# ASPECTS OF THE CATEGORY SKB OF SKEW BRACES 

DOMINIQUE BOURN, ALBERTO FACCHINI, AND MARA POMPILI


#### Abstract

We examine the pointed protomodular category SKB of left skew braces. We study the notion of commutator of ideals in a left skew brace. Notice that in the literature, "product" of ideals of skew braces is often considered. We show that Huq=Smith for left skew braces. Finally, we give a set of generators for the commutator of two ideals, and prove that every ideal of a left skew brace has a centralizer.


## Introduction

Braces appear in connections to the study of set-theoretic solutions of the YangBaxter equation. A set-theoretic solution of the Yang-Baxter equation is a pair $(X, r)$, where $X$ is a set, $r: X \times X \rightarrow X \times X$ is a bijection, and $(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id})=$ $(\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$ [13]. Set-theoretic solutions of the Yang-Baxter equation appear, for instance, in the study of representations of braid groups, and form a category SYBE, whose objects are these pairs $(X, r)$, and morphisms $f:(X, r) \rightarrow$ $\left(X^{\prime}, r^{\prime}\right)$ are the mappings $f: X \rightarrow X^{\prime}$ that make the diagram

commute.
One way to produce set-theoretic solutions of the Yang-Baxter equation is using left skew braces.

Definition [15] A (left) skew brace is a triple $(A, *, \circ)$, where $(A, *)$ and $(A, \circ)$ are groups (not necessarily abelian) such that

$$
\begin{equation*}
a \circ(b * c)=(a \circ b) * a^{-*} *(a \circ c) \tag{B}
\end{equation*}
$$

for every $a, b, c \in A$. Here $a^{-*}$ denotes the inverse of $a$ in the group $(A, *)$. The inverse of $a$ in the group ( $A, \circ$ ) will be denoted by $a^{-\circ}$.

[^0]A brace is sometimes seen as an algebraic structure similar to that of a ring, with distributivity warped in some sense. But a better description of a brace is probably that of an algebraic structure with two group structures out of phase with each other.

For every left skew brace $(A, *, \circ)$, the mapping

$$
r: A \times A \rightarrow A \times A, \quad r(x, y)=\left(x^{-*} *(x \circ y),\left(x^{-*} *(x \circ y)\right)^{-\circ} \circ x \circ y\right)
$$

is a non-degenerate set-theoretic solution of the Yang-Baxter equation ([15) Theorem 3.1] and [19, p. 96]). Here "non-degenerate" means that the mappings $\pi_{1} r\left(x_{0},-\right): A \rightarrow A$ and $\pi_{2} r\left(-, y_{0}\right): A \rightarrow A$ are bijections for every $x_{0} \in A$ and every $y_{0} \in A$.

The simplest examples of left skew braces are:
(1) For any associative ring $(R,+, \cdot)$, the Jacobson radical $(J(R),+, \circ)$, where $\circ$ is the operation on $J(R)$ defined by $x \circ y=x y+x+y$ for every $x, y \in J(R)$.
(2) For any group $(G, *)$, the left skew braces $(G, *, *)$ and $\left(G, *, *^{o p}\right)$.

Several non-trivial examples of skew braces can be found in [23]. A complete classification of braces of low cardinality has been obtained via computer [19].

A homomorphism of skew braces is a mapping which is a group homomorphism for both the operations. This defines the category SKB of skew braces.

From [15], we know that in a skew brace the units of the two groups coincide. So, SKB appears as a fully faithful subcategory $\mathrm{SKB} \hookrightarrow$ DiGp of the category DiGp of digroups, where a digroup is a triple $(G, *, \circ)$ of a set $G$ endowed with two group structures with same unit. This notion was introduced in [8] and devised during discussions between the first author and G. Janelidze.

There are two forgetful functors $U_{i}: \mathrm{DiGp} \rightarrow \mathrm{Gp}, i \in\{0,1\}$, associating respectively the first and the second group structures. They both reflect isomorphisms. Since $U_{0}$ is left exact and reflects isomorphisms, it naturally allows the lifting of the protomodular aspects of the category Gp of groups to the category DiGp . In turn, the left exact fully faithful embedding SKB $\hookrightarrow$ DiGp makes SKB a pointed protomodular category. The protomodular axiom was introduced in [5] in order to extract the essence of the homological constructions and in particular to induce an intrinsic notion of exact sequence.

In this paper, after recalling the basic facts about protomodular categories, we study the "protomodular aspects" of left skew braces, in particular in relation to the category of digroups. We study the notion of commutator of ideals in a left skew brace (in the literature, "product" of ideals of skew braces is often considered). We show that Huq=Smith for left skew braces. Notice that Huq $\neq$ Smith for digroups and near-rings [18. We give a set of generators for the commutator of two ideals, and prove that every ideal of a left skew brace has a centralizer.

## 1. Basic recalls on protomodular categories

In this work, any category $\mathbb{E}$ will be supposed finitely complete, which implies that it has a terminal object 1 . The terminal map from $X$ is denoted $\tau_{X}: X \rightarrow 1$. Given any map $f: X \rightarrow Y$, the equivalence relation $R[f]$ on $X$ is produced by the pullback of $f$ along itself. The map $f$ is said to be a regular epimorphism in $\mathbb{E}$ when $f$ is the quotient of $R[f]$. When it is the case, we denote it by a double head arrow $X \rightarrow Y$.
1.1. Pointed protomodular categories. The category $\mathbb{E}$ is said to be pointed when the terminal object 1 is initial as well. Let us recall that a pointed category $\mathbb{A}$ is additive if and only if, given any split epimorphism $f: X \rightleftarrows Y, f s=1_{Y}$, the following downward pullback:

is an upward pushout, namely if and only if $X$ is the direct sum (= coproduct) of $Y$ and $\operatorname{Ker} f$. Let us recall the following:
Definition 1.1. [5] A pointed category $\mathbb{E}$ is said to be protomodular when, given any split epimorphism as above, the pair $\left(k_{f}, s\right)$ of monomorphisms is jointly strongly epic.

This means that the only subobject $u: U \mapsto X$ containing the pair $\left(k_{f}, s\right)$ of subobjects is, up to isomorphism, $1_{X}$. It implies that, given any pair $(f, g): X \rightrightarrows Z$ of arrows which are equalized by $k_{f}$ and $s$, they are necessarily equal (take the equalizer of this pair). Pulling back the split epimorphisms along the initial map $0_{Y}: 1 \multimap Y$ being a left exact process, the previous definition is equivalent to saying that this process reflects isomorphisms.

The category $G p$ of groups is clearly pointed protomodular. This is the case of the category Rng of rings as well, and more generally, given a commutative ring $R$, of any category $R$-Alg of any given kind of $R$-algebras without unit, possibly nonassociative. This is in particular the case of the category $R$-Lie of Lie $R$-algebras. Even for $R$ a non-commutative ring, in which case $R$-algebras have a more complex behaviour (they are usually called $R$-rings, see [2, p. 36] or [14, p. 52]), one has that the category $R$-Rng of $R$-rings is pointed protomodular, as can be seen from the fact that the forgetul functor $R$-Rng $\rightarrow A b$ reflects isomorphisms and $A b$ is protomodular.

The pointed protomodular axiom implies that the category $\mathbb{E}$ shares with the category Gp of groups the following well-known Five Principles:
(1) a morphism $f$ is a monomorphism if and only if its kernel $\operatorname{Ker} f$ is trivial [5];
(2) any regular epimorphism is the cokernel of its kernel, in other words any regular epimorphism produces an exact sequence, which determines an intrinsic notion of exact sequences in $\mathbb{E}$ [5;
(3) there a specific class of monomorphisms $u: U \longmapsto X$, the normal monomorphisms [7], see next section ;
(4) there is an intrinsic notion of abelian object [7, see section 3.1.1,
(5) any reflexive relation in $\mathbb{E}$ is an equivalence relation, i.e. the category $\mathbb{E}$ is a Mal'tsev one [6].

So, according to Principle (1), a pointed protomodular category is characterized by the validity of the split short five lemma. Generally, Principle (5) is not directly exploited in Gp; we shall show in Section 3.4 .2 how importantly it works out inside a pointed protomodular category $\mathbb{E}$. Pointed protomodular varieties of universal algebras are characterized in [12].

### 1.2. Normal monomorphisms.

Definition 1.2. [7] In any category $\mathbb{E}$, given a pair $(u, R)$ of a monomorphism $u: U \longmapsto X$ and an equivalence relation $R$ on $X$, the monomorphism $u$ is said to be normal to $R$ when the equivalence relation $u^{-1}(R)$ is the indiscrete equivalence relation $\nabla_{X}=R\left[\tau_{X}\right]$ on $X$ and, moreover, any commutative square in the following induced diagram is a pullback:


In the category Set, provided that $U \neq \emptyset$, these two properties characterize the equivalence classes of $R$. By the Yoneda embedding, this implies the following:

Proposition 1.3. Given any equivalence relation $R$ on an object $X$ in a category $\mathbb{E}$, for any map $x$, the following upper monomorphism $\check{x}=d_{1}^{R} . \bar{x}$ is normal to $R$ :


In a pointed category $\mathbb{E}$, taking the initial map $0_{X}: 1 \rightarrow X$ gives rise to a monomorhism $\iota_{R}: I_{R} \mapsto X$ which is normal to $R$. This construction produces a preorder mapping $\iota^{X}:$ Equ $_{X} \mathbb{E} \rightarrow$ Mon $_{X} \mathbb{E}$ from the preorder of the equivalence relations on $X$ to the preorder of subobjects of $X$ which preserves intersections. Starting with any map $f: X \rightarrow Y$, we get $I_{R[f]}=\operatorname{Ker} f$ which says that any kernel map $k_{f}$ is normal to $R[f]$. Principle (3) above is a consequence of the fact [7] that in a protomodular category a monomorphism is normal to at most one equivalence relation (up to isomorphism). So that being normal, for a monomorphism $u$, becomes a property in this kind of categories. This is equivalent to saying that the preorder homomorphism $\iota_{X}: \mathrm{Equ}_{X} \mathbb{E} \rightarrow$ Mon $_{X} \mathbb{E}$ reflects inclusions; so, the preorder $\operatorname{Norm}_{X}$ of normal subobjects of $X$ is just the image $\iota^{X}\left(\mathrm{Equ}_{X}\right) \subset \mathrm{Mon}_{X}$.
1.3. Regular context. Let us recall from [1] the following:

Definition 1.4. A category $\mathbb{E}$ is regular when it satisfies the two first conditions, and exact when it satisfies all the three conditions:
(1) regular epimorphisms are stable under pullbacks;
(2) any kernel equivalence relation $R[f]$ has a quotient $q_{f}$;
(3) any equivalence relation $R$ is a kernel equivalence relation.

Then, in the regular context, given any map $f: X \rightarrow Y$, the following canonical factorization $m$ is necessarily a monomorphism:


This produces a canonical decomposition of the map $f$ in a monomorphism and a regular epimorphism which is stable under pullbacks. Now, given any regular
epimorhism $f: X \rightarrow Y$ and any subobject $u: U \mapsto X$, the direct image $f(u)$ : $f(U) \longmapsto Y$ of $u$ along the regular epimorphism $f$ is given by $f(U)=\operatorname{Im}_{f . u} \mapsto Y$.

Any variety in the sense of Universal Algebra is exact and regular epimorphisms coincide with surjective homomorphisms.
1.4. Homological categories. The significance of pointed protomodular categories grows up in the regular context since, in this context, the split short five lemma can be extended to any exact sequence. Furthermore, the $3 \times 3$ lemma, Noether isomorphisms and snake lemma hold; they are all collected in 3. This is the reason why a regular pointed protomodular category $\mathbb{E}$ is called homological.

## 2. Protomodular aspects of Skew braces

2.1. Digroups. From [8, we get the characterization of normal monomorphisms in DiGp:

Proposition 2.1. A suboject $i:(G, *, \circ) \longmapsto(K, *, \circ)$ is normal in the category DiGp if and only if the three following conditions hold:
(1) $i:(G, *) \mapsto(K, *)$ is normal in Gp ,
(2) $i:(G, \circ) \longmapsto(K, \circ)$ is normal in Gp ,
(3) for all $(x, y) \in K \times K, x^{-*} * y \in G$ if and only if $x^{-\circ} \circ y \in G$.
2.2. Skew braces. The following observation is very important:

Proposition 2.2. Let $(G, *, \circ)$ be any skew brace. Consider the mapping $\lambda: G \times$ $G \rightarrow G$ defined by $\lambda(a, u)=a^{-*} *(a \circ u)$. Then:
(1) $\lambda_{a}=\lambda(a,-)$ is underlying a group homomorphism $(G, \circ) \rightarrow \operatorname{Aut}(G, *)$, and this condition is equivalent to (B);
(2) we have

$$
\begin{equation*}
\lambda\left(a^{-\circ}, u\right)=\left(a^{-\circ}\right)^{-*} *\left(a^{-\circ} \circ u\right)=a^{-\circ} \circ(a * u) \tag{1}
\end{equation*}
$$

Proof. For (1), see [15]. For (2), we have $\left(a^{-\circ} \circ a\right) *\left(a^{-\circ}\right)^{-*} *\left(a^{-\circ} \circ u\right)=\left(a^{-\circ}\right)^{-*} *$ $\left(a^{-\circ} * u\right)=\lambda\left(a^{-\circ}, u\right)$.
2.3. First properties of skew braces. The following observation is straightforward:
Proposition 2.3. SKB is a Birkhoff subcategory of DiGp.
This means that any subobject of a skew brace in DiGp is a skew brace and that, given any surjective homomorphism $f: X \rightarrow Y$ in DiGp, the digroup $Y$ is a skew brace as soon as so is $X$. In this way, any equivalence relation $R$ in DiGp on a skew brace $X$ actually lies in SKB since it determines a subobject $R \subset X \times X$ in DiGp and, moreover, its quotient in SKB is its quotient in DiGp. The first part of this last sentence implies that any normal subobject $u: U \rightharpoondown X$ in DiGp with $X \in$ SKB is normal in SKB.

We are now going to show that the normal subobjects in SKB coincide with the ideals of [15].

Proposition 2.4. A subobject $i:(G, *, \circ) \longmapsto(K, *, \circ)$ is normal in the category SKB if and only if the three following conditions hold:
(1) $i:(G, *) \mapsto(K, *)$ is normal in Gp ,
(2) $i:(G, \circ) \mapsto(K, \circ)$ is normal in Gp ,
$\left(3^{\prime}\right) \lambda_{x}(G)=G$ for all $x \in K$.

Proof. Suppose (1) and (2). We are going to show (3) $\Longleftrightarrow\left(3^{\prime}\right)$, with (3) given in Proposition 2.1 .
(i) $x^{-\circ} \circ y \in G \Rightarrow x^{-*} * y \in G$ if and only if $\lambda_{x}(G) \subset G$, setting $y=x \circ u, u \in G$.
(ii) from (11): $x^{-*} * y \in G \Rightarrow x^{-\circ} \circ y \in G$ if and only if $\lambda_{x^{-\circ}}(G) \subset G$, setting $y=x * u, u \in G$.
Finally $\lambda_{x}(G) \subset G$ for all $x$ is equivalent to $\lambda_{x}(G)=G$.

Corollary 2.5. A subobject $i:(G, *, \circ) \mapsto(K, *, \circ)$ is normal in the category SKB if and only if it is an ideal in the sense of [15], namely is such that:

1) $i:(G, \circ) \longmapsto(K, \circ)$ is normal, 2) $G * a=a * G$ for all $a \in K$, 3) $\lambda_{a}(G) \subset G$ for all $a \in K$.

Proof. Straightforward.

Being a variety in the sense of Universal Algebra, SKB is finitely cocomplete; accordingly it has binary sums (called coproducts as well). So, SKB is a semiabelian category according to the definition introduced in [17:

Definition 2.6. A pointed category $\mathbb{E}$ is said to be semi-abelian when it is protomodular, exact and has finite sums.

From the same [17, let us recall the following observation which explains the choice of the terminology: a pointed category $\mathbb{E}$ is abelian if and only if both $\mathbb{E}$ and $\mathbb{E}^{o p}$ are semi-abelian.
2.4. Internal skew braces. Given any category $\mathbb{E}$, the notion of internal group, digroup and skew brace is straightforward, determining the categories $\mathrm{Gp} \mathbb{E}, \mathrm{DiGp} \mathbb{E}$ and $\operatorname{SKBE}$. Since $G p \mathbb{E}$ is protomodular, so are the two others. An important case is produced with $\mathbb{E}=$ Top the category of topological spaces. Although Top is not a regular category, so is the category $\mathrm{Gp} T o p$, the regular epimorphisms being the open surjective homomorphisms. So GpTop is homological but not semi-abelian.

Now let $f: X \rightarrow Y$ be any map in DiGpTop. Let us show that $R[f]$ has a quotient in DiGpTop. Take its quotient $q_{R[f]}: X \rightarrow Q_{f}$ in DiGp, then endow $Q_{f}$ with the quotient topology with respect to $R[f]$; then $q_{R[f]}$ is an open surjective homomorphism since so is $U_{0}\left(q_{R[f]}\right)$. Accordingly, a regular epimorphism in DiGpTop is again an open surjective homomorphism. Moreover this same functor $U_{0}:$ DiGpTop $\rightarrow$ GpTop being left exact and reflecting the homeomorphic isomorphisms, it reflects the regular epimorphisms; so, these regular epimorphisms in DiGpTop are stable under pullbacks. Accordingly the category DiGpTop is regular. Similarly the category SKBTop is homological as well, without being semi-abelian. As any category of topological semi-abelian algebras, both DiGpTop and SKBTop are finitely cocomplete, see [4].

## 3. Skew braces and their commutators

### 3.1. Protomodular aspects.

3.1.1. Commutative pairs of subobjects, abelian objects. Given any pointed category $\mathbb{E}$, the protomodular axiom applies to the following specific downward pullback:

where the monomorphisms are the canonical inclusions. This is the definition of a unital category [6. In this kind of categories there is an intrisic notion of commutative pair of subobjects:

Definition 3.1. Let $\mathbb{E}$ be a unital category. Given a pair $(u, v)$ of subobjects of $X$, we say that the subobjects $u$ and $v$ cooperate (or commute) when there is a (necessarily unique) map $\varphi$, called the cooperator of the pair $(u, v)$, making the following diagram commute:


We denote this situation by $[u, v]=0$. A subobject $u: U \rightharpoondown Y$ is central when $\left[u, 1_{X}\right]=0$. An object $X$ is commutative when $\left[1_{X}, 1_{X}\right]=0$.

Clearly $\left[1_{X}, 1_{X}\right]=0$ gives $X$ a structure of internal unitary magma, which, $\mathbb{E}$ being unital, is necessarily underlying an internal commutative monoid structure. When $\mathbb{E}$ is protomodular, this is actually an internal abelian group structure, so that we call $X$ an abelian object [7]. This gives rise to a fully faithful subcategory $A b(\mathbb{E}) \hookrightarrow \mathbb{E}$, which is additive and stable under finite limits in $\mathbb{E}$. From that we can derive:

Proposition 3.2. [7] A pointed protomodular category $\mathbb{E}$ is additive if and only if any monomorphism is normal.
3.1.2. Connected pairs $(R, S)$ of equivalence relations. Since a protomodular category is necessarily a Mal'tsev one, we can transfer the following notions. Given any pair $(R, S)$ of equivalence relations on the object $X$ in $\mathbb{E}$, take the following rightward and downward pullback:

where $l_{R}$ and $r_{S}$ are the sections induced by the maps $s_{0}^{R}$ and $s_{0}^{S}$. Let us recall the following definition from [10]:

Definition 3.3. In a Mal'tsev category $\mathbb{E}$, the pair $(R, S)$ is said to be connected when there is a (necessarily unique) morphism

$$
p: R \vec{×}_{X} S \rightarrow X, x R y S z \mapsto p(x R y S z)
$$

such that $p r_{S}=d_{1}^{S}$ and $p l_{R}=d_{0}^{R}$, namely such that the following identities hold: $p(x R y S y)=x$ and $p(y R y S z)=z$. This morphism $p$ is then called the connector of the pair, and we denote the situation by $[R, S]=0$.

From [11, let us recall that:
Lemma 3.4. Let $\mathbb{E}$ be a Mal'tsev category, $f: X \rightarrow Y$ any map, $(R, S)$ any pair of equivalence relations on $X,(\bar{R}, \bar{S})$ any pair of equivalence relations on $Y$ such that $R \subset f^{-1}(\bar{R})$ and $S \subset f^{-1}(\bar{S})$. Suppose moreover that $[R, S]=0$ and $[\bar{R}, \bar{S}]=0$. Then the following diagram necessarily commutes:

where $\tilde{f}$ is the natural factorization induced by $f^{-1}(\bar{R})$ and $S \subset f^{-1}(\bar{S})$.
A pointed Mal'tsev category is necessarily unital. From [10], in any pointed Mal'sev category $\mathbb{E}$, we have necessarily

$$
\begin{equation*}
[R, S]=0 \Rightarrow\left[I_{R}, I_{S}\right]=0 \tag{2}
\end{equation*}
$$

In this way, the "Smith commutation" [22] implies the "Huq commutation" 16].
3.2. Huq=Smith. The converse is not necessarily true, even if $\mathbb{E}$ is pointed protomodular, see Proposition 3.6 below. When this is the case, we say that $\mathbb{E}$ satisfies the (Huq=Smith) condition. Any pointed strongly protomodular category satisfies (Huq=Smith), see [10. (Huq=Smith) is true for Gp by the following straighforward:

Proposition 3.5. Let $(R, S)$ be a pair of equivalence relations in $G p$ on the group $(G, *)$. The following conditions are equivalent:
(1) $\left[I_{R}, I_{S}\right]=0$;
(2) $p(x, y, z)=x * y^{-1} * z$ defines a group homomorphism $p: G \times G \times G \rightarrow G$;
(3) $[R, S]=0$.

Proposition 3.6. The category DiGp of digroups does not satisfy (Huq=Smith).
Proof. We can use the counterexample introduced in [8] for another purpose. Start with an abelian group $(A,+)$ and an object $a$ such that $-a \neq a$. Then define $\theta: A \times A \rightarrow A \times A$ as the involutive bijection which leaves fixed any object $(x, y)$ except $(a, a)$ which is exchanged with $(-a, a)$. Then defined the group structure $(A \times A, \circ)$ on $A \times A$ as the transformed along $\theta$ of $(A \times A,+)$. So, we get:

$$
(x, z) \circ\left(x^{\prime}, z^{\prime}\right)=\theta\left(\theta(x, z)+\theta\left(x^{\prime}, z^{\prime}\right)\right)
$$

Clearly we have $(a, a)^{-\circ}=(a,-a)$. Since the second projection $\pi: A \times A \rightarrow$ $A$ is such that $\pi \theta=\pi$, we get a digroup homomorphism $\pi:(A \times A,+, \circ) \rightarrow$ $(A,+,+)$ whose kernel map is, up to isomorphism, $\iota_{A}:(A,+,+) \rightharpoondown(A \times A,+, \circ)$ defined by $\iota(a)=(a, 0)$. The commutativity of the law + makes $\left[\iota_{A}, \iota_{A}\right]=0$ inside

DiGp. We are going to show that, however we do not have $[R[\pi], R[\pi]]=0$. If it was the case, according to the previous proposition and considering the images by $U_{0}$ and $U_{1}$ of the desired ternary operation, we should have, for any triple $(x, y) R[\pi]\left(x^{\prime}, y\right) R[\pi]\left(x^{\prime \prime}, y\right)$ :

$$
(x, y)-\left(x^{\prime}, y\right)+\left(x^{\prime \prime}, y\right)=(x, y) \circ\left(x^{\prime}, y\right)^{-\circ} \circ\left(x^{\prime \prime}, y\right)
$$

namely $(x, y) \circ\left(x^{\prime}, y\right)^{-\circ} \circ(x ", y)=\left(x-x^{\prime}+x ", y\right)$. Now take $y=a=x^{\prime}$ and $a \neq x \neq-a$. Then we get:
$(x, a) \circ(a, a)^{-\circ} \circ(x ", a)=(x, a) \circ(a,-a) \circ(x ", a)=(x+a, 0) \circ(x ", a)$
$=(x+a+x ", a)$, if moreover $a \neq x " \neq-a$. Now, clearly we get $x+a+x " \neq x-a+x "$ since $a \neq-a$.

However we have the following very general observation:
Proposition 3.7. Let $\mathbb{E}$ be any pointed Mal'tsev satisfying (Huq=Smith). So is any functor category $\mathcal{F}(\mathbb{C}, \mathbb{E})$.

Proof. Let $(R, S)$ be a pair of equivalence relation on an object $F \in F(\mathbb{C}, \mathbb{E})$. We have $[R, S]=0$ if and only if for each object $C \in \mathbb{C}$ we have $[R(C), S(C)]=0$ since, by Lemma 3.4 the naturality follows. In the same way, if $(u, v)$ is a pair of subfunctors of $F$, we have $[u, v]=0$ if and only if for each object $C \in \mathbb{C}$ we have $[u(C), v(C)]=0$. Suppose now that $\mathbb{E}$ satisfies (Huq=Smith), and that $\left[I_{R}, I_{S}\right]=0$. So, for each object $C \in \mathbb{C}$ we have $\left[I_{R}(C), I_{S}(C)\right]=0$, which implies $[R(C), S(C)]=0$. Accordingly $[R, S]=0$.

Let $\mathbb{T}$ be any finitary algebraic theory, and denote by $\mathbb{T}(\mathbb{E})$ the category of internal $\mathbb{T}$-algebras in $\mathbb{E}$. Let us recall that, given any variety of algebras $\mathbb{V}(\mathbb{T})$, we have a Yoneda embedding for the internal $\mathbb{T}$-algebras, namely a left exact fully faithful factorization of the Yoneda embedding for $\mathbb{E}$ :

where $\mathcal{U}: \mathbb{V}(\mathbb{T}) \rightarrow$ Set is the canonical forgetful functor.
Theorem 3.8. Let $\mathbb{T}$ be any finitary algebraic theory such that the associated variety of algebras $\mathbb{V}(\mathbb{T})$ is pointed protomodular. If $\mathbb{V}(\mathbb{T})$ satisfies (Huq=Smith), so does any category $\mathbb{T}(\mathbb{E})$.

Proof. If $\mathbb{V}(\mathbb{T})$ satisfies (Huq=Smith), so does $\mathcal{F}\left(\mathbb{E}^{o p}, \mathbb{V}(\mathbb{T})\right)$ by the previous proposition. Accordingly, $\bar{Y}_{\mathbb{T}}$ being left exact and fully faithful, so does $\mathbb{T}(\mathbb{E})$.
3.3. Any category SKBE does satisfy (Huq=Smith).

Proposition 3.9. Given any pair $(U, V)$ of subobjects of $X$ in SKB, the following conditions are equivalent:
(1) $[U, V]=0$;
(2) for all $(u, v) \in U \times V$, we get $u \circ v=u * v$ and this restriction is commutative;
(3) for all $(u, v) \in U \times V, \lambda_{u}(v)=v,\left[U_{0}(U), U_{0}(V)\right]=0$ and $\left[U_{1}(U), U_{1}(V)\right]=0$.

Accordingly, an abelian object in SKB is necessarily of the form $(A,+,+)$ with $(A,+)$ abelian.

Proof. Straightforward, setting $\varphi(u, v)=u+v$ and using an Eckmann-Hilton argument.

Proposition 3.10 (SKB does satisfy (Huq=Smith)). Let $R$ and $S$ be two equivalence relations on an object $X \in S K B$. The following conditions are equivalent:
(1) $\left[I_{R}, I_{S}\right]=0$;
(2) $\left[U_{0}(U), U_{0}(V)\right]=0,\left[U_{1}(U), U_{1}(V)\right]=0$ and $x * y^{-*} * z=x \circ y^{-\circ} \circ z$ for all $x R y S z$;
(3) $[R, S]=0$.

Proof. The identity $x * y^{-*} * z=x \circ y^{-\circ} \circ z$ is equivalent to

$$
\begin{gathered}
y^{-\circ} \circ z=x^{-\circ} \circ\left(x * y^{-*} * z\right)=\left(x^{-\circ} \circ x\right) *\left(x^{-\circ}\right)^{-*} *\left(x^{-\circ} \circ\left(y^{-*} * z\right)\right)= \\
=\left(x^{-\circ}\right)^{-*} *\left(x^{-\circ} \circ\left(y^{-*} * z\right)\right),
\end{gathered}
$$

which, in turn, is equivalent to

$$
\lambda_{x^{-\circ}}\left(y^{-*} * z\right)=y^{-\circ} \circ z
$$

Suppose $x R y S y$. Setting $z=y * v, v \in I_{S}$, this is equivalent to $\lambda_{x^{-} \circ}(v)=y^{-\circ} \circ$ $(y * v)=\lambda_{y^{-\circ}}(v)$ by (11). This in turn is equivalent to $\lambda_{y} \circ \lambda_{x^{-} \circ}(v)=\lambda_{y \circ x-\circ}(v)=v$, $v \in I_{S}$. Setting $y=u \circ x, u \in I_{R}$, this is equivalent to $\lambda_{u}(v)=v,(u, v) \in I_{R} \times I_{S}$.

Now, by Proposition [3.9, $\left[I_{R}, I_{S}\right]=0$ is equivalent to: for all $(u, v) \in I_{R} \times I_{S}$, we get $\lambda_{u}(v)=v,\left[U_{0}(U), U_{0}(V)\right]=0$ and $\left[U_{1}(U), U_{1}(V)\right]=0$. So we get $\left.\left.[1) \Longleftrightarrow 2\right)\right]$.

Suppose (2). From $\left[U_{0}(U), U_{0}(V)\right]=0$, we know by Proposition 3.9 that $p(x, y, z)=$ $x * y^{-*} * z$ is a group homomorphism $\left(R \vec{×}_{X} S, *\right), \rightarrow(X, *)$, and from $\left[U_{1}(U), U_{1}(V)\right]=$ 0 that $q(x, y, z)=x \circ y^{-\circ} \circ z$ is a group homomorphism $\left(R \vec{x}_{X} S, \circ\right) \rightarrow(X, \circ)$. If $p=q$, this produces the desired $R \overrightarrow{\times}_{X} S \rightarrow X$ in SKB showing that $[R, S]=0$. Whence $[(2) \Rightarrow(3)]$. We have already noticed that the last implication $[(3) \Rightarrow(1)]$ holds in any pointed category.

According to Theorem 3.8, we get the following:
Corollary 3.11. Given any category $\mathbb{E}$, the category SKBE satisfies (Huq= Smith). This is the case in particular for the category SKBTop of topological skew braces.

### 3.4. Homological aspects of commutators.

3.4.1. Abstract Huq commutator. Suppose now that $\mathbb{E}$ is any finitely cocomplete regular unital category. In this setting, we gave in [9], for any pair $u: U \mapsto X$, $v: V \mapsto X$ of subobjects, the construction of a regular epimorphism $\psi_{(u, v)}$ which universally makes their direct images cooperate. Indeed consider the following diagram, where $Q[u, v]$ is the limit of the plain arrows:


The map $\psi_{(u, v)}$ is necessarily a regular epimorphism and the map $\bar{\psi}_{(u, v)}$ induces the cooperator of the direct images of the pair $(u, v)$ along $\psi_{(u, v)}$. This regular
epimorphism $\psi_{(u, v)}$ measures the lack of cooperation of the pair $(u, v)$ in the sense that the map $\psi_{(u, v)}$ is an isomorphism if and only if $[u, v]=0$. We then get a symmetric tensor product: $I_{R\left[\psi_{(-,-)}\right]}: \operatorname{Mon}_{X} \times \operatorname{Mon}_{X} \rightarrow$ Mon $_{X}$ of preordered sets.

Since the map $\psi_{(u, v)}$ is a regular epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is measured by the kernel equivalence relation $R\left[\psi_{(u, v)}\right]$. Accordingly, in the homological context, it is meaningful to introduce the following definition, see also [20]:
Definition 3.12. Given any finitely cocomplete homological category $\mathbb{E}$ and any pair $(u, v)$ of subobjects of $X$, their abstract Huq commutator $[u, v]$ is defined as $I_{R\left[\psi_{(u, v)}\right]}$ or equivalently as the kernel map $k_{\psi_{(u, v)}}$.

By this universal definition, in the category Gp , this $[u, v]$ coincides with the usual $[U, V]$.
3.4.2. Abstract Smith commutator. Suppose $\mathbb{E}$ is a regular category. Then, given any regular epimorphism $f: X \rightarrow Y$ and any equivalence relation $R$ on $X$, the direct image $f(R) \mapsto Y \times Y$ of $R \mapsto X \times X$ along the regular epimorphism $f \times f$ : $X \times X \rightarrow Y \times Y$ is reflexive and symmetric, but generally not transitive. Now, when $\mathbb{E}$ is a regular Malt'sev category, this direct image $f(R)$, being a reflexive relation, is an equivalence relation.

Suppose moreover that $\mathbb{E}$ is finitely cocomplete. Let $(R, S)$ be a pair of equivalence relations on $X$, and consider the following diagram, where $Q[R, S]$ is the colimit of the plain arrows:


Notice that, here, in consideration of the pullback defining $R \vec{×}_{X} S$, the role of the projections $d_{0}$ and $d_{1}$ have been interchanged. This map $\chi_{(R, S)}$ measures the lack of connection between $R$ and $S$, see [9]:
Theorem 3.13. Let $\mathbb{E}$ be a finitely cocomplete regular Mal'tsev category. Then the map $\chi_{(R, S)}$ is a regular epimorphism and is the universal one which makes the direct images $\chi_{(R, S)}(R)$ and $\chi_{(R, S)}(S)$ connected. The equivalence relations $R$ and $S$ are connected (i.e. $[R, S]=0$ ) if and only if $\chi_{(R, S)}$ is an isomorphism.

Since the map $\chi_{(R, S)}$ is a regular epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel equivalence relation $R\left[\chi_{(R, S)}\right]$. Accordingly, we give the following definition:

Definition 3.14. Let $\mathbb{E}$ be any finitely cocomplete regular Mal'tsev category. Given any pair $(R, S)$ of equivalence relations on $X$, their abstract Smith commutator $[R, S]$ is defined as the kernel equivalence relation $R\left[\chi_{(R, S)}\right]$ of the map $\chi_{(R, S)}$.

In this way, we define a symmetric tensor product $[-,-]=R\left[\chi_{(-,-)}\right]: \mathrm{Equ}_{X} \times$ $\mathrm{Equ}_{X} \rightarrow \mathrm{Equ}_{X}$ of preorered sets. It is clear that, with this definition, we get
$[R, S]=0$ in the sense of connected pairs if and only if $[R, S]=\Delta_{X}$ (the identity equivalence relation on $X$ ) in the sense of this new definition. This is coherent since $\Delta_{X}$ is effectively the 0 of the preorder $\mathrm{Equ}_{X}$. Let us recall the following:

Proposition 3.15. Let $\mathbb{E}$ be a pointed regular Mal'tsev category. Let $f: X \rightarrow Y$ be a regular epimorphism and $R$ an equivalence relation on $X$. Then the direct image $f\left(I_{R}\right)$ of the normal subjobject $I_{R}$ along $f$ is $I_{f(R)}$.

From that, we can assert the following:
Proposition 3.16. Let $\mathbb{E}$ be a finitely cocomplete homological category. Given any pair $(R, S)$ of equivalence relations on $X$, we have $\left[I_{R}, I_{S}\right] \subset I_{[R, S]}$.

Proof. From (2), we get

$$
\left[\chi_{(R, S)}(R), \chi_{(R, S)}(S)\right]=0 \Rightarrow\left[I_{\chi_{(R, S)}(R)}, I_{\chi_{(R, S)}(S)}\right]=0
$$

By the previous proposition we have:

$$
0=\left[I_{\chi_{(R, S)}(R)}, I_{\chi_{(R, S)}(S)}\right]=\left[\chi_{(R, S)}\left(I_{R}\right), \chi_{(R, S)}\left(I_{S}\right)\right] .
$$

Accordingly, by the universal property of the regular epimorphism $\psi_{\left(I_{R}, I_{S}\right)}$ we get a factorization:

which shows that $\left[I_{R}, I_{S}\right] \subset I_{[R, S]}$.
Theorem 3.17. In a finitely cocomplete homological category $\mathbb{E}$ the following conditions are equivalent:
(1) $\mathbb{E}$ satisfies (Huq=Smith);
(2) $\left[I_{R}, I_{S}\right]=I_{[R, S]}$ for any pair $(R, S)$ of equivalence relations on $X$.

Under any of these conditions, the regular epimorphisms $\chi_{(R, S)}$ and $\psi_{\left(I_{R}, I_{S}\right)}$ do coincide.

Proof. Suppose (2). Then $\left[I_{R}, I_{S}\right]=0$ means that $\psi_{\left(I_{R}, I_{S}\right)}$ is an isomorphism, so that $0=\left[I_{R}, I_{S}\right]=I_{[R, S]}$. In a homological category $I_{[R, S]}=0$ is equivalent to $[R, S]=0$. Conversely, suppose (1). We have to find a factorization:

namely to show that $\left[\psi_{\left(I_{R}, I_{S}\right)}(R), \psi_{\left(I_{R}, I_{S}\right)}(S)\right]=0$. By (1) this is equivalent to $0=$ $\left[I_{\psi_{\left(I_{R}, I_{S}\right)}(R)}, I_{\psi_{\left(I_{R}, I_{S}\right)}(S)}\right]$, namely to $0=\left[\psi_{\left(I_{R}, I_{S}\right)}\left(I_{R}\right), \psi_{\left(I_{R}, I_{S}\right)}\left(I_{S}\right)\right]$ by Proposition 3.15. This is true by the universal property of the regular epimorphism $\psi_{\left(I_{R}, I_{S}\right)}$.
3.5. Skew braces and their commutators. Since the categories SKB and SKBTop are finitely cocomplete homological categories, all the results of the previous section concerning commutators do apply and, in particular, thanks to the (Huq=Smith) condition, the two notions of commutator are equivalent. It remains now to make explicit the description of the Huq commutator.

We will determine a set of generators for the Huq commutator of two ideals in a skew brace:
Proposition 3.18. If $I$ and $J$ are two ideals of a left skew brace $(A, *, \circ)$, their Huq commutator $[I, J]$ is the ideal of A generated by the union of the following three sets:
(1) the set $\left\{i \circ j \circ(j \circ i)^{-\circ} \mid i \in I, j \in J\right\}$, (which generates the commutator $[I, J]_{(A, \circ)}$ of the normal subgroups $I$ and $J$ of the group $(A, \circ)$ );
(2) the set $\left\{i * j *(j * i)^{-*} \mid i \in I, j \in J\right\}$, (which generates the commutator $[I, J]_{(A, *)}$ of the normal subgroups $I$ and $J$ of the group $\left.(A, *)\right)$; and
(3) the set $\left\{(i \circ j) *(i * j)^{-*} \mid i \in I, j \in J\right\}$.

Proof. Assume that the mapping $\mu: I \times J \rightarrow A / K, \mu(i, j)=i * j * K$ is a skew brace morphism for some ideal $K$ of $A$. Then

$$
\begin{aligned}
& (i \circ j) \circ K=(i \circ K) \circ(j \circ K)=(i * K) \circ(j * K)= \\
& \quad=\mu(i, 1) \circ \mu(1, j)=\mu((i, 1) \circ(1, j))=\mu(i, j)=\mu((1, j) \circ(i, 1))= \\
& \quad=\mu((1, j) \circ \mu(i, 1))=(j * K) \circ(i * K)=(j \circ K) \circ(i \circ K)=(j \circ i) \circ K
\end{aligned}
$$

This proves that the set (1) is contained in $K$.
Similarly,

$$
\begin{aligned}
& (i * j) * K=(i * K) *(j * K)=\mu(i, 1) * \mu(1, j)=\mu((i, 1) *(1, j))=\mu(i, j)= \\
& \quad=\mu((1, j) *(i, 1))=\mu((1, j) * \mu(i, 1))=(j * K) *(i * K)=(j * i) * K
\end{aligned}
$$

Thus the set (2) is contained in $K$.
Also,

$$
\begin{aligned}
& (i \circ j) * K=(i \circ j) \circ K=(i \circ K) \circ(j \circ K)=(i * K) \circ(j * K)= \\
& \quad=\mu(i, 1)) \circ \mu(1, j)=\mu((i, 1) \circ(1, j))=\mu(i, j)=\mu((i, 1) *(1, j))= \\
& \quad=\mu(i, 1) * \mu(1, j)=(i * K) *(j * K)=(i * j) * K
\end{aligned}
$$

Hence the set (3) is also contained in $K$.
Conversely, let $K$ be the ideal of $A$ generated by the union of the three sets. It is then very easy to check that he mapping $\mu: I \times J \rightarrow A / K, \mu(i, j)=i * j * K$ is a skew brace morphism.

It the literature, great attention has been posed in the study of product $I \cdot J$ of two ideals $I, J$ of a (left skew) brace $(A, *, \circ)$. This product is with respect to the product • in the brace $A$ defined, for every $x, y \in A$ by $x \cdot y=y^{-*} * \lambda_{x}(y)$. Then, for every $i \in I$ and $j \in J, i \cdot j=j^{-*} * \lambda_{i}(j)=j^{-*} * i^{-*} *(i \circ j)=(i * j)^{-*} *(i \circ j)$. Hence the ideal of $A$ generated by the set $I \cdot J$ of all products $i \cdot j$ coincides with the ideal of $A$ generated by the set (3) in the statement of Proposition 3.18.

Clearly, for a left skew brace $A$, the Huq commutator $[I, J]$ is equal to the Huq commutator $[J, I]$. Also, $I \cdot J=(J \cdot I)^{-*}$, so that the left annihilator of $I$ in $(A, \cdot)$ is equal to the right annihilator of $I$ in $(A, \cdot)$. Moreover, the condition " $I \cdot J=0$ " can be equivalently expressed as " $J$ is contained in the kernel of the group homomorphism $\left.\lambda\right|^{I}:(A, \circ) \rightarrow \operatorname{Aut}(I, *)$.

Proposition 3.19. For an ideal I of a left skew brace $A$, there is a largest ideal of A that centralizes I (the centralizer of I).

Proof. The zero ideal centralizes $I$ and the union of a chain of ideals that centralize $I$ centralizes $I$. Hence there is a maximal element in the set of all the ideals of $A$ that centralize $I$. Now if $J_{1}$ and $J_{2}$ are two ideals of $A$, then $J_{1} * J_{2}=J_{1} \circ J_{2}$ is the join of $\left\{J_{1}, J_{2}\right\}$ in the lattice of all ideals of $A$. Now $J_{1}$ centralizes $I$ if and only if (1) $J_{1} \subseteq C_{(A, *)}(I)$, the centralizer of the normal subgroup $I$ in the group $(A, *)$; (2) $J_{1} \subseteq C_{(A, \circ)}(I)$, the centralizer of the normal subgroup $I$ in the group $(A, \circ)$; and (3) $J_{1}$ is contained in the kernel of the group morphism $\left.\lambda\right|^{I}:(A, \circ) \rightarrow \operatorname{Aut}(I, *)$, which is a normal subgroup of $(A, \circ)$. Similarly for $J_{2}$. Hence if both $J_{1}$ and $J_{2}$ centralize $I$, then $J_{1} * J_{2} \subseteq C_{(A, *)}(I)$, and $J_{1} \circ J_{2} \subseteq C_{(A, \circ)}(I) \cap$ ker $\left.\lambda\right|^{I}$. Therefore $J_{1} * J_{2}=J_{1} \circ J_{2}$ centralizes $I$. It follows that the set of all the ideals of $A$ that centralize $I$ is a lattice. Hence the maximal element in the set of all the ideals of $A$ that centralize $I$ is the largest element in that set.

In particular, the centralizer of the improper ideal of a left skew brace $A$ is the center of $A$.

A description of the free left skew brace over a set $X$ is available, in a language very different from ours, in [21].

## References

[1] M. Barr, Exact categories, Lecture Notes in Math. 236 (1971), Springer, Berlin, pp. 1-120.
[2] G. M. Bergman, Modules over coproducts of rings, Trans. Amer. Math. Soc. 200 (1974), 1-32.
[3] F. Borceux and D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Math. and Its Applications 566, Kluwer, 2004.
[4] F. Borceux and M. M. Clementino, Topological semi-Abelian algebras, Advances in Math. 190 (2005), 425-453.
[5] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, Lecture Notes in Math. 1488 (1991), Springer, Berlin, pp. 43-62.
[6] D. Bourn, Mal'tsevCategories and fibration of pointed objects, Applied categorical structures 4 (1996), 302-327.
[7] D. Bourn, Normal subobjects and abelian objects in protomodular categories, J. Algebra 228 (2000), 143-164.
[8] D. Bourn, Normal functors and strong protomodularity, Theory Appl. Categ. 7(9) (2000), 206-218.
[9] D. Bourn, Commutator theory in regular Mal'tsev categories, in: "Galois theory, Hopf algebras, and Semiabelian categories", G. Janelidze, B. Pareigis and W. Tholen Eds, Fields Institute Communications 43, Amer. Math. Soc. (2004), pp. 61-75.
[10] D. Bourn and M. Gran, Centrality and normality in protomodular categories, Theory and Applications of Categories 9 (2002), 151-165.
[11] D. Bourn and M. Gran, Centrality and connectors in Maltsev categories, Algebra Universalis 48 (2002), 309-331.
[12] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, Theory Appl. Categ. 11(6) (2003), 143-147.
[13] V. G. Drinfel'd, On some unsolved problems in quantum group theory, in "Quantum groups (Leningrad, 1990)", P. P. Kulish Ed., Lecture Notes in Math. 1510, Springer, Berlin, 1992, pp. 1-8.
[14] A. Facchini, "Semilocal categories and modules with semilocal endomorphism rings", Progress in Math. 331, Birkhäuser/Springer, Cham, 2019.
[15] L. Guarnieri and L. Vendramin, Skew braces and the Yang-Baxter equation, Math. Comp. 86(307) (2017), 2519-2534.
[16] S.A. Huq, Commutator, nilpotency and solvability in categories, Quart. J. Oxford 19, 1968, 363-389.
[17] G. Janelidze, L. Márki and W. Tholen, Semi-abelian categories, J. Pure Appl. Alg. 168 (2002), 367-386.
[18] G. Janelidze, L. Márki and S. Veldsman, Commutators for near-rings: Huq $\neq$ Smith, Algebra Universalis 76(2) (2016), 223-229.
[19] A. Konovalov, A. Smoktunowicz and L. Vendramin, On skew braces and their ideals, Exp. Math. 30 (2021), no. 1, 95-104.
[20] S. Mantovani and G. Metere, Normalities and commutators, J. Algebra 324 (2010), 2568 2588.
[21] J. Orza, A construction of the free skew brace, https://arxiv.org /abs/2002.12131.
[22] J. D. H. Smith, "Mal'tsevVarieties", Lecture Notes in Math. 554, Springer, Berlin-New York, 1976.
[23] A. Smoktunowicz and L. Vendramin, On skew braces (with an appendix by N. Byott and L. Vendramin), J. Comb. Algebra 2(1) (2018), 47-86.

Univ. Littoral Côte d'Opale, UR 2597, LMPA, Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France

Email address: bourn@univ-littoral.fr
Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, 35121 Padova, Italy

Email address: facchini@math.unipd.it
Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, 35121 Padova, Italy

Email address: mara.pompili@studenti.unipd.it


[^0]:    Key words and phrases. Brace; Skew brace; Yang-Baxter equation; Mal'tsev category; Protomodular category.
    2020 Mathematics Subject Classification. Primary 16T25, 18E13, 20N99.
    The second author is partially supported by Ministero dell'Istruzione, dell'Università e della Ricerca (Progetto di ricerca di rilevante interesse nazionale "Categories, Algebras: RingTheoretical and Homological Approaches (CARTHA)"), Fondazione Cariverona (Research project "Reducing complexity in algebra, logic, combinatorics - REDCOM" within the framework of the programme Ricerca Scientifica di Eccellenza 2018), and the Department of Mathematics "Tullio Levi-Civita" of the University of Padua (Research programme DOR1828909"Anelli e categorie di moduli").

