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REVIEW ARTICLE

Ashtekar formulation of general relativity and loop-space non-perturbative quantum gravity: a report

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Abstract. The formulation of general relativity discovered by Ashtekar and the recent results obtained in non-perturbative quantum gravity using loop-space techniques are reviewed. The new formulation is based on the choice of a set of Lagrangian (and Hamiltonian) variables, instead of the spacetime metric. In terms of these new variables, the dynamical equations are remarkably simplified and a structural identity between general relativity and the Yang–Mills theories is revealed. The formalism has proven to be useful in numerous problems in gravitational physics. In quantum gravity, the new formalism has overcome long-standing difficulties and led to unexpected results. A non-perturbative approach to quantum theory has been constructed in terms of the Wilson loops of the Ashtekar connection. This approach, denoted as loop-space representation, has led to the complete solution of the quantum diffeomorphism constraint in terms of knot states, to the discovery of an infinite-dimensional class of solutions to the quantum gravitational dynamics, and to certain surprising indications on the existence of a discrete structure of spacetime around the Planck length. These results are presented here in a compact self-contained form. The basic Ashtekar formalism is presented and its applications are outlined. The loop-space representation and the non-perturbative knot states of quantum gravity are described in detail, with particular regard to their physical interpretation and to the information they may provide on the microstructure of spacetime.

1. Classical theory

1.1. *Why new variables?*

The discordance between our basic theory of mechanics—quantum theory—and our basic theory of spacetime—general relativity—is a prime open problem in fundamental physics. In 1986 Abhay Ashtekar introduced a reformulation of general relativity in terms of a set of variables that replace the spacetime metric. In the following five years, this formalism has been used in a large number of problems in gravitational physics. Ashtekar's target was the quantum gravity issue. And, indeed, the new-variables formalism has opened a novel line of approach to this problem.

In conjunction with new techniques for dealing with non-perturbative quantum field theories—in particular, the loop representation—the new formalism has overcome long-standing difficulties in traditional approaches to quantum gravity. The new results have brought new drive to the field and have raised hopes for the solution

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of the quantum gravity puzzle. The present report is a review of the new variables formalism and of these recent developments in quantum gravity.

General relativity is characterized by its great beauty. Einstein's idea of interpreting the gravitational force as a modification of the spacetime metric geometry is so compelling that it is legitimate to ask why would we want to describe the gravitational field in terms of some other kind of variable than the spacetime metric. In the Ashtekar reformulation, general relativity is more similar to the rest of theoretical physics than in the old formulation; however, as I will try to make clear in this review, far from challenging the beauty of the theory, the new formalism sheds new light and reveals new aspects of it. General relativity is still capable of providing surprises and wonders.

This review appears five years on from Ashtekar's introduction of the new variables. The formalism is now quite settled, but the applications are still emerging. This review is far from being definitive; rather, it represents a snapshot of the present state of the art of the research. As far as the basic new formalism is concerned, this review contains a synthetic, but complete, description of the theory (section 1). Applications in classical physics are briefly outlined and the main results are mentioned, at least as far as I understand them. A few applications, arbitrarily chosen on the grounds of taste and of their relevance for the quantization, are described in a little more detail (section 2).

The new approach to non-perturbative quantum gravity is described in detail. The focus is more on ideas than on technicalities. My aim is to present a coherent overall view of the work done so far; the reader should refer to the specific papers for the technical details. The main problem is how to define a quantum field theory in the absence of a background metric geometry. The loop representation is an approach to the solution of this problem. This representation is first introduced in the context of well known theories (section 3). The main results in quantum gravity are then described, including recent (unpublished) results on the physical interpretation of the exact solutions to the quantum gravitational equations—the knot states—and on the emergence of a discrete structure at the Planck scale (section 4).

There are two books related to the subject of the present report. The first [1] was published shortly after the introduction of the new formalism. It contains a didactic introduction to the new ideas, and very useful background material. The second book [2] is based on a series of lectures that Ashtekar gave in Poona, India, in July–August 1989. There is a certain overlap with the present paper; the present paper is much more compact, is written in a language more oriented to a standard physics audience, and is based on a different perspective. It also includes certain recent developments. The book contains more details, including all the demonstrations, and develops topics just touched upon here. As far as the main results are concerned, the present review paper is complete and essentially self-contained.

1.2. Lagrangian theory

General relativity can be reformulated in terms of two fields: a (real) tetrad field e^I_μ, and a complex connection ${}^4A^{IJ}_\mu$. Here the indices $\mu, \nu, \ldots$ are spacetime indices and run from 0 to 3, and the indices $I, J, K, \ldots$ are internal indices, which also run from 0 to 3 and are raised and lowered with the Minkowski metric $\eta^{IJ} = [-1, 1, 1, 1]$. The connection ${}^4A^{IJ}_\mu$ is defined to be self-dual, namely to satisfy

$$ {}^4A^{MN}_\mu = -\tfrac{1}{2}\mathrm{i}\epsilon^{MN}{}_{IJ}\ {}^4A^{IJ}_\mu \qquad (1.1)$$

where $\epsilon^{MN}{}_{IJ}$ is the completely antisymmetric tensor. The action is

$$S[e, {}^4A] = \int \mathrm{d}^4x \; e_{\mu I} e_{\nu J} \; {}^4F^{IJ}_{\tau\sigma} \; \epsilon^{\mu\nu\tau\sigma} \tag{1.2}$$

where ${}^4F^{IJ}_{\mu\nu}$ is the Yang–Mills field strength of ${}^4A^{IJ}_{\mu}$:

$${}^4F^{IJ}_{\mu\nu} = \partial_\mu {}^4A^{IJ}_{\nu} - \partial_\nu {}^4A^{IJ}_{\mu} + {}^4A^{IM}_{\mu} \; {}^4A_{\nu M}{}^{J} - {}^4A^{IM}_{\nu} \; {}^4A_{\mu M}{}^{J}. \tag{1.3}$$

The equations of motion that follow from the action (1.2) are

$$\epsilon^{\mu\nu\rho\sigma} e_{\nu J} F^{IJ}_{\rho\sigma} = 0 \tag{1.4}$$

$$(\delta^{KI}\delta^{LJ} + \tfrac{1}{2}\mathrm{i}\epsilon^{KLIJ})\epsilon^{\mu\nu\rho\sigma}\mathcal{D}_\rho(e_{\mu I} e_{\nu J}) = 0 \tag{1.5}$$

where $\mathcal{D}_\rho$ is the covariant derivative defined by ${}^4A$.

The following theorem then holds.

Theorem. If $(e^I_\mu(x), {}^4A^{IJ}_\mu(x))$ satisfy the equations of motion of the theory (1.4) and (1.5), the metric

$$g_{\mu\nu}(x) = e^I_\mu(x) \; e^J_\nu(x) \; \eta_{IJ} \tag{1.6}$$

is a solution of the vacuum Einstein equations. Similarly, every solution of the vacuum Einstein equations can be written in terms of the solution of the theory (1.2) as in (1.6).

This is the Ashtekar theory in the Lagrangian formalism.

Some comments follow.

(i) The advantages of this formulation, and the simplification that this formulation brings, will become clear. In particular, it is the Hamiltonian theory that descends from the action (1.2) that will make results especially simple.

(ii) By emphasizing the role of the connection ${}^4A^{IJ}_\mu$ over the role of the metric, the similarity between general relativity and Yang–Mills theories is underlined. Again, it is in the canonical theory that this will become more evident.

(iii) A peculiar feature of this formulation is the use of complex numbers. Complex numbers enter in this formulation in two distinct way. First, the variables are complex; second, the action is complex. The puzzling aspect is the second. A complex action is (to my knowledge) a novel feature in mechanics. As far as the Lagrangian formalism is concerned, it will be shown in the next section that the imaginary part of the action (1.2) has no effect on the equations of motion. Thus the imaginary part of the action is harmless. However, this imaginary term affects the canonical framework. In constructing the canonical formalism it will be necessary to deal with the fact that action is complex. This problem will be discussed in detail in section 1.4 and in appendix A.

(iv) Ashtekar introduced the new formalism [3] in the canonical framework and using spinors. The bridge between the spinorial formalism and the one used here is straightforward; it will be given in section 1.4. The Lagrangian formulation was constructed by Samuel, and by Jacobson and Smolin [4]. Several slightly different action formulations, all leading to the same canonical theory, have then appeared. Among these, there is quite an interesting action written purely in terms of a connection (without tetrads fields) introduced by Capovilla *et al* [5].

(v) In this paper, space is assumed to be compact. Therefore boundary terms are systematically disregarded. For an analysis of the field fall-off conditions required for a consistent definition of the theory in the open-space case, see [2].

1.3. Relation with the Einstein formulation and the geometrical meaning of the new variables

In this section, the relation between the new variables formalism and the standard formulation of general relativity is constructed. I start from the metric formulation, and construct the transformation to the new variables through two intermediate steps, both of which are well known. The first is the Palatini, or first-order, form of the theory; the second is the use of tetrads. The relation to the standard formalism proves the theorem of the previous section and elucidates the geometrical meaning of the Ashtekar variables ${}^4A_\mu^{IJ}$ and e_μ^I.

1.3.1. Palatini and tetrads formalism. As Palatini realized, general relativity admits a first-order formulation: it is possible to take the Einstein–Hilbert action

$$S[g] = \int \mathrm{d}^4x \, \sqrt{g} g^{\mu\nu} R_{\mu\nu}[g] \tag{1.7}$$

and consider the metric and the affine connection $\Gamma^\sigma_{\mu\nu}$ as independent variables:

$$S[g, \Gamma] = \int \mathrm{d}^4x \, \sqrt{g} g^{\mu\nu} R_{\mu\nu}[\Gamma]. \tag{1.8}$$

By varying Γ it follows that $\Gamma = \Gamma[g]$, namely an equation that fixes Γ as the metric affine connection defined by g (the Christoffel symbol). By varying g we get the vacuum Einstein equations.

The tetrad formalism consists in substituting the metric with four linearly independent covariant vector fields e_μ^I, related to the metric by equation (1.6). They define the $so(3,1)$ connection $\omega_\mu^{IJ}[e]$, usually denoted as the spin connection, by

$$\partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu J}[e] e^J_{\nu]} = 0 \tag{1.9}$$

(this is the second Cartan structure equation). The Einstein–Hilbert action can be rewritten in terms of the tetrads:

$$S[e] = \int \mathrm{d}^4x \; e \; e^\mu_I e^{\nu I} \; R_{\mu\nu}[g[e]] \tag{1.10}$$

where e is the determinant and e^μ_I the inverse of the matrix e^I_μ. The Riemann curvature is related to the curvature of the spin connection by

$$R^\mu_{\nu\tau\sigma}[g[e]] = e^\mu_I e_{\nu J} \; R^{IJ}_{\tau\sigma}[\omega[e]] \tag{1.11}$$

so that

$$S[e] = \int \mathrm{d}^4x \; e \; e^\mu_I e^\nu_J \; R^{IJ}_{\mu\nu}[\omega[e]]. \tag{1.12}$$

The first-order formalism, *á la* Palatini, and the use of the tetrads can be combined: tetrads e and spin connection ω can be considered as independent variables. The action

$$S[e, \omega] = \int \mathrm{d}^4x \; e \; e^\mu_I e^\nu_J \; R^{IJ}_{\mu\nu}[\omega] \tag{1.13}$$

is again equivalent to the Einstein–Hilbert action. The equations of motion for ω are $\omega = \omega[e]$; namely ω is fixed to be the spin connection defined by e (via equation (1.9)). The equations for e are the vacuum Einstein equations.

Geometrically, we are dealing with a vector Lorentz bundle over the spacetime manifold. The I, J indices denote the vector components in the Lorentz fibre. ω is a connection on the bundle, and e is a soldering form, namely a one-to-one mapping between the fibre and the tangent space, at every point of the spacetime manifold. These formulations are well known; in the next section I define the transformation to the Ashtekar formulation.

1.3.2. The transformation to the new variables. The integral

$$T[e,\omega] = \int \mathrm{d}^4x \; e \; e^\mu_I e^\nu_J \; \epsilon^{IJ}{}_{MN} \; R^{MN}_{\mu\nu}[\omega] \tag{1.14}$$

can be added to the action (1.13) without affecting the equations of motion. This follows from the fact that the integral $T[e] = T[e,\omega[e]]$ is a topological term, i.e. is invariant under local variations of e.

Let us subtract the integral $T[e,\omega]$, multiplied by one half the imaginary unit, to $S[e,\omega]$:

$$S'[e,\omega] = S[e,\omega] - \tfrac{1}{2}\mathrm{i}T[e,\omega] = \int \mathrm{d}^4x \; e \; e^\mu_I e^\nu_J \; \left(R^{IJ}_{\mu\nu}[\omega] - \tfrac{1}{2}\mathrm{i}\epsilon^{IJ}{}_{MN} R^{MN}_{\mu\nu}[\omega]\right). \tag{1.15}$$

The imaginary term has no effect on the equations of motion, therefore $S'[e,\omega]$ is another good action for general relativity. Now, given the spin connection ω, consider the complex quantity

$${}^4A^{IJ}_\mu[\omega] = \omega^{IJ}_\mu - \tfrac{1}{2}\mathrm{i}\,\epsilon^{IJ}{}_{MN}\,\omega^{MN}_\mu \tag{1.16}$$

which is denoted as the self-dual spin connection. The curvature of this self-dual spin connection ${}^4A$ will be denoted ${}^4F^{IJ}_{\mu\nu}$. The key observation is that this curvature is related to the curvature of ω simply by

$$F^{IJ}_{\mu\nu}[{}^4A[\omega]] = R^{IJ}_{\mu\nu}[\omega] - \tfrac{1}{2}\mathrm{i}\epsilon^{IJ}{}_{MN} R^{MN}_{\mu\nu}[\omega]. \tag{1.17}$$

This means that the curvature of the self-dual spin connection is the self-dual part of the curvature. Now, this self-dual part of the curvature is precisely the term in parenthesis in (1.15). So the action can be rewritten as

$$S'[e,\omega] = \int \mathrm{d}^4x \; e e^\mu_I e^\nu_J \; F^{IJ}_{\mu\nu}[{}^4A[\omega]] = \int \mathrm{d}^4x \; e_{\mu I} e_{\nu J} \; F^{IJ}_{\sigma\tau}[{}^4A[\omega]] \; \epsilon^{\mu\nu\sigma\tau}. \tag{1.18}$$

Finally, consider a change of Lagrangian variables from (e,ω) to $(e,{}^4A)$. The action becomes

$$S[e,{}^4A] = \int \mathrm{d}^4x \; e_{\mu I} e_{\nu J} \; {}^4F^{IJ}_{\tau\sigma}[{}^4A] \; \epsilon^{\mu\nu\tau\sigma} \tag{1.19}$$

which is precisely the action (1.2). The new Lagrangian variable ${}^4A$ is a complex variable, but it is not an arbitrary complex number, because it has to be related to a real ω by equation (1.16). It is straightforward to see that this requirement is equivalent to the self-duality condition equation (1.1).

General relativity has been re-expressed in terms of a self-dual connection ${}^4A$ and a tetrad field e, with action (1.2): in this way, the Lagrangian Ashtekar formalism is recovered, and its equivalence with general relativity is demonstrated.

1.3.3. Geometry. The above derivation displays the geometrical meaning of the (four-dimensional) Ashtekar connection: ${}^4A$ is the self-dual part of the spin connection. In fact, the second equation of motion (1.5) fixes ${}^4A$ to be the self-dual part of the spin-connection defined by e, namely it is equivalent to

$$ {}^4A^{IJ}_\mu[e] = \omega^{IJ}_\mu[e] - \tfrac{1}{2}\mathrm{i}\epsilon^{IJ}{}_{MN}\,\omega^{MN}_\mu[e]. \tag{1.20}$$

where $\omega^{MN}_\mu[e]$ is defined in equation (1.9). By inserting ${}^4A^{IJ}_\mu[e]$ in the other equation of motion (1.4), it follows that the Ricci tensor vanishes.

From the geometrical point of view, the formalism exploits the fact that the complexified Lorentz algebra $so(3,1;C)$ (or $so(4,C)$) is the direct sum of two complex $so(3,C)$ algebras (its self-dual and antiself-dual part). In other words, the generators X^{IJ} of the Lorentz group have the property that the two sets $W^{IJ} = X^{IJ} + \mathrm{i}\epsilon^{IJ}_{MN}X^{MN}$ and $Z^{IJ} = X^{IJ} - \mathrm{i}\epsilon^{IJ}_{MN}X^{MN}$ commute one with the other. Thus, considering a $so(3,1;C)$, rather than $so(3,1)$, vector bundle over the spacetime, the connection splits into two independent components, the self-dual and the antiself-dual components. The fact that they are independent is the reason for the fact that the self-dual part of the curvature is the curvature of the self-dual connection (1.17).

The Ashtekar formalism is based on the interplay between two facts. The first is that there is a self-dual connection ${}^4A$ for every real connection ω (equation (1.16)). Thus, the self-dual connection rather than the real connection can be used as a Lagrangian variable. The second is that it is possible to directly substitute in the action (1.13) the real curvature with the self-dual curvature because the difference is the topological integral $T[e,\omega]$.

Finally, a note on self-duality may be useful to avoid misunderstandings. The self-duality considered here is the self-duality with respect to the internal indices. This should not be confused with a different notion of self-duality. There is an independent notion of self-duality on the spacetime indices: it is possible to define the spacetime self-dual (and antiself-dual) part of the curvature tensor $R^{IJ}_{\mu\nu}$ by

$$ R^{\pm}{}^{IJ}_{\mu\nu}[\omega] = R^{IJ}_{\mu\nu}[\omega] \pm \tfrac{1}{2}\mathrm{i}\epsilon_{\mu\nu}{}^{\sigma\tau}R^{IJ}_{\sigma\tau}[\omega]. \tag{1.21}$$

For a generic complex spin connection, the two notions of self-duality are independent. By considering both notions, the curvature splits into four components: $R^+[{}^4A]$, $R^-[{}^4A]$, $R^+[{}^4\bar{A}]$, $R^-[{}^4\bar{A}]$. Here $R^+[{}^4A]$ is the spacetime self-dual part of the curvature of the self-dual spin connection, and so on†.

1.4. Canonical theory

The action (1.2) is complex. The standard Hamiltonian formalism has to be extended in order to deal with complex actions. Since to my knowledge this extension has

† By using these definitions, it is easy to prove that the vacuum Einstein equations are equivalent to the requirement that the internal-self-dual spacetime-antiself-dual component of the connection vanishes: $R^-[{}^4A] = 0$. This equation is equal to the self-dual Yang–Mills equation for the connection ${}^4A$. Therefore the four-dimensional Ashtekar connection satisfies the self-dual Yang–Mills equations. General relativity is equivalent to a self-dual Yang–Mills theory defined on a curved background (the gauge algebra being the self-dual Lorentz algebra), plus the requirement that there is relation between the curved background and the Yang–Mills potential (the relation is the following: the Yang–Mills potential is equal to the self-dual spin connection). This curious interpretation of general relativity is, for instance, at the root of the lightcone-cuts formulation of the theory recently developed by Kozameh and Newman [6].

never been studied, in appendix A I develop the Hamiltonian framework for theories with complex action in general. The conclusion of appendix A is that the equations of the real Hamiltonian formalism are still valid as complex equations, but certain caveats have to be considered. The reader not interested in this technical aspect of the problem may follow the derivation given here, and refer to the appendix only in case of confusion.

To construct the Hamiltonian theory, one may work in a given coordinate system, and develop the Hamiltonian formalism in the $\partial/\partial x^0$ direction. Alternatively, one may work in a coordinate independent formalism and develop the Hamiltonian formalism along an arbitrary vector field $\boldsymbol{n}$. This second alternative is more rigorous, more elegant, and is used by Ashtekar in [1, 2], but the two formalisms are equivalent. Here I use a coordinate formalism.

An important remark on general covariance is the following. It is often stated that the Hamiltonian formalism breaks general covariance because space and time are treated in a different fashion. This is not correct. Only *explicit* general covariance is broken in the Hamiltonian formalism. It may be shown that the phase space can be identified as the space of the solutions of the Euler–Lagrange equations, and all the structures over the phase space admit an explicitly general covariant formulation (see for instance [7]). Once more, however, I am not interested here in fancy formulations that make general covariance manifest, and I use the standard form of the Hamiltonian theory.

Finally, I should add for clarity that the derivation of the canonical theory given here is different from derivations given elsewhere.

1.4.1. The Legendre transform to the canonical theory. In order to construct the canonical theory it is convenient to go to a second-order formalism. By inserting the solution (1.20) of the equation of motion (1.5) back into the action, we obtain an equivalent form of the action (1.2), where the only independent variable is the tetrad

$$S[e] = \int \mathrm{d}^4x \; e_{\mu I} e_{\nu J} \; {}^4F^{IJ}_{\tau\sigma}[{}^4A[e]] \; \epsilon^{\mu\nu\tau\sigma}. \tag{1.22}$$

This will be the starting point for the canonical theory.

Let me begin by introducing space indices $a = 1, 2, 3$ and by splitting, *á la* ADM, the Lagrangian variables e as follows:

$$N^a = e^a_I e^{0I} \tag{1.23}$$

$$N = \frac{e}{\sqrt{q}} \tag{1.24}$$

$$E^a_I = e^a_I - N^a e^0_I. \tag{1.25}$$

In the second equation, q is the determinant of the 3-metric $q_{ab} = g_{ab}[e]$. It is useful to introduce densitized triads

$$\tilde{E}^a_I = \sqrt{q} E^a_I \tag{1.26}$$

since they will be the natural canonical variables. Following a standard notation, I put n tildes over the quantities that transform as densities of weight n. One more structure is needed. I define the antisymmetric 3-indices tensor

$$\epsilon^{IJK} = e^0_L \epsilon^{LIJK} \tag{1.27}$$

in terms of which the 1-index connection

$$A_a^I[e] = \epsilon^I_{JK}\, {}^4A_a^{JK}[e] \tag{1.28}$$

is defined. It is easy to show (using the self-duality of the curvature) that in terms of these variables the action (1.22) can be written as:

$$S[\tilde{E}, N, N^a, e_I^0] = \int \mathrm{d}^4x \; \mathrm{i}A_a^I[e]\dot{\tilde{E}}_I^a + \mathrm{i}A_o^I[e]D_b\tilde{E}_I^b + \mathrm{i}N^a\tilde{E}_J^bF_{ab}^J + N\tilde{E}_I^a\tilde{E}_J^bF_{ab}^{IJ} \tag{1.29}$$

where D_a is the covariant derivative and F_{ab}^I the curvature defined by $A[e]$. From this form, it is clear that the only dynamical variable is $\tilde{E}_I^a$. The other variables, namely N, N^a, e_I^0, can be taken as Lagrange multipliers and freely fixed. In particular, part of the internal gauge symmetry can be used to fix $e_i^0 = 0$ for $i = 1, 2, 3$. In this gauge, which will be used in what follows, the non-vanishing components of ϵ^{IJK} are given by the usual totally antisymmetric three-dimensional tensor ϵ^{ijk} and the non-vanishing components of $A_a^I[e]$ are $A_a^i[e]$, so that the action can be written as

$$S[\tilde{E}, N, N^a, e_I^0] = \int \mathrm{d}^4x \; \mathrm{i}A_a^i[e]\dot{\tilde{E}}_i^a + \mathrm{i}A_o^i[e]C_i + \mathrm{i}N^aC_a + NC \tag{1.30}$$

where

$$C_i = D_b\tilde{E}_i^b \tag{1.31}$$

$$C_a = \tilde{E}_i^bF_{ab}^i \tag{1.32}$$

$$C = \tilde{E}_i^a\tilde{E}_i^bF_{ab}^{ij}. \tag{1.33}$$

Now the momenta can be defined as

$$p_a^i = \frac{\partial L}{\partial \dot{\tilde{E}}_i^a} = \mathrm{i}A_a^i[e] \tag{1.34}$$

(following the formalism of appendix A, these are complex quantities). A_a^i is the three-dimensional projection of ${}^4A_\mu^{IJ}$, and is usually referred to as the Ashtekar connection. The (complex) phase space is coordinatized by $(p_a^i, \tilde{E}_i^a)$. The basic Poisson brackets are

$$\{p_a^i(x), E_j^b(y)\} = -\delta_a^b\delta_j^i\delta^3(x, y). \tag{1.35}$$

(In the appendix, the precise meaning of a 'Poisson structure' in a half-complex and half-real space is elucidated.)

The form of the action shows that the canonical Hamiltonian vanishes weakly, as always in a time-reparametrization-invariant theory (because the Hamiltonian is the generator of time translations, time translations are gauges, the generator of a gauge transformation is a first-class constraint and therefore vanishes weakly). The remaining task is to write the constraints. There are two kind of constraints. The first kind is given by the primary constraints that follow from the definition of the momenta. To find them, let us separate the real and the imaginary part of these momenta. The real part is (using the definitions (1.20) and (1.28))

$$\text{Real } p^i_a = \text{Real } [\mathrm{i}\epsilon^i_{jk} A^{jk}_a[e]] = \epsilon^i_{jk}\epsilon^{jk}_{MN}\omega^{MN}_\mu[e] = \omega^{oi}_\mu[e]. \tag{1.36}$$

Since the real part of the action is just the standard tetrad action, the real momentum turns out to be, as can be checked, just the standard momentum of the tetrad formalism, which is related to the extrinsic curvature k_{ab} (and, in turn, to the standard ADM momentum) by

$$k_{ab} = \text{Real } p^i_{(a}e^i_{b)}. \tag{1.37}$$

The imaginary part of the momentum is

$$\text{Im } p^i_a = \text{Im } [\mathrm{i}\epsilon^i_{jk} A^{jk}_a[e]] = \epsilon^i_{jk}\ \omega^{jk}_a[e]. \tag{1.38}$$

Now, from the definition (1.9) of the spin connection, it follows that $\omega^{jk}_a[e]$ does not depend on time derivatives of e; it is indeed just the three-dimensional spin connection of the three-dimensional frame field (triad) $\tilde{E}^a_i$. Thus the imaginary part of the momentum is entirely constrained. In other words we get the primary constraint

$$\text{Im } p^i_a - \epsilon^i_{jk}\omega^{jk}_\mu[\tilde{E}] = 0. \tag{1.39}$$

This had to be expected, since the imaginary part of the action is a topological term, and thus the I-theory (see appendix A) must be entirely non-dynamical. This concludes the derivation of the canonical theory, the basic formulae of which are summarized in the next section.

1.4.2. The Ashtekar theory in canonical form. According to the rules of Hamiltonian mechanics, the momentum is to be considered as an independent canonical variable. Following the standard notation, I use A^i_a rather than p^i_a, but now A^i_a is to be considered as an independent canonical variable. The phase space has coordinates $(A^i_a, \tilde{E}^a_i)$, and the basic Poisson brackets are

$$\{A^i_a(x), \tilde{E}^b_j(x)\} = \mathrm{i}\ \delta^b_a\delta^i_j\delta^3(x,y). \tag{1.40}$$

In terms of A^i_a the primary constraint (1.39) is

$$\bar{A}^i_a = A^i_a + 2\mathrm{i}\epsilon^i_{jk}\omega^{jk}_a[\tilde{E}]. \tag{1.41}$$

This constraint is denoted as the *reality condition*. As will be discussed later, in the quantization this constraint plays a different role from the other constraints.

The other constraints follow from the presence of the Lagrange multipliers (N, N^a, e^0_i). Namely (see equations (1.31)–(1.33)),

$$C_i[A,E] = 0 \qquad C_a[A,E] = 0 \qquad C[A,E] = 0. \tag{1.42}$$

(As usual, they can be more rigorously obtained as secondary constraints by treating (N, N^a, e^0_i) as genuine dynamical variables and eliminating the redundant sector of the phase space by gauge fixing.) They are denoted the gauge constraint, the vector constraint and the Hamiltonian constraint. An explicit computation shows [1] that all the constraints, including the reality condition, are first class.

Note that the constraints (1.42) are polynomial (at worst quadratic) in each of the canonical variables. The reality condition, as written in equation (1.41), is not polynomial; however, it can be equivalently [2] rewritten in polynomial form as follows:

$$\mathrm{Re}\ D_c(\tilde{E}_i^{[c}\tilde{E}_j^{(a]})\tilde{E}_k^{b)}\ \epsilon^{ijk} = 0. \tag{1.43}$$

(The parentheses on the indices indicate symmetrization in (ab) and antisymmetrization in $[ca]$.)

The phase space with canonical coordinates $\tilde{E}_i^a$ and A_a^i, with Poisson brackets (1.40) and constraints (1.43) and (1.42) defines the Ashtekar theory in canonical form.

1.4.3. Physical interpretation of the constraints. The constraint C_i is the standard Gauss-law constraint of non-Abelian gauge theories. The Poisson algebra of the C_i is the $so(3)$ current algebra

$$\{C_i(x), C_j(y)\} = \epsilon_{ij}^k C_k(x)\delta^3(x,y). \tag{1.44}$$

The physical origin of this $SO(3)$ gauge symmetry is not related to the introduction of the new variables; rather, it follows from the use of tetrads. Physically, this gauge symmetry reflects the freedom in choosing the locally Euclidean reference system defined by the triad. Indeed, in the Hamiltonian formalism derived from the tetrad action $S[e]$ there are the constraints

$$C_i' = \epsilon_{ik}^j\ e_j^a\ p_a^k \tag{1.45}$$

which generate the same $SO(3)$ invariance as the C_i. The new feature of the Ashtekar variables is that the Ashtekar constraints C_i have precisely the same form as the gauge constraints in Yang–Mills theory. This is due to the fact that the variable A_a^i transforms as a connection, precisely as the Yang–Mills potential

$$\delta_\lambda A_a^i = \{A_a^i, C(\lambda)\} = D_a\lambda^i \tag{1.46}$$

where

$$C(\lambda) = \int \mathrm{d}^3x\lambda^i(x)C_i(x). \tag{1.47}$$

Thus, the phase space of general relativity in the Ashtekar formulation has the very same structure as the phase space of a non-Abelian gauge theory.

The fact that the real part of A_a^i transforms as a connection is no surprise; after all, it is the spin connection. But the imaginary part is essentially the extrinsic curvature. How does it transform? Note that, since λ^i is real, the non-homogeneous term $\partial_a\lambda^i$ does not affect the imaginary part of A_a^i. Thus, this transforms homogeneously, as it should.

Next, consider the vector constraint C_a. It is shown in [1] that the following linear combination of vector and gauge constraints

$$C(N) = \int \mathrm{d}^3xN^a(x)(C_a(x) - A_a^i(x)C_i(x)) \tag{1.48}$$

is the generator of three-dimensional diffeomorphism transformations. Namely

$$\delta_N f(A,E) = \{f(A,E), C(\vec{N})\} = \mathcal{L}_{\vec{N}} f(A,E) \tag{1.49}$$

where $\mathcal{L}_{\vec{N}}$ is the Lie derivative along $\vec{N}$, namely the variation under the infinitesimal coordinate transformation $\vec{x} \longmapsto \vec{x} + \vec{N}(x)$. The combination (1.48) is denoted as the diffeomorphism constraint.

Finally, given any four-dimensional solution of the equations of motion, the scalar constraint

$$C(N) = \int \mathrm{d}^3x \; N(x) C(x) \tag{1.50}$$

generates the evolution in the Lagrangian parameter time [1]

$$\{f(A,E), C(N)\} = N \frac{\partial}{\partial x^0} f(A,E). \tag{1.51}$$

This constraint plays the same role as the Hamiltonian ADM constraint in the ADM canonical formulation.

It is well known that the evolution generated by the Einstein equations can be interpreted in the ADM formalism as the motion of a particle in an infinite-dimensional configuration space (superspace) with metric given by the DeWitt supermetric and potential given by the three-dimensional Ricci scalar curvature. In the Ashtekar formalism this interpretation is not only preserved, but it is also remarkably simplified. Consider the infinite-dimensional space of the connections A^i_a as the Ashtekar superspace, equipped with the (super-) metric

$$\mathrm{d}s^2 = \int \mathrm{d}^3x \int \mathrm{d}^3y \; G^{iajb}(x) \delta^3(x,y) \; \mathrm{d}A_{ia}(x) \, \mathrm{d}A_{jb}(y) \tag{1.52}$$

where $G^{iajb}(x)$ is the inverse of the (9×9) matrix

$$G_{iajb}(x) = \epsilon_{kij} \; F^k_{ab}(x). \tag{1.53}$$

The motion generated by the Einstein equations can be interpreted as the motion of a *massless particle* (without any potential) in this geometry. Thus, the problem of solving the Einstein equations is equivalent to the problem of finding the null geodesics of the metric (1.53).

1.4.4. Difference between the Einstein theory and the Ashtekar theory. The fact that the dynamical equations are polynomial has an important physical consequence. In the standard formulation of general relativity, the action contains the Ricci scalar, which is obtained by contracting the Ricci tensor with the *inverse* of the metric tensor. Similarly, the ADM Hamiltonian formulation requires the Ricci scalar of the three-dimensional metric to be defined. In order to have an inverse, the metric tensor must be non-degenerate. Thus, we are forced to restrict ourselves to non-degenerate metrics in order to define the theory.

In the Ashtekar formulation, this same requirement can be imposed. Namely it is possible to require that $\tilde{E}^a_i$ be non-degenerate. If this is required, the Ashtekar theory is equivalent to general relativity.

However, since all the equations are polynomials, and in particular the inverse of $\tilde{E}_i^a$ does not enter in the equations, the Ashtekar equations also make sense if $\tilde{E}_i^a$ is degenerate. Thus, we are not *forced* to require that the metric be non- degenerate. Therefore, general relativity admits an extension in which degenerate metrics are allowed: by dropping the requirement that $\tilde{E}_i^a$ be non-degenerate from the new variables theory, a theory is defined that has all the solutions of general relativity plus additional solutions in which the metric is degenerate.

One may take different attitudes towards this possibility. The extended theory can be seen as physically irrelevant. But it is also possible to take the theory seriously and consider the physical hypothesis that the extended theory does describe physical configurations of the gravitational field.

To provide an analogy, given the equation $\ddot{y} = \omega^2 y + 2y^{-1}\dot{y}$, we may say that this equation rapidly blows up and looses sense because y goes to infinity; but we can also recognize that in terms of the variable $x = y^{-1}$ the equation becomes $\ddot{x} = -\omega^2 x$ and is therefore very well behaved for every t: it allows us to continue $y(t)$ 'over infinity'.

It has been suggested that by using the Ashtekar equations one may continue a solution of the Einstein equations beyond certain singularities, and maybe discuss topology change in the classical theory. This possibility has still to be explored. The possibility of having a degenerate metric in the theory (more precisely, a degenerate inverse metric) is a crucial ingredient of the quantization attempts.

1.4.5. Spinorial formalism, parallel transport of fermions and Wilson loops. In the original Ashtekar papers, a spinorial formalism was used. This formalism will also be used in the rest of this paper. The translation to the spinorial formalism is straightforward. Consider the Pauli matrices divided by $\sqrt{2}$: $\tau_i{}^A{}_B$, $A, B = 1, 2$, and let

$$A_a^A{}_B = A_a^i \; \tau_i{}^A{}_B \tag{1.54}$$

$$\tilde{E}^{aA}{}_B = \tilde{E}^{ai} \; \tau_i{}^A{}_B. \tag{1.55}$$

The spinorial indices can be suppressed in most of the equations, the matrix product being understood. In this notation the constraints become

$$C_i = \mathrm{Tr}[D_a \tilde{E}^a] \tag{1.56}$$

$$C_a = \mathrm{Tr}[F_{ab} \tilde{E}^b] \tag{1.57}$$

$$C = \mathrm{Tr}[F_{ab} \tilde{E}^a \tilde{E}^b]. \tag{1.58}$$

The connection $A_a^A{}_B$ takes values in the spin-$\frac{1}{2}$ representation of the $so(3)$ Lie algebra. This connection has an interesting physical interpretation. If we want to couple spinor fields to general relativity, a tetrad formalism must be used. A spinor field lives in an internal spinor space which can be identified (under the mapping given by the gamma matrices) with the internal Lorentz space provided by the tetrad formalism. The decomposition of the Lorentz algebra into its self-dual and antiself-dual components corresponds to the invariant splitting of a 4-spinor into its left- and right-handed components.

It follows that the four-dimensional Ashtekar connection ${}^4A_\mu^{IJ}$ is the connection that defines the parallel transport of a left-handed spinor in spacetime. Since the three-dimensional connection A_a^i is the space restriction of ${}^4A_\mu^{IJ}$, the Ashtekar connection

A_a^i is the geometrical object that specifies the parallel transport for left-handed spinors ψ^A in space.

Let us write explicitly the parallel transport operator $U^A{}_B$. Let $\alpha : [0, 2\pi] \to \Sigma$ be a curve in the three-dimensional physical space Σ. Let us denote the parameter along the curve as s and the coordinates of the curve as $\alpha^a(s)$. Then, by denoting $\dot{\alpha}^a(s) = (\mathrm{d}/\mathrm{d}s)\alpha^a(s)$, it follows that

$$U_\alpha(0, 2\pi)_B^A = \left(\mathcal{P}\ \exp \int_\alpha A\right)_B^A = \left(\mathcal{P}\ \exp \int_0^{2\pi} \mathrm{d}s\ \dot{\alpha}^a(s) A_a^i(\alpha(s))\tau_i\right)_B^A. \tag{1.59}$$

Here $\mathcal{P}$ means path ordering, namely

$$\left(\mathcal{P}\ \exp \int_\alpha A\right)_B^A = 1_B^A + \int_0^{2\pi} \mathrm{d}s\ \dot{\alpha}^a(s) A_a^i(\alpha(s))\tau_{i\,B}^{\ A}$$
$$+ \int_0^{2\pi} \mathrm{d}s \int_s^{2\pi} \mathrm{d}s'\ \dot{\alpha}^a(s)\dot{\alpha}^b(s') A_a^i(\alpha(s))\tau_{i\,C}^{\ A} A_b^j(\alpha(s'))\tau_{j\,B}^{\ C} + \ \ldots. \tag{1.60}$$

Consider a *closed* curve α. Let $U_\alpha{}^A{}_A(s)$ be the matrix of the parallel transport around the loop, starting from the point s. The trace of this parallel transport operator $T[\alpha] = U_\alpha{}^A{}_A(s)$ (which is independent of the origin of the loop) is *the Wilson loop of the Ashtekar connection*. This object will play a very crucial role in the rest of this paper.

1.5. *T variables*

The similarity of the Ashtekar formalism with Yang–Mills theories has suggested bringing certain typical gauge theories techniques to general relativity. The concept of the Wilson loop has been particularly useful in the quantization. Here I define a class of observables on the $(A_a^i, \tilde{E}_i^a)$ phase space, denoted as the loop observables (introduced by Smolin and the author [8]), which will be the main tool in the second part of this review. The first of these observables is the Wilson loop of the Ashtekar connection

$$T[\alpha] = \mathrm{Tr}\ \mathcal{P}\ \exp\left(\oint_\alpha A\right). \tag{1.61}$$

The second observable also depends on the $\tilde{E}_i^a$ variables. It is defined as follows:

$$T^a[\alpha](s) = \mathrm{Tr}[U_\alpha(s)\tilde{E}^a(\alpha(s))]. \tag{1.62}$$

The T and T^a variables have a set of remarkable properties that are listed here without proof (see [8]).

(i) They contain all the gauge invariant information contained in A_a^i and $\tilde{E}_i^a$. More precisely, they are invariant under internal gauge transformations, and are good local coordinates on the phase space reduced by the gauge constraint.

(ii) There is a generalization to objects with more indices. For instance

$$T^{ab}[\alpha](s,t) = \mathrm{Tr}[U_\alpha(s,t)\ \tilde{E}_a(\alpha(t))\ U_\alpha(t, s+2\pi)\ \tilde{E}_b(\alpha(s))]. \tag{1.63}$$

($U_\alpha(t, s+2\pi)$ is the parallel transport from t to the origin and then from the origin to s).

(iii) Relevant objects of the theory can be expressed in terms of these loop observables; the expression of local objects implies a limit. For instance, the (densitized controvariant) space metric, which is related to $\tilde{E}^a_i$ by $\tilde{\tilde{q}}^{ab} = \tilde{E}^a_i \tilde{E}^b_i$ can be expressed in terms of the t observables as

$$\tilde{\tilde{q}}^{ab}(x) = -\tfrac{1}{2} \lim_{\epsilon \to 0} (T^{ab}[\alpha^\epsilon_x](0, \pi)) \tag{1.64}$$

where α^ϵ_x is a loop centred in x with (coordinate-) area ϵ.

The diffeomorphism constraint is given by

$$C_a(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \, \epsilon_{abc} \, T^b[\alpha_{x,c,\epsilon}](0) \tag{1.65}$$

where $\alpha_{x,c,\epsilon}$ is a loop centred in x with area ϵ, and lying on a surface normal to the c direction. (By normal to the c direction, I mean here the area element $dx^a dx^b \epsilon_{abc}$. The expression normal, in this context is not very appropriate, since no metric is involved.) The Hamiltonian constraint is

$$C(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{abc} \epsilon^{abc} (T^{ab}[\alpha_{x,c,\epsilon}](0, \epsilon^2)). \tag{1.66}$$

These results follow from the well known expansion

$$U_{\alpha_{x,c,\epsilon}}(s) = 1 + \epsilon \, F_{ab}(x) \epsilon^{abc} + o(\epsilon). \tag{1.67}$$

(iv) The following properties hold.

(a) Invariance under inversion:

$$T[\alpha] = T[\alpha^{-1}]. \tag{1.68}$$

(b) The so-called spinor identity, which encodes the fact that the relevant algebra is $su(2)$:

$$T[\alpha]T[\beta] = T[\alpha \# \beta] + T[\alpha \# \beta^{-1}]. \tag{1.69}$$

Here the loop $\alpha \# \beta$ is defined as follows. If α and β intersect in a point P, it is the loop obtained starting from P, going through α, then through β, and finally closing at P. Equation (1.69) holds (and makes sense) only if α and β intersect.

(c) The 'retracing' identity:

$$T[\alpha] = T[\alpha \cdot l \cdot l^{-1}] \tag{1.70}$$

where l is a line with one end on α and $\alpha \cdot l \cdot l^{-1}$ is the loop obtained by going around α, then along the line, and then back along the line to α.

(d)

$$\lim_{\epsilon \to 0} T[\alpha^\epsilon_x] = 2. \tag{1.71}$$

The last four equations allow a complete characterization of the T observables [9]. Related properties hold for the T^a observables.

Finally, the properties of the loop observables under the Poisson brackets operation are described in the next section.

1.5.1. The loop algebra and its geometry. The most important property of the loop observables is that they have a closed Poisson algebra:

$$\{T[\alpha], T[\beta]\} = 0 \tag{1.72}$$

$$\{T^a[\alpha](s), T[\beta]\} = -\tfrac{1}{2}\mathrm{i}\Delta^a[\beta, \alpha(s)]\,(T[\alpha\#_s\beta] - T[\alpha\#_s\beta^{-1}]) \tag{1.73}$$

$$\begin{aligned}\{T^a[\alpha](s), T^b[\beta](t)\} = &-\tfrac{1}{2}\mathrm{i}\Delta^a[\beta, \alpha(s)]\,(T^b[\alpha\#_s\beta](t) - T^b[\alpha\#_s\beta^{-1}](t)) \\ &+ \tfrac{1}{2}\mathrm{i}\Delta^b[\alpha, \beta(t)]\,(T^a[\beta\#_t\alpha](s) - T^a[\beta\#_t\alpha^{-1}](s))\end{aligned} \tag{1.74}$$

where

$$\Delta^a[\beta, x] = \oint \mathrm{d}t\ \dot\beta^a(t)\ \delta^3(\beta(t), x) \tag{1.75}$$

and the notation $\#_s$ indicates that the breaking and rejoining of the loops happens at the intersection where the parameter is s (if the loops do not intersect the Poisson bracket is zero). This algebra is called the loop algebra. It is a gauge invariant version of the Poisson algebra of the phase space of a Yang–Mills theory.

The loop algebra has a remarkable geometrical meaning. To reveal this geometrical content of the algebra we must get rid of the distributional character of the structure constants Δ†. There are several ways of smearing the loop observables. A smearing that emphasizes the geometrical character of the loop algebra has been introduced by Smolin [10]. Let us consider a one-parameter family of loops β_t, $t \in [0,1]$ such that they form a ribbon R;

$$R^a(s,t) = \beta^a_t(s). \tag{1.76}$$

A prime will denote the derivative with respect to the parameter t. Now the following smeared version of the T^a observable can be defined:

$$T[R] = \int_0^1 \mathrm{d}t \oint_0^{2\pi} \mathrm{d}s\ R'^a(s,t)\dot R^b(s,t)\ \epsilon_{abc}\ T^c[\beta_t](s). \tag{1.77}$$

The ribbon is oriented: by reversing the orientation of the ribbon by $R^{-1}(s,t) = R(-s,t)$ it follows that $T[R^{-1}] = -T[R]$. The observables are now the Wilson loops $T[\alpha]$ and the 'Wilson ribbons' $T[R]$. The Poisson algebra of these objects is surprising. Given a ribbon R and a loop α, the loop may or may not intersect the ribbon. If it does, let us denote by $\alpha\#R$ the loop $\alpha\#\beta_t$, where t is the coordinate of the intersection point. If there is more than one intersection, the intersections will be labelled by an index i, and $\alpha\#_i\beta_t$ will be the loop obtained by considering the intersection i. Moreover, given two ribbons R and S, I denote by $R\#S$ the ribbon formed by all the loops $R_t\#S$, namely by all the loops that intersect.

The loop algebra then takes this very compact form:

$$\{T[\alpha], T[\beta]\} = 0 \tag{1.78}$$

$$\{T[R], T[\alpha]\} = \sum_i \epsilon_i\ T[\alpha\#_i R] - T[\alpha\#_i R^{-1}] \tag{1.79}$$

$$\{T[R], T[S]\} = \sum_i \epsilon_i\ T[R\#_i S] - T[R\#_i S^{-1}]. \tag{1.80}$$

† The Poisson brackets of any classical field theory contain distributions.

Here ϵ_i is either $+\frac{1}{2}\mathrm{i}$ or $-\frac{1}{2}\mathrm{i}$, according to the orientation of the intersection. If the loop does not intersect the ribbon, or in the degenerate case in which the loop is tangent to the ribbon, the result of the Poisson bracket vanishes. These operations are very natural geometrical operations defined in terms of breaking and rejoining of loops and ribbons at their intersections. The structure of the loop algebra is entirely geometrical: it can be entirely coded in the geometry of the splitting and recombining of intersecting loops and ribbons.

In conclusion, the entire structure of Hamiltonian general relativity (phase space, constraints, Poisson brackets) can be expressed in terms of the loop variables. This form of the Hamiltonian theory will be used for the quantization.

An analysis of this structure in the classical framework has not yet been performed. For instance, the Hamiltonian evolution for the loop observables (which would allow the reconstruction of spacetime) has not been studied. Another interesting open problem is whether there is a Lagrangian theory corresponding to this Hamiltonian loop formulation.

2. Applications of the formalism

2.1. Survey of applications

The formalism described in the previous section has been used in a wide variety of contexts, and has been generalized in several directions. The reader interested only in the quantum gravity issue may go directly to the next section. In this section I present a synthetic overview of other applications.

Exact solutions of the classical theory have been investigated in different forms. Gravitational instantons have been studied by Samuel [11] and by Capovilla *et al* [12]. An interesting result on the instantons of self-dual general relativity has been obtained, by using the new variables, by Torre [13]. Torre has shown that in the presence of a positive cosmological constant the moduli space of the instanton solutions is zero-dimensional (discrete). In the negative cosmological constant case, the dimension of the moduli space is controlled by the Atiyah–Singer index theorem, which in the present context means the Euler number and Hirzebruch signature. This analysis has then been extended in [14].

The solution of the classical constraints has been studied by Capovilla *et al* [5], with the surprising result that the Hamiltonian and diffeomorphism constraint can be solved exactly in local form. I describe this result below in section 2.2.1.

The new variables are particularly suitable for cosmological models. Indeed, the new variables open a new point of view on these models. The key observation is the observation made after equation (1.53), above: the Hamiltonian constraint defines the motion of a *free* massless particle in a space of a given geometry. It is difficult to make use of this observation in the full theory, because the space in which the hypothetical particle is moving is infinite dimensional. But in the cosmological models the configuration space is finite dimensional. Thus, the solution of the equations of motion of any cosmological model can be reformulated as the study of the null geodesics of an assigned geometry (Ashtekar and Pullin [15]). See also Kodama [16]. Related to the cosmological models is the strong coupling limit of the theory, studied by Goldberg [17].

The reduction of general relativity obtained by requiring the existence of two or one Killing vector fields has been studied in the new variables by Smolin and Husain

[18] and by Pullin and Husain [19]. The BRST structure of general relativity in the new variables has been analysed by Ashtekar *et al* [20]. The result is that the BRST charge is similar to that of metric general relativity, but it is completely polynomial.

On the mathematical side, since the four-dimensional Ashtekar connection ${}^4A_\mu^{IJ}$ is self-dual, the formalism offers a natural framework for analysing the self-dual Einstein equations. These equations have been extensively studied along different lines by Penrose, Newman and Plebansky in recent years, to the point where the general structure of the general solution is quite well understood. Ashtekar *et al* [21] studied the problem in the new variables and found a remarkably simple local formulation of the basic equations. Robinson has related this formulation to hyperKähler structures that naturally exist in half-flat spaces [22]. Newman and Mason [23] used the new formulation for an interesting analysis of a relation between general relativity and Yang–Mills theories with infinite-dimensional gauge groups.

A well known open problem in general relativity is the definition of an internal variable which could be reasonably identified (maybe within some approximation) with the non-general-relativistic physical observed time. The new variables provide an elegant solution to this problem. There is a functional $T[A]$ of the Ashtekar connection which can be identified with the physical Minkowski time up to second order in an expansion around Minkowski space. This result is due to Ashtekar [2].

The basic formalism has been extended in several directions. Matter can be naturally included in the formalism. Ashtekar *et al* [24] have constructed the relevant Lagrangians and have shown that the inclusion of matter does not spoil the crucial result that the constraints are polynomial. Jacobson found the fermions Lagrangian [25] and reformulated supergravity in the new variables with analogous results [26]. Thus, the formalism is 'robust', in the sense that the main results, in particular the polynomiality of the constraints, survive modifications of the basic theory. The inclusion of matter in the theory is reviewed in detail in Ashtekar's Poona lectures [2].

The extension of the formalism to other dimensions has been studied by Bengtsson [27]. There is a natural new variables formulation of general relativity in $(2+1)$ dimensions, which has been used by Witten [28] for solving the quantum theory. I describe this formulation below in section 2.2.2. The BRST structure of the $(2+1)$ theory has been studied by Gonzalez and Pullin [29]. In the same paper, it is shown that in the path integral formulation of the quantum theory there is a suitable gauge fixing of the BRST action that reproduces the action used by Witten for studying the topology-changing amplitudes.

The formalism itself has suggested several new models. Among these, I find particularly interesting a generally covariant theory in 3+1 dimensions, which is a simplified version of general relativity developed by Kuchař and Husain [30] and a particular form of the weak coupling limit of the theory suggested by Smolin [31]. These are described below in section 2.2.3.

On the numerical side, it has been repeatedly suggested that the formalism may be useful for computer calculations, but I am not aware of any such work. Algebraic computing for the new variables has been developed by Giannopoulos and Daftardar [32].

In the context of quantum gravity, besides the loop-representation of the nonperturbative theory, which is described in the rest of this report, Renteln and Smolin [33] constructed a lattice formulation of canonical quantum gravity using the new variables, and Renteln [34] studied numerically the algebra of the lattice quantum constraints. Husain constructed the quantum theory in the strong coupling limit

[35]. Ashtekar *et al* [36] studied possible qualitative predictions of a quantum gravity theory. They considered that the internal gauge may provide the topological setting for topological and θ-angle-like violations of classical invariances. A detailed analysis of the topological properties of the phase space of the theory led them to formulate the hypothesis of quantum gravitational CP violations.

2.2. Selected applications

2.2.1. The general solution to the classical diffeomorphism and Hamiltonian constraint. In terms of the new variables, it is possible to solve the gauge constraint by going to the loop variables. Alternatively, it is also possible to solve the diffeomorphism and Hamiltonian constraint. A general solution of all the constraints together is still missing in the classical theory. The solution of the Hamiltonian and diffeomorphism constraints is the following.

Let us choose an arbitrary connection $A_a^i(x)$ and an arbitrary symmetric traceless tensor field $\phi^{ij}(x)$. Then we may construct the triad field

$$\tilde{E}^{ai}[A,\phi] = (\phi^{-1})^i_j \epsilon^{abc} F^j_{bc} \tag{2.1}$$

where the notation ϕ^{-1} refers to the inversion of the 3×3 matrix ϕ^{ij}. We have the following theorem. The fields $(\tilde{E}^{ai}[A,\phi],\, A_a^i)$ solve the diffeomorphism and Hamiltonian constraint. The proof is very straightforward: it suffices to insert (2.1) in the constraint and work out the algebra of the three-dimensional matrices. This surprisingly simple solution of the two constraint equations was overlooked for several years, and then discovered by Capovilla *et al* [5].

The fields $(\tilde{E}^{ai}[A,\phi],\, A_a^i)$ solve two of the three constraints in terms of the independent variables A, ϕ. In order to have the complete solution of all the constraints, we must solve the gauge constraint. The gauge constraint, written in terms of the independent variables A, ϕ looks as follows

$$\epsilon^{abc} F^i_{bc} D_c(\phi_{ik}\phi^k_j - \tfrac{1}{2}\delta_{ij}\phi^{2k}_{\ k}) = 0. \tag{2.2}$$

This equation can be read as a reformulation of the constraint equations of general relativity.

2.2.2. (2 + 1)-dimensional theory. The vacuum Einstein equations $R_{\mu\nu} = 0$ in three spacetime dimensions are trivial. In three dimensions, in fact, the Riemann curvature is entirely determined by the Ricci tensor, therefore the Einstein equations imply that spacetime is flat. The only solution, up to gauge, is the three-dimensional Minkowski metric.

However, if we assume that the topology of the two-dimensional space Σ is non-trivial, then there is more than one solution to the theory, because flat spaces may be globally non-isometric. Therefore, the theory defined on a topologically non-trivial Σ has a non-trivial dynamics with a finite number of degrees of freedom. These degrees of freedom are global, in the sense that locally all the solutions are gauge equivalent. The theory is very interesting because it is a solvable diffeomorphism-invariant theory, and therefore it represents a good exercise-room for studying the quantization of diffeomorphism-invariant theories.

The theory admits a formulation in terms of Ashtekar variables. The Lagrangian fields are three covariant vector fields e^i_μ and an $SO(2,1)$ connection A^i_μ. Here the

Greek spacetime indices run from zero to three, and the Latin internal indices also run from zero to three. The action is

$$S[e,A]=\int \mathrm{d}^3x\; e^i_\mu\; F^i_{\nu\sigma}\; \epsilon^{\mu\nu\sigma}. \tag{2.3}$$

The canonical formalism is defined on the phase space with canonical coordinates E^a_i, A^i_a (here $a=1,2$), by the constraints

$$C_i = D_a E^a_i \tag{2.4}$$

$$D^i = F^i_{ab}\epsilon^{ab}. \tag{2.5}$$

The first one is the standard Gauss law, which implements the internal $SO(2,1)$ invariance, and the second one requires that the space connection A^i_a is flat. The physical configuration space is therefore given by the moduli space of A^i_a, namely by the gauge-inequivalent flat connections.

The constraints close, and their algebra is the algebra of the $(2+1)$-dimensional Poincaré group. There is an equivalent formulation of the theory as a Chern–Simons theory of the Poincaré group, in which the triad e^i_μ and the Lorentz connection A^i_μ are considered as components of a Poincaré connection A^I_μ, $I=0,5$. With a suitable choice of the Poincaré trace, the action written above can be rewritten as a Chern–Simons action for this Poincaré connection.

Finally, it is possible to define the loop observables. I write here the smeared version of them, which will be used in section 3 for the quantization of the theory. The T loop observable can be defined precisely as its three-dimensional counterpart

$$T[\gamma] = \mathrm{Tr}\; \mathcal{P}\; \exp\left(\oint_\gamma A\right) \tag{2.6}$$

while the analogue of the smeared T^a observable, namely of the 'ribbon' observable, is

$$T[R] = \int \mathrm{d}s \dot{R}^b(s)\epsilon_{ba}T^a[R](s) = \int \mathrm{d}s\dot{R}^b(s)\epsilon_{ba}\mathrm{Tr}\; [U_R(s)E^a(R(s))] \tag{2.7}$$

where, here, $R^a(s)$ is a loop. What is going on is the following. In order to smear T^a we need to contract its free index. To preserve diffeomorphism covariance the index can only be contracted with the totally antisymmetric tensor. The other indices of the totally antisymmetric tensor must be contracted with the area element of the surface over which we smear. Therefore T^a must be smeared over a surface with one dimension less that the space. In three dimensions this surface was the ribbon. In two dimensions the ribbon is replaced by a $(2-1=1)$-dimensional object, namely a loop. Thus in two dimensions the smeared T^a observables also depend on a loop, as the T observables. In spite of that, I keep the notation R for these loops, to remind that they are the two-dimensional analogues of the three-dimensional ribbons.

2.2.3. Other models. Consider a Lie group G, a Yang–Mills connection A_μ with values on the Lie algebra $\mathcal{G}$ of G, and a covariant vector field e_μ taking values in $\mathcal{G}$. Choose an invariant trace on $\mathcal{G}$. Then the action (1.2) can be generalized to

$$S[e,A] = \int \mathrm{d}^4x\; \mathrm{Tr}\; [e_\mu e_\nu\; {}^4F_{\tau\sigma}]\; \epsilon^{\mu\nu\tau\sigma}. \tag{2.8}$$

Kuchař and Husain [30] studied the theory defined by the group $SO(3)$. The theory has three physical degrees of freedom per space point (rather than two, as in general relativity), but its equations of motion can be solved exactly. The canonical formulation of the theory turns out to be precisely given by the canonical formulation of the Ashtekar theory without the Hamiltonian constraint. Thus, the theory has all the features of general relativity, but without the 'difficult' part of it. It represents a non-trivial generally covariant model, which can be used as a laboratory for any attempt to quantize gravity.

Smolin [31] studied the theory defined by the group $U(1) \times U(1) \times U(1)$. He has shown that this theory is equivalent to the limit in which the Newton constant is sent to zero keeping the canonical variables fixed. The resulting theory is equivalent to linearized general relativity if the standard reality conditions are imposed. However, if Euclidean reality conditions are imposed (all the fields real) then the theory represents the self-dual Einstein theory plus the linearization of the antiself-dual part. Smolin suggested that this form of the linearized theory can be used as a starting point for a perturbation expansion.

To my knowledge, other theories in this class, obtained by using other groups G, have not been studied. These theories are diffeomorphism invariant, and are non-trivial in the sense that they have an infinite number of degrees of freedom. Thus they are 'infinite-dimensional topological field theories' in the sense that they are defined on a manifold with no metric structure, like the topological field theories recently discussed in the literature [37]; however, unlike the topological field theories, they are genuine field theories with infinite degrees of freedom. Note that the existence of these theories contradicts the widespread assumption that any field theory with no fixed or dynamical metric has a finite number of degrees of freedom. I think that it would be very interesting to study this class of theories, both as classical and as quantum theories.

2.2.4. Hamilton-Jacobi theory. Finally, I would like to point out a potentially interesting direction of investigation. In the Hamilton–Jacobi framework, the classical dynamics of general relativity is essentially contained in the Hamilton–Jacobi equation

$$\epsilon_{ijk}\ F^i_{ab}(x)\ \frac{\delta S[A]}{\delta A^j_a(x)}\frac{\delta S[A]}{\delta A^k_b(x)} = 0. \tag{2.9}$$

This equation follows from the Hamiltonian constraint by considering $\tilde{E}^a_i(x)$ as the momentum and by replacing it with the derivative of the Hamilton–Jacobi function $S[A]$.

Now, an *exact* solution of this equation, depending on an infinite set of constants, was found by Jacobson and Smolin in [38]. This solution is

$$S[A,\alpha] = \mathrm{Tr}\ \mathcal{P}\exp\left(\oint_\alpha A\right) \tag{2.10}$$

where the loop α is differentiable and without intersections. (The loop α can be considered as the set of the Hamilton–Jacobi constants.) This result follows from a straightforward computation of the functional derivative:

$$\frac{\delta}{\delta A^i_a(x)}\mathrm{Tr}\mathcal{P}\exp\left(\oint_\alpha A\right) = \oint \mathrm{d}s\ \dot\alpha^a(s)\ \delta^3(\alpha(s),x)\ \mathrm{Tr}\ [U_\alpha(s)\tau^i] \tag{2.11}$$

and from the antisymmetry of F^i_{ab}.

The relevance of this surprising result to the investigation of the Einstein equations, or, more precisely, to the Ashtekar version of the Einstein equations (which allows also degenerate metrics) has not yet been explored.

3. Quantum field theory on manifolds: the loop representation

In this section, I describe the general approach and the main techniques that will be used in the next section for quantum gravity. In section 3.1, I introduce the quantum gravity problem, discuss the motivations for seeking a non-perturbative quantization, and illustrate the general quantization scheme used, which is a modification of Dirac's technique of quantization of first-class constraints.

The loop representation is introduced in section 3.2. It is introduced in the familiar context of Maxwell theory and non-Abelian Yang–Mills theory, in order to separate the description of the loop technique itself from the difficulties of gravity. The advantage of the loop quantization is that it handles diffeomorphism invariance in a natural way. To illustrate this point, I describe the use of the loop quantization in the quantization of general relativity in $(2+1)$ dimensions. Certain conceptual questions related to the construction of a quantum theory on a manifold are discussed in the last part of this section.

3.1. Quantum general relativity: ideas and hopes

3.1.1. The need for a non-perturbative theory. General relativity and the $SU(3) \times SU(2) \times U(1)$ standard model constitute a theoretical framework which, in principle, predicts the behaviour of any physical system in any physical circumstance, *except in one case.* This 'hole' is constituted by the phenomena in which the quantum properties of the gravitational interaction cannot be disregarded.

This theoretical framework will perhaps (probably?) be challenged by future experiments, and, due for instance to the number of free parameters, it may be considered aesthetically unsatisfactory and more or less likely to be incomplete. However, to have a fundamental theory which is not contradicted by any known physical fact is a novel situation in the history of modern physics. In such a situation, it is natural to concentrate on the open 'hole', and focus on the single crack of the present theory.

The crack is more substantial than the mere impossibility of calculating Planck-scale phenomena. Indeed, quantum field theories of the standard model on the one hand, and general relativity on the other, provide two strikingly different pictures of nature. So different that one wonders how physics students may accept such a schizophrenic description of nature as a reasonable one. Clearly, there is a contradiction here at the basic level in the present description of the world. Of course, to face contradictions at the fundamental level has always been a vital tool that has led to major advances in theoretical physics. (The contradiction between Galilean invariance and the Maxwell equations motivated special relativity; the one between Newtonian gravity and special relativity motivated general relativity; the one between the Galilean earth physics and the Keplerian celestial physics motivated the Newtonian synthesis, and so on.) Thus, the problem of the quantum description of gravity is at the heart of today's physics.

Let me analyse the problem more closely. There is an important observation to be made: one should distinguish quantum mechanics, which is a general mechanical theory, from the standard formalism of quantum field theory, which is a particular application of quantum mechanics to certain systems with an infinite number of degrees of freedom. General relativity is incompatible with the standard formalism of quantum field theory, ***but this does not necessarily imply that general relativity is incompatible with quantum mechanics.***

Standard quantum field theory relies on the existence of a fixed causal structure on the spacetime manifold, as well as on the Poincaré invariance of such a structure (without a fixed causal structure one cannot define the local quantum field operators as operators commuting at *spacelike* separations). General relativity does not allow any non-dynamical causal structure and is not Poincaré invariant. The reason for the failure of all the attempts to construct quantum gravity within the standard framework of quantum field theory appears clear, at least *a posteriori*: in order to squeeze general relativity into the standard formalism, we are forced to artificially incorporate into the theory a background Minkowski metric $\eta_{\mu\nu}$, and assume (against general relativity itself) that the physical causal structure is defined by $\eta_{\mu\nu}$, rather than by $g_{\mu\nu}$. (Quantum field operators are then defined as operators that commute when they are spacelike separated, where 'spacelike' is defined by $\eta_{\mu\nu}$.)

Thus, the problem of quantum gravity is the following: is it possible to construct a quantum theory for an infinite-dimensional system without assuming the existence of a background causal structure? Such a quantum field theory should be radically different from usual Poincaré-invariant quantum field theories†.

In gravity there is a subtle interplay between this addition of a fictitious background metric, and the use of a perturbation expansion. Since the causal structure is defined by the dynamical variable $g_{\mu\nu}$ itself, a perturbation expansion around the Minkowski solution implies that we are using the Minkowski fictitious metric as a background causal structure. By defining the local operators in perturbation expansion, we commit them to respect the unperturbed (and therefore fictitious) commutation relations. Thus, while in usual quantum field theories the perturbation expansion is a method for solving the theory, in gravity it is a method for *defining* the theory.

A formulation which does not use the unphysical Minkowski metric in order to define the theory is therefore needed. Any such formulation will be denoted 'non-perturbative'‡. Thus, the problem is to develop a quantum field theory without background geometry, namely a field theory on a differentiable manifold rather than on a metric space. In section 2.2, I describe a formulation of quantum field theory which avoids any reference to the background geometry, and which is in a position to han-

† A few examples: the concept of particle is intimately related to Poincaré invariance; there are no well defined quantum particles in a quantum field theory on a manifold. Similarly, there is no concept of vacuum in a theory with no Hamiltonian. And so on.

‡ The need for a non-perturbative theory is reflected in another peculiarity of the gravitational field. Quantum gravitational effects appear only at the Planck scale. Therefore any perturbation expansion should reach this scale in order to be physically meaningful. Perturbation expansions in quantum field theory (or string theory) are in general divergent. Since the Planck constant is the only dimensional quantity, the perturbation expansion most likely diverges at the Plank scale. But then it never reaches the region where the physics is. If the theory is defined via a perturbation expansion, the perturbation expansion has to be convergent, otherwise the theory is meaningless: a renormalizable perturbation expansion (or even finite order by order) is not a solution of the problem. In a sense, the difficulty of string theory of providing substantial physical information on what happens at 10^{-35} cm or at 10^{21} GeV is a manifestation of this problem.

dle this diffeomorphism invariance. This formulation derives from the early work in canonical quantum gravity of Wheeler and DeWitt [39] and, more recently, on the work of Kuchař [40] and especially of Isham [41]. It was introduced by Smolin and by the author [8], following results obtained by Jacobson and Smolin [38] and has been developed by Ashtekar, Smolin and the author [2, 42–44]. In the context of Yang–Mills theories an essentially analogous formulation had been (independently) introduced by Gambini and Trias [45]. This formulation is based on the possibility of coding the information about the quantum field on a space $\mathcal{L}$ over which we have a certain control of the action of the diffeomorphism group. This space is the space of the loops over the manifold.

The fact that a gauge theory can be expressed in terms of loops has been known for a long time. There is a persistent line of thought, that advocates that loops are the natural objects in terms of which a gauge theory should be described [46, 47]. Among others, it includes Polyakov, Mandelstam, Wilson, and it dates back to Faraday. At the same time, loops have the remarkable feature that their diffeomorphism-invariant properties are simple: they are captured by the way the loops are knotted and linked.

The Ashtekar formulation of general relativity provides an unexpected bridge between these two characters of the loops (that they describe gauge theories and they capture diffeomorphism-invariant properties): by representing gravity in terms of a Yang–Mills-like connection, the Ashtekar formalism furnishes an object, namely the Wilson loop of this connection, which captures the gravitational field and that, at the same time, has a manageable behaviour under diffeomorphisms.

3.1.2. Dirac quantization and its problems. Many aspects of quantum gravity follow from the fact that the canonical Hamiltonian of general relativity vanishes weakly. This fact is not accidental: rather, it is deeply rooted in the physics of the gravitational field. The Hamiltonian is the generator of time evolution. The physical meaning of the general covariance of general relativity is that space distances and time intervals have no meaning *a priori*, but can only be defined by the dynamics of the gravitational field itself. Therefore evolution in a pre-assigned universal time is unphysical in general relativity. Accordingly, there is no Hamiltonian in the theory.

In spite of the vanishing of the Hamiltonian, the canonical formalism does provide a viable framework. The canonical formalism to be used is not the standard Hamiltonian one, but the generalization provided by Dirac's constrained-systems theory, or, in modern terms, by presymplectic dynamics. The constraints in gravity are not just a nuisance, like the Gauss law constraints of canonical Maxwell theory; rather, they encode the physical content of the theory.

Accordingly, the main instrument for the quantization is Dirac's theory of quantization of first-class constraint systems (not to be confused with the Dirac theory of second-class constraint systems—the theory of the Dirac brackets). The Dirac theory is well known: quantize the system as if there were no constraint, pick up the subspace $\mathcal{H}_{\rm Ph}$ of the Hilbert space $\mathcal{H}$ defined by $\hat{C}_i\Psi = 0$, where $\hat{C}_i$ are the quantum operators corresponding to the constraints, and choose a set of observables $\hat{O}_n$ that commute with the constraints. Then $\mathcal{H}_{\rm Ph}$ and the $\hat{O}_n$ operators define the quantum theory.

There are two problems in this approach. The first one is the difficulty of recognizing how the quantum theory describes the physical time evolution; this issue will be discussed in section 3.3.2. The second problem is that Dirac's theory is incomplete in the following sense. In general, the scalar product in $\mathcal{H}$ does not induce a viable scalar product in $\mathcal{H}_{\rm Ph}$ because (when, as usually happens, the $\hat{C}_i$ have continuum spectrum)

$\mathcal{H}_{\rm Ph}$ is in general formed by improper vectors of $\mathcal{H}$, and the scalar product of $\mathcal{H}$ is not defined on these vectors†. In order to apply Dirac's quantization scheme to general relativity, we have to supplement it with a method for choosing the physical scalar product.

The difficulty with the scalar product has raised a certain confusion in the literature, including claims that the theory cannot be defined in a standard Hilbert space framework, that the theory is under-defined because we do not know how to pick the physical product, or that a theory without time cannot have a scalar product. A straightforward way out from this difficulty is discussed in the next section.

3.1.3. The reality conditions determine the inner product. The linear structure of a vector space is well defined by itself and is independent of any eventual scalar product that one may define on that space. On the same linear space different scalar products can be defined. By using this observation, a theory can be quantized in two steps. The first is to pick a linear space $\mathcal{H}$, and linear operators $\hat{O}_n$ corresponding to classical observables O_n (with the correct commutation relations, for instance the canonical commutation relations). The second step is the choice of the scalar product. This is a shift in perspective with respect to the usual procedure, in which one starts from a Hilbert space and then chooses *self-adjoint* operators $\hat{O}_i$.

The question is: how to choose the scalar product? Of course, in order to get the correct final answer, the requirement on the choice of the scalar product is that the linear operators $\hat{O}_n$ must be self-adjoint with respect to the scalar product chosen. (The definition of the adjoint operation depends on the scalar product.) Thus, the strategy is first to work out the linear structure entirely, and then choose the scalar product that makes the observables self-adjoint. If the programme is completed, the final result is clearly independent of the procedure followed.

The advantage of this procedure is that we may solve the quantum Dirac constraint at the linear level, and only later are we concerned with the inner product. But then it is clear which is the condition that determines the choice of the inner product in $\mathcal{H}_{\rm Ph}$: the condition is that the operators corresponding to the physical observables $\hat{O}_n$ must be Hermitian in the chosen scalar product. This is a highly non-trivial requirement on the choice of the scalar product. In general, if the operators $\hat{O}_i$ are 'enough', this requirement fixes the scalar product *uniquely*. Thus, there *is* a precise criterion for the selection of the scalar product on $\mathcal{H}_{\rm Ph}$: the Hermiticity of the gauge invariant $\hat{O}_i$.

Now, in the new variables formulation there is an additional issue to be addressed. We work with complex classical observables, namely with complex linear combinations of classical observables: $O = f(x,p) + \imath g(x,p)$, where f and g are real observables. Obviously f and g must be Hermitian, not O. At the linear level, we can simply quantize O in term of a linear operator $\hat{O}$, but then at the moment of choosing the scalar product the adjoint of $\hat{O}$ must have suitable properties, such that f and g be Hermitian.

Specifically, the Ashtekar connection is a complex linear combination of two real observables: $A = p + \mathrm{i}\omega[e]$. In order to have real observables represented by Hermitian operators the inner product should be such that

$$\hat{A}_a^\dagger = \hat{A}_a - 2\mathrm{i}\omega_a[\hat{e}]. \tag{3.1}$$

† This is a well known problem in the quantization of any theory with a gauge invariance. In standard Yang–Mills theories there are known ways around it (for instance the introduction of ghosts). Moreover, there is a guiding principle that is used for fixing the scalar product: Poincaré invariance.

Equation (3.1) is the quantum version of the reality conditions equation (1.41). The inner product should be such that equation (3.1) holds. Thus, *the inner product is determined by the reality conditions* [1, 8].

Let me be more precise. The variable A is not gauge invariant, $\hat{A}$ does not commute with the constraints, and it is not defined on $\mathcal{H}_{\mathrm{Ph}}$. What we have to do is to compute the reality conditions for the gauge-invariant observables O_n that follow from the reality conditions for A and $\tilde{E}^a_i$. Let these reality conditions be

$$\bar{O}_i = f_i(O_j). \tag{3.2}$$

Then the inner product on $\mathcal{H}_{\mathrm{Ph}}$ is determined by the corresponding quantum reality conditions:

$$\hat{O}_i^\dagger = f_i(\hat{O}_j). \tag{3.3}$$

Namely, the scalar product is determined by the condition

$$\langle \hat{O}_i \psi | \phi \rangle = \langle \psi | f_i(\hat{O}_j) \phi \rangle. \tag{3.4}$$

Examples of this procedure for fixing the inner product are the quantization of Maxwell field with self-dual variables in appendix 3, and the quantization of linearized gravity in the next section.

3.2. Loops

3.2.1. Maxwell: quantum Faraday lines. The idea of using loops as the objects for describing a gauge theory has been concretized in several forms. Here a canonical theory is defined. I follow the work of Ashtekar and the author in [42].

Maxwell theory is a free-field theory and the standard quantization is straightforward. For instance, one may fix the radiation gauge $\partial_a A^a = 0$ and decompose the fields in Fourier modes. Each mode is an harmonic oscillator for which creation (or positive frequency) and annihilation (or negative frequency) operators can be defined.

The positive-frequency field, which I denote by ${}^+A_a$ is given in terms of the real Maxwell connection A_a and its momentum (the electric field) E^a by

$${}^+A_a(k) = \frac{1}{\sqrt{2}}\left(A_a(k) - \mathrm{i}|k|^{-1}E_a(k)\right). \tag{3.5}$$

The negative-frequency field, which I denote here by ${}^-E_a$, is

$${}^-E_a(k) = \frac{1}{\sqrt{2}}\left(A_a(k) + \mathrm{i}|k|^{-1}E_a(k)\right). \tag{3.6}$$

These two fields satisfy the standard canonical commutation relations and can be represented in the quantum theory in terms of creations and annihilation operators on a Fock space. By doing so, one discovers that the Hamiltonian is diagonal on the Fock basis of the n-photon states. The basic prediction of the theory is the existence of the photons.

The loop representation of Maxwell theory is a different representation of the theory, which turns out to be equivalent to the Fock representation.

Consider the ^+A_a field in coordinate space. Consider its Wilson loop. More precisely, consider the (Abelian) holonomy of $-\mathrm{i}$ times $^+A_a(x)$, and denote it as $T[\gamma]$:

$$T[\gamma] = \exp\left(-\mathrm{i}\oint_\gamma {}^+A\right) \tag{3.7}$$

where the line integral along the loop is defined as (this is the standard line integral of 1-forms)

$$\oint_\gamma {}^+A = \oint \mathrm{d}s\ \dot\gamma^a(s)\ {}^+A_a(\gamma(s)). \tag{3.8}$$

$T[\gamma]$ is the first relevant variable to be used for the quantization. It is of course the Abelian version of the gravitational T observable defined in section 1.5†. The second relevant variable is $^-E^a(k)$.

The Poisson algebra of T and ^-E closes:

$$\{T[\gamma], {}^-E^a(k)\} = \mathrm{i}F^a[\gamma, k] \tag{3.9}$$

and is denoted *Abelian loop algebra*. Here, $F^a[\gamma, k]$, called the form factor, is defined as

$$F^a[\gamma, k] = \oint \mathrm{d}s\, \dot\gamma^a(s)\, \mathrm{e}^{\mathrm{i}k\cdot\gamma(s)}. \tag{3.10}$$

The form factor $F^a[\gamma, k]$ will be a very important object in what follows. Its main property is that its Fourier transform is the real function (distribution) with support on the loop itself $\Delta^a[\gamma, x]$ defined above in equation (1.75). Note that the holonomy T can also be written as

$$T(\gamma) = \exp\left(-\mathrm{i}\int \mathrm{d}^3x\ \Delta^a[\gamma, x]A_a(x)\right). \tag{3.11}$$

The key idea of the loop representation is the following. Rather than looking for a representation of the canonical creation–annihilation algebra, namely a representation of the Poisson algebra of ^+A and ^-E, we look for a representation of the Poisson algebra of T and ^-E. In other words, we search for two operators $\hat T[\gamma]$ and $\hat E^a(k)$ (I drop the superscript $^-$ from the electric field operator) which satisfy the commutation relations

$$[\hat T[\gamma], \hat E^a(k)] = \hbar F^a[\gamma, k]. \tag{3.12}$$

The idea that non-canonical algebras are better suited for non-perturbative quantization has been advocated by Isham [41]. As we shall see, this change of basic algebra is harmless in simple theories, but has far reaching consequences in gravity.

The loop representation is based on the fact that there is a representation of the loop algebra (3.9) in terms of functions on a loop space. A single loop is here a piecewise smooth closed curve $\alpha^a(s)$ in a fixed three-dimensional manifold. A multiple loop is a collection of a finite number of single loops. Multiple loops will also be denoted with Greek letters: $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$. Let $\mathcal{L}$ be the space of all these multiple loops.

† The i in the definition is just a matter of convention.

The wavefunctions that represent the unconstrained quantum states of the theory will be complex functions on $\mathcal{L}$. They assign a complex amplitude to every multiple loop α

$$\text{wavefunction} \longleftrightarrow \Psi(\alpha). \tag{3.13}$$

On the space of these wavefunctions there is a representation of the observables $T[\gamma]$ and $^{-}E^a(k)$, defined as follows:

$$\hat{T}[\gamma]\,\Psi(\alpha) = \Psi(\alpha \cup \gamma) \tag{3.14}$$

$$\hat{E}^a(k)\,\Psi(\alpha) = \hbar F^a[\gamma, k]\,\Psi(\alpha). \tag{3.15}$$

These operators satisfy equation (3.12). They answer the quantization problem in the same sense in which the creation and annihilation operators on a Fock space, or the x and $-i\hbar\partial/\partial x$ operators do. Thus, we have a linear space of states, and a basic set of operators on this space, with the correct commutation relations for the definition of the quantum theory.

Notice that the wavefunction $\Psi(\alpha)$ is not a wavefunction on a configuration space. Rather, it has to be thought of as an abstract vector in a linear space. To provide an analogy, consider a hydrogen atom quantum state $|\Psi\rangle$, and its components on the basis $|n, l, m\rangle$ which diagonalizes energy, total angular momentum and one component of the angular momentum. Such components are $\Psi(n, l, m) \equiv \langle n, l, m|\Psi\rangle$. The quantities $\Psi(n, l, m)$ provide a representation of the physics of the hydrogen atom. Here n, m, l is not a classical configuration variable, and the $\Psi(n, l, m)$ picture is not very intuitive. However, the set of the $\Psi(n, l, m)$ plus a definition of the action over them of the relevant operators provide a complete description of the physics of the atom. Later I will discuss the physical interpretation of the term α that appears as argument of the wavefunction.

The classical T observables are not independent. They are invariant under (monotonic) reparametrizations of the loops and, as their gravitational analogues, they satisfy the following relations:

$$\lim_{\epsilon \to 0} T[\alpha_x^\epsilon] = 1 \tag{3.16}$$

$$T[\alpha \circ l \circ l^{-1}] = T[\alpha] \tag{3.17}$$

(see section 1.5 for the notation) Because of the Abelian character of the group, the spinor identity (1.69) is replaced by a simpler relation: anytime a loop α has a self-intersection which breaks it into two loops β and γ, we have

$$T[\alpha] = T[\beta]T[\gamma]. \tag{3.18}$$

If the basic observables chosen for the quantization are not independent, the resulting quantum theory may be reducible ('larger' than necessary). To fix this, we have to implement these relations in the quantum theory. We impose these relations as operator equations. This is equivalent to restricting ourselves to the states that satisfy

$$\Psi(\delta \cup \alpha) = \Psi(\delta \cup \beta \cup \gamma) \tag{3.19}$$

$$\Psi(\alpha \cup \gamma \circ l \circ l^{-1}) = \Psi(\alpha \cup \gamma) \tag{3.20}$$

$$\lim_{\epsilon \to 0} \Psi(\gamma_x^\epsilon) = 1 \tag{3.21}$$

and which have the same value on loops related by a reparametrization. The first two of these equations imply that Ψ is completely determined by its restriction on the single loops.

In order to solve the theory, we have to consider gauge invariance and to find the eigenvectors of the Hamiltonian. Gauge invariance is straightforward: since both T and E are gauge invariant, we are in fact already working in the physical gauge invariant phase space. Thus, we do not have to take into account gauge invariance any more. I will come back to this point later.

The Hamiltonian can be written in terms of the basic operators of the loop representation. The classical Hamiltonian, written in terms of ${}^{+}A$ and ${}^{-}E$ is

$$H = \int \mathrm{d}^3k \; {}^{-}E^a(-k)\, {}^{+}B_a(k) \tag{3.22}$$

where ${}^{+}B$ is the magnetic field of ${}^{+}A$. We need the operator that corresponds to the classical observable ${}^{+}B$. Note that for a small loop $\gamma_{\epsilon,a,x}$ which is centred in x, has area ϵ and is oriented in the plane normal to the a direction

$$T[\gamma_{\epsilon,a,x}] = 1 + \mathrm{i}\epsilon\, {}^{+}B_a(x) + \mathrm{O}(\epsilon^2) \tag{3.23}$$

so that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}(T[\gamma_{\epsilon,a,x}] - 1) = \mathrm{i}\, {}^{+}B_a(x). \tag{3.24}$$

Accordingly, the quantum operator $\hat{B}$ corresponding to the classical observable ${}^{+}B$ (again I drop the superscript $^{+}$ in the operator) is defined by

$$\begin{aligned}
\hat{B}_a(x)\Psi(\alpha) &\equiv -\mathrm{i}\lim_{\epsilon\to 0}\frac{1}{\epsilon}(T[\gamma_{\epsilon,a,x}] - 1)\Psi(\alpha) \\
&= -\mathrm{i}\lim_{\epsilon\to 0}\frac{\Psi(\alpha\cup\gamma_{\epsilon,a,x}) - \Psi(\alpha)}{\epsilon} \\
&\equiv -\mathrm{i}\frac{\delta}{\delta\gamma^a_x}\Psi(\alpha).
\end{aligned} \tag{3.25}$$

In the last line, the notation $\delta/\delta\gamma^a_x$ has been introduced in order to emphasize that the operator is a derivative operator. Note, however, that it is not a functional derivative.

In terms of this operator the Hamiltonian is

$$\hat{H} = \int \mathrm{d}^3k E^a(-k)\, \hat{B}_a(k) \tag{3.26}$$

and the time-independent Schrödinger equation is

$$\int \mathrm{d}^3k\, F^a(-k)\, \frac{\delta}{\delta\gamma^a_k}\Psi_{(n)}(\alpha) = E_{(n)}\Psi_{(n)}(\alpha). \tag{3.27}$$

It is a straightforward calculation to check that this equation is satisfied by the states

$$\begin{aligned}
&\Psi_0(\alpha) = 1 \\
&\Psi_{(k,\epsilon)}(\alpha) = \epsilon_a F^a[\alpha, k] \\
&\Psi_{(k,\epsilon;p,\sigma)}(\alpha) = \epsilon_a F^a[\alpha, k]\; \sigma_b F^b[\alpha, p].
\end{aligned} \tag{3.28}$$

It is easy to recognize that these states are the n-photon states. Indeed, the corresponding energy eigenvalues are $0, |k|, |k| + |p|, \ldots$. Here, ϵ, σ are the polarizations vectors.

These Fock states can be written in coordinate space as follows. Consider a one-photon state with one-particle wavefunction $f_a^{(1)}(x)$. Then from equation (3.28) and the definition of the form factor it follows that

$$\Psi_{f^{(1)}}(\alpha) = \oint_\alpha f^{(1)}. \tag{3.29}$$

Thus, *the loop functional representing a one-particle state is the line integral along the loop of the one-particle wavefunction f*. Note that the line integral depends only on the *transverse* component of $f_a(x)$. This is how the loop representation naturally enforces gauge invariance. A two-photon state with two-particle wavefunction $f_{ab}^{(2)}(x,y)$ is given by the loop functional

$$\Psi_{f^{(2)}}(\alpha) = \oint_\alpha \oint_\alpha f^{(2)} \tag{3.30}$$

where one line integral is on x and one on y.

Some comments follow.

(i) Gauge invariance. The transversality of the Maxwell field follows from the following key property of the form factors:

$$k_a F^a[\alpha, k] = 0 \tag{3.31}$$

which in turn follows immediately from the definition. Because of this property, there are only two independent photons for every momentum, as required by gauge invariance. The main advantage of the entire formalism is that it allows us to work on the space of transverse A fields without breaking manifest Lorentz covariance.

(ii) What we have done is simply to give a different representation of the Fock space. This is a different representation in the same sense that the momentum representation and the coordinate representation of a particle are two equivalent representations. We may look at the choice of this representation as at the choice of a basis $|\alpha\rangle$, in the Hilbert space of the theory:

$$\Psi(\alpha) = \langle \alpha | \Psi \rangle. \tag{3.32}$$

(iii) Since we have the explicit form of all the Fock states, we know the inner product. I refer to [42] for a detailed discussion of this point, but I give here the basic expression for the scalar product in loop space. First, it may be shown [43] that every loop functional ψ in the physical state space determines a (polynomial) functional ϕ over the space of the (transverse) functions of k, $F^a(k)$, via

$$\Psi(\alpha) = \phi[F^a[k, \alpha]]. \tag{3.33}$$

Using this, the scalar product is defined by

$$\langle \Psi | \Psi' \rangle = \int \mathrm{d}[F] \exp\left(- \int \frac{\mathrm{d}^3 k}{|k|} \bar{F}^a(k) F_a(k) \right) \bar{\phi}[F]\, \phi'[F]\,. \tag{3.34}$$

This expression reproduces the standard inner product for Maxwell theory. For more details see [42] and [2].

(iv) Equation (3.28) provides the matrix elements of the operator $\langle\alpha|p_1\epsilon_n,\ldots,p_n\epsilon_n\rangle$ that defines the change of basis from the Fock representation to the loop representation. Besides the Fock, or particle, representation, there is another well known representation of a free field. This is the functional Bargmann representation [48], in which the quantum states are represented by holomorphic functionals $\Psi[A]$ of the (positive-frequency) Maxwell connection, and the Fock-basis states are represented by power functionals. How are the Bargmann and the loop representation related? The answer can be worked out completely. Indeed, one can show that the relation is given by the following functional integral transform:

$$\Psi(\alpha) = \int \mathrm{d}\mu[A] \exp\left(\mathrm{i}\oint_\gamma A\right) \Psi[A]. \tag{3.35}$$

The measure in the integral is the well defined free-field measure (Gaussian), and one can demonstrate [42] that the integral exists and converges for all the physical states. It is not difficult to check that the Bargmann n-photon states are mapped in the states (3.28); this is just an exercise in Gaussian functional integration. The integral transform (3.35) is a unitary one-to-one mapping from the Bargmann representation to the loop representation. It is an infinite-dimensional analogue of the Fourier transform that maps the coordinate representation into the momentum representation.

The loop representation can be *introduced* by first constructing the Bargmann representation and then defining the loop transform (3.35). For a discussion of this approach, see the book [2].

(v) The main motivation for the loop representation, presented at the beginning of this section, was the need for a quantum theory defined without any reference to the background metric. In this section, this goal does not seem to have been achieved, because the separation between positive frequency and negative frequency field relies on the existence of the background metric. However, the entire formalism can be reproduced starting from the self-dual connection, rather than from the positive-frequency connection. (The split between self-dual and antiself-dual connection is metric independent.) The idea of replacing positive–negative frequency with self-duality in the quantization was introduced and discussed by Ashtekar [49]. The loop quantization of the Maxwell field in terms of the self-dual potential is described in appendix 3.

(vi) Let me discuss the physical interpretation of the loop states $|\alpha\rangle$. As it is clear from its definition (equation (3.15)), the positive-frequency electric-field operator is diagonal in the loop representation. Since this is the annihilation operator, it follows that the loop states $|\alpha\rangle$ are eigenstates of this operator, namely they are coherent states. The corresponding classical configuration is given by their eigenvalue. The eigenvalue is the form factor, which in coordinate space is real, and is given by the distribution $\Delta^a[\alpha, x]$ (equation (1.75)) with support on the loop itself.

Thus, the state $|\alpha\rangle$ is the coherent state corresponding to the classical field configuration in which the magnetic field is zero, and there is a (distributional) electric field concentrated along the loop α and proportional to the tangent of the loop.

(vii) Note that a gauge has not been fixed, but the formalism is gauge invariant. Because of gauge invariance, the electric field has to be divergenceless. The simplest excitation of a divergenceless vector field cannot be a point excitation but has to 'continue' and (in absence of charges) has to be loop-like. Thus, loops are the simplest excitations of a divergenceless vector field, namely the *loops are the simplest gauge invariant excitations of the electric field.* In this sense, the loop representation is a natural representation of a gauge theory. Moreover, gauge theories *originated* from Faraday's idea of description of electric and magnetic force in terms of loops that fill the space: *the loops of the loop representation are precisely the quantum version of Faraday's 'force lines', which historically gave birth to gauge theories.*

(viii) In conclusion, the loop representation provides a consistent and complete quantum theory for the Maxwell theory, which is equivalent to the Fock representation.

3.2.2. Yang–Mills. In order to generalize the loop representation to non-Abelian theories, the problem is that the electric field is no longer gauge invariant. The solution is provided by the T^a observables (or the $T[R]$ smeared ribbon observables) defined in section 1.5.

Consider a Yang–Mills theory, where $A_a(x)$ is the Yang–Mills potential, which takes values in the adjoint representation of a Lie algebra $\mathcal{G}$, and $E^a(x)$ is the electric field. We pick for concreteness the group $SU(2)$, which is the one relevant for gravity. We consider the observables $T[\alpha]$ and $T[R]$ defined in section 1.5, and their Poisson algebra, which is the non-Abelian loop algebra introduced in section 1.5 (equation (1.40)).

We quantize the theory by considering again the space of loop functionals $\Psi(\alpha)$, and the following two quantum operators:

$$\hat{T}[\gamma]\Psi(\alpha) = \Psi(\alpha \cup \gamma) \tag{3.36}$$

$$\hat{T}[R]\Psi(\alpha) = \hbar \sum_i \epsilon_i \left(\Psi(\alpha \#_i R) - \Psi(\alpha \#_i R^{-1})\right) \tag{3.37}$$

the notation is described in section 1.5.1. Once again, the commutator algebra of these operators reproduces $-i\hbar$ times the corresponding classical Poisson algebra†.

The unsmeared form of the operator $\hat{T}^a[\gamma](s)$ is

$$\hat{T}^a[\gamma](s)\Psi(\alpha) = \frac{i\hbar}{2} \sum_i \Delta^a[\alpha, \gamma(s)] \left(\Psi(\alpha \#_i \gamma) - \Psi(\alpha \#_i \gamma^{-1})\right). \tag{3.38}$$

Note that the operator $\hat{T}^a[\gamma](s)$ acts on a loop α only if α and γ intersect at the point $\gamma(s)$. The action consists in inserting the loop γ in α, starting from the intersection. The point $\gamma(s)$ on the loop γ is denoted the *hand* of the operator $T^a[\gamma](s)$, and the action of the operator is denoted as the *grasping* of the hand over the loop α.

It is easy to define also loop operators corresponding to the loop operators with more than one index defined in section 1.5. The action of these operators is given by the sum of the grasping of each of their hands [8].

As in the Maxwell case, the loop functionals must satisfy the conditions (3.16) and (3.17) for irreducibility, but now the group is non-Abelian and the simple condition (3.18) is replaced by the condition

$$\Psi(\alpha \cup \beta) = \Psi(\alpha \# \beta) + \Psi(\alpha \# \beta^{-1}) \tag{3.39}$$

† At this point, the reason for choosing the set of piecewise smooth closed loops should be clear. On everywhere-differentiable (or smooth) loops the non-Abelian algebra would not close. There is a smaller set of loops which may be chosen (and maybe must be chosen in order to have an irreducible representation). These are all the loops obtained from smooth loops by rerouting at the intersections.

which implements the spinor identity (equation (1.69)) satisfied by the classical $SU(2)$ holonomies T. As in the Abelian case, it is not difficult to show that a loop functional that satisfies all these properties is entirely determined by its restriction on the single loops.

The relation between the loop representation and a functional Schrödinger representation is given by a non-Abelian analogue of the Abelian loop transform (3.35):

$$\Psi(\alpha) = \int \mathrm{d}\mu[A]\, \mathrm{Tr}\mathcal{P} \exp\left(\oint_\alpha A\right)\ \Psi[A]. \tag{3.40}$$

Unlike its Abelian version, this loop transform is far from being well defined because we do not have at our disposal a well defined gauge invariant measure $\mu[A]$†. In spite of these problems, the loop transform is a very useful device. It can be used as an heuristic trick. Indeed, the form of most of the loop operators has not been pulled out of the air, but it has been suggested by formal manipulations on the loop transform (see [50]). Moreover, shortly after the definition of this loop transform, Witten (with different motivations) has been able to construct a definition of the integral good enough to actually *compute* the integral in certain cases. I will come back to this later.

Non-Abelian Yang–Mills theories in this non-perturbative canonical loop formalism have not been extensively studied (see, however, [45] and [51]). The main difficulty that one may expect is related to the renormalization of the Hamiltonian operator. The B^2 term in the Hamiltonian may be defined by using a limiting procedure analogous to the one used in the Abelian case. In this way there is a built-in regularization of the operator. One expects standard ultraviolet divergences in the limit. However, there is a surprising result in perturbative quantum field theory which may be related to the formalism I am describing. In spite of the fact that the T observables are complicated non-polynomial operators, integrated in only one dimension, their expectation value is multiplicatively renormalizable *for all orders in perturbation theory* [52]. This result raised a certain interest several years ago, but, to my knowledge, this direction of research has not been pursued.

3.2.3. Lattice Yang–Mills. The loop quantum theory defined by the previous equations has a natural version on the lattice. The lattice version of the loop representation of Yang–Mills theories has been studied in detail. The lattice theory has a finite number of degrees of freedom, and its definition is completely under control. Indeed, the loop transform (3.40) is well defined on the lattice, where the gauge invariant measure is known. The transform defines a well defined change of basis in the Hilbert space. A complete construction of the theory along these lines is given by Smolin and the author in [53].

The new basis that defines the loop representation is the (overcomplete) basis formed by all the (spacelike) Wilson-loop states, which were introduced by Wilson and Suskind in the first investigations of Hamiltonian lattice gauge theories [47]. Therefore, the loop representation is the continuum limit of the Wilson-loops Hamiltonian formalism.

† This same functional integral transform will relate the loop representation of quantum general relativity with the connection representation; in this case the integral is even less defined than in the Yang–Mills case, because we need to assume the measure to be also diffeomorphism invariant.

It has been suggested that this loop lattice formulation may provide a numerical calculation method as an alternative to standard Monte Carlo methods. This suggestion has been tested by Brügmann in [54], where a $(2+1)$-dimensional $SU(2)$ theory has been numerically analysed using the loop representation, with results in very good agreement with the ones obtained in other ways.

3.2.4. Gravity in $2+1$ *dimensions.* The application of the loop quantization technique to general relativity in $2+1$ dimensions is a simple illustration of the natural way in which the loop representation deals with diffeomorphism invariance. The quantum theory was first constructed by Witten [28], using an Ashtekar-like classical formulation similar to the one described in section 2.2.2, and geometrical techniques. Witten's results were obtained again, using the loop representation, by Ashtekar, Husain, Samuel, Smolin and the author in [55]. I do not not discuss here the entire formulation of the theory, but only the key conceptual point which will be used again in the full theory.

The quantum representation of the loop observables (2.2.2) is given by the operators

$$\hat{T}[\gamma]\Psi(\alpha) = \Psi(\gamma \cup \alpha) \tag{3.41}$$

$$\hat{T}[R]\Psi(\alpha) = \hbar \sum_i \epsilon_i \left(\Psi(R\#_i\alpha) - \Psi(R\#_i\alpha^{-1})\right) \tag{3.42}$$

where now the multiple loops live in a two-dimensional space with non-trivial topology, and the 'ribbons' R, I recall, are (in two dimensions) just standard loops†.

The only remaining constraint is $D^i = 0$ (see (2.4)) or $F^i_{ab} = 0$. In terms of the T observables, this constraint is equivalent to the requirement that the holonomy of any two loops that can be smoothly deformed one into the other is the same, namely

$$T[\alpha] - T[\phi \cdot \alpha] = 0 \tag{3.43}$$

for any diffeomorphism ϕ in the connected component of the identity. The quantum constraint is therefore

$$\left(\hat{T}[\alpha] - \hat{T}[\phi \cdot \alpha]\right)\Psi(\alpha) = 0. \tag{3.44}$$

The solution is given by any state for which

$$\Psi(\alpha) = \Psi(\phi \cdot \alpha). \tag{3.45}$$

Does this mean that the state is constant everywhere on loop space? The answer is no, because two loops that wrap around the manifold in different ways, namely which are in two distinct homotopy classes of the manifold cannot be smoothly deformed one into the other. Thus, the physical states have the form

$$\Psi(\alpha) = \Psi(h(\alpha)) \tag{3.46}$$

where h is the homotopy class of the (multiple) loop α.

† In [55], the representation is introduced directly in terms of its restriction on the single loops sector.

The conclusion is that the physical quantum states of the theory can be represented as functions $\Psi(h)$ on the set of the homotopy classes h of the two-dimensional manifold. This is indeed the same conclusion reached by the previous investigation of this theory.

Equivalently, we may introduce a state $|h\rangle$ that has value one for the loops in the homotopy class h, and otherwise vanishes; and we may represent a physical state of the theory as a linear combination of homotopy classes

$$|\Psi\rangle = \sum_h c_h \, |h\rangle. \tag{3.47}$$

3.2.5. Other works on the loop representation. The loop representation was first constructed by Gambini and Trias [45]. The form of the loop representation developed by Gambini and Trias has certain interesting differences from the form described here. These authors consider the group structure that is given on the space of all the loops in a manifold (with a common base point) by the composition operation (at the base point). This group is denoted as the group of loops. The inverse is the loop with reversed orientation. They assume the existence of a norm on this loop space, and they consider a loop derivative defined in terms of this norm. The loop derivative is essentially the generator of the group of loops, and can be essentially identified with the $\hat{B}_a(x)$ operator defined above. This interesting construction could be very useful in gravity.

Other theories have been studied in the loop representation. Husain and Smolin have considered the quantization of general relativity with two Killing fields [18]. Chern–Simon theories have been studied by Miao Li [56]. Preliminary investigation of the loop representation for continuum Yang–Mills theories has been considered by Loll [57] and the author [51]. Loll [57] has discovered a way to get rid of the redundancy in the loop observables due to the spinor identity (3.39). She defines certain linear combinations of the T observables, denoted L observables, which solve the spinor identity. She studies the quantization of the L observables algebra, and finds an interesting generalization of the loop representation.

Rayner [58] has studied the possibility of a rigorous mathematical construction of the loop representation. He has introduced a natural scalar product in loop space and studied certain natural self-adjoint operators.

Manojlovic [59] has noted that the loop observables are invariant under change of sign of both E and A (and therefore are not good global observables in phase space) and has developed an alternative version of the loop observables which cure the problem. He has applied this version of the loop representation to $2+1$ gravity, by making use of Isham group theoretical quantization, and considered the application to the full theory.

Nayak [60] has studied the problem of time in $2+1$ gravity using the loop representation, and has considered the possibility writing the action directly in terms of loop observables. This is a very interesting open problem.

The loop representation has been rederived from a 'highest weight' Verma-module type representation of the loop-algebra by Aldaya and Navarro-Salas [61]. Using these techniques, these authors also define a modified representation of the loop algebra, and, in the context of quantum gravity, study the problem of the solution of the quantum constraints.

Works on the loop representation specifically devoted to quantum gravity are referred to in the next section.

3.3. Quantum theory without background metric

Before facing the technical difficulties of quantum general relativity, I discuss here some conceptual issues which raised a certain confusion in the literature. The subject of this section is controversial. I do not intend to describe the different solutions proposed; rather, I present an overall point of view, which may constitute a possible (but not the only possible) conceptual ground for the technical construction of the next section. The point of view presented here is not shared by all the people working on non-perturbative quantum gravity.

3.3.1. Observables. The physical interpretation of a classical dynamical theory with a gauge invariance requires that only observables which are gauge-invariant (have weakly vanishing Poisson brackets with all the constraints) have physical meaning. In the quantum theory, this requirement becomes stringent, for an operator is well defined on the space of the physical states if and only if it commutes (weakly) with the quantum constraints.

It has been suggested that this rule should be relaxed for general relativity, on the grounds that measurements require a physical reference frame, and the gauge of general relativity just reflects the freedom in choosing this reference frame. While it is certainly possible to have in the formalism objects that are not gauge invariant, still all the quantities that can be *predicted* by the theory—and therefore the quantities to which physical quantum operators can be associated—must be gauge invariant quantities. These quantities are the physical observables, in the sense of Dirac.

This comment is relevant for the interpretation of the quantum theory. One should be careful, indeed, to give physical meaning only to gauge-invariant properties of the wavefunction. For instance, to say that (in the metric representation) $|\Psi[g]|^2$ (where g is a '3-geometry', namely an equivalent class of 3-metrics under three-dimensional diffeomorphisms) represents *the probability that the geometry be* g, is certainly incorrect, because this statement is not invariant under the transformations generated by the scalar constraint, namely it is not invariant under four-dimensional diffeomorphisms. For a detained analysis of the problem of the observability in quantum gravity, see [62].

3.3.2. Time. A subproblem of the problem of defining the observables is the issue of time. To deal with the problem of time in gravity forces us to slightly extend standard quantum mechanics [63].

According to the discussion in the previous section, physical time evolution in gravitational physics should be expressed in a gauge-invariant fashion. It may be shown that *it is indeed possible* to write gauge-invariant observables that express evolution [62]. This evolution, however, need not be a standard Hamiltonian evolution. In other words, the kind of 'evolution' described by a constrained system with vanishing Hamiltonian may be a genuine extension of the evolution generated by a Hamiltonian, and, in general, cannot be reduced to it. Physically this reflects the absence of the 'absolute clock' postulates in Newtonian (or special relativistic) dynamics [64].

As far as the classical theory is concerned, we have enough physical intuition about solutions of Einstein equations to be content with the theory, even if it has a vanishing Hamiltonian. In the quantum theory, on the contrary, we are used to having at our disposal a Schrödinger equation (namely a Hamiltonian). But the Schrödinger equation and the Hamiltonian operator are equivalent to the assumption of the existence of the external absolute clock, which is in contradiction with the physics of the gravitational field.

It has been suggested that, because of this problem, general relativity and the Hilbert space formulation of quantum mechanics are intrinsically contradictory. This, I believe, is not necessarily the case. The constraint formulation of classical canonical theory is a genuine extension of Hamiltonian canonical mechanics [65], because there are systems that admit a formulation in terms of constraints, but not in terms of a Hamiltonian [66]. More precisely, there is an extension of symplectic mechanics—presymplectic mechanics—which has the advantage of treating time (clocks) on the same ground as other variables [63]. In a completely analogous fashion, the quantum theory of a constrained system is a genuine extension of the quantum theory of a Hamiltonian system. The corresponding quantum mechanics is a standard quantum mechanics, where, however, the axiom on the existence of the Hamiltonian is dropped and, in its place, the Hamiltonian constraints define evolution in implicit form.

The standard interpretation of quantum mechanics can be applied in the general case (finite norm states, self-adjoint operators which commute with the constraints, probability, projection of the wave function...), even in the absence of the Hamiltonian operator. For a detailed discussion, see [63, 66].

In conclusion, the physics of general relativity forces us to extend quantum mechanics by dropping the postulate of the existence of the Hamiltonian. The rest of standard quantum mechanics is still completely viable [63]. Evolution should be expressed by suitable gauge-invariant operators which represent evolution in spite of the absence of an absolute external clock.

3.4. Conclusions

The results described in this section can be summarized as follows.

(i) Every theory written in terms of a connection admits a representation in terms of functionals on a space of loops. The representation is defined by the loop operators (3.36) and (3.37).

(ii) These loops represent the gauge invariant quantum excitations of the electric field of the theory (the Faraday lines).

(iii) In the cases in which we have a complete control of the theory, like the non-Abelian lattice theory and the Abelian continuum theory, it is possible to rigorously prove the complete equivalence between the loops methods and standard formulations.

(iv) For non-Abelian connections, the action of the basic observables is related to rerouting of the loops at the intersections.

(v) In $2+1$ gravity, diffeomorphism invariance reduces the functionals of loops to functionals of homotopy classes, in agreement with other independent treatments of the theory.

(vi) The difficulty in constructing a meaningful quantum theory of gravity is the difficulty of constructing a quantum field theory on a manifold without a fixed metric structure. The loop representation does not require a background causal structure to be defined, and deals naturally with diffeomorphism invariance.

Armed with all this, we face quantum general relativity.

4. Non-perturbative quantum general relativity

In this section, I describe the present stage of the construction of a non-perturbative quantum gravity theory. Following the discussion at the beginning of section 3, the hypothesis here is the following. That the difficulties of *perturbative* quantum general relativity do not follow from any intrinsic incompatibility between general relativity and quantum mechanics; rather, they reflect the inadequacy of the standard Poincaré-invariant perturbative formulation of quantum-field-theory for a generally covariant theory as general relativity.

The project of a non-perturbative canonical quantization of general relativity dates back to the work of Wheeler and DeWitt [39] in the sixties. In their approach, the quantum dynamics of general relativity is encoded in the (Dirac) quantum version of the ADM constraints. These are the quantum constraint that implement three-dimensional diffeomorphism invariance and the quantum Hamiltonian constraint, also denoted as the Wheeler–DeWitt equation. These constraints are expressed as functional equations for the quantum states in a Schrödinger-like representation of the quantum theory. In spite of intense efforts in this direction, the complexity of these equations has long prevented substantial developments.

Ashtekar's formulation of general relativity in terms of the phase space of a Yang–Mills theory allows us to apply to quantum gravity the non-perturbative loop quantization technique developed in the previous section. As anticipated, the loop representation is in a position of handling diffeomorphism invariance. Indeed, the first result of the loop formulation is to provide the complete solution of the quantum diffeomorphism constraint. This is described in section 4.1.1.

The surprising result of the loop representation, however, is that the Hamiltonian constraint (namely the loop-analogue of the Wheeler–DeWitt equation) can also now be solved. More precisely an infinite-dimensional class of solutions of the entire set of constraints are known. These solutions are represented in terms of knots classes; they are described in section 4.1.2.

In order to understand the physical content of these knot states, it is necessary to relate them with the classical field (the spacetime geometry) and with the concept of gravitons. Work is in progress in this direction; its present state is described in section 4.2. The main result is a relation established between the graviton Fock-space of the linearized quantum theory and the knot states. At least in principle, it is possible to express the quantum vacuum state corresponding to flat Minkowski spacetime (and any n-graviton state) in terms of a linear combination of the knot states. In this way a physical interpretation of the exact knot states is established.

In the process of establishing this relation, certain surprising indications of a discrete structure of space around the Planck scale appear. These will be described in section 4.2.5. Finally, in section 4.3.2, I summarize the results obtained in the loop-representation of general relativity, discuss the open problems and the overall picture of quantum gravity which is emerging.

4.1. Loop quantum gravity

We choose as basic variables, to which we want to associate the quantum operators, the T observables (1.61) and (1.77). We associate to these observables the operators $\hat{T}$ defined in equations (3.36) and (3.37), which are defined on the space of loop functionals $\Psi(\alpha)$. The task is to solve the quantum constraints equations.

The internal gauge constraint is solved by using the gauge-invariant loop variables, and we do not have to worry about it.

4.1.1. Diffeomorphism constraints: the knot states. The diffeomorphism constraint is written in terms of loop variables in (1.65). The quantum diffeomorphism constraint

$\hat{C}(N)$ is obtained by substituting the classical T variable with the quantum $\hat{T}$ operators. A straightforward calculation (see for instance [67]), shows that the operator that one obtains is the generator of diffeomorphisms:

$$\hat{C}(N)\Psi(\alpha) = \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\phi_t \cdot \alpha) \tag{4.1}$$

where ϕ_t is the one-parameter family of diffeomorphisms generated by the vector field N. This is the loop representation version of the standard result (due to Higgs) that the ADM vector constraint is the generator of diffeomorphism invariance.

In exponentiated form, the first constraint equation is therefore equivalent to the requirement of diffeomorphism invariance on the loop functional

$$\Psi(\alpha) = \Psi(\phi \cdot \alpha). \tag{4.2}$$

This equation can be exactly solved in closed form. This is because the orbits of the diffeomorphism group in the space of the loops are well known: they are the knot classes. A knot, in fact, can be defined as an equivalence class of loops under diffeomorphisms: two loops can be mapped one into the other by a diffeomorphism (in the connected component of the identity) if and only if they are knotted in the same way. Thus, the general solution of the first quantum constraint is

$$\Psi(\alpha) = \Psi(K(\alpha)) \tag{4.3}$$

where K is the knot class of the loop α†. Equivalently, as we did in $2+1$ dimensions, we may introduce states $\Psi_K(\alpha)$ or, *á la* Dirac, simply $|K\rangle$, picked on the knot class K

$$\Psi_K(\alpha) \equiv \langle\alpha|K\rangle \equiv \begin{cases} 1 & \text{if } \alpha \text{ is in the knot class } K \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

and represent the general diffeomorphism-invariant state in quantum gravity as

$$|\Psi\rangle = \sum_K c_K \, |K\rangle. \tag{4.5}$$

A technical point needs to be emphasized here. In order to represent the loop algebra, loops with intersections and corners (points where the loop is non-differentiable) are also required (otherwise the algebra does not close); therefore, the knot classes K must be defined as the equivalence classes under diffeomorphisms of loops which may have also intersections and corners. These are denoted *generalized knot classes*, to distinguish them from the standard knot classes studied in knot theory (see [68]) which are the knot classes formed by the loops with no intersections and corners. I denote the loops with no intersections and no corners as *regular loops*, and the standard knot classes of knot theory as *regular knot classes*. What was shown above is that a diffeomorphism-invariant quantum state of the gravitational field can be represented as a linear combination of generalized knot-class states.

† If we consider the states as functions of multiloops, then we should consider link classes rather than knot classes. A link class is an equivalence class of multiple loops under diffeomorphisms. However, as stressed above (after equation (3.39)), the value of the wavefunction on multiple loops (and therefore on links) is determined by its value on the single loops (and therefore on knots).

This is the first basic result of the loop representation for quantum general relativity.

Some comments follow.

(i) This result is based on two physical ideas. The first is that loops are the natural objects for a Yang–Mills theory. The second is diffeomorphism invariance. As was already observed, the Ashtekar formalism provides the bridge, by expressing general relativity as a gauge theory. The physical excitations of the gravitational field can be represented by loops and, because of diffeomorphism invariance, these loops are only distinguished by the way they are entangled. Namely, the physical states of the quantum gravitational field may be described in terms of knots. From this perspective the result is quite natural.

(ii) In the metric representation, too, the vector constraint requires the state functional $\Psi[g]$ to be diffeomorphism invariant (here g is the three-dimensional metric). As sometimes stated, $\Psi[g]$ must be function of the '3-geometry' alone, namely it must have the same value for any two g related by an (active) diffeomorphism. While the physical meaning of this requirement is transparent, the explicit solution of the constraint is unknown because very little is known about the '3-geometries', namely about the orbits of the diffeomorphism group on the space of the metrics.

(iii) An example: consider the connection representation of quantum gravity [1,2], in which states are functionals $\Psi[A]$ of the connection. As discussed in section 3.2.2, the loop representation is related to the connection representation by the formal integral transform (3.40).

If we include the cosmological constant λ in the theory, there is one known solution to all the constraint equations in the connection representation. This is the exponent of the Chern–Simons integral of A_a^i

$$\Psi^{(\mathrm{CS})}[A] = \exp\left(-\frac{1}{\lambda}CS[A]\right) = \exp\left(-\frac{1}{\lambda}\int \mathrm{Tr}\,(A_a\partial_b A_c + \frac{2}{3}A_aA_bA_c)\epsilon^{abc}\right). \qquad (4.6)$$

This state is gauge invariant and diffeomorphism invariant by inspection, and it is easy to check that it satisfies the Hamiltonian constraint (this was pointed out by Kodama [69]).

The state $\Psi^{(\mathrm{CS})}[A]$ should be represented in the loop representation by

$$\Psi^{(\mathrm{CS})}(\alpha) = \int \mathrm{d}\mu[A]\,\mathrm{Tr}\,\mathcal{P}\exp\left(\int_\alpha A\right)\,\exp\left(-\frac{1}{\lambda}CS[A]\right). \qquad (4.7)$$

In spite of the difficulty of defining this functional integral, the integral has been calculated. The calculation is a celebrated calculation performed by Witten in [28]. The result is

$$\Psi^{(\mathrm{CS})}(\alpha) = J_\lambda(K(\alpha)) \qquad (4.8)$$

where $J_\lambda(K)$ is the Jones polynomial [68] in the variable λ. The Jones polynomial is a well known and well studied function on the space of the knot classes. Thus, there is a state that can be written in the two representations, and which is an exact solution of the Hamiltonian constraint in the presence of a cosmological constant. Note, however, that the Witten calculation does not specify entirely the loop state $\Psi^{(\mathrm{CS})}$, because in the calculation α is assumed to be regular.

(iv) A surprising feature of the general solution of the diffeomorphism constraint is that it is given in terms of a *discrete* basis. A discrete basis is unusual in field theory, but there is nothing particularly strange about it. The Fock space of *any* quantum field theory is a separable Hilbert space and therefore it is easy to construct a discrete basis in it.

4.1.2. Quantum dynamics: the Hamiltonian constraint. There is a surprising number of versions of the loop-representation Hamiltonian constraint that appeared in the literature.

(i) In the original paper on the loop representation [8], the quantum Hamiltonian constraint was constructed in terms of the limiting procedure defined in section 1.5.

(ii) In the same paper, it was suggested that there should be a simple operator in loop space with a direct geometrical meaning, denoted the shift-operator, which represents the Hamiltonian constraint. An incomplete definition of this operator was suggested and some preliminary calculations that indicated that the operator agreed with the one defined by the limiting procedure were given.

(iii) Later, Blencowe [67] suggested an alternative definition in terms of loop derivatives, which has the advantage of a larger domain of definition in loop space.

(iv) Gambini and Trias [45] in turn suggested a definition in terms of the generator of the 'loop group' defined in their formalism.

(v) Brügmann and Pullin [70] suggested that the limiting procedure that defines the constraint in the original paper is exactly equivalent to the Gambini and Trias operator, and this, in turn, to the shift operator.

The reasons for this diversity† is partially related to the fact that there are many ways of defining a quantum operator with a prescribed classical limit. A complete demonstration of the equivalence of these different approaches, is still lacking. In this review, I do not describe the details of the calculations, I refer for these to the quoted papers. I just present the main result on the solution of the Hamiltonian constraint and a sketch of the proof. The result is the following.

Theorem. If a loop state $\Psi(\alpha)$ has support only on the *regular* loops (namely if $\Psi(\alpha) = 0$ for every loop α which has corners or intersections), then $\Psi(\alpha)$ satisfies the quantum Hamiltonian constraint.

This is the second main result of the loop representation.

The classical constraint is $\mathrm{Tr}[F_{ab}\tilde{E}^a\tilde{E}^b]$. F_{ab}, being a curvature, is antisymmetric in the indices ab. In the quantum theory, the $\tilde{E}^a$ corresponds to the 'hand' in the $\hat{T}^{ab}$ operators. These hands act on the argument α of the loop functional $\Psi(\alpha)$ by breaking and rejoining (see equation (3.38) and the following comments). Any time they act, they produce a multiplicative coefficient proportional to $\dot{\alpha}^a$ in front of the result. The two hands corresponding to $\tilde{E}^a$ and $\tilde{E}^b$ produce two multiplicative factors $\dot{\alpha}^a$ and $\dot{\alpha}^b$. Since the ab indices must be antisymmetrized, there is a term $\dot{\alpha}^{[a}\dot{\alpha}^{b]}$ in the result of the action of the Hamiltonian constraint. The loop α must have, in at least one point, two different tangents $\dot{\alpha}$ and $\dot{\alpha}'$, in order $\dot{\alpha}^{[a}\dot{\alpha}^{b]}$ not to vanish. This may happen, for instance, if the loop intersects itself. But if the loop is regular there is no point of this kind, and therefore the Hamiltonian constraint, acting on that loop,

† To complicate the matter, Blencowe pointed out a technical mistake in the paper [8], which was corrected by slightly changing the definition of the quantum constraint [71] (the final result is not affected by the mistake or by the change).

gives zero†.

Having a set of solutions of the Hamiltonian constraint and the general solution of the diffeomorphism constraint, we may look for solutions of the entire set of constraints. This is easy, because the set of the regular loops transforms into itself under diffeomorphisms. Thus we have the following final result.

Any quantum state of the form

$$|\Psi\rangle = \sum_{\text{regular } K} c_K \, |K\rangle \tag{4.9}$$

is a solution of all the quantum constraint equations, namely it is a physical state of the quantum gravitational field. These states are denoted physical knot states.

Some comments follow.

(i) The theorem does not make any statement about the *general* solution of the equation. Little is known about that. The set of solutions described form an infinite-dimensional space. This space is a sector of the physical space of quantum gravity, it is denoted as *regular knot sector*. It is likely that there are other solutions of the full set of constraints. Indeed, solutions of the Hamiltonian constraints corresponding to loops with intersections have been found in the connection representation by Husain [72] and Brügmann and Pullin [73]. Brügmann and Pullin have developed a computer code which can construct solutions for intersections of any order.

(ii) The physical knot states are solutions of the quantum *dynamics* of the gravitational field. To clarify this point consider the analogy with the quantum mechanics of a free relativistic particle. The quantum Hamiltonian constraint is the analogue to the Klein–Gordon equation. In fact, the Klein–Gordon equation too can be obtained as a Dirac quantum constraint that quantizes the classical constraint $p_\mu p^\mu - m^2$ which defines the dynamics of a free particle. The physical knot states $|K_{\text{regular}}\rangle$ (which solve the Hamiltonian constraint) are the analogue of the quantum states

$$\psi_{\boldsymbol{k}}(\boldsymbol{x}, t) = \exp\left(\mathrm{i}\boldsymbol{k}\bot\boldsymbol{x} - \sqrt{|\boldsymbol{k}|^2 + m^2 t}\right) \tag{4.10}$$

which solve the Klein–Gordon equation. These states contain also the entire evolution (t-dependence) of the state.

(iii) This analogy can be extended: the unconstrained quantum state space of a free particle is spanned by the states $|k^\mu\rangle$. The solutions of the Klein–Gordon equation are spanned by the subset of these states which are on the mass shell: $|k^\mu_{\text{mass shell}}\rangle$. This is completely analogous to the $|K\rangle$ and $|K_{\text{regular}}\rangle$ structure. In the momentum representation ($\langle k^\mu|\psi\rangle = \psi(k^\mu)$), the Klein–Gordon equation is a statement on the support of $\psi(k^\mu)$ (mass shell). In the loop representation ($\langle K|\Psi\rangle = \Psi(K)$), the Hamiltonian constraint is a statement on the support of $\Psi(K)$ (regular knots). (The analogy is partial, since we do not know the general solution of the Hamiltonian constraint.) Thus, the loop representation 'diagonalizes' the Hamiltonian constraint equation (partially), in the same sense in which the momentum representation diagonalizes the Klein–Gordon operator.

(iv) The fact that the Hamiltonian constraint turns out to be diagonal in the loop representation, namely that we may find solutions simply by restricting the support of the wave function, is quite surprising at this point. Historically, the development

† Note the analogy with the computation in section 2.2.4.

followed a different path. The fact that the states $|\alpha\rangle$, where α is a regular loop, satisfies the Hamiltonian constraint was discovered by Jacobson and Smolin in the connection representation [38] before the definition of the loop representation. This result was the starting point of the loop representation: the idea was to take the Wilson loop states $|\alpha\rangle$ as basis states. In this basis, the Hamiltonian operator was expected to be diagonal, and the functions with support on the regular loops were expected to satisfy the constraint. The loop representation is the realization of this programme.

The 'miraculous' aspect of the constraint solutions in the loop representation is the fact that the *same* basis is the basis that 'diagonalizes' the Hamiltonian constraint *and* the basis that allows us to immediately solve the diffeomorphism constraint. (By 'miraculous', as usual in theoretical physics, I denote something we like but do not understand.)

(v) It is difficult to judge to what extent the discovery of these solutions brings us closer to the construction of a consistent non-perturbative quantum gravity. Major problems are open. Among these, the construction of the physically observable quantities and the definition of the inner product. In any case, in order to understand these solutions, the first step to take is to unravel the physics they contain. This is the argument of the following section.

4.2. Interpretation: the linearization problem

To get some understanding of the physics contained in the knot states, we need to relate them with usual concepts in terms of which gravity is described. Since we are dealing with pure gravity, we expect the theory to describe, in some approximation, a state in which there are gravitons wandering around some background geometry. How can this physics be described in the loop picture? Equivalently, *how can we get a metric manifold and metric relations from the purely topological world of the knots?*

To describe gravitons on a background geometry, I introduce the background metric $g^{(0)}$. The physical metric g will be given by the background metric $g^{(0)}$ plus small dynamical fluctuations. I choose $g^{(0)}$ to be flat, and fix a coordinate system in which it is the Euclidean metric. The Einstein equations can be linearized around this flat metric. The corresponding quantum theory describes two (traceless transverse) gravitons, namely a spin-2 particle. There should be some limit (low-energy, or long-distance limit) in which the full theory reproduces this free-graviton theory.

We are interested to find the description of these free gravitons *within the knot framework*. Note that this point of view is reversed with respect to the standard one: the problem is not how the gravitons describe the full theory, but the way the full theory may describe states that look like gravitons, at least at large distances.

To achieve this result, a mapping has to be found between the space of the gravitons states and the space of the quantum knot states. If $|p,\sigma\rangle$ is, say, a one-graviton state (with momentum p and polarization σ), we want to calculate the coefficients of the expansion

$$|p,\sigma\rangle = \sum_K c_K^{(p,\sigma)} |K\rangle. \tag{4.11}$$

In particular, the quantum state corresponding to the (free-field vacuum on the) background geometry $g^{(0)}$ will also be a linear combination of knots:

$$|0\rangle = \sum_K c_K^{(0)} |K\rangle. \tag{4.12}$$

The task is to calculate the coefficients $c_K^{(0)}$ that represent this flat geometry.

The first step is to choose a convenient way to represent the free graviton theory. If $\mathcal{H}_{\rm L}$ (L for linear) is the Hilbert space of this theory, the second step is to find the mapping $\mathcal{M}$, that relates $\mathcal{H}_{\rm L}$ with the knot states space $\mathcal{K}$.

Assuming that there is no degeneracy, this mapping is uniquely determined as follows. If the same classical observable O is represented by the operator $\hat{O}_{\rm L}$ on $\mathcal{H}_{\rm L}$ and by the operator $\hat{O}$ on $\mathcal{K}$, then $\mathcal{M}$ must send $\hat{O}_{\rm L}$ in $\hat{O}$. Namely

$$\hat{O}_{\rm L} = \mathcal{M}\,\hat{O}\,\mathcal{M}^{-1}. \tag{4.13}$$

To simplify the determination of $\mathcal{M}$, it is convenient to start by a formulation of the free theory as similar as possible to the full theory. Thus, a loop representation of linearized general relativity is needed.

4.2.1. Quantum linear gravity. The loop representation of quantum linear general relativity has been constructed by Ashtekar, Smolin and the author [43]. Here, I give only a brief account of the construction. The classical linear theory is given by expanding the canonical variables around the flat-space solution

$$\tilde{E}_i^a = \delta_i^a + G\ e_i^a \tag{4.14}$$

$$A_a^i = 0 + G\ a_a^i \tag{4.15}$$

(G is the Newton constant). Internal indices can be transformed to space indices, and vice versa, using the background metric. From now on, lower case letters indicate the objects in the linear theory. The theory is given by the linearized constraints

$$c^i = \partial_a e^{ai} + a_k^j \epsilon^{ijk} \tag{4.16}$$

$$c_a = f_{ai}^i \tag{4.17}$$

$$c \ \ = f_{jk}^i \epsilon_i^{jk} \tag{4.18}$$

where f_{ab}^i is the *Abelian* curvature of a_a^i. The internal gauge constraints c_i have vanishing Poisson brackets with each other, and the connection a_a^i transforms as an Abelian connection under their action. Indeed, in the linear limit the group $SO(3)$ reduces to $U(1) \times U(1) \times U(1)$.

The Hamiltonian that generates the standard Minkowski time evolution is

$$H = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[\left(e(-k)_a^a - \frac{G}{k^2} b(-k)^a{}_a \right) b(k)_c^c - \left(e(-k)_{ka} - \frac{G}{k^2} b(-k)_{ka} \right) b(k)^{ak} \right] \tag{4.19}$$

where b_a^i is the magnetic field: $b_a^i = 1/2\epsilon_a^{bc} f_{bc}^i$. This Hamiltonian is analogous to three copies of the Hamiltonian of the self-dual formulation of the Maxwell theory.

There are two problems, not present in Maxwell theory, that have to be addressed in order to construct the loop representation of linearized gravity. The first problem is how to deal with the internal index $i = 1, 2, 3$. Since the kinematics of linear gravity is like three copies of Maxwell theory, it is natural to consider the triple tensor product of the Maxwell state space with itself. This means that it is natural to choose the functions of *triplets* of multiple loops as quantum states†.

The second problem is that the Ashtekar connection is self-dual. In order to remain as close as possible to the formalism of the full theory, we want to use the holonomy of the self-dual connection. Now the loop quantization of Maxwell theory in terms of the self-dual (rather than positive frequency) connection is described in appendix 3. The formalism is similar to the positive-frequency case, but the difference in the Hamiltonian and in the reality conditions results in the presence of certain Gaussian exponentials of the form factors in the states. These exponentials are divergent. To cure these divergences a standard regularization procedure does not seem to be sufficient [43]. The way out is provided by the use of the holonomies 'averaged' over small tubes around the loops, which is described in appendix 2. As we shall see, this technical point has far-reaching consequences.

Because of the use of the self-dual connection, the new variables formalism is chiral asymmetric (of course there is a spicular formalism in term of the antiself-dual connection). As a consequence, gravitons of the two opposite helicities turn out to be described in this formalism in a remarkably asymmetric fashion. Again, see appendix 3 where the same is true in the quantization of the Maxwell field in terms of the self-dual connection. This curious difference in the description of the left- and right-handed gravitons appears also in other formalisms aimed toward quantum gravity, in particular in Penrose twistor approach [74], and in the Kozameh–Newman light-cone cuts formalism [6].

Generalizing the quantization of Maxwell field given in appendix 3, we define the three Abelian holonomies

$$t^i[\gamma] = \exp\left(\oint_\gamma a^i\right). \tag{4.20}$$

Note that this holonomy, in spite of being gauge invariant under internal gauge transformations, has an i index, and note the absence of the path ordering and trace. As in the Maxwell case, the loops can be smeared in terms of a universal function. The smeared holonomy is

$$t^i_\epsilon[\gamma] = \exp\left(\int \mathrm{d}^3x F^a_\epsilon[\gamma, x] a^i_a(x)\right). \tag{4.21}$$

The basic variables for the quantization will be these holonomies and the *symmetrized* linearized triads $h^{ai} = \frac{1}{2}(e^{ai} + e^{ia})$. The antisymmetric part of the triad is gauge. The loop algebra is

$$\{t^i_\epsilon[\gamma], h^{ja}(k)\} = i\delta^{i(j} F^{a)}_\epsilon[\gamma, k] \tag{4.22}$$

The loop representation of this loop algebra is defined on the space of states of the form $\psi(\alpha_1, \alpha_2, \alpha_3)$ where each α_i is a multiple loop. I use the notation

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \tag{4.23}$$

and the notation

$$\alpha \cup_1 \beta = (\alpha_1 \cup \beta, \alpha_2, \alpha_3). \tag{4.24}$$

† The quantum theory of three particles is given in terms of states $\psi(x_1, x_2, x_3)$.

The representation is obtained by taking the tensor product of the Maxwell representation with itself:

$$\hat{t}^i_\epsilon[\gamma]\,\psi(\alpha) = \psi(\alpha \cup_i \gamma) \tag{4.25}$$

$$\hat{h}^{ai}(k)\,\psi(\alpha) = \hbar\; F^{(a}_\epsilon[\alpha^{i)}, k]\;\psi(\alpha). \tag{4.26}$$

This algebra, as can be directly checked, reproduces the classical loop algebra.

Now we have to solve the constraints. The detailed calculation can be found in [43]. The result is that a state satisfies all the constraints if and only if it is a function of

$$\sigma_{ab}(k)\; F^a_\epsilon[\alpha^b, k] \tag{4.27}$$

where $\sigma_{ab}(k)$ is symmetric, transverse and traceless. This is the standard result on the physical degrees of freedom of the graviton. Therefore, as we did for Maxwell, we can introduce the two independent physical components of the form factor (see the appendix 3 for the definition of the transverse basis $m^a(k), \bar{m}^a(k), k^a/|k|$)

$$F^+_\epsilon[\alpha, k] = m_a(k) m_b(k)\; F^a_\epsilon[\alpha^b, k] \tag{4.28}$$

$$F^-_\epsilon[\alpha, k] = \bar{m}_a(k)\bar{m}_b(k)\; F^a_\epsilon[\alpha^b, k]. \tag{4.29}$$

Finally the eigenstates of the Hamiltonian are completely analogous to the Maxwell case. The vacuum is

$$\psi_0(\alpha) = \exp\left(-\hbar \int \frac{\mathrm{d}^3 k}{(2\pi)^3} F^+_\epsilon[\alpha, -k] F^+_\epsilon[\alpha, k]\right). \tag{4.30}$$

The right-handed n-graviton *states are given by homogeneous polynomials of degree* n in F^+ times the vacuum. The left-handed n-graviton states are given by Hermite polynomials in F^- times the vacuum. The final result is entirely equivalent to the usual Fock space representation.

4.2.2. The mapping $\mathcal{M}$. In this section, I describe the relation between the graviton states $\psi(\alpha)$ and the knot states. This and the following sections describe recent work by Ashtekar, Smolin and the author [44], which is still in progress. Therefore the present and following sections should be considered as a progress report of developing ideas.

I begin by working in the space of the unconstrained states $\Psi(\alpha)$, and I deal later with the restriction to diffeomorphism-invariant knot states.

The key to identify the mapping between the linear theory and the full theory is equation (4.13) above. In order to use this equation it is necessary to write the linearized loop observables t^i_ϵ and h^{ab} in terms of the full loop observables T and T^a. This cannot be done exactly, because we may write only $SO(3)$ invariant quantities in terms of T and T^a, and the linearized loop observables are not $SO(3)$ invariant. However, this can be done to first order in the Newton constant, and this is enough here, because the identification between the full theory and the linear theory should only hold to first order in G (the linear theory is meaningless beyond first order in G).

To first order in G we have

$$t^i_\epsilon[\gamma] = 1 + G\int \mathrm{d}^3x F^a_\epsilon[\gamma, x] A^i_a(x) = 1 + G\int \mathrm{d}^3x\; f_\epsilon(x)\; \oint_{(\gamma+x)} A^i \tag{4.31}$$

where $\gamma + x$ is the loop obtained by displacing rigidly γ by an amounts x (this makes sense because there is the background geometry). In the full theory we have, again to first order in G,

$$T^a[\gamma](s) = 2 + G\,\delta_i^a \oint_\gamma A^i. \tag{4.32}$$

Therefore, up to order G, the relation between the two is

$$t_\epsilon^i[\gamma] = \delta_i^a \int \mathrm{d}^3x\ f(x)\ T^a[\gamma + x](0) - 1. \tag{4.33}$$

This equation is sufficient for our aim.

Now, we are looking for a map $\mathcal{M} : \Psi \longmapsto \psi$ such that

$$\hat{t}_\epsilon^i[\gamma]\ \mathcal{M} = \mathcal{M}\left(\delta_a^i \int \mathrm{d}^3x\ f_\epsilon(x)\ \hat{T}^a[\gamma + x](0) - 1\right) + \mathrm{O}(G^2). \tag{4.34}$$

This is the basic equation for the determination of $\mathcal{M}$. The map $\mathcal{M}$ that satisfies equation (4.34) is given below in equation (4.39). In the following, it is constructed step-by-step from equation (4.34).

I start from the following ansatz: that there exists a (multiple) loop Δ in the multiple loop space, such that

$$\psi(\emptyset,\emptyset,\emptyset) = \Psi(\Delta) \tag{4.35}$$

whenever $\psi = \mathcal{M}\Psi$. (Here $\emptyset$ is the no-loop multiple loop.)

In equation (4.34), the operator $\hat{t}_\epsilon^i[\gamma]$ in the LHS creates a γ^i loop in the argument of ψ, while the operator $\hat{T}^a[\gamma]$ on the RHS attaches a loop γ to the argument of the loop functional Ψ. By acting three times with the equation (4.34) on a state Ψ, then evaluating in Δ and using the ansatz we may build the entire $\mathcal{M}$.

The loop Δ may be specified by requiring that also h^{ab} transforms appropriately. Here, I do not try to derive the properties of Δ; rather, I postulate these properties and study their consequences.

Let Δ be the union of three multiple loops: $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$, which have the following property

$$\dot{\Delta}_i^a(s) = \delta_i^a \tag{4.36}$$

and assume that the single loops that form one component Δ_i (which are parallel because of the last equation) are equally spaced. Finally, assume that Δ does not contain self intersections. These properties determine Δ almost completely. The multiple loop Δ is denoted *the weave*. The weave is formed by a three-dimensional cubic lattice of non-intersecting lines. A key quantity not yet specified is the 'lattice spacing' a, namely the distance between the single parallel loops. The following section will be devoted to a discussion of this quantity.

Let me work out the mapping $\mathcal{M}$ determined by Δ. If $\psi = \mathcal{M}\Psi$, then

$$
\begin{aligned}
\psi(\alpha,\emptyset,\emptyset) = (t^1[\alpha]\psi)(\emptyset,\emptyset,\emptyset) &= (t^1[\alpha]\mathcal{M}\Psi)(\emptyset,\emptyset,\emptyset) \\
&= \left[\mathcal{M}\left(\int d^3x\ f(x)\ \delta^1_a\hat{T}^a[\alpha+x](0) - 1\right)\Psi\right](\emptyset,\emptyset,\emptyset) \\
&= \left(\int d^3x\ f_\epsilon(x)\ \delta^1_a\hat{T}^a[\alpha+x](0) - 1\right)\Psi(\Delta) \\
&= \int d^3x\ f_\epsilon(x)\ \delta^1_a F^a_\epsilon[\Delta,(\alpha+x)(0)] \\
&\quad \times [\Psi(\Delta\#(\alpha+x)) - \Psi(\Delta\#(\alpha+x)^{-1})] \\
&= \int d^3x\ f_\epsilon(x)\int ds\ \delta^3(\Delta_1(s),(\alpha+x)(0)) \\
&\quad \times \Psi(\Delta_1\#^\pm(\alpha+x)\cup\Delta_2\cup\Delta_3).
\end{aligned}
$$

I have used the definition of t^1, the fact that $\psi = \mathcal{M}\Psi$, the basic equation (4.34), the ansatz, the definition of $\hat{T}^a$ and the property of the tangent of the weave. In the last line I have introduced the notation

$$\Psi(\alpha\#^\pm\beta) = \Psi(\alpha\#\beta) - \Psi(\alpha\#\beta^{-1}). \tag{4.37}$$

The final quantity is finite and well defined.

The result can be described as follows. The zero loop in the linear theory 'corresponds' to the weave in the full theory; the loop α_1 corresponds to the weave plus the loop α attached (in the two possible ways) to the Δ_1 component of the weave. More precisely, to a linear combinations of such loops, in each one of which α is attached to the weave in a slightly different position.

A second run of the same calculation gives

$$
\begin{aligned}
\psi(\alpha,\beta,\emptyset) = \int f_\epsilon(x)\int ds\ \delta^3(\Delta_1(s),(\alpha+x)(0))\int f_\epsilon(y)\int\ dt\ \delta^3(\Delta_2(t),(\beta+y)(0)) \\
\times\ \Psi(\Delta_1\#^\pm(\alpha+x)\cup\Delta_2\#^\pm(\beta+y)\cup\Delta_3) + \mathrm{O}(a/\epsilon)
\end{aligned}
\tag{4.38}
$$

where the last term comes from the grasping over α and is as small as a/ϵ (recall that a is the lattice spacing of the weave). And similarly for a third run with the third component of the linear loop. These equations define $\mathcal{M}$. Note that for consistency the term in a/ϵ must be small in the approximation that we are using. This follows from the fact that the $\hat{t}^i_\epsilon$ operators commute, and this term breaks the commutativity. Therefore, we must choose a version of the linear theory in which the smearing ϵ is much larger than the weave lattice spacing a.

The picture that emerges is the following. The loops α_i that describe gravitons in the linear theory are related to complex loops in the full theory, which are obtained by inserting the α_i on the weave, in the specified manner. In terms of abstract states the result can be written as

$$
\begin{aligned}
\mathcal{M}^\dagger|\alpha\rangle = \int d^3x f_\epsilon(x)\int ds\ \delta^3(\Delta_1(s),(\alpha_1+x)(0))\int d^3y f_\epsilon(y) \\
\times\int\ dt\ \delta^3(\Delta_2(t),(\alpha_2+y)(0))\int d^3z f_\epsilon(z)\int du\ \delta^3(\Delta_3(u),(\alpha_3+t)(0)) \\
|\Delta_1\#^\pm(\alpha_1+x)\cup\Delta_2\#^\pm(\alpha_2+y)\cup\Delta_3\#^\pm(\alpha_3+p)\rangle\!\rangle + \mathrm{O}(a/\epsilon).
\end{aligned}
\tag{4.39}
$$

Using this equation, we can express the linearized $|\alpha\rangle$ states in terms of the full theory $|\alpha\rangle$ states.

Essentially a state $|\alpha\rangle$ corresponds to a linear combination of full theory states obtained by inserting the three α^i loops on the weave. These insertions look like an embroidery over the weave.

Given a quantum state $\Psi(\alpha)$ in the full theory, the equations above that define $\mathcal{M}$ produce a unique corresponding quantum state $\psi(\alpha)$ in the linear theory, which represents the same physics described in the linearized variables. Note that this construction is simply the loop space version of the equation

$$\psi(h) = \Psi(\eta + h) \tag{4.40}$$

which relates the metric-representation quantum state $\Psi(g)$ with the linear-gravity Schrödinger-representation quantum state $\psi(h)$. The weave Δ plays the role of the background 3-geometry η.

4.2.3. Gravitons from knots: the embroidery. At this point, the last step can be taken, by considering diffeomorphism invariance. Assume that a state $\Psi(\alpha)$ depends only on the knot class of α. A key observation is then the following one.

The states $\psi(\alpha)$ depend on the actual position in space of the loops α_i. Namely $\psi(\alpha)$ changes under any displacement of α_i; the states $\Psi(\alpha)$, on the contrary, depend only on the way α is knotted. Is this a source of inconsistency for the relation developed above between the two descriptions of the quantum field?

If α_i is displaced to, say, α'_i in $\psi(\alpha)$, then, under the $\mathcal{M}$ mapping, α'_i is entangled around the weave Δ in a different way than α_i. Therefore a 'shifted' linear loop does correspond to an inequivalent knot, unless the shifting is smaller than the weave lattice spacing. Thus, by postulating that $\Psi(\alpha)$ is a knot state, only information on the linearized states at scales smaller that the lattice spacing a is lost.

The space position of the loops that represent gravitons—in the linear theory—is coded in the entangling of these loops with the weave—in the full theory. The weave translates between metric properties and topological relations. It allows metric relations to emerge from the purely topological world of the knots. These metric relations, however, exist only at scales larger than a.

The picture that emerges recalls an embroidery. Embroidery is the art of constructing pictures (which have metric properties) by using only the knotting of a thread (topology). In the embroidery, we have a one-dimensional object, the thread, which first builds up an higher-dimensional space, the weave, by self-entangling; then, the thread may draw shapes by getting entangled with the weave.

Up to the a scale, it is possible to reproduce any state of linear gravity in terms of states that depend only on knots. In particular, recall that the linear vacuum on Minkowski spacetime is the loop functional (4.30). We have all the ingredients to calculate the coefficient $c_K^{(o)}$ in the knot basis of this state:

$$\exp\left(-\int \frac{\mathrm{d}^3k}{(2\pi)^3} F_\epsilon^+[\alpha,-k] F_\epsilon^+[\alpha,k]\right) = \langle\alpha|\sum_K c_K^{(o)}|K\rangle. \tag{4.41}$$

Similarly, we can define the linear combinations of knots that correspond to n-graviton states. Smolin [10] has shown that the requirement that the state $\Psi(\alpha)$ is function of a knot classes translates under $\mathcal{M}$ (to first order in G) to the statement that there are no longitudinal gravitons.

4.2.4. Other background geometries. A conjecture. Up to now, I have considered only a flat background geometry. The relation between the weave Δ and the flat metric $g^{(0)}_{ab}(x) = \delta_{ab}$, or more precisely, the flat triad $E^{(0)a}_{\ i}(x) = \delta^a_i$, is given by equation (4.36). Given an arbitrary background geometry, defined by the metric $g^{(0)}_{ab}(x)$, or by the triad $E^{(0)a}_{\ i}(x)$, it is natural to assume that the corresponding state in the full theory is constructed around the knot defined by the 'distorted weave' defined by

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$$
$$\dot{\Delta}^a_i(s) = E^{(0)a}_{\ i}(\Delta_i(s)). \tag{4.42}$$

It is easy to show that if two triads $E^{(0)a}_{\ i}(x)$ and $E'^{(0)a}_{\ i}(x)$ are related by a diffeomorphism, then they define the same knot via equation (4.42).

It seems reasonable to conjecture that two metrics which are *not* related by a diffeomorphism define, via equation (4.42), two *different* knots, provided that the lattice spacing a is taken sufficiently small.

This observation suggests that it may be possible to establish a correspondence between knot classes of a manifold and equivalence classes of metrics (3-geometries) of the manifold. Consider for simplicity a three-dimensional manifold with boundaries and consider all the possible metrics on the manifold that go to the flat metric on the boundary sufficiently fast. Assume that a unique way to associate a triad field to every metric has been chosen. Consider a metric g. Fix a lattice of points, with lattice spacing a, on the boundary, and, originating from these points, integrate the triad fields, as in equation (4.42), up to the point in which the integral line reaches the boundary again (assuming that the metric is sufficiently regular so that every integral line emerges from the interior). The integral lines define a knot (more precisely, a braid), which I denote $K_a[g]$. It is clear that diffeomorphic-equivalent metrics define the same knot. The conjecture is that, provided that a is taken small enough, if two metrics g and g' are not diffeomorphic-equivalent, then $K_a[g]$ and $K_a[g']$ are different knots.

If this construction can be made precise, and the conjecture is correct, then it is possible to characterize a 3-geometry, up to any given scale, by assigning the corresponding knot.

4.2.5. Emergence of the Planck scale structure. What is the value of the lattice spacing a? Recent calculations seem to indicate that the theory fixes the value of a. These are preliminary results, and the content of this section is still speculative.

Consider the three-dimensional metric q_{ab}. The metric is not well defined as an operator in the loop representation, for it is a product of local operators $\tilde{\tilde{q}}^{ab}(x) \sim E^{ia}(x)E^{ib}(x)$. However, it is possible to define an integrated version $q(F)$ of the metric, by smearing $E^a(x)$ and $E^b(x)$ over a region of finite radius, with a smearing function $F_{ab}(x)$ which varies only a certain large scale. Let $\delta(F)$ be the value of the flat Euclidean metric smeared with F.

It is possible to construct an operator $\hat{q}(F)$ in the loop representation such that its classical limit is the smeared metric $q(F)$. Then we have the following result [44].

The loop state $|\Delta\rangle$ is an eigenstate of the operator $\hat{q}(F)$ with eigenvalue $\delta(F)$

$$\hat{q}(F)\ |\Delta\rangle = \delta(F)\ |\Delta\rangle \tag{4.43}$$

if and only if the lattice spacing is exactly the Planck length:

$$a = \sqrt{\frac{G\hbar}{c^3}}. \tag{4.44}$$

Some comments follow.

(i) The demonstration of this result involves a careful construction of the smeared metric operator, and a delicate calculation with the loop operators. The result is still at some preliminary stage, but the reason for the result is perhaps intuitive: every thread of the weave carries a certain 'flux' of metric. The weave gives a certain approximation of a flat metric. One may think that it is possible to obtain a better approximation by having a thinner lattice, but it is not so, for if we double the number of threads, we get a doubled number of elementary excitation of the 'flux' of the 3-metric, and therefore we do not get the Euclidean metric, but the double of the Euclidean metric.

(ii) The result is coordinate invariant, and in a sense, scale invariant: if we double the number of the loops, then the coordinate distance between each single thread is half of the Planck length. But the resulting (inverse, densitized) metric is four times the Euclidean metric. As a consequence, the *invariant* distance between the threads is *still* the Planck length. The result, therefore, can be stated by saying that at whatever distance we put the threads of the weave one from the other, they always turn out to be at a *physical* Planck distance. More precisely, they *determine* a (Planck-length) unit of distance.

(iii) The emergence of the Planck length may seem surprising. The Plank constant comes from the quantization (there is the usual Planck constant in the definition of the quantum operators); but how does the Newton constant enter the game, given that there is no Newton constant in the vacuum Einstein equations? Consider the classical theory of a free (non-relativistic) particle. The equation of motion ($\ddot{x} = 0$) does not contain the mass. There is no way to measure the mass by observing the classical motion of a free particle. However, the mass does enter in the quantum theory: the Schrödinger equation contains the mass, and the spread of the wave packet (or the Compton wavelength of the particle) depends on the mass. By measurements on the quantum particle, we may measure its mass. The specification of the mass is required in order to write the Lagrangian and the Hamiltonian theory (the mass appears as a multiplicative overall factor in the action). Physically, the Heisenberg indetermination relations know the mass, because they are defined between position and momentum, and the kinematical (measurable) indetermination between position and velocity is the Plank constant divided by the mass.

In pure gravity, the Newton constant follows the same pattern as the mass for the particle. The classical equations of motions do not depend on G, but $1/G$ comes in front of the Lagrangian and enters the definition of the momenta. The Heisenberg indetermination relations between the 3-metric and the extrinsic curvature depend on the Planck constant multiplied by G. It may not be the more efficient way, but in principle it is possible to measure G in pure-quantum-gravitational experiments.

(iv) The physically interesting state is not the loop state $|\Delta\rangle$, but the knot state $|K_\Delta\rangle$, where K_Δ is the knot to which the weave Δ belongs. If the ideas developed so far are correct, we expect that the outcome of any diffeomorphism invariant measurement of the geometry on $|K_\Delta\rangle$ should be flat space, provided that the geometry is tested only on scales much larger than the Planck length.

(v) A discrete structure at the Planck level is intriguing. The existence of a discrete structure has been suggested many times, but here the structure emerges from the theory, without artificial inputs. At this stage it is not clear how we should take this result. In particular, it is not clear what is precisely the physical meaning of the state $|\Delta\rangle$. It is clearly related to flat space, but how? Note that the quantum field theoretical vacuum of the linear theory is 'peaked' on the knot state $|K_\Delta\rangle$.

(vi) The weave Δ was introduced in order to discuss the relation with the linear theory, but the result described in this section is unrelated to the linear theory. As far as the the linear theory is concerned, recall that the mapping from knot space to the graviton states is consistent only provided that in the linear theory we use a smearing ϵ much larger than a. Since a is the Planck length, it follows that the graviton picture makes sense only at scales much larger than the Plank length. The origin of the divergences in perturbation theory are integrals at small distances, namely the assumption that the graviton picture makes sense at every scale. In the non-perturbative theory, the result on the discrete structure at the Planck scale may be a concrete indication of how perturbation theory goes wrong.

(vii) It is not clear to what extent a similar discrete structure occurs in the loop quantization of the Maxwell field. The single elementary excitations of the Maxwell field are loop-like and quantized. Thus, it seems that the magnetic flux through a fixed surface should be quantized (in units of Bohr magnetons). This is not unreasonable: it makes sense, for instance, to interpret the quantization of the magnetic flux measured by a SQUID magnetometer as a quantum property of the electromagnetic field itself. To my knowledge, a rigorous analysis of the spectrum of the magnetic-flux field-operator in the free Maxwell theory, using standard formalism, has never been performed. (The difficulties come from the boundaries of the 2-surface. If the 2-surface has no boundary, the flux *is* quantized, but the quantization can be attributed to topological effects.) On the other side, in the loop representation there are technical differences between Maxwell theory and gravity (see [43]) that indicate that the flux quantization is peculiar of gravity.

4.3. Concluding remarks

4.3.1. Open problems. The construction outlined is preliminary. Some of the open problems are the following.

(i) The definition of the physical observables. The linearization may help to find physical observables. The linearization around flat space by itself does not break diffeomorphism invariance. (It is the use of the background metric to fix the causal structure that breaks the invariance.) Provided that the wavefunction is (in the same sense) concentrated around flat spacetime, the linearized gauge-invariant observables (transverse traceless components of the graviton) do represent diffeomorphism invariant properties of the full solution. In the embroidery construction, the transverse-traceless linear observables can be carried to the full theory quantum space. Here they should read out invariant (topological) properties of the knots, namely the way the embroidery loops are entangled on the weave.

An alternative way for getting observables is to couple matter to general relativity. By coupling a finite amount of matter, concentrated in a small region of space, we obtain a theory with two regions: an external vacuum region, where the constraints can be solved using the techniques described in this section, and an internal region where matter provides physical gauge invariant observables and a well defined 'clock' evolution. A model of this kind is constructed in [75].

These are possible directions for constructing the physical observables but the problem is entirely open.

(ii) The inner product must be defined on the space of the physical states. As discussed in section 3, the inner product is determined by the Hermiticity condition on the real physical observables. The linear theory may provide indications. In fact, the linear scalar product must be the scalar product inherited from the full-theory scalar product through $\mathcal{M}$. Since the linear scalar product is known, the full scalar product may be be deduced from it, at least up to the approximation in which the relation between the two theories makes sense.

(iii) It is not clear to what extent the regular knot sector alone can represent interesting physics. Other solutions involving intersections should be investigated. It should not be difficult to recover in the loop representation the solutions with intersections discovered in the connection representation by Brügmann and Pullin [73]. Two important open questions regarding intersecting-solutions are the following. (a) Are other solutions constructed only in terms of particular linear combinations of intersecting loops *with the same support*, or should they involve loops with different support? (b) Are linear combinations of loops with intersections of an *infinite* number of components required?

(iv) Brügmann and Pullin [73] noted that in the connection representation the known solutions of the Hamiltonian constraint satisfy the Hamiltonian constraint for every value of the cosmological constant. This result is disturbing, and its significance is not understood.

(v) A related problem is the relation between the different proposed forms of the Hamiltonian constraint. To fix a unique and simple definition for the Hamiltonian constraint is also necessary in order to study the problem of the quantum closure of the constraints. Note that the very existence of common solutions to all the constraints shows that there are no anomalies proportional to the identity in the quantum commutator of the constraints. However, it would be interesting to explicitly calculate these commutators.

(vi) Perturbation theory. Having a construction of gravitons within the full theory, in a context in which the continuum breaks down to a discrete structure at the Planck energy, suggests that at this point one could be able to reconsider 'perturbative' graviton–graviton scattering. The full theory should modify the (approximate) linearized theory by providing a physical cut-off at the Planck length.

(vii) The regular way the weave Δ has been constructed is perhaps a first approximation. We may expect the weave to look more like a tangle than like an ordered weave. Note, however, that the relevant object is not the weave, but its knot class K_Δ, which is an equivalence class of many very 'disordered' loops.

4.3.2. An overall picture. In this section, the present stage of the construction of a non-perturbative quantization of general relativity has been outlined. The main results are the following.

(i) Quantum general relativity admits a representation in which the quantum states are represented by functionals on a loop space $\Psi(\alpha)$, and the loop variables are represented by operators $\hat{T}, \hat{T}^a$ that act by creating loops and breaking and rejoining loops at intersections.

(ii) The diffeomorphism invariant states are given by linear combinations of knot states $|K\rangle$. These constitute the general solution to the quantum diffeomorphism constraint.

(iii) An infinite-dimensional space of physical states, which describe solutions to the quantum dynamics, is given by (linear combinations of) the regular-knot states $|K_{\text{regular}}\rangle$.

(iv) Preliminary results indicate that one of these knot states, the 'weave-knot' $|K_{\Delta}\rangle$, is related to flat space. It has the property of being an eigenstate of smeared diffeomorphism invariant operators, provided that the smearing is taken on a large scale. Its eigenvalues correspond to a flat Euclidean metric. By measuring the metric over large regions in the state $|K_{\Delta}\rangle$, the outcome of the measurement corresponds to the flat metric. Different weaves (entangled in a different way) should correspond (at large scales) to different geometries.

(v) By measuring the metric in the state $|K_{\Delta}\rangle$ at smaller scales, some roughness appears, and the continuum structure breaks down completely at the Planck length, where the metric has a discontinuous distributional structure.

(vi) The quantum field theoretical vacuum of the linearized theory around flat space is represented within the full theory by a Gaussian-like linear combination of knot classes, peaked around the weave $|K_{\Delta}\rangle$. Gravitons are represented in terms of small deformations of $|K_{\Delta}\rangle$ obtained by attaching loops (embroidery loops) to the threads of the weave. The spatial position is determined in the full theory by the entangling of the embroidery loops with the weave. Position is determined only up to the Planck length.

This picture is certainly incomplete. Until a complete theory is defined, or until concrete calculations can be performed, the main question—which is whether or not a quantum theory of general relativity exists—does not yet have an answer. However, the indications are promising, and the hope is that we are not too far from calculating finite amplitudes above the Planck energy.

The reason for the failure of perturbative quantum gravity now seems clear, and the non-perturbative methods presented here reveal an unsuspected richness of structures which could not have been caught in perturbation theory. In spite of the intricacy of the technicalities, the results that are emerging are surprisingly simple and intuitive.

In conclusion, I would like to emphasize an important characteristic of the approach I have described. The results presented here follow from applying standard quantum mechanics to standard general relativity. No additional physical principle, or additional hypothesis, has been added to these two theories, which are both firmly supported by observations. In a sense, the construction described in this report is an attempt to grasp the microstructure of spacetime by building on relatively solid grounds: the physical assumptions are only general relativity and quantum mechanics, which summarize so much of the present understanding of the physical world.

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Appendix 1. Complex Hamiltonian mechanics

In this appendix, I study the extension of Hamiltonian mechanics to the case of complex actions. This extension provides a geometrical framework in which the phase space of the Ashtekar theory can be interpreted. In particular, I clarify the meaning of the Poisson structure (1.40), which is defined on a space that is half-real and half-complex.

Let me assume, in general, that we have a configuration space C with N (real) variables q_i, which I represent by the single vector q, and the following complex action is given

$$S[q] = S^{\mathrm{R}}[q] + \mathrm{i}S^{\mathrm{I}}[q] \tag{A1.1}$$

where S^{R} and S^{I} are two real functionals of $q(t)$. The assumption is that the motion is given by the $q(t)$ such that

$$\delta S[q] = 0. \tag{A1.2}$$

Is there a canonical description of this set of equations?

Since S is complex, (A1.2) has two components, the real one and the imaginary one. There are $2N$ equations of motions for the N variables q. It is useful to think that we are dealing with two dynamical theories for the same system, namely for the same set of variables. The first dynamical theory is defined by Re $\delta S[q] = \delta S^{\mathrm{R}}[q] = 0$, namely by the real part of the action. I denote this theory as the R-dynamical theory. The second dynamical theory is defined by Im $\delta S[q] = \delta S^{\mathrm{I}}[q] = 0$, and will be denoted as the I-dynamical theory. The physical motions of the system must satisfy the equations of motion of *both* theories.

Since we have two standard dynamical theories, we may use the standard machinery of analytical mechanics, by running it twice, in parallel. By doing that, we may forget the imaginary unit, and work entirely in terms of real quantities.

Let me construct the Hamiltonian formulation of both theories. I define two momenta

$$p^{\mathrm{R}} = \frac{\partial L^{\mathrm{R}}}{\partial \dot{q}} \qquad p^{\mathrm{I}} = \frac{\partial L^{\mathrm{I}}}{\partial \dot{q}}. \tag{A1.3}$$

(L^{R} and L^{I} are the two Lagrangians) and consider two phase spaces: the phase space $\mathcal{S}^{\mathrm{R}}$ with coordinates (p^{R}, q) and the phase space $\mathcal{S}^{\mathrm{I}}$ with coordinates (p^{I}, q). The two spaces are two copies of the cotangent space of C. We denote $\mathcal{S}^{\mathrm{R}}$ and $\mathcal{S}^{\mathrm{I}}$ the R phase space and the I phase space. On each of these spaces the Hamiltonian theory is defined by the standard Legendre transformation. There is a symplectic form ω^{R} in $\mathcal{S}^{\mathrm{R}}$ and a symplectic form ω^{I} in $\mathcal{S}^{\mathrm{I}}$:

$$\omega^{\mathrm{R}} = \mathrm{d}p^{\mathrm{R}} \wedge \mathrm{d}q \qquad \omega^{\mathrm{I}} = \mathrm{d}p^{\mathrm{I}} \wedge \mathrm{d}q \tag{A1.4}$$

a Hamiltonian H^{R} and a Hamiltonian H^{I}, and, possibly, each one of the two theories may have first-class constraints. (They better have to, if the total theory has to be non-trivial, since otherwise the $2N$ independent equations of motions for the N fields tend to be overdetermined.) On the constraint surfaces, the Hamiltonians define the Hamiltonian vector fields X^{R} and X^{I}. These are defined by the standard Hamilton equations, which in this language are

$$i_{X^{\mathrm{R}}}\omega^{\mathrm{R}} = -\mathrm{d}H^{\mathrm{R}} \qquad i_{X^{\mathrm{I}}}\omega^{\mathrm{I}} = -\mathrm{d}H^{\mathrm{I}} \tag{A1.5}$$

where i denotes the contraction of the 2-form with the vector field. The Hamiltonian vector fields are partially under-determined if there are gauges: we can always add to them a vector tangent to the gauge orbits. The integral lines of the vector field X^{R} are the motions of the R system, and so for the I system.

Now we have to recall that the physical motions must be a motion for *both* dynamical theories. What does this mean in the Hamiltonian picture that has been constructed? They are the q coordinates in the two spaces that have to be identified. Consider a motion in $\mathcal{S}^{\mathrm{R}}$. Let $q(t)$ be the projection of the motion on the q subspace of $\mathcal{S}^{\mathrm{R}}$. This projection fixes the motion entirely: it is a solution of the Lagrange equations of the R theory. The question is: is there a motion in the I theory such that its projection on its q subspace is also $q(t)$?

From a geometrical point of view, we have to consider the linear space with coordinates $(p^{\mathrm{R}}, p^{\mathrm{I}}, q)$. This is a $3N$-dimensional space, where N is the dimension of the configuration space. This is an appropriate space for the Hamiltonian dynamics of complex actions. This space will be denoted $\mathcal{S}$. There is a natural projection from $\mathcal{S}$ to $\mathcal{S}^{\mathrm{R}}$ (namely $(p^{\mathrm{R}}, p^{\mathrm{I}}, q) \rightarrow (p^{\mathrm{R}}, 0, q)$), and a similar one to $\mathcal{S}^{\mathrm{I}}$. A motion $(p^{\mathrm{R}}(t), p^{\mathrm{I}}(t), q(t))$ in $\mathcal{S}$, projects to $\mathcal{S}^{\mathrm{R}}$ and $\mathcal{S}^{\mathrm{I}}$ in such a way that both motions in turn project to the same motion in the configuration space. A motion in $\mathcal{S}$ such that both its projections satisfy the respective dynamical equations will be a solution of the Lagramge equations.

All the structure in $\mathcal{S}^{\mathrm{R}}$ and in $\mathcal{S}^{\mathrm{I}}$ extends immediately to $\mathcal{S}$, because we may pull back ω^{R} and ω^{I}, H^{R} and H^{I}, by using the projections. More simply, everything is naturally defined everywhere in $\mathcal{S}$.

We then have the following straightforward theorem.

Theorem. A physical motion (a solution of the Lagrange equations) is given by a curve in $\mathcal{S}$, such that its tangent X satisfies both the equations

$$i_X \omega^{\mathrm{R}} = -\mathrm{d}H^{\mathrm{R}} \qquad i_X \omega^{\mathrm{I}} = -dH^{\mathrm{I}} \tag{A1.6}$$

and stays on the constraint surface.

It is clear that if both these equations are satisfied on $\mathcal{S}$, then the projection of X in $\mathcal{S}^{\mathrm{R}}$ and $\mathcal{S}^{\mathrm{I}}$ will satisfy the respective Hamilton equations, and, by construction of $\mathcal{S}$, their projection on the configuration space is the same.

Note that ω^{R} and ω^{I} are not symplectic, due to the fact that $\mathcal{S}$ is odd-dimensional. $\mathcal{S}$ is geometrically defined as follows. The two actions $S^{\mathrm{R}}[q]$ and $S^{\mathrm{I}}[q]$ define two different mappings from the cotangent space $T^{\#}C$ of the configuration space C, to the tangent space TC. Thus, we may consider two cotangent structures over C, each one equipped with its own mapping on the tangent space.

At this point I may reinsert the complex numbers, and make use of the compactness of notation that they allow. I use complex numbers in two different ways. The first is to write complex equations simply as a compact form for a couple of real equations. The second is to use complex coordinates for the spaces introduced in the previous section. The interplay of the two uses of complex numbers simplifies the notation. The two equations (A1.6) of the previous section can be written as a single complex equation, by defining a complex 2-form

$$\omega = \omega^{\mathrm{R}} + \mathrm{i}\omega^{\mathrm{I}} \tag{A1.7}$$

and complex Hamiltonian

$$H = H^{\mathrm{R}} + \mathrm{i}H^{\mathrm{I}}. \tag{A1.8}$$

Then (A1.6) becomes just

$$i_X\omega = -\mathrm{d}H \tag{A1.9}$$

I then introduce complex variables. I define

$$z = p^{\mathrm{R}} + \mathrm{i}p^{\mathrm{I}}. \tag{A1.10}$$

In terms of these variables, the complex 2-form ω is

$$\omega = \mathrm{d}z \wedge \mathrm{d}q. \tag{A1.11}$$

By recalling that there was originally an i connecting the two theories, I may note that

$$z = \frac{\partial L}{\partial \dot{q}} \tag{A1.12}$$

The Hamiltonian theory is now defined by the formulae (A1.12), (A1.11), (A1.9). These equations are precisely the same equations that define the Hamiltonian theory of a real action. Thus, we reached the following result.

The standard equations of Hamiltonian mechanics can be used also for complex actions, without visible changes.

However, one should not be confused by this apparent simplicity. In particular, ω is *not* symplectic, (nor are its real and imaginary parts symplectic), the phase space S has three times the dimensions of the configuration space, and so on. The following terminology may be useful. I denote the dynamical systems with a complex action as *complex dynamical systems*, the phase space S as *complex phase space*. I denote ω as *complex symplectic form* and z as *complex momentum*. This terminology, indeed, has more or less been used in the Ashtekar formalism and is very natural; but one should be careful not to be confused by it: the complex phase space is not a complex space and is not the direct sum of the R and I phase spaces, the complex symplectic form is not symplectic (it is presymplectic), and a complex momentum corresponds to a single real canonical coordinate.

Finally, note that in general the complex Hamiltonian system (S, ω, H, C) cannot be interpreted as a standard real Hamiltonian system: we are dealing here with a genuine extension of standard Hamiltonian mechanics.

A particular case of a complex Hamiltonian system is given when the imaginary part of the action has no effect on the Lagrange equations. It is worth considering this case in detail because the Ashtekar theory belongs to it. In this case, the dynamical system is physically equivalent to its real sector (the evolution in the configuration space is the same). But the Hamiltonian description that one gets from the complex action is different from the Hamiltonian description that one gets from the real action (Ashtekar's Hamiltonian theory is different than ADM theory). To study this particular case, I consider, as a specific example, a one-dimensional harmonic oscillator

$$S^{\mathrm{R}}[q] = \int \mathrm{d}t\ \tfrac{1}{2}(\dot{q}^2 - q^2) \tag{A1.13}$$

and add to this action an imaginary part with no effect on the Lagrange equations:

$$S^{\mathrm{I}}[q] = \int \mathrm{d}t\ \tfrac{1}{2}\dot{q}q. \tag{A1.14}$$

The R dynamical system is the well known one: the R phase space has coordinates (p^{R}, q), with

$$H^{\mathrm{R}} = \tfrac{1}{2}(p^2 + q^2) \tag{A1.15}$$

The I dynamical system has momentum

$$p^{\mathrm{I}} = \frac{\partial L^{\mathrm{I}}}{\partial \dot{q}} = q \tag{A1.16}$$

and vanishing canonical Hamiltonian. The I phase space is the (p^{I}, q) space, with the primary first class constraint

$$C = p^{\mathrm{I}} - q = 0 \tag{A1.17}$$

which defines a one-dimensional constraint surface in $\mathcal{S}^{\mathrm{I}}$. The restriction of the I symplectic form to the constraint surface vanishes, namely the single direction along the constraint surface is a gauge direction.

The complex phase space $\mathcal{S}$ of the theory is the space $(p^{\mathrm{R}}, p^{\mathrm{I}}, q)$. The theory is defined by the complex symplectic form $\omega = \mathrm{d}p^{\mathrm{R}}\mathrm{d}q + \mathrm{i}\ \mathrm{d}p^{\mathrm{I}}\mathrm{d}q$ by the real Hamiltonian H^{R} (its complex part vanishes) and by the constraint (A1.17). The imaginary part of ω vanishes in restricting to the constraint surface, so that only the real component of the Hamilton equation survives. The solution of (A1.9) that stays in the constraint surface is unique

$$X = p^{\mathrm{R}} \frac{\partial}{\partial q} - q \frac{\partial}{\partial p^{\mathrm{R}}} - p^{\mathrm{R}} \frac{\partial}{\partial p^{\mathrm{I}}}. \tag{A1.18}$$

If we project the integral lines of this vector field on the configuration space we have the motions of the oscillator.

I now repeat the analysis in terms of complex coordinates. The momentum is complex and is given by

$$z = \frac{\partial S[q]}{\partial \dot{q}} = p^{\mathrm{R}} + \mathrm{i}p^{\mathrm{I}}. \tag{A1.19}$$

The phase space S is the space (z, q), with three real dimensions, and the complex symplectic form on the phase space is $\omega = \mathrm{d}z \wedge \mathrm{d}q$. The Hamiltonian is

$$H = z^2 - \mathrm{i}zq. \tag{A1.20}$$

The constraint is $\bar{z} = z - 2\mathrm{i}q$. The rest goes as above. It is important to note that in this formulation $\bar{z}$ appears in the theory only through the 'reality condition constraint'. The Hamiltonian is a holomorphic function of z.

In general, when the imaginary part of the action has no effect on the equations of motion, the I system is given by N constraints which define an N-dimensional constraint surface which is a unique gauge orbit. In the complex phase space these N constraints define a $2N$-dimensional surface which is isomorphic to the R phase space. These constraints are denoted as *reality conditions* and their constraint surface is denoted the *real phase space*. Note that this 'real phase space' is not the R phase space, but it is isomorphic to it.

The analogy between the one-dimensional harmonic oscillator treated above and the Ashtekar theory is complete. The identifications are

$$q \leftrightarrow E^a_i(x) \tag{A1.21}$$

$$p^{\mathrm{R}} \leftrightarrow p^i_a(x) \tag{A1.22}$$

$$p^{\mathrm{I}} \leftrightarrow \omega^i_a(x) \tag{A1.23}$$

$$z \leftrightarrow p^i_a(x) + \mathrm{i}\omega^i_a(x) = A^i_a(x) \tag{A1.24}$$

$$\bar{z} = z - 2\mathrm{i}q \leftrightarrow \bar{A} = A - 2\mathrm{i}\omega. \tag{A1.25}$$

Appendix 2. Maxwell 2: a smeared version

In this appendix, a different version of the loop quantization of Maxwell theory is discussed. This version has certain advantages with respect to the version introduced in section 3, and is a model for the quantization of linearized gravity.

Let me start by fixing a universal smearing function $f_\epsilon(\boldsymbol{x})$ which I choose as follows. It is smooth, it has compact support in a region of radius ϵ around $\boldsymbol{x} = 0$, and its integral is one. In terms of this function, the 'smeared form factor' is defined as

$$F^a_\epsilon(\alpha, \boldsymbol{x}) = \int \mathrm{d}^3y\ f_\epsilon(\boldsymbol{y} - \boldsymbol{x})\Delta^a(\alpha, \boldsymbol{y}). \tag{A2.1}$$

This is a real vector field with support on an ϵ-small tube around the loop α, and which points along the tangent of α. The loop quantization can be performed, by using the smeared form factor rather than the unsmeared one. The smeared holonomy observable is

$$T_\epsilon(\alpha) = \exp\left(-\mathrm{i}\int \mathrm{d}^3x\ F^a_\epsilon[\alpha, \boldsymbol{x}]A_a(\boldsymbol{x})\right) \tag{A2.2}$$

(cf (3.11)). The loop algebra to be quantized is the T_ϵ, E algebra. The quantization is achieved by picking the same space of loop functionals as in the unsmeared case (now I denote them $\Psi_\epsilon(\alpha)$) and by defining the two operators

$$\hat{T}[\gamma]\,\Psi_\epsilon(\alpha) = \Psi_\epsilon(\alpha \cup \gamma) \tag{A2.3}$$

$$\hat{E}^a_\epsilon(\boldsymbol{k})\,\Psi_\epsilon(\alpha) = \hbar\ F^a_\epsilon[\gamma, \boldsymbol{k}]\ \Psi_\epsilon(\alpha). \tag{A2.4}$$

Something curious is going on here: the smeared classical loop algebra in which the holonomy is smeared is quantized by an unsmeared $\hat{T}$ operator and by a smeared $\hat{E}_\epsilon$ operator. The smearing shifts from T to E in going to the quantum theory. It is easy to check that the commutators reproduce the correct Poisson brackets. One should not be confused by the notation, which becomes a bit tricky because of the shift in the position of ϵ: $\hat{T}$ is the operator that corresponds to T_ϵ and $\hat{E}_\epsilon$ is the operator that corresponds to E.

We may repeat the previous definition of the Hamiltonian, and we discover now the eigenstates of the Hamiltonian have the same form as the states in (3.28) above, with the form factor replaced by the smeared form factor.

Also the mapping to the Bargmann representation can be generalized to the present case. We have

$$\Psi_\epsilon(\alpha) = \int \mathrm{d}\mu[A]\ \exp\left(-\mathrm{i}\int \mathrm{d}^3x F^a_\epsilon[\alpha, x] A_a(x)\right)\Psi[A]. \tag{A2.5}$$

Note that the same linear space of loop functions carries both the unsmeared representation and the smeared one. More precisely, it carries a one-parameter (ϵ) family of representations. The same loop function represents different physical states in two representations corresponding to a different ϵ.

A direct advantage of the smeared formalism is a simple definition of the scalar product. In fact, consider the loop states $|\alpha_\epsilon\rangle$ that define the representation:

$$\Psi_\epsilon(\alpha) = \langle\alpha_\epsilon|\Psi\rangle. \tag{A2.6}$$

On these states, the action of the basic operators is

$$\hat{T}_\epsilon[\gamma]\,|\alpha_\epsilon\rangle = |(\alpha\cup\gamma)_\epsilon\rangle \tag{A2.7}$$

$$\hat{E}^a(k)\,|\alpha_\epsilon\rangle = \hbar\ F^a_\epsilon[\gamma, k]\ |\alpha_\epsilon\rangle. \tag{A2.8}$$

We want to define the scalar product by using the reality conditions. The reality conditions follow from the fact that A and E are real. In terms of the unsmeared positive and negative frequency fields they are ${}^+\bar{A}(x) = {}^+E(x)$. They can be written in terms of the smeared loop variables as

$$B^{\epsilon\, a}(x) \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}(T_\epsilon[\gamma_{\epsilon,a,x}] - 1) = \int \mathrm{d}^3y\ f_\epsilon(x-y)\ \bar{E}^a(y). \tag{A2.9}$$

They are implemented in the quantum theory as operator equations (with the complex conjugation replaced by adjoint operation) if the adjoint operation is defined by the scalar product

$$\langle\alpha_\epsilon|\beta_\epsilon\rangle = \exp\left(-\hbar\int \mathrm{d}^3x F^a_\epsilon[\alpha, x] F_{a\epsilon}[\beta, x]\right). \tag{A2.10}$$

Since the smeared form factor is bounded and has compact support, the integral is well defined and finite, and this equation provides a consistent definition of the scalar product directly in loop space.

Appendix 3. Maxwell 3: self duality; how to prefer left photons over right photons

In this appendix, the loop quantization of the Maxwell field obtained by starting from the self-dual, rather than the positive-frequency, connection is described. This form of the theory mimics the treatment of gravity. As stressed by Ashtekar, while the distinction between positive frequency and negative frequency is meaningless in a generally covariant framework, the distinction between self-dual and antiself-dual sector *remains* meaningful. This is one of the key reasons for which the quantization methods developed in section 3 may work also in the absence of Poincaré invariance.

In order to mimic Ashtekar theory I choose as elementary variables the real electric field E^a and the self-dual Maxwell connection A_a^{sd}, which is defined, up to a gauge that will be soon irrelevant, by

$$\epsilon^{abc}\partial_b A_c^{sd} = B^{a\ sd} \equiv B^a + iE^a \tag{A3.1}$$

where E and B are the real electric and magnetic fields. The self-dual component of the field is formed by the positive frequency components of the positive helicity sector plus the negative frequency components of the negative helicity sector. I define the self-dual holonomy

$$T_\epsilon^{sd}[\gamma] = \exp\left(\int d^3x\, F_\epsilon^a[\gamma, x]\, A_a^{sd}(x)\right). \tag{A3.2}$$

In order to keep track of the two helicities, it is convenient to split the form factor into its positive an negative helicity components. In terms of the standard transverse unit basis vectors $m(k)^a, \bar{m}(k)^a$ and $\hat{k}^a$, defined by

$$m(k)_a k^a = 0 \qquad m(k)_a m(k)^a = 0 \qquad m(k)_a \bar{m}(k)^a = 1 \qquad \frac{\hat{k}^a = k^a}{|k|} \tag{A3.3}$$

it is possible to define

$$F_\epsilon^+[\gamma, k] = m_a(k) F_\epsilon^a[\gamma, k] \tag{A3.4}$$

$$F_\epsilon^-[\gamma, k] = \bar{m}_a(k) F_\epsilon^a[\gamma, k]. \tag{A3.5}$$

The T_ϵ^{sd}, E^a algebra can be quantized in terms of the standard space of loop states, now denoted Ψ_{sd}, and the usual loop operators

$$\hat{T}_\epsilon[\gamma]\, \Psi_{sd}(\alpha) = \Psi_{sd}(\alpha \cup \gamma) \tag{A3.6}$$

$$\hat{E}^a(k)\, \Psi_{sd}(\alpha) = \hbar\ F_\epsilon^a[\alpha, k]\ \Psi_{sd}(\alpha). \tag{A3.7}$$

Note that we have precisely the same space and the same operators as in the positive-frequency case. How does the theory know that now the same operators represent different observables? The answer is that the reality conditions and the Hamiltonian are different.

The reality conditions are now

$$\hat{B}^a(x)^\dagger = \hat{B}^a(x) - 2i \int d^3y\ f_\epsilon(x-y) E^a(y). \tag{A3.8}$$

In order to have these reality condition implemented in the quantum theory, we are forced to define the scalar product as

$$\langle \alpha_\epsilon | \beta_\epsilon \rangle = \exp\left(\frac{1}{2}\int d^3x (F_\epsilon^+[\alpha, x] F_\epsilon^+[\beta, x] - F_\epsilon^-[\alpha, x] F_\epsilon^-[\beta, x])\right). \tag{A3.9}$$

The classical Hamiltonian, written in terms of the basic variables, is

$$H = \int d^3x\ (B^{sd})^2 - 2iE_a B^{sd\ a}. \tag{A3.10}$$

The Schrödinger equation is

$$\left(\int d^3x\ \hat{B}^{sd}_a \hat{B}^{a\,sd} - 2iF^a_{\epsilon a}\hat{B}^{a\,sd}\right)\Psi_{sd}(\alpha) = E\Psi_{sd}(\alpha). \tag{A3.11}$$

A straightforward calculation shows that the vacuum is

$$\Psi^{(0)}_{sd}(\alpha) = \exp\left(-\int d^3x\ F^+_\epsilon[\alpha, \boldsymbol{x}]\ F^+_\epsilon[\alpha, \boldsymbol{x}]\right). \tag{A3.12}$$

The n-photon states are given by the following loop functionals. The positive-helicity n-photon states are homogeneous polynomials (of degree n) in F^+ multiplied by the vacuum. The negative-helicity photons are Hermite polynomials in F^- times the vacuum. Thus to work with a self-dual connection and a real electric field, produces a mixed representation, which is a Bargmann representation in the positive-helicity sector and a Schrödinger representation in the negative-helicity sector. This hybrid situation is not a consequence of the loop representation, but just of the fact that the variables we use are not symmetric under parity.

More details on this mixed representation can be found in [43].

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