ASKEY-WILSON POLYNOMIALS FOR ROOT SYSTEMS OF TYPE BC

Tom H. Koornwinder

Abstract. This paper introduces a family of Askey-Wilson type orthogonal polynomials in n variables associated with a root system of type BC_n. The family depends, apart from q, on 5 parameters. For $n = 1$ it specializes to the four-parameter family of one-variable Askey-Wilson polynomials. For any n it contains Macdonald's two three-parameter families of orthogonal polynomials associated with a root system of type BC_n as special cases.

1. Introduction

In recent years, some families of orthogonal polynomials associated with root systems were introduced. The families studied by Heckman & Opdam [6], [4], [5] become Jacobi polynomials for root system BC_1. The families studied by Macdonald (see [11] for root system A_n and [12] for general root systems) become continuous q-ultraspherical polynomials for root system A_1 and continuous q-Jacobi polynomials for root system BC_1 (see Askey & Wilson [1, §4]). For all root systems Macdonald's polynomials tend to the Heckman-Opdam polynomials as q tends to 1.

This paper introduces a family of Askey-Wilson type polynomials for root system BC_n which depends, apart from q, on 5 parameters. For $n = 1$ it specializes to the four-parameter family of Askey-Wilson polynomials. For any n it contains Macdonald's two three-parameter families as special cases: for the pair (BC_n, B_n) directly and for the pair (BC_n, C_n) when q is replaced by q^2. Moreover, the weight function integrated over the orthogonality domain was explicitly evaluated by Gustafson [3] as a generalization of Selberg's beta integral.

The proofs in this paper are very much inspired by Macdonald's proofs in [12], in particular by his proofs in case of root systems E_8, F_4, G_2, where there is no minuscule fundamental weight available.

The contents of this paper are as follows. Section 2 summarizes Macdonald's results. The special case BC_n of these results is discussed in §3 and the further specialization to BC_1 in §4. The long section 5 introduces Askey-Wilson polynomials for root system BC_n, shows that these polynomials are eigenfunctions of a certain difference operator and establishes the full orthogonality of the polynomials. Finally, in §6, special cases and open problems are discussed

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2. Summary of Macdonald's results

In this section we summarize Macdonald's [12] results on orthogonal polynomials associated with root systems. See Humphreys [7] and Bourbaki [2, Chap.6] for preliminaries on root systems. Let V be a finite dimensional real vector space with inner product $\langle\,.\,,\,.\,\rangle$. Write $|v| := \langle v, v\rangle^{1/2}$ for the norm of $v \in V$. Write

$$v^\vee := 2v/|v|^2, \quad 0 \neq v \in V.$$

Let R be a not necessarily reduced root system spanning V. Let S be a reduced root system in V such that the set of lines $\{\mathbb{R}\alpha \mid \alpha \in R\}$ equals $\{\mathbb{R}\alpha \mid \alpha \in S\}$. Then the pair (R, S) is called *admissible* and R and S have the same Weyl group W. Now, for each $\alpha \in R$, there is a (unique) $u_\alpha > 0$ such that $\alpha_* := u_\alpha^{-1}\,\alpha \in S$. Assume that R is irreducible. It can be arranged, after possible dilation of R and S, that u_α takes values in $\{1, 2\}$ or in $\{1, 3\}$.

Let $0 < q < 1$. Put $q_\alpha := q^{u_\alpha}$. Let $\alpha \mapsto t_\alpha$ be a W-invariant function on R, taking values in $(0, 1)$ (for convenience). Then t_α only depends on $|\alpha|$. Put $t_\alpha := 1$ if $\alpha \in V\backslash R$. Let $k_\alpha \geq 0$ be such that $q_\alpha^{k_\alpha} = t_\alpha$.

Let R^+ be a choice for the set of positive roots in R. Let

$$Q := \mathbb{Z}\text{-Span}(R), \quad Q^+ := \mathbb{Z}_+\text{-Span}(R^+).$$

Here, and throughout the paper, $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. Let

$$P := \{\lambda \in V \mid \langle \lambda, \alpha^\vee\rangle \in \mathbb{Z} \quad \forall \alpha \in R\}, \quad P^+ := \{\lambda \in V \mid \langle \lambda, \alpha^\vee\rangle \in \mathbb{Z}_+ \quad \forall \alpha \in R^+\}$$

be respectively the weight lattice of R and the cone of dominant weights. Define a partial order on P by $\lambda \geq \mu$ iff $\lambda - \mu \in Q^+$.

For $\lambda \in P$ let e^λ be the function on V defined by

$$e^\lambda(x) := e^{i\langle \lambda, x\rangle}, \quad x \in V.$$

Extend this holomorphically to $V + iV$. If f is a function on V then put $(wf)(x) := f(w^{-1}x)$ for $w \in W$, $x \in V$. Hence $we^\lambda = e^{w\lambda}$. Let A be the complex linear span of the e^λ $(\lambda \in P)$. Let A^W denote the space of W-invariants of A. Put

$$m_\lambda := |W_\lambda|^{-1} \sum_{w \in W} e^{w\lambda} = \sum_{\mu \in W\lambda} e^\mu, \quad \lambda \in P^+.$$

Here W_λ denotes the stabilizer of λ in W. The m_λ $(\lambda \in P^+)$ form a basis of A^W. Note that $\overline{m_\lambda(x)} = m_\lambda(-x)$ $(\lambda \in P^+,\ x \in V)$. If $-\text{id} \in W$ then $f(x) = f(-x)$ for $f \in A^W$. In particular, we will then have that m_λ is real-valued on V.

Let $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$ be the root system dual to R. Let $Q^\vee := \mathbb{Z}\text{-Span}(R^\vee)$. Then $T := V/(2\pi Q^\vee)$ is a torus. Let $\dot{x}$ be the image in T of $x \in V$. Let $d\dot{x}$ be the normalized Haar measure on T. For $\lambda \in P$ the function $\dot{x} \mapsto e^\lambda(x)$ is well-defined on T. For $a, a_1, \ldots, a_k \in \mathbb{C}$ put

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j), \quad (a_1,\ldots,a_k;q)_\infty := \prod_{i=1}^{k}(a_i;q)_\infty.$$

Define

$$\Delta := \prod_{\alpha\in R} \frac{(t_{2\alpha}^{1/2}\, e^\alpha; q_\alpha)_\infty}{(t_\alpha\, t_{2\alpha}^{1/2}\, e^\alpha; q_\alpha)_\infty}, \quad \Delta^+ := \prod_{\alpha\in R^+} \frac{(t_{2\alpha}^{1/2}\, e^\alpha; q_\alpha)_\infty}{(t_\alpha\, t_{2\alpha}^{1/2}\, e^\alpha; q_\alpha)_\infty}.$$

Then $\Delta = \Delta^+\,\overline{\Delta^+}$. Define a hermitian inner product on A^W by

$$\langle f, g\rangle := |W|^{-1} \int_T f(\dot{x})\,\overline{g(\dot{x})}\,\Delta(x)\,d\dot{x}.$$

Definition 2.1. For $\lambda \in P^+$ let $P_\lambda \in A^W$ be characterized by the two conditions

(i) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda,\mu}\, m_\mu$ for certain complex coefficients $u_{\lambda,\mu}$;

(ii) $\langle P_\lambda, m_\mu\rangle = 0$ if $\mu < \lambda$.

Theorem 2.2.

$$\langle P_\lambda, P_\mu\rangle = 0 \quad \textit{if } \lambda \neq \mu.$$

Define

$$(T_v f)(x) := f(x - i(\log q)v), \quad x, v \in V,$$

for functions f being analytic on a suitable subset of $V + iV$ containing V. Hence

$$T_v e^\lambda = q^{\langle v,\lambda\rangle}\, e^\lambda, \quad \lambda \in P.$$

Let $\sigma \in V$ be such that $\langle \sigma, \alpha_*\rangle$ takes just two values 0 and 1 as α runs through R^+ (σ is a so-called minuscule fundamental weight for $S^\vee$). Such σ exists for all S not being of type E_8, F_4 or G_2. In these last three cases we can choose σ such that $\langle \sigma, \alpha_*\rangle$ takes values 0, 1 and 2 as α runs through R^+. Now put

$$\Phi_\sigma := \frac{T_\sigma \Delta^+}{\Delta^+},$$

$$E_\sigma f := |W_\sigma|^{-1} \sum_{w\in W} w(\Phi_\sigma\, T_\sigma f),$$

$$D_\sigma f := |W_\sigma|^{-1} \sum_{w\in W} w(\Phi_\sigma(T_\sigma f - f)),$$

$$\widetilde{m}_\sigma(\lambda) := |W_\sigma|^{-1} \sum_{w\in W} q^{\langle w\sigma,\lambda\rangle},$$

$$\rho_k := \tfrac{1}{2} \sum_{\alpha\in R^+} k_\alpha\, \alpha.$$

Theorem 2.3. *D_σ maps A^W into itself. The P_λ are eigenfunctions of D_σ with eigenvalue*

$$q^{\langle\sigma,\rho_k\rangle}\,(\widetilde{m}_\sigma(\lambda+\rho_k)-\widetilde{m}_\sigma(\rho_k)).$$

If S is not of type E_8, F_4 or G_2 then E_σ maps A^W into itself, the P_λ are also eigenfunctions of E_σ with eigenvalue

$$q^{\langle\sigma,\rho_k\rangle}\,\widetilde{m}_\sigma(\lambda+\rho_k).$$

and

$$D_\sigma = E_\sigma - E_\sigma(1)$$

with $E_\sigma(1)$ scalar.

3. The case $R = BC_n$

Identify V with $\mathbb{R}^n$ and let $\varepsilon_1,\ldots,\varepsilon_n$ be its standard basis. Consider in V the root systems

$R := \{\pm\varepsilon_j\}\cup\{\pm2\varepsilon_j\}\cup\{\pm\varepsilon_i\pm\varepsilon_j\}_{i<j}$ of type BC_n,

$S_B := \{\pm\varepsilon_j\}\cup\{\pm\varepsilon_i\pm\varepsilon_j\}_{i<j}$ of type B_n.

$S_C := \{\pm\varepsilon_j\}\cup\{\frac{1}{2}(\pm\varepsilon_i\pm\varepsilon_j)\}_{i<j}$ of type C_n.

Then S_B and S_C are reduced, R, S_B and S_C have the same Weyl groups and in the mappings $\alpha\mapsto u_\alpha^{-1}\alpha$ of R onto S_B and onto S_C, u_α take the values 1 and 2.

Note that $R^\vee = R$ and that the weight lattice P and the root lattice Q of R are both given by

$$P = Q = \{m_1\varepsilon_1+\cdots+m_n\varepsilon_n \mid m_1,\ldots,m_n\in\mathbb{Z}\}.$$

Take

$$R^+ := \{\varepsilon_j\}\cup\{2\varepsilon_j\}\cup\{\varepsilon_i\pm\varepsilon_j\}_{i<j}.$$

Then

$$P^+ = \{m_1\varepsilon_1+\cdots+m_n\varepsilon_n \mid m_1\ge m_2\ge\ldots\ge m_n\ge 0,\quad m_1,\ldots,m_n\in\mathbb{Z}\},$$
$$Q^+ = \{m_1(\varepsilon_1-\varepsilon_2)+\cdots+m_{n-1}(\varepsilon_{n-1}-\varepsilon_n)+m_n\varepsilon_n \mid m_1,\ldots,m_n\in\mathbb{Z}_+\}.$$

The torus $T := V/(2\pi Q^\vee)$ becomes $\mathbb{R}^n/(2\pi\mathbb{Z}^n)$. Recall that we have a partial ordering on P such that $\lambda\ge\mu$ iff $\lambda-\mu\in Q^+$.

For the pair (R, S_B) we have

$$q_{\pm\varepsilon_j} = q,\quad q_{\pm2\varepsilon_j} = q^2,\quad q_{\pm\varepsilon_i\pm\varepsilon_j} = q,$$

and there are three different parameters t_α, which we write as

$$a := t_{\pm\varepsilon_j},\quad b := t_{\pm2\varepsilon_j},\quad t := t_{\pm\varepsilon_i\pm\varepsilon_j}.$$

(Recall that $t_\alpha = 1$ if $\alpha\notin R$.) Thus

$$\Delta^+ = \Delta_1^+\,\Delta_2^+, \tag{3.1}$$

where

$$\Delta_1^+ := \prod_{j=1}^n \frac{(b^{1/2}\,e^{\varepsilon_j};q)_\infty}{(a\,b^{1/2}\,e^{\varepsilon_j};q)_\infty}\,\frac{(e^{2\varepsilon_j};q^2)_\infty}{(b\,e^{2\varepsilon_j};q^2)_\infty} = \prod_{j=1}^n \frac{(e^{2\varepsilon_j};q)_\infty}{(q^{1/2}\,e^{\varepsilon_j},-q^{1/2}\,e^{\varepsilon_j},a\,b^{1/2}\,e^{\varepsilon_j},-b^{1/2}\,e^{\varepsilon_j};q)_\infty} \tag{3.2}$$

and

$$\Delta_2^+ := \prod_{\alpha=\varepsilon_i\pm\varepsilon_j;\,i<j} \frac{(e^\alpha;q)_\infty}{(t\,e^\alpha;q)_\infty}\,. \tag{3.3}$$

For the pair (R, S_C) we have

$$q_{\pm\varepsilon_j} = q, \quad q_{\pm2\varepsilon_j} = q^2, \quad q_{\pm\varepsilon_i\pm\varepsilon_j} = q^2,$$

and there are three different parameters t_α, which we write as

$$a := t_{\pm\varepsilon_j}, \quad b := t_{\pm2\varepsilon_j}, \quad t := t_{\pm\varepsilon_i\pm\varepsilon_j}.$$

Thus (3.1) holds with

$$\Delta_1^+ := \prod_{j=1}^n \frac{(b^{1/2}\,e^{\varepsilon_j};q)_\infty}{(a\,b^{1/2}\,e^{\varepsilon_j};q)_\infty}\,\frac{(e^{2\varepsilon_j};q^2)_\infty}{(b\,e^{2\varepsilon_j};q^2)_\infty} = \prod_{j=1}^n \frac{(e^{2\varepsilon_j};q^2)_\infty}{(a\,b^{1/2}\,e^{\varepsilon_j},q\,a\,b^{1/2}\,e^{\varepsilon_j},-b^{1/2}\,e^{\varepsilon_j},-q\,b^{1/2}\,e^{\varepsilon_j};q^2)_\infty} \tag{3.4}$$

and

$$\Delta_2^+ := \prod_{\alpha=\varepsilon_i\pm\varepsilon_j;\,i<j} \frac{(e^\alpha;q^2)_\infty}{(t\,e^\alpha;q^2)_\infty}\,. \tag{3.5}$$

Since $-\mathrm{id} \in W$ in case of root system BC_n, m_λ will be real-valued and we can read for condition (ii) of Definition 2.1 that

$$\int_T P_\lambda(x)\,m_\mu(x)\,\Delta(x)\,d\dot{x} = 0 \quad \text{if } \mu < \lambda.$$

For the element σ of §2 we can take ε_1 in the case S_B and $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$ in the case S_C. In both cases σ is minuscule. So Theorem 2.3 is valid with these choices of σ. In particular, in the case S_B the polynomial P_λ is eigenfunction of D_{ε_1} with eigenvalue

$$\sum_{j=1}^n \bigl(ab\,t^{2n-j-1}\,(q^{\lambda_j}-1) + t^{j-1}\,(q^{-\lambda_j}-1)\bigr). \tag{3.6}$$

The choice $\sigma := 2\varepsilon_1$ in the case S_C would give values 0, 1 and 2 for $\langle\sigma,\alpha_*\rangle$ as α runs through R^+. It will turn out in §6.1 that, in case S_C, the P_λ are not only eigenfunctions of $E_{\varepsilon_1+\cdots+\varepsilon_n}$ but also of $D_{2\varepsilon_1}$.

4. The case $R = BC_1$

For $n = 1$ the two root systems S_B and S_C coincide and the results of §3 specialize as follows. We have $T = \mathbb{R}/(2\pi\mathbb{Z})$, $P = Q = \mathbb{Z}$, $P^+ = \mathbb{Z}_+$, the partial order on P is the ordinary total order on $\mathbb{Z}$,

$$m_l(x) = \begin{cases} e^{ilx} + e^{-ilx}, & l = 1, 2, \ldots, \\ 1, & l = 0. \end{cases}$$

and

$$\begin{aligned}\Delta^+(x) &= \frac{(e^{2ix}; q)_\infty}{(q^{1/2}\, e^{ix}, -q^{1/2}\, e^{ix}, a\, b^{1/2}\, e^{ix}, -b^{1/2}\, e^{ix}; q)_\infty} \\ &= \frac{(e^{2ix}; q^2)_\infty}{(a\, b^{1/2}\, e^{ix}, q\, a\, b^{1/2}\, e^{ix}, -b^{1/2}\, e^{ix}, -q\, b^{1/2}\, e^{ix}; q^2)_\infty}\,.\end{aligned}$$

The inner product for W-invariant functions f, g becomes an integral over the period of 2π-periodic even functions, so it can be written as

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^\pi f(x)\, \overline{g(x)}\, \Delta(x)\, dx.$$

Askey-Wilson polynomials $p_n(y; a, b, c, d \,|\, q)$ $(n \in \mathbb{Z}_+)$ are defined, up to a constant factor, as polynomials of degree n in y which satsify the orthogonality relations

$$\int_0^\pi (p_n p_m)(\cos x; a, b, c, d \,|\, q) \left| \frac{(e^{2ix}; q)_\infty}{(ae^{ix}, be^{ix}, ce^{ix}, de^{ix}; q)_\infty} \right|^2 dx = 0, \quad n \neq m. \tag{4.1}$$

See Askey & Wilson [1]. Here a, b, c, d are real, or if complex, appear in complex conjugate pairs, and $|a|, |b|, |c|, |d| \le 1$, but the pairwise products of a, b, c, d are not ≥ 1. When the condition $|a|, |b|, |c|, |d| \le 1$ on the parameters is dropped, finitely many discrete terms have to be added to the orthogonality relation (4.1).

When we compare the expression for $\Delta^+(x)$ with the Askey-Wilson weight function we see that Macdonald's polynomials for root system BC_1 coincide, up to a constant factor, with Askey-Wilson polynomials

$$p_l(\cos x; q^{1/2}, -q^{1/2}, ab^{1/2}, -b^{1/2} \,|\, q).$$

By Askey & Wilson [1, (4.16), (4.17), (4.20)] the continuous q-Jacobi polynomials in M. Rahman's notation can be expressed in terms of Askey-Wilson polynomials by

$$\begin{aligned}P_l^{(\alpha,\beta)}(\cos x; q) &= \text{const.}\ p_l(\cos x; q^{1/2}, -q^{1/2}, q^{\alpha+1/2}, -q^{\beta+1/2} \,|\, q) \\ &= \text{const.}\ p_l(\cos x; q^{\alpha+1/2}, q^{\alpha+3/2}, -q^{\beta+1/2}, -q^{\beta+3/2} \,|\, q^2).\end{aligned}$$

Thus, if we put $a := q^\alpha$, $b := q^{2\beta}$, then Macdonald's polynomials for root system BC_1 coincide, up to a constant factor, with continuous q-Jacobi polynomals $P_l^{(\alpha+\beta-1/2,\beta-1/2)}(\cos x; q)$. This observation was already made by Macdonald [12, §9].

If $n=1$ then, with $\sigma := 1$,

$$\Phi_\sigma(x) = \frac{(1-a\,b^{1/2}\,e^{ix})\,(1+b^{1/2}\,e^{ix})}{1-e^{2ix}}\,.$$

Thus, if we write

$$R_l(e^{ix}) := P_l(x)$$

then Theorem 2.3 yields

$$\Phi_\sigma(-x)\,R_l(q^{-1}e^{ix}) + \Phi_\sigma(x)\,R_l(qe^{ix}) = (abq^l + q^{-l})\,R_l(e^{ix}).$$

Compare this with Askey & Wilson [1, (5.7), (5.8), (5.9)]:

$$A(-x)\,\big(R_l(q^{-1}e^{ix}) - R_l(e^{ix})\big) + A(x)\,\big(R_l(qe^{ix}) - R_l(e^{ix})\big) \\ = -(1-q^{-l})\,(1-q^{l-1}abcd)\,R_l(e^{ix}),$$

where

$$R_l(e^{ix}) := \text{const.}\ p_l(\cos x; a,b,c,d\,|\,q)$$

and

$$A(x) := \frac{(1-ae^{ix})\,(1-be^{ix})\,(1-ce^{ix})\,(1-de^{ix})}{(1-e^{2ix})\,(1-qe^{2ix})}\,.$$

If $c=q^{1/2}$, $d=-q^{1/2}$ (the continuous q-Jacobi case) then

$$A(x) = \frac{(1-ae^{ix})\,(1-be^{ix})}{1-e^{2ix}}$$

and

$$A(x)+A(-x) = 1-ab,$$

so

$$A(-x)\,R_l(q^{-1}e^{ix}) + A(x)\,R_l(qe^{ix}) = (q^{-l}-q^l ab)\,R_l(e^{ix}).$$

Thus Macdonald's difference equation for P_l in case $R=BC_1$ coincides with the continuous q-Jacobi case of the difference equation for Askey-Wilson polynomials.

5. Askey-Wilson polynomials for root system BC_n

We use the notation of §2 and §3. Let

$$R_1^+ := \{2\varepsilon_j\}_{j=1,\dots,n}, \quad R_2^+ := \{\varepsilon_i \pm \varepsilon_j\}_{1\le i<j\le n},$$

$$R_1 := R_1^+ \cup (-R_1^+), \quad R_2 := R_2^+ \cup (-R_2^+),$$

$$R_\ell^+ := R_1^+ \cup R_2^+, \quad R_\ell := R_1 \cup R_2 = R_\ell^+ \cup (-R_\ell^+).$$

Let R be the root system of type BC_n of §3. Then $R_\ell = \{\alpha \in R \mid 2\alpha \notin R\}$, a root system of type C_n in V with subsystems R_1 of type nA_1 and R_2 of type D_n. (The subscript ℓ stands for 'long'.) Let W be the Weyl group of R_ℓ. It is a semidirect product of the group of permutations of the coordinates and the group of sign changes of the coordinates. Let ρ, ρ_1, ρ_2 denote half the sum of the positive roots of R_ℓ, R_1, R_2, respectively. Then $\rho = \rho_1 + \rho_2$ and

$$\rho_1 = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n, \quad \rho_2 = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}.$$

Let P, P^+ and the partial order $\le$ be as in §3. Write $\varepsilon(w) := \det(w)$ $(w \in W)$. Let $A^{W,\varepsilon}$ consist of all $f \in A$ such that $wf = \varepsilon(w) f$ $(w \in W)$. Write

$$J_\lambda := \sum_{w \in W} \varepsilon(w)\, e^{w\lambda}, \quad \lambda \in P.$$

The $J_{\lambda+\rho}$ $(\lambda \in P^+)$ form a basis of $A^{W,\varepsilon}$. In particular, put

$$\delta := J_\rho = \prod_{\alpha \in R_\ell^+} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}) = e^\rho \prod_{\alpha \in R_\ell^+} (1 - e^{-\alpha})$$

and

$$\chi_\lambda := \delta^{-1}\, J_{\lambda+\rho}, \quad \lambda \in P.$$

The χ_λ $(\lambda \in P^+)$ are in A^W and form a basis of A^W. We have

$$\chi_\lambda = m_\lambda + \sum_{\mu \in P^+;\, \mu < \lambda} a_{\lambda,\mu}\, m_\mu, \quad \lambda \in P^+,$$

for certain complex $a_{\lambda,\mu}$.

Fix $q \in (0,1)$ and $a, b, c, d, t \in \mathbb{C}$. Let

$$\Delta^+ := \prod_{\alpha \in R_1^+} \frac{(e^\alpha; q)_\infty}{(ae^{\frac{1}{2}\alpha}, be^{\frac{1}{2}\alpha}, ce^{\frac{1}{2}\alpha}, de^{\frac{1}{2}\alpha}; q)_\infty} \prod_{\alpha \in R_2^+} \frac{(e^\alpha; q)_\infty}{(te^\alpha; q)_\infty} \tag{5.1}$$

and

$$\Delta(x) := \Delta^+(x)\, \Delta^+(-x). \tag{5.2}$$

We are now ready to introduce Askey-Wilson polynomials for root system BC_n.

Definition 5.1. Assume a, b, c, d are real, or if complex, appear in conjugate pairs, and that $|a|, |b|, |c|, |d| \le 1$, but the pairwise products of a, b, c, d are not ≥ 1. Assume $-1 < t < 1$. Let $T := [-\pi, \pi]^n \subset V$. For $\lambda \in P^+$ define $P_\lambda \in A^W$ by the two conditions

(i) $P_\lambda = m_\lambda + \sum_{\mu \in P^+;\, \mu < \lambda} u_{\lambda,\mu}\, m_\mu$ for certain coefficients $u_{\lambda,\mu}$;
(ii) $\int_T P_\lambda(x)\, m_\mu(x)\, \Delta(x)\, dx = 0$ for $\mu \in P^+$, $\mu < \lambda$.

We will generalize the case $R = BC_n$ of Theorem 2.2 by showing that the P_λ are orthogonal on T with respect to the weight function Δ. The proof will be based on two lemmas, the first one giving the action of a suitable difference operator on the m_λ, and the second one showing self-adjointness of this operator with respect to Δ on T, when acting on A^W.

Let $\sigma := \varepsilon_1$, similarly as in §3 for the pair (BC_n, B_n). Define Φ_σ and D_σ as in §2, with Δ^+ being given by (5.1). Thus

$$\Phi_\sigma := \frac{T_\sigma \Delta^+}{\Delta^+}, \tag{5.3}$$

and

$$\begin{aligned} D_\sigma f :=& |W_\sigma|^{-1} \sum_{w \in W} w(\Phi_\sigma\, (T_\sigma f - f)) \\ =& |W_\sigma|^{-1} \sum_{w \in W} (w\Phi_\sigma)\, (T_{w\sigma} f - f), \quad f \in A^W. \end{aligned} \tag{5.4}$$

Lemma 5.2.

$$D_\sigma m_\lambda = \sum_{\mu\in P^+;\,\mu\le\lambda} a_{\lambda,\mu}\, m_\mu$$

with

$$a_{\lambda,\lambda} = \sum_{j=1}^{n}\big(q^{-1}\, abcd\, t^{2n-j-1}\,(q^{\lambda_j}-1) + t^{j-1}\,(q^{-\lambda_j}-1)\big). \tag{5.5}$$

Here a,b,c,d,t may be arbitrarily complex.

Proof. It will be convenient to replace a, b, c, d in the expression (5.1) for Δ^+ by a, $-b$, $q^{\frac12}c$, $-q^{\frac12}d$, respectively. Thus

$$\Delta^+ := \prod_{\alpha\in R_1^+} \frac{(e^\alpha;q)_\infty}{(ae^{\frac12\alpha},-be^{\frac12\alpha},q^{\frac12}ce^{\frac12\alpha},-q^{\frac12}de^{\frac12\alpha};q)_\infty} \prod_{\alpha\in R_2^+}\frac{(e^\alpha;q)_\infty}{(te^\alpha;q)_\infty}\,. \tag{5.6}$$

By substitution of (5.6) in (5.3) we obtain

$$\begin{aligned}
\Phi_\sigma &= \frac{(1-ae^{\varepsilon_1})\,(1+be^{\varepsilon_1})\,(1-q^{\frac12}ce^{\varepsilon_1})\,(1+q^{\frac12}de^{\varepsilon_1})}{(1-e^{2\varepsilon_1})\,(1-qe^{2\varepsilon_1})} \prod_{\alpha=\varepsilon_1\pm\varepsilon_l;\,l=2,\ldots,n}\frac{1-te^\alpha}{1-e^\alpha}\\
&= abcd\,t^{2(n-1)}\,\frac{(1-a^{-1}e^{-\varepsilon_1})\,(1+b^{-1}e^{-\varepsilon_1})\,(1-q^{-\frac12}c^{-1}e^{-\varepsilon_1})\,(1+q^{-\frac12}d^{-1}e^{-\varepsilon_1})}{(1-e^{-2\varepsilon_1})\,(1-q^{-1}e^{-2\varepsilon_1})}\\
&\qquad\times\prod_{\alpha=\varepsilon_1\pm\varepsilon_l;\,l=2,\ldots,n}\frac{1-t^{-1}e^{-\alpha}}{1-e^{-\alpha}}\\
&= abcd\,t^{2(n-1)}\prod_{\alpha\in R_2^+}\frac{1-t^{-\langle\sigma,\alpha\rangle}e^{-\alpha}}{1-e^{-\alpha}}\prod_{j=1}^{n}\Big[\frac{(1-a^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})\,(1+b^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})}{(1-e^{-2\varepsilon_j})}\\
&\qquad\times\frac{(1-q^{-\frac12}c^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})\,(1+q^{-\frac12}d^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})}{(1-q^{-1}e^{-2\varepsilon_j})}\Big].
\end{aligned}$$

Hence

$$\Phi_\sigma = \delta^{-1}\,\delta_q^{-1}\,\Psi_\sigma,$$

where

$$\begin{aligned}
\Psi_\sigma := (abcd)^{\langle\sigma,\rho_1\rangle}\,t^{\langle\sigma,2\rho_2\rangle}\,e^{\rho+2\rho_1}\prod_{\alpha\in R_2^+}(1-t^{-\langle\sigma,\alpha\rangle}e^{-\alpha})\\
\times\prod_{j=1}^{n}\big[(1-qe^{-2\varepsilon_j})\,(1-a^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})(1+b^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})\\
\times(1-q^{-\frac12}c^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})\,(1+q^{-\frac12}d^{-\langle\sigma,\varepsilon_j\rangle}e^{-\varepsilon_j})\big]
\end{aligned} \tag{5.7}$$

and δ_q is the following element of A^W:

$$\begin{aligned}
\delta_q &:= \prod_{j=1}^{n}(q^{-\frac12}e^{\varepsilon_j}-q^{\frac12}e^{-\varepsilon_j})\,(q^{\frac12}e^{\varepsilon_j}-q^{-\frac12}e^{-\varepsilon_j})\\
&= e^{2\rho_1}\prod_{j=1}^{n}(1-qe^{-2\varepsilon_j})\,(1-q^{-1}e^{-2\varepsilon_j}).
\end{aligned} \tag{5.8}$$

If $E \subset (\frac{1}{2}R_1) \cup R_2$ then write

$$||E|| := \sum_{\alpha \in E} \alpha.$$

Expansion of (5.7) yields

$$\Psi_\sigma = \sum_{E_0,\dots,E_4 \subset \frac{1}{2}R_1^+} \sum_{F \subset R_2^+} c_{E_0,\dots,E_4,F}\, e^{\rho+2\rho_1-2||E_0||-||E_1||-\cdots-||E_4||-||F||}, \tag{5.9}$$

where

$$\begin{aligned} c_{E_0,\dots,E_4,F} &:= (-1)^{|E_0|+|E_1|+|E_3|+|F|}\, q^{|E_0|-\frac{1}{2}|E_3|-\frac{1}{2}|E_4|} \\ &\quad \times a^{\langle \sigma,\rho_1-||E_1||\rangle}\, b^{\langle \sigma,\rho_1-||E_2||\rangle}\, c^{\langle \sigma,\rho_1-||E_3||\rangle}\, d^{\langle \sigma,\rho_1-||E_4||\rangle}\, t^{\langle \sigma,2\rho_2-||F||\rangle}. \end{aligned} \tag{5.10}$$

Now we can rewrite (5.4) as

$$D_\sigma f = \delta^{-1}\, \delta_q^{-1}\, \widetilde{D}_\sigma f$$

with

$$\widetilde{D}_\sigma f := |W_\sigma|^{-1} \sum_{w \in W} \varepsilon(w)\, (w\Psi_\sigma)\, (T_{w\sigma} f - f). \tag{5.11}$$

Consider (5.11) with $f := m_\lambda$ ($\lambda \in P^+$) and substitute (5.9). Then

$$\begin{aligned} \widetilde{D}_\sigma m_\lambda = |W_\sigma|^{-1}\, |W_\lambda|^{-1} \sum_{w_1,w_2 \in W} \sum_{E_0,\dots,E_4 \subset \frac{1}{2}R_1^+} \sum_{F \subset R_2^+} c_{E_0,\dots,E_4,F}\; \varepsilon(w_1) \\ \times (q^{\langle w_1\sigma, w_2\lambda\rangle} - 1)\, e^{w_1(\rho+2\rho_1-2||E_0||-||E_1||-\cdots-||E_4||-||F||)+w_2\lambda}. \end{aligned}$$

Put $w_2 = w_1 w$. Then

$$\begin{aligned} \widetilde{D}_\sigma m_\lambda = |W_\sigma|^{-1}\, |W_\lambda|^{-1} \sum_{w \in W} \sum_{E_0,\dots,E_4 \subset \frac{1}{2}R_1^+} \sum_{F \subset R_2^+} c_{E_0,\dots,E_4,F}\, (q^{\langle \sigma, w\lambda\rangle} - 1) \\ \times J_{w\lambda+\rho+2\rho_1-2||E_0||-||E_1||-\cdots-||E_4||-||F||}. \end{aligned} \tag{5.12}$$

Hence $\widetilde{D}_\sigma m_\lambda \in A^{W,\varepsilon}$. Now the J-function in (5.12) is either 0 or $\varepsilon(w')\,\delta\,\chi_\nu$, where $w' \in W$, $\nu \in P^+$ and

$$w'(\nu+\rho) = w\lambda + \rho + 2\rho_1 - 2||E_0|| - ||E_1|| - \cdots - ||E_4|| - ||F||,$$

so that

$$\begin{aligned} \nu + \rho &= (w')^{-1} w\lambda \\ &\quad + (w')^{-1}(3\rho_1 - 2||E_0|| - ||E_1|| - \cdots - ||E_4||) + (w')^{-1}(\rho_2 - ||F||). \end{aligned} \tag{5.13}$$

Now

$$(w')^{-1}(3\rho_1 - 2||E_0|| - ||E_1|| - \cdots - ||E_4||) = (w')^{-1} \sum_{j=1}^n k_j \varepsilon_j = \sum_{j=1}^n k_j' \varepsilon_j \le 3\rho_1 \tag{5.14}$$

with $k_j, k'_j \in \{-3,-2,-1,0,1,2,3\}$,

$$(w')^{-1}(\rho_2 - ||F||) = (w')^{-1} \sum_{\alpha\in R_2^+} k_\alpha \alpha = \sum_{\alpha\in R_2^+} k'_\alpha \alpha \le \rho_2 \tag{5.15}$$

with $k_\alpha, k'_\alpha = \pm\frac{1}{2}$, and

$$(w')^{-1} w\lambda \le \lambda. \tag{5.16}$$

Substitution of (5.14), (5.15), (5.16) in (5.13) yields

$$\nu + \rho \le \lambda + 3\rho_1 + \rho_2 = \lambda + \rho + 2\rho_1. \tag{5.17}$$

Hence

$$\widetilde{D}_\sigma m_\lambda = \sum_{\nu\in P^+;\, \nu\le\lambda+2\rho_1} b_\nu \, J_{\nu+\rho} \tag{5.18}$$

for certain coefficients b_ν.

In order to compute $b_{\lambda+2\rho_1}$ observe that equality in (5.17) holds iff equality holds in (5.14), (5.15), (5.16), i.e., iff $(w')^{-1}w \in W_\lambda$ and

$$||E_0|| = ||E_1|| = \cdots = ||E_4|| = \tfrac{1}{2}(\rho_1 - w'\rho_1), \tag{5.19}$$

$$||F|| = \rho_2 - w'\rho_2. \tag{5.20}$$

Hence, by (5.12),

$$b_{\lambda+2\rho_1} = |W_\sigma|^{-1}\, |W_\lambda|^{-1} \sum_{w,w'\in W;\, (w')^{-1}w\in W_\lambda} \varepsilon(w')\, c_{E_0,\ldots,E_4,F}\, (q^{\langle\sigma,w\lambda\rangle} - 1), \tag{5.21}$$

where $E_0,\ldots,E_4,F$ are determined by (5.19), (5.20). It follows from (5.19), (5.20) that

$$2||E_0|| + ||F|| = \rho - w'\rho, \quad \text{hence} \quad (-1)^{|E_0|+|F|} = \varepsilon(w').$$

Substitution of (5.19), (5.20) into (5.10) now yields:

$$c_{E_0,\ldots,E_4,F} = \varepsilon(w')\,(abcd)^{\frac{1}{2}(1+\langle (w')^{-1}\sigma,\rho_1\rangle)}\, t^{n-1+\langle (w')^{-1}\sigma,\rho_2\rangle}.$$

When we substitute this last expression into (5.21) then we obtain

$$\begin{aligned} b_{\lambda+2\rho_1} &= |W_\sigma|^{-1}\, |W_\lambda|^{-1} \\ &\times \sum_{w,w'\in W;\, (w')^{-1}w\in W_\lambda} (abcd)^{\frac{1}{2}(1+\langle (w')^{-1}\sigma,\rho_1\rangle)}\, t^{n-1+\langle (w')^{-1}\sigma,\rho_2\rangle}\, (q^{\langle (w')^{-1}\sigma,\lambda\rangle} - 1) \\ &= |W_\sigma|^{-1} \sum_{w\in W} (abcd)^{\frac{1}{2}(1+\langle w\sigma,\rho_1\rangle)}\, t^{n-1+\langle w\sigma,\rho_2\rangle}\, (q^{\langle w\sigma,\lambda\rangle} - 1). \end{aligned}$$

Hence

$$b_{\lambda+2\rho_1}=\sum_{j=1}^n\sum_{\varepsilon=\pm1}(abcd)^{\frac12(1+\varepsilon)}\,t^{n-1+\varepsilon(n-j)}\,(q^{\varepsilon\lambda_j}-1),\tag{5.22}$$

which is (5.5), when we take in account the replacement made for a,b,c,d.

Next we show that, for $f\in A^W$, $\widetilde D_\sigma f$ given by (5.11) is divisible by δ_q. In view of (5.8) this will follow if we can show that, for each $w\in W$, $(w\Psi_\sigma)\,(T_{w\sigma}f-f)$ is divisible by the $4n$ prime factors $1\pm q^{\pm\frac12}e^{-\varepsilon_j}$. By (5.7), all but the two factors $1\pm q^{-\frac12}e^{-w\sigma}$ are divisors of $w\Psi_\sigma$. We will show that these two factors are divisors of $T_{w\sigma}f-f$. Write f as a Laurent polynomial $F(e^{\varepsilon_1},\dots,e^{\varepsilon_n})$, invariant under the transformations $e^{\varepsilon_j}\mapsto e^{-\varepsilon_j}$. If $w\sigma=\varepsilon_j$ then

$$T_{w\sigma}f-f=F(e^{\varepsilon_1},\dots,qe^{\varepsilon_j},\dots,e^{\varepsilon_n})-F(e^{\varepsilon_1},\dots,e^{\varepsilon_j},\dots,e^{\varepsilon_n})$$

becomes 0 for $e^{\varepsilon_j}=\pm q^{-\frac12}$, hence it is divisible by $1\pm q^{-\frac12}e^{-\varepsilon_j}$. A similar argument is valid for $w\sigma=-\varepsilon_j$.

By (5.18),

$$\delta^{-1}\,\widetilde D_\sigma m_\lambda=\sum_{\nu\in P^+;\,\nu\le\lambda+2\rho_1}c_\nu\,m_\nu\tag{5.23}$$

for certain coefficients c_ν, with $c_{\lambda+2\rho_1}=b_{\lambda+2\rho_1}$ given by (5.22). Also, $\delta^{-1}\,\widetilde D_\sigma m_\lambda$ will still be divisible by δ_q. By (5.8), $\delta_q\in A^W$ with highest term $m_{2\rho_1}$. Hence $D_\sigma m_\lambda=\delta^{-1}\,\delta_q^{-1}\,\widetilde D_\sigma m_\lambda$ will be in A^W with highest term $b_{\lambda+2\rho_1}\,m_\lambda$. □

We have

$$\Delta=\prod_{\alpha\in R_1}\frac{(e^\alpha;q)_\infty}{(ae^{\frac12\alpha},be^{\frac12\alpha},ce^{\frac12\alpha},de^{\frac12\alpha};q)_\infty}\prod_{\alpha\in R_2}\frac{(e^\alpha;q)_\infty}{(te^\alpha;q)_\infty}.$$

Hence

$$\Delta(x)=(w\Delta)(x)=(w\Delta^+)(x)\,(w\Delta^+)(-x),\quad w\in W.$$

Lemma 5.3. *With the assumptions of Definition 5.1 we have*

$$\int_T(D_\sigma f)(x)\,g(x)\,\Delta(x)\,dx=\int_T f(x)\,(D_\sigma g)(x)\,\Delta(x)\,dx,\quad f,g\in A^W.\tag{5.24}$$

Proof. Since $-\mathrm{id}\in W$, $f(x)=f(-x)$ and $g(x)=g(-x)$. By (5.4) and (5.3), formula (5.24) can be equivalently written as

$$\begin{aligned}\sum_{w\in W}\int_T(T_{w\sigma}(w\Delta^+))(x)\,\big((T_{w\sigma}f)(x)-f(x)\big)\,(w\Delta^+)(-x)\,g(-x)\,dx\\ =\sum_{w\in W}\int_T(w\Delta^+)(x)\,f(x)\,(T_{w\sigma}(w\Delta^+))(-x)\,\big((T_{w\sigma}g)(-x)-g(-x)\big)\,dx.\end{aligned}\tag{5.25}$$

Since T, f and g are W-invariant, formula (5.25) will be implied by the two identities

$$\int_T(\Delta^+f)(x-i(\log q)\sigma)\,(\Delta^+g)(-x)\,dx=\int_T(\Delta^+f)(x)\,(\Delta^+g)(-x-i(\log q)\sigma)\,dx$$

and

$$\sum_{w\in W}(T_\sigma\Delta^+)(w^{-1}x)\,\Delta^+(-w^{-1}x)=\sum_{w\in W}\Delta^+(w^{-1}x)\,(T_\sigma\Delta^+)(-w^{-1}x).$$

The second identity is obvious, since $-\mathrm{id}\in W$. For the first identity observe that the integral

$$\int_{\mathcal{C}}(\Delta^+f)(z-i(\log q),x_2,\ldots,x_n)\,(\Delta^+g)(-z,-x_2,\ldots,-x_n)\,dz$$

over the contour

$$\mathcal{C}=[-\pi,\pi]\cup[\pi,\pi+i\log q]\cup[\pi+i\log q,-\pi+i\log q]\cup[-\pi+i\log q,-\pi]$$

vanishes by Cauchy's theorem. (By the assumptions on a,b,c,d,t there are no singularities inside the contour.) Now the result follows, since Δ^+f and Δ^+g are invariant under translations by $2\pi\sigma$. □

It follows now immediately from Lemmas 5.2 and 5.3 that:

Theorem 5.4. *$D_\sigma P_\lambda=a_{\lambda,\lambda}\,P_\lambda$ with $a_{\lambda,\lambda}$ given by (5.5).*

Now we are ready for the main theorem.

Theorem 5.5. *If $\lambda,\mu\in P^+$, $\lambda\neq\mu$, then*

$$\int_T P_\lambda(x)\,P_\mu(x)\,\Delta(x)\,dx=0.$$

Proof. All integrals

$$\int_T m_\lambda(x)\,m_\mu(x)\,\Delta(x)\,dx$$

are continuous in a,b,c,d,t. Hence the coefficients $u_{\lambda,\mu}$ in Definition 5.1 are continuous in a,b,c,d,t. This implies that

$$\int_T P_\lambda(x)\,P_\mu(x)\,\Delta(x)\,dx$$

is continuous in a,b,c,d,t. By Theorem 5.4 and Lemma 5.3,

$$\int_T P_\lambda(x)\,P_\mu(x)\,\Delta(x)\,dx=0$$

if $a_{\lambda,\lambda}\neq a_{\mu,\mu}$. Fix distinct λ and μ it follows from (5.5) that, for fixed nonzero a,b,c,d, the eigenvalues $a_{\lambda,\lambda}$ and $a_{\mu,\mu}$ are distinct as polynomials in t. This implies the orthogonality of P_λ and P_μ for a,b,c,d,t in a dense subset of the parameter domain under consideration. Hence, by continuity, the theorem follows. □

The method of proof in this last theorem is different from the method used in similar situations by Macdonald [12]. While Macdonald leaves the parameters fixed and shows that equality of eigenvalues for all q implies (in most cases) equality of weights, the above proof leaves q fixed and shows that equality of eigenvalues for all parameter values implies equality of weights.

6. Discussion of results

6.1. Special cases. When we compare (5.1) with (3.1), (3.2) and (3.3), then it is clear that Askey-Wilson polynomials for root system BC_n with a, b, c, d, t replaced by $q^{\frac{1}{2}}, -q^{\frac{1}{2}}, ab^{\frac{1}{2}}, -b^{\frac{1}{2}}, t$ become Macdonald's polynomials for the pair (BC_n, B_n). The operator D_σ given by (5.4) then specializes to the operator for which Theorem 2.3 is valid in case (BC_n, B_n), and the eigenvalue (5.5) specializes to the eigenvalue in Theorem 2.3, cf. (3.6). We can also work then with E_σ instead of D_σ.

Next, when we compare (5.1) with (3.1), (3.4) and (3.5) then it is clear that our polynomials with a, b, c, d, t, q replaced by $ab^{\frac{1}{2}}, qab^{\frac{1}{2}}, -b^{\frac{1}{2}}, -qb^{\frac{1}{2}}, t, q^2$ become Macdonald's polynomials for the pair (BC_n, C_n). The operator D_σ given by (5.4) then becomes the operator $D_{2\varepsilon_1}$ for the pair (BC_n, C_n). Theorem 2.3 does not say anything about eigenfunctions of this operator, but Theorem 5.4 implies that Macdonald's polynomials for the pair (BC_n, C_n) are eigenfunctions of $D_{2\varepsilon_1}$. This corresponds nicely with the cases E_8, F_4, G_2 of Theorem 2.3, where $\langle \sigma, \alpha_* \rangle$ takes values $0, 1, 2$ as α runs through R^+ and we have to work with D_σ instead of E_σ. It would be interesting to consider if P_λ might also be eigenfunction of D_σ for other "quasi-minuscule" σ.

Comparison of (5.1) and (4.1) makes it evident that the BC_n Askey-Wilson polynomials reduce to the one-variable Askey-Wilson polynomials for $n = 1$.

6.2. A Selberg-type integral and a conjectured quadratic norm. Let Δ be given by (5.2) and (5.1) and let the parameters satisfy the inequalities of Definition 5.1. Gustafson [3, (2)] evaluated the Selberg type integral

$$\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta(x)\, dx = 2^n n! \prod_{j=1}^{n} \frac{(t, t^{n+j-2}abcd; q)_\infty}{(t^j, q, abt^{j-1}, act^{j-1}, \ldots, cdt^{j-1}; q)_\infty}.$$

On the other hand, Macdonald [12, (12.6)] conjectured an explicit expression for the quadratic norm $\langle P_\lambda, P_\lambda \rangle$ for polynomials P_λ associated with any admissible pair (R, S). It can be shown that Macdonald's conjecture in case $\lambda = 0$ and $(R, S) = (BC_n, B_n)$ coincides with Gustafson's formula for $(a, b, c, d) = (q^{\frac{1}{2}}, -q^{\frac{1}{2}}, ab^{\frac{1}{2}}, -b^{\frac{1}{2}})$. In October 1991, when prof. Macdonald was visiting The Netherlands, first the author has given a conjectured expression for $\langle P_\lambda, P_\lambda \rangle / \langle 1, 1 \rangle$, where P_λ is an Askey-Wilson polynomial for root system BC_n, and next Macdonald [13] has rewritten this as a conjectured expression for $\langle P_\lambda, P_\lambda \rangle$. On the same occasion, Macdonald [13] has also extended his other conjectures in [12, §12] to the BC_n Askey-Wilson case.

6.3. The Askey-Wilson hierarchy for BC_n. It is very probable that all specializations and limit cases of one-variable Askey-Wilson polynomials have their analogues in the case of BC_n. Someone should certainly write down the orthogonality relations and difference operators with explicit eigenvalues for all these specializations. In some cases these explicit formulas may be rigorously proved by straightforward limit transition from the general Askey-Wilson case. In other cases, the limit transition may only give a formal proof and, for a rigorous derivation, the proofs of the present paper will have to be imitated.

q-Racah-type polynomials for root system BC_n should also be obtained. Here analytic continuation from the BC_n Askey-Wilson polynomials will be needed and residues, possibly higher dimensional, will have to be taken. Similar problems will arise when the condition $|a|, |b|, |c|, |d| \leq 1$ is dropped in Definition 5.1. In the corresponding one-variable case discrete terms are then added to the orthogonality relations.

6.4. Quantum group interpretations. It is known from work by Koornwinder [9], [10], Koelink [8] and Noumi & Mimachi [15], [16] that one-variable Askey-Wilson polynomials have an interpretation on the quantum group $SU_q(2)$. Noumi [14] announces an interpretation of Macdonald's polynomials for root system A_{n-1} as zonal spherical functions on the quantum analogues of the homogeneous spaces $GL(n)/SO(n)$ and $GL(2n)/Sp(2n)$. According to Noumi, this was already done for the quantum analogue of $SL(3)/SO(3)$ by Ueno & Takebayashi. It would be interesting to find quantum group interpretations of Macdonald's polynomials in case of all root systems, and also of the BC_n Askey-Wilson polynomials considered in the present paper.

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CWI, P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Current address: University of Amsterdam, Faculty of Mathematics and Computer Science, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

E-mail address: thk@fwi.uva.nl