterize the equilibrium for the entire class of preferences for which the equilibrium exists. Second, this auction is, in a sense, equivalent to the limit of the auction analyzed in our previous paper when the number of bidders increases. Moreover, the characterization of the solutions is the same as that of a continuous ascending bid auction with a continuum of types when the solution of such auction game exists.

We model ascending bid auctions as extensive form games with incomplete information. Aspects of the game that are common knowledge include the number of active bidders at each stage; the way in which the outcome of the game depends on the actions of the players; and the feasible sets of actions of each player. We assume that beliefs are consistent in the sense of Harsanyi (1967-68), i.e., there exists a prior probability distribution on the set of types (preferences) such that each bidder's conditional distribution, given his own type, is identical to the distribution that would have been computed from the given prior according to Bayes' rule. The prior distribution is the same for all agents and this fact is common knowledge. We assume also that the utility of each player depends only on his own type. The solution concept that we employ in the analysis of the game is the Bayesian-Nash equilibrium among the agents.

The plan of the remainder of this paper is as follows: In the next section we present the ascending bid auction game to be analyzed. In section 3 we prove the existence and characterize the nature of a Bayesian equilibrium of the game when the players have quasi-concave preferences. In section 4 we discuss the significance of this game from the viewpoint of the theory of auctions.

2. **An Ascending Bid Auction Game**

2.1 The Rules. Consider a discrete, noncooperative, ascending bid auction, in which an object, $\bar{A}$, is being auctioned to J, $(J \geq 2)$, bidders. Let [0,M] be the possible range of prices that bidders may pay for $\bar{A}$, i.e., the probability that anyone is ready to pay more than M is zero, and the probability that someone is ready to pay more than 0 is 1.[3] Let $Y = (y_1, y_2, \ldots, y_n)$, $y_1 = 0$, and $y_1 = M$, be a partition of [0,M]. The points y_i, $i = 1, \ldots, n$, represent prices announced by the auctioneer. We shall refer to these prices as decision points.

The auctioneer starts the auction by announcing y_1 as the selling price. He then proceeds to increase the price. At each point, every bidder must declare whether he is in the game or not. A decision to withdraw is irreversible. A decision to remain in the game means that the bidder is ready to pay the price announced for the object being auctioned. The game terminates when either only one bidder remains in the game, or when all **eligible bidders**, i.e., bidders who, at the previous decision point were in the game, announce their withdrawal at the same point. In the former case the remaining bidder gets the object and is charged the price corresponding to the first decision point at which he is the sole bidder. In the latter case none of the bidders get the object which remains in the possession of the original owner. This last rule is the main change in the structure of the game from the ascending bid auction analyzed in Karni and Safra (1987). There, we assumed that the tie is broken and the winner is determined by a random draw from the set of eligible bidders. The present rule is not meant to be a description of an observed practice, rather it is intended to capture situations in which, for all practical purposes, withdrawal from the auction means that the bidder does not win the object. These situations include the case in which the price goes up by small increments and the participants in the auction are drawn from a continuous distribution on the set of admissible preferences (to be defined below). We shall elaborate on these points in section 4.

2.2. Types and Beliefs. Let L be the set of cumulative probability distributions on R endowed with the topology of weak convergence. Let Ω be the set of binary relations on L that satisfy weak order, continuity, monotonicity with respect to first-order stochastic dominance, the reduction of compound lottery axiom, and smoothness, in the sense of having a Gateaux differentiable representation $V: L \to R$. Let $T \subset \Omega$ be a finite set. T is the set of types. Each bidder j in J is characterized by his type (i.e., his preferences), which is known only to himself, and his initial wealth, which is common knowledge. Without loss of generality we assume that all bidders have the same level of initial wealth, w, $w > M$.

The beliefs that bidder k in J holds regarding the preferences of the other bidders, given that he is of type t, are summarized by a conditional probability distribution, $\mu(\xi^{-k}|t)$ defined on T^{J-1}. Following the approach of Harsanyi (1967-68), we assume that these beliefs are **consistent** in the sense that they are derived from a joint probability distribution μ on T^J using Bayes' rule and that this is common knowledge. Formally, let $\xi = (\xi^{-k},t)$ then

(1) $\mu(\xi^{-k}|t) = \mu(\xi)/\mu(t)$ for all t in T and μ^{-k} in T^{J-1},

where

$$\mu(t) = \sum_{\xi^{-k}\epsilon T^{J-1}} \mu(\xi).$$

$\mu(t)$ represents the initial beliefs that every bidder k has on the type of bidder j $(j \neq k)$. We assume that for all t in T $\mu(t) > 0$.

2.3. The Players. We distinguish between bidders and players. In particular, we associate with each bidder a set of agents, one agent for each decision point in Y. **Agent** (ji) represents bidder j at the decision point y_i. The set of players, N, consists of the Jn agents, namely, $N = \{(11), (21),\ldots,(Jn)\}$. By definition each agent in the subset of agents corresponding to a given bidder knows the preferences of all the other agents belonging to this subset. Unlike Selten's (1975) formulation of extensive games in agent normal form, our representation is motivated by the need to capture the fact that the same bidder at different decision points **evaluates the strategies** differently, and must therefore be regarded as a different player.

2.4. **Announcements and Decisions**. At each decision point the respective agent must announce an element from the set $\{0,1\}$. The announcements are made simultaneously. An announcement of 0 by bidder j at the point y_i in Y indicates that agent j withdraws from the auction at this point. Since the rules of the auction prohibit re-entry, an announcement of 0 by (ji) terminates the game for $(j(i+k))$, $k = 1,\ldots,n-i$. Similarly, an announcement of 1 indicates that agent (ji) is in the auction, i.e., bidder j is ready to pay the price y_i for the object. A **decision** for player (ji) is an element of the set $[0,1]$, i.e., $\alpha \in [0,1]$ is a decision to announce 1 with probability α and to announce 0 with probability $(1-\alpha)$.

2.5. **Histories**. For all i, $i = 2,\ldots,n+1$, a history h_i is a J-tuple, where $(h_i)^j$ in $\{y_1, y_2,\ldots,y_{i-1}\}$ is the last decision point at which bidder j, acting through his representing agents, announced that he is in the game. Because there is no history at y_1, we define $h_1 = 0$. At each decision point y_i, h_i is common knowledge. Let H_i be the set of all histories at the decision point y_i and let $H_i^j \subset H_i$ be the subset of histories such that $(h_i)^j = y_{i-1}$.

2.6. Strategies and Beliefs. A strategy for player (ji) is a function $s^{ji}: TxH_i^j \to [0,1]$. Thus, $s^{ji}(t,h_i)$ is the probability that player (ji) of type t will announce 1 given the history h_i. Let S^{ji} be the set of all the strategies of player (ji), $S^j = S^{j1} \times S^{j2} \times \ldots \times S^{jn}$, and $S = S^1 \times S^2 \times \ldots \times S^J$. The sets S^{ji} can be identified with a (finite) Cartesian product of intervals [0,1]. Note that S^{ji} are compact and convex. The sets of strategies of all players are common knowledge.

For any strategy $s \in S$ and a history $h_i \in H_i$ we assign a measure $\tilde{\mu}^{ki}(\cdot;s,h_i)$ on T in the following manner:

$$\tilde{\mu}^{k1}(\tau;s,h_1,t) = \sum_{\xi \epsilon T^J} \mu(\xi^{-kj}|\tau,t)$$

(2) $\tilde{\mu}^{ki}(\tau;s,h_i,t) = 0$ if $h_i \notin H_i^k$; otherwise

$$\tilde{\mu}^{ki}(\tau;s,h_i,t) = \frac{\tilde{\mu}^{k(i-1)}(\tau;s,h_{i-1},t)s^{k(i-1)}(\tau,h_{i-1})}{\sum_{r \epsilon T} \tilde{\mu}^{k(i-1)}(r;s,h_{i-1},t)s^{k(i-1)}(r,h_{i-1})}$$

where $(\xi^{-kj}|\tau,t) \in T^J$ is the ξ such that τ and t are the types of k and j respectively.

The measure $\tilde{\mu}^{ki}(\cdot;s,h_i,t)$ is the posterior probability that any other player (ji), $j \neq k$, of type t has for the event that (ki) is of type τ, given s and h_i. If h_i is not in H_i^k then k is not in the game and thus $\tilde{\mu}^{ki}(\tau;s,h_i,t)$ was defined to be zero. We shall see later that the measure defined in (2) exists.

Next, we define the functions $q^{ki}: S \times H_i \to [0,1]$ that assign to every strategy $s \in S$ and history $h_i \in H_i$ the probability that player (ji) of type t assigns to the event that players (ki) announced 1 at y_i. These functions are defined as follows:

(3) $$q^{ki}(s,h_i,t) = \sum_{\tau \in T} \tilde{\mu}^{ki}(\tau;s,h_i,t)s^{ki}(\tau,h_i).$$

Finally, let J^{ji} be the subset of $J\backslash\{j\}$ of bidders k for whom $(h_{i+1})^k = y_i$. Define

(4) $$p^{ji}(h_{i+1}|h_i,s,t) = \Pi_{k \in J^{ji}} q^{ki}(s,h_i,t) . \Pi_{\substack{k \notin J^{ji} \\ k \neq j}} [1-q^{ki}(s,h_i,t)]$$

where $p^{ji}(h_{i+1}|h_i,s,t)$ or $p^{ji}(h_{i+1}|h_i)$ for short, is (ji)'s probability of reaching the history h_{i+1} given the current history h_i and the strategy s. We also denote the history h_{i+1} where all $k \neq j$ announced zero by $(h_i,0)$.

2.7. Consequences. Assume that $\bar{A}$ is in L, i.e., the object being auctioned is a lottery ticket. Insofar as player (ji) is concerned, the outcome of the game as a function of his own decisions may be determined as follows: Given h_i, if he announced 0 -- that is, if he withdraws from the game -- then the outcome is δ_w in L where, for all $x \in R$, δ_x denotes the degenerate lottery that assigns the entire probability mass to x (i.e., δ_x is the element of L defined by $\delta_x(z) = 0$ for $z < x$, and $\delta_x(z) = 1$ otherwise.) If he announces 1, that is, if he remains in the auction, then either (a) the auction terminates and he wins $(A - y_i)$, (where, for all F in L, $(F - x)$ is defined by $(F - x)(t) = F(x + t)$, and $A = (\bar{A}+w)$) or (b) the auction does not terminate at this stage, in which case the outcome is a lottery with prizes $\{\delta_w, (A - y_{i+1}),\ldots,(A - y_n)\}$ with probabilities that reflect the decisions of the subsequent players.

At each decision point the **consequences** of a decision to a player at that point is a lottery in L. The consequences to player (ji) of type t of his decision α at y_i, for any history $h_i \in H_i$, and any strategies $s \in S$, is given by the function G^{ji}: $[0,1] \times T \times H_i \times S \to L$, defined for $j = 1,\ldots,J$, $i = 1,\ldots,n+1$ as follows:

$$G^{ji}(\alpha,t,h_i,s) = \delta_w \text{ if } h_i \notin H_i^j, \text{ otherwise}$$

$$G^{ji}(\alpha,t,(h_{i-1},0),s) = A-y_{i-1}$$

$$(5) \qquad G^{ji}(\alpha,t,h_i,s) = (1-\alpha)\delta_w + \alpha \sum_{h_{i+1}\in H_{i+1}} p^{ji}(h_{i+1}|h_i)G^{j(i+1)}(s^{j(i+1)}(t,h_{i+1}), t,h_{i+1},s)$$

and, if $h_{n+1} \neq (h_n, 0)$,

$$G^{j(n+1)}(\alpha,t,h_{n+1},s) = \delta_w.$$

The first equation indicates that if j is not eligible at y_i then he cannot win the lottery A. The second says that if j is the only remaining bidder, he wins the lottery A for the price y_{i-1}. The third equation specifies the general form of the lottery in which (ji) participates in case he chooses to play α. Finally, the last equation specifies the resolution of the game if none of the agents representing bidder j withdraws from the auction. The outcome in this case is that bidder j is awarded the auctioned object for the price y_n if no other bidder is in the game and δ_w otherwise.

2.8. **Payoffs**. To complete the description of the game, we define the payoffs. Let V^t be a real-valued functional representation of the preferences of type t, then, given the strategies of the other players, the history h_i and the decision α, the **payoff to the (ji) player of type t** is $V^t(G^{ji}(\alpha,t,h_i,s))$.

2.9. **The Game**. We are now in a position to define the ascending bid auction game, Γ, formally as follows:

$$\Gamma = [N, (T)_{j\epsilon J}, (S^{ji})_{ji\epsilon N}, (V^t)_{t\epsilon T}, \mu).$$

3. Bayesian Equilibrium with Behaviorally Consistent Bidders

3.1. **Behavioral Consistency**. A behaviorally consistent bidder is rational in the sense that he never deceives himself by choosing a course of action from which he knows in advance he will deviate when the time to implement this course of action comes. In other words, a behaviorally consistent bidder j of type t is a set of players $\{(ji)\}_{i=1}^n$ that choose their moves optimally, given their type and assuming that the other players in the set do the same. Formally, define a **behavior for bidder j of type t**, b^j, to be an element of S^j restricted to t in T.

Definition 1: A behavior b^j of bidder j of type t is **consistent** if, for $i = 1,\dots,n$, for all $h_i \in H_i$, and for all $\alpha \in [0,1]$, $V^t(G^{ji}(b^{ji},t,h_i,s)) \geq V^t(G^{ji}(\alpha,t,h_i,s))$.

We denote by $B^j(t)$ the set of all consistent behaviors of bidder j of type t. A **behaviorally consistent bidder** is a set of players that make their decisions optimally assuming that the other players in the set do the same.

3.2. Betweenness and Revelation. The **value** of a lottery A to a bidder of type t is given by the function $v^t\colon L \to R$ defined by $V^t((A-v^t)) = V^t(\delta_w)$. If V^t is linear in the probabilities, then a consistent, value-revealing, behavior constitutes a dominant strategy for bidders of type t. In general, behavioral consistency does not imply revelation in the sense that $b^{ji} = 1$ for i such that $y_i < v^t(A)$ and $b^{ji} = 0$ otherwise. There is, however, a class of preference relations in Ω for which behavioral consistency implies revelation in this sense regardless of the strategies of the other players. These preferences are characterized by a property called betweenness. A preference relation in Ω satisfies **betweenness** if and only if for all F and H in L such that F is strictly preferred to H, F is strictly preferred to the mixture $\alpha F + (1-\alpha)H$ and this mixture is strictly preferred to H, for all $\alpha \in (0,1)$.[4]

Theorem 1: If V^t satisfies betweenness, then there exists b^j in $B^j(t)$ such that $b^{ji} = 0$ for $y_i \geq v^t(A)$ and $b^{ji} = 1$ for $y_i < v^t(A)$.

Proof: The proof consists of two parts. In the first part, we show that $b^{ji} = 0$ for all i such that $y_i \geq v^t(A)$. Then, using this result, we show that $b^{ji} = 1$ for all i such that $y_i < v^t(A)$.

(a) Consider $y_i \geq v^t(A)$ and any $h_i \in H_i^j$. Let

$$L_1 = \delta_w = G^{ji}(0,t,h_i,s)$$

and

$$L_2 = p^{ji}(h_i,0 \mid h_i)(A-y_i) + (1-p^{ji}(h_i,0 \mid h_i))Q = G^{ji}(1,t,h_i,s)$$

where Q is a convex combination of δ_w and $(A-y_{i+k})$ $k = 1,\ldots,n-i$. Thus, L_2 is a convex combination of δ_w and $(A-y_{i+k})$, $k = 0,1,\ldots,n-i$, and $V^t(\delta_w) \geq V^t(A-y_{i+k})$. Hence, by betweenness, $V^t(L_2) \leq V^t(\delta_w) = V^t(L_1)$ and, again by betweenness, for all $\alpha \in [0,1]$, $V^t(L_1) \geq V^t((1-\alpha)L_1 + \alpha L_2)$. Hence, $\alpha = 0$ is a dominant strategy and $b^{ji} = 0$

(b) For any $h_i \in H_i^j$ suppose that $y_i < v^t(A)$ and b^j is followed at $y_k \geq v^t(A)$. Then, L_2 defined above is a convex combination of δ_w and $A-y_{i+k}$ where $y_{i+k} \leq v^t(A)$. By betweenness, $V^t(L_2) \geq V^t(\delta_w) = V^t(L_1)$. Thus, for all $\alpha \in [0,1]$, $V^t(L_2) \geq V^t((1-\alpha)L_1+\alpha L_2)$ and $\alpha = 1$ is a dominant strategy. Hence $b^{ji} = 1$. □

3.3 Equilibrium. The equilibrium concept that we use to analyze the game described in section 2 is a Bayesian-Nash equilibrium among the agents.

Definition 2: s in S is a **Bayesian-Nash Equilibrium** for Γ if for all (ji) in N, for all t in T, and for all h_i in H_i: for all $\alpha \in [0,1]$,

$$V^t(G^{ji}(s^{ji}(t,h_i),\ t,h_i,s) \geq V^t(G^{ji}(\alpha,t,h_i,s)).$$

Note that this definition implies that in equilibrium bidders employ behaviorally consistent strategies.

3.4 Existence. A sufficient condition for the existence of a Bayesian-Nash equilibrium in Γ is that the preferences of the bidders are quasi-concave.

Theorem 2: If for all $t \in T$ V^t is quasi-concave then Γ has a Bayesian-Nash equilibrium.

Proof: Follows directly from Friedman (1971), Proposition 1.

□

3.5. Characterization. If the set of types consists of preference relations that satisfy betweenness then an immediate implication of Theorems 1 and 2 is that there exists an equilibrium of the ascending bid auction game Γ in agent-dominant, behaviorally-consistent strategies. Furthermore, the equilibrium has the revelation property that each bidder is in the auction as long as the price is short of his personal evaluation of the object and he withdraws from the auction at the point when the price exceeds his value. Consequently, the outcome of the auction is Pareto optimal in the sense that the bidder who values the object most is the winner.

To characterize an equilibrium for the case where quasi-concave preferences are admitted, we introduce the following additional notation. Let $u_w^t: R \to R$ be the local utility function of V^t at δ_w. The existence and continuity of u_w^t follow from the Gateaux differentiability of V^t. The assumption that V^t satisfies first-order stochastic dominance implies that

u_w^t is monotonic increasing. Let $\bar{V}^t$: $L \to R$ be defined as $\bar{V}^t(F) = \int u_w^t(x)dF(x)$ for all F in L, and denote by v_w^t the function v_w^t: $L \to R$ defined by $\bar{V}^t((A-v_w^t)) = \bar{V}_w^t(\delta_w)$. Thus, $v_w^t(A)$ is the value of A for the preferences induced by the linear approximation of V^t at δ_w. We shall show that for bidders with quasi-concave preferences there exists an equilibrium in behaviorally-consistent strategies such that $s^{ji}(t,h_i) = 0$ if $y_i \geq v_w^t(A)$ and $s^{ji}(t,h_i)$ belongs to $(0,1]$ for $y_i < v_w^t(A)$.

Theorem 3: If V^t is quasi-concave and Y includes $v_w^t(A)$ then: (a) For any $s \in S$ the agent-optimal strategy $s^j \in S^j$ of bidders j of type t requires that $s^{ji} = 0$ for all i such that $y_i \geq v_w^t(A)$ and, (b) if s is an equilibrium then s^{ji} belongs to $(0,1]$ for all $y_i < v_w^t(A)$.

Proof: (a) Consider $y_i \geq v_w^t(A)$ and any history $h_i \in H_i$. Let L_1 and L_2 be defined as in the proof of theorem 1. L_2 is a convex combination of δ_w and of lotteries $(A-y_{i+k})$ $k = 0,1,\ldots,n-i$ that are stochastically dominated by $(A-v_w^t(A))$. Thus, $\bar{V}^t(\delta_w) \geq \bar{V}^t((A-y_{i+k}))$ and, by the linearity of $\bar{V}^t$, for all $\alpha \in [0,1]$ $\bar{V}^t(\delta_w) \geq \bar{V}^t((1-\alpha)L_1 + \alpha L_2)$. We shall show that this implies that $V^t(\delta_w) \geq V^t((1-\alpha)L_1 + \alpha L_2)$ for all $\alpha \in [0,1]$. Suppose, by way of negation, that there exists $\alpha \in (0,1]$ such that $V^t(\delta_w) < V^t((1-\alpha)L_1 + \alpha L_2)$. Then, by quasi-concavity of V^t, for all $r \in (0,1]$ $V^t(\delta_w) < V^t(L_1 + r[(1-\alpha)L_1 + \alpha L_2 - L_1]) = V^t(\delta_w + r[(1-\alpha)L_1 + \alpha L_2 - \delta_w])$. Thus, at δ_w, the Gateaux derivative of V^t in the direction $[(1-\alpha)L_1 + \alpha L_2 - \delta_w]$ is positive. By the Gateaux differentiability of V^t this implies

$$\int u_w^t(x)d[(1-\alpha)L_1 + \alpha L_2 - \delta_w](x) > 0,$$

and, equivalently, $\bar{V}^t(\delta_w) < \bar{V}^t((1-\alpha(L_1 + \alpha L_2)$, a contradiction. Hence, for all $\alpha \in (0,1]$ $V^t(L_1) = V^t(\delta_w) \geq V^t((1-\alpha)L_1 + \alpha L_2)$, and a behaviorally consistent strategy s^j may be chosen such that $s^{ji} = 0$ for $y_i \geq v_w^t(A)$.

(b) Consider $y_i < v_w^t(A)$ and any history $h_i \in H_i$. Let s^j be such that $s^{ji} = 0$ for all $y_i \geq v_w^t(A)$. In this case L_2 is a convex combination of δ_w and of lotteries $(A-y_k)$, $y_i \leq y_k \leq v_w^t(A)$, which dominate $(A-v_w^t(A))$ according to first-order stochastic dominance. Following the above procedure, it can be shown that $\bar{V}^t(L_2) \geq \bar{V}^t(\delta_w) = \bar{V}^t(L_1)$. Notice that here we may have $V^t(L_2) < V^t(\delta_w)$. However, if one of the lotteries $(A-y_k)$, $y_i \leq y_k < v_w^t(A)$ appears in L_2 with strictly positive probability, then $\bar{V}^t(L_2) > \bar{V}^t(\delta_w)$, otherwise $L_2 = \delta_w$. In the latter case for all $\alpha \in [0,1]$ $V^t(L_2) \geq V^t((1-\alpha)L_2 + \alpha L_1)$ and $s^{ji}(t,h_i)$ may be chosen to be equal to 1. In the former case we get $\int u_w^t(x)d[L_2 - \delta_w](x) > 0$. Hence, the Gateaux derivative of V^t at δ_w in the direction L_2 is positive. Thus, there exists $\alpha \in (0,1]$ such that $V^t((1-\alpha)L_1 + \alpha L_2) > V^t(L_1)$ and $s^{ji}(t,h_i)$, which is equal to the best of those α's, is positive. □

Notice that with strictly quasi-concave preferences the equilibrium strategies are not value-revealing and the outcome of the game is not necessarily Pareto optimal in the sense described above.

Finally, we are in a position to define M. Let $M = \max\{v_w^t(A) \mid t \in T\}$. Because M is defined by the set of types, the object A and by δ_w, all of which are independent of the definition of the game, it is clear that no circularity was introduced into the definition of M by postponing it until now. Furthermore, from the definition of M it follows that for all y_i in Y the prior probability that, in equilibrium, the auction will reach the point y_i is strictly positive. Thus, the beliefs in equation (2) and the Bayesian-Nash equilibrium are well defined.

4. Concluding Remarks

One aspect of the auction game of section 2, namely the rule that if all the eligible participants withdraw from the auction at the same point then nobody gets the object, is not observed in actual auctions. Yet, this rule is responsible for the great simplification of the analysis and for many of the results, including the characterization of equilibrium. (For comparison see Karni and Safra (1987) where a tie in the auction is resolved by a chance mechanism). Thus, the real issue is not the realism of this rule but to what extent this model and the results presented here constitute a reasonable approximation to actual situations. To answer this question we need to understand the source of difficulty with ties. Any other form of resolution of ties implies that a decision to withdraw from the auction will have a lottery as a consequence and the exact nature of this lottery depends on the number of bidders that are in the game at that decision point. Thus, the analysis cannot be anchored to δ_w. The critical issue is, therefore, how important are ties in actual auctions. If ties are not important for individual decisions, then no essential loss is entailed and significant simplification is gained by assuming that ties are resolved according to our rule. Situations in which ties are not important include the case in which the participants in the auction are drawn from a distribution with positive density everywhere on the set of admissible preference relations Ω, and the partition Y is sufficiently refined. In this case the probability of an actual tie occurring at any given decision point is sufficiently small and a withdrawal from the auction yields δ_w almost surely. In this case the ascending bid auction described in section 2 is a good approximation.

FOOTNOTES

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1. In the case of decision making under risk, these theories include: Kahneman & Tversky (1979), Machina (1982), Quiggin (1982), Fishburn (1983), Chew and MacCrimmon (1979), Chew (1981), (1983), Yaari (1987), and Dekel (1986). For the case of uncertainty there is Schmeidler (1984) and its extension in Gilboa (1985).

2. Because of the structure of the games considered below the criticism of Strotz by Peleg and Yaari (1973) does not apply. For the definition of a sequential equilibrium, see Kreps & Wilson (1982).

3. M will be defined in the sequel.

4. The functional representation of a preference relation in Ω that satisfies the betweenness property is both quasi-concave and quasi-convex on L. For axiomatizations of preference relations in this class see Fishburn (1983), Chew and MacCrimmon (1979), Chew (1981), and Dekel (1986).

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