Aspects of Wireless Network Modeling based on Poisson Point Processes

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1 Introduction

In these notes we discuss stochastic models for a simplified wireless network that consists of a collection of spatially distributed stations equipped with emitters and/or receivers for transmission over a common communication channel. The modeling approach is based on using Poisson point processes for the spatial locations as well as for other signaling characteristics o f the network nodes.

Throughout we work from the premises that the transmission of signals is syncronized and slotted in time so that in a fixed time slot each sender attempts to emit the equivalent of one symbol. The signal power is affected by random fading and attenuation proportional to the traveled distance. Using simple Poisson models and superpositioning the effect of all stations in one slot it is possible to obtain some insights into the balance between node density and node interference.

For traffic sessions over a fixed or random number of slots and under assumptions of Rayleigh fading, we propose a modeling scenario based on the Lévy gamma subordinator process and its relation to complex Gaussian waweforms. These models can be extended to sessions which are Poisson in both space and time, and have short-tailed or heavy-tailed random session duration times. It is possible also to include lognormal fading. This is a mechanism supposed to act on the slow time scale of sessions, which is in contrast to Rayleigh fading that generate random variation on the fast scale of slots. For the traffic session models it is possible to perform a scaling approximation to analyze the fluctuations that build up in the interference field.

1.1 Poisson integral calculus

We recall briefly some of the basic tools for handling stochastic integrals with respect to Poisson mesaures, see e.g. Kingman [Ki]. For general theory, see e.g. Kallenberg

[Ka]. We consider a Poisson point measure $N = \sum_j \delta_{X_j}$ defined on a measurable state space $\mathbb{X}$. The intensity measure (or mean measure) is a σ-finite measure n also defined on $\mathbb{X}$. For any set $A \subset \mathbb{X}$, the number of points in A, $N(A) = \sum_j I\{X_j \in A\}$, has the Poisson distribution with mean $n(A)$ (if $n(A)$ is infinite, $N(A)$ is countably infinite with probability one). For any disjoint sets $A_1, \ldots, A_n$ in $\mathbb{X}$ the variables $N(A_1), \ldots, N(A_n)$ are independent.

Let $f : \mathbb{X} \to \mathbb{R}$ be a measurable function. The stochastic integral of f with respect to N,

$$\int_{\mathbb{X}} f(x)\, N(dx) = \sum_j f(X_j),$$

exists (the sum is absolutely convergent) with probability one if and only if

$$\int_{\mathbb{X}} \min(|f(x)|, 1)\, n(dx) < \infty.$$

For such functions f, the distribution of the Poisson integral is determined by the characteristic function

$$E \exp\Big\{ i\theta \int_{\mathbb{X}} f(x)\, N(dx) \Big\} = \exp\Big\{ \int_{\mathbb{X}} (e^{i\theta f(x)} - 1)\, n(dx) \Big\}, \quad \theta \in \mathbb{R}.$$

In particular,

$$E \int_{\mathbb{X}} f(x)\, N(dx) = \int_{\mathbb{X}} f(x)\, n(dx)$$

and

$$\mathrm{Var} \int_{\mathbb{X}} f(x)\, N(dx) = \int_{\mathbb{X}} f(x)^2\, n(dx)$$

The compensated (or centered) stochastic integral

$$\int_{\mathbb{X}} f(x)\, N(dx) - E \int_{\mathbb{X}} f(x)\, N(dx) = \int_{\mathbb{X}} f(x)\, (N(dx) - n(dx))$$

with respect to the compensated measure $\widetilde{N}(dx) = N(dx) - n(dx)$, has the characteristic function

$$E \exp\Big\{ i\theta \int_{\mathbb{X}} f(x)\, \widetilde{N}(dx) \Big\} = \exp\Big\{ \int_{\mathbb{X}} (e^{i\theta f(x)} - 1 - i\theta f(x))\, n(dx) \Big\}.$$

The integral $\int_{\mathbb{X}} f(x)\, \widetilde{N}(dx)$ exists in $L^1(\mathbb{X})$ if and only if

$$\int_{\mathbb{X}} \min(|f(x)|, f(x)^2)\, n(dx) < \infty.$$

This condition may hold even for functions f such that $\int_{\mathbb{X}} f(x)\, N(dx)$ diverges.

2 One-slot models

2.1 Connectivity model

Let $F_R(r)$, $r \geq 0$, denote a distribution function on $\mathbb{R}_+$ and let $N(dx, dr)$ be a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $n(dx, dr) = \lambda dx\, F_R(dr)$. The Poisson points (X_j, R_j) correspond to the nodes of a wireless network. The nodes consist of radio transmitters with random locations X_j in space and random transmission radius R_j, which is independent between nodes. In a given transmission slot a signal from node j reaches any other node within distance R_j. By definition, for any set $A \subset \mathbb{R}^d \times \mathbb{R}_+$ such that $\int_A n(dx, dr)$ is finite the integral $\int_A N(dx, dr)$, which gives the number of network nodes with their positions and transmission capacities in accordance to A, has the Poisson distribution with expected value $\int_A n(dx, dr)$. Moreover, the two Poisson integrals $\int_{A_1} N(dx, dr)$ and $\int_{A_2} N(dx, dr)$ are independent whenever the sets A_1 and A_2 are disjoint.

Place a receiver at $y \in \mathbb{R}^d$ and consider such a collection of network nodes. The number of successful one-slot transmissions received at y is given by

$$M_1(y) = \sum_j \mathrm{I}\{R_j > |X_j - y|\} = \int_{\mathbb{R}^d} \int_0^\infty \mathrm{I}\{r > |x - y|\}\, N(dx, dr),$$

and $\{M_1(y), y \in \mathbb{R}^d\}$ is a stationary integer-valued random field. Write $B(x, r)$ for the unit ball in $\mathbb{R}^d$ with center x and radius r and let $|B(x, r)| = |B(x, 1)|\, r^d$ denote its volume. Assume that the volume of the random radius ball $B(x, R)$ has finite expected value, that is $ER^d < \infty$. Then the number of signals picked up by a reciever at the origin, $M_1 = M_1(0)$, has finite expected value

$$\begin{aligned} EM_1 &= \int_{\mathbb{R}^d} \int_0^\infty \mathrm{I}\{r > |x|\}\, n(dx, dr) = \lambda \int_0^\infty \int_{\mathbb{R}^d} \mathrm{I}\{r > |x|\}\, dx\, F_R(dr) \\ &= \lambda \int_0^\infty |B(0, r)|\, F_R(dr) = \lambda |B(0, 1)|\, E(R^d). \end{aligned}$$

The moment generating function is

$$\begin{aligned} \log E e^{\theta M_1} &= \int_{\mathbb{R}^d} \int_0^\infty e^{\theta \mathrm{I}\{r > |x|\}} - 1)\, n(dx, dr) \\ &= \lambda(e^\theta - 1) \int_{\mathbb{R}^d} \int_0^\infty \mathrm{I}\{r > |x|\}\, n(dx, dr) \\ &= \lambda(e^\theta - 1)|B(0, 1)|\, E(R^d) \end{aligned}$$

(similarly for characteristic function), which verifies that M_1 itself is Poisson distributed. For fixed transmission radius $R = r$, the probability that the origin is connected to at least one network node equals

$$P(M_1 \geq 1) = 1 - e^{-\lambda |B(0,1)| r^d}.$$

2.2 Pathloss model

Again the network nodes are located according to a spatial Poisson process with intensity λdx in $\mathbb{R}^d$. The nodes are stations equipped with transmitters which emit signals at a power level that is random and independent between nodes. Pathloss is an attenuation effect which results in a reduction of the signal power in proportion to the propagation distance between sender and receiver. It is typically assummed that the expected received power from a sender at distance x is determined by a decreasing attenuation function $a(x)$ of $|x|$. A common choice is to consider the specific power law function $a_0(x) = |x|^{-\beta}$, where $\beta > d$ is a parameter for certain physical aspects of the environment of the communication system, [S, IHV, YP, BBM]. More realistically the attenuation function should be bounded in zero. Indeed, the amplification effect for emitters close to the receiver which is a result of the singularity of a_0 at the origin is absurd. To begin with, we use a_0 despite of these shortcomings since then certain calculations become more explicit. Later we will work with $a_1(x) = (1+x)^{-\beta}$.

The signal is received correctly if the signal to noise ratio exceeds a given threshold value. Hence, assume that each node is marked with a signal power $S \geq 0$ with distribution $F_S(ds)$ which corresponds to the transmission of one symbol in a unit slot of time. If a node located in x has power S, due to path loss the remaining power received at the origin is $Sa(x)$. Let W be a fixed or random noise variable, such as the energy of thermal noise, common for all nodes in the network and independent of the radio transmitters. We fix the threshold T and say that a transmission from point x to the origin is successful if the signal to noise ratio exceeds the threshold, that is

$$\mathrm{SNR} = \frac{S\,a(x)}{W} > T.$$

To see how this model relates to the connectivity model, we rephrase in terms of a Poisson point process $N(dx, ds)$ with intensity measure $n(dx, ds) = \lambda dx\, F_S(ds)$ defined on $\mathbb{R}^d \times \mathbb{R}_+$. The number of nodes with signals successfully received at the origin is given by

$$M_2 = \sum_j \mathrm{I}\{S_j a(X_j) > TW\} = \int_{\mathbb{R}^d} \int_0^\infty \mathrm{I}\{s\, a(x) > TW\}\, N(dx, ds).$$

Using $a_0(x) = |x|^{-\beta}$,

$$M_2 = \int_{\mathbb{R}^d} \int_0^\infty \mathrm{I}\{(s/TW)^{1/\beta} > |x|\}\, N(dx, ds).$$

Thus, in this case the pathloss model is equivalent to the connectivity model with radius

$$R = (S/TW)^{1/\beta}, \quad F_R(r) = P(S < TWr^\beta) = F_S(TWr^\beta),$$

and so

$$EM_2 = \lambda |B(0,1)|\, E[(S/TW)^{d/\beta}] = \lambda |B(0,1)|\, T^{-d/\beta} E(S^{d/\beta})\, E(W^{-d/\beta}).$$

This shows that the basic assumption to impose on S is the moment condition $E(S^{d/\beta}) < \infty$. Since $\beta > d$ it suffices for this to assume the finite mean condition

$ES < \infty$. The additional moment condition for external noise seen to be necessary to keep EM_2 finite, is somewhat artificial and can be attributed to the singularity of a_0.

Rayleigh fading

The Rayleigh fading pathloss model takes for S the exponential distribution. The motivation for this comes from the underlying picture of the signal as a complex waveform $Z = X + iY$ with Gaussian real and imaginary parts. If we assume in fact that X and Y are independent zero mean Gaussian random variables with variance σ^2 then the power of the wave is given by the squared amplitude $X^2 + Y^2$, which has an exponential distribution with mean $2\sigma^2$.

Under the Rayleigh fading assumption with exponential signal power $F_S(s) = 1 - e^{-\mu s}$ applied to the pathloss model we obtain the corresponding radius distribution

$$P(R > r) = EP(S > TWr^\beta | N_0) = E(e^{-\mu TWr^\beta}), \quad r \geq 0.$$

For $W = w$ constant this is a Weibull distribution with

$$ER = \beta^{-1}\Gamma(1/\beta)\,\frac{1}{(\mu wT)^{1/\beta}}$$

For W exponential with mean w:

$$P(R > r) = \frac{1}{1 + \mu wTr^\beta}, \quad r \geq 0.$$

2.3 Multicast model

Consider users with random spatial locations in $\mathbb{R}^d$ determined by a Poisson point process with intensity measure λdx. The users are potential receivers of a signal of power S which is emitted at the origin. The transmission is subject to attenuation pathloss according to $a_0(x) = |x|^{-\beta}$ and occurs under thermal noise W. The number of users that recieve the transmitted message equals

$$M_3 = \sum_j \mathrm{I}\{S\,a(X_j) > TW\} = \int_{\mathbb{R}^d} \mathrm{I}\{S\,a(x) > TW\}\,N(dx).$$

The distribution of M_3 is determined by the characteristic function

$$\begin{aligned} E(e^{i\theta M_3}) &= E\exp\left\{\lambda(e^{i\theta} - 1)\int_{\mathbb{R}^d} \mathrm{I}\{S > WT|x|^\beta\}\,dx\right\} \\ &= E\exp\{\lambda(e^{i\theta} - 1)|B(0,1)|(S/WT)^{d/\beta}\}. \end{aligned}$$

Thus, M_3 is a mixed Poisson random variable with a random intensity which depends on the non-fading signal to noise ratio S/W. The expected value remains the same as for the pathloss model, $EM_3 = EM_2$.

2.4 Interference model

Using the same notation as for the pathloss model we introduce the field of Poisson interferers as the shot noise process

$$I_\lambda(y) = \sum_j S_j a(X_j - y) = \int_{\mathbb{R}^d} \int_0^\infty s\, a(x-y)\, N(dx, ds), \quad y \in R^d,$$

which gives the total contribution to interference noise generated by all signals emitted from the network and observed at a point y after pathloss power reduction according to attenuation a. The distribution of $I_\lambda(y)$ is stationary over $y \in \mathbb{R}^d$.

The expected value of the interference field equals

$$EI_\lambda(y) = \int_{\mathbb{R}^d} \int_0^\infty s\, a(x-y)\, n(dx, ds) = \int_{\mathbb{R}^d} a(x)\, dx\, E(S), \quad y \in R^d,$$

which is finite if we take for instance $a_\kappa(x) = (\kappa + |x|^\beta)^{-1}$, $\beta > d$, with $\kappa > 0$. However, we begin with the infinite mean value model $a(x) = a_0(x)$.

Singular attenuation, stable field of interferers For $I_\lambda = I_\lambda(0)$ and $a = a_0$ we compute the logarithmic characteristic function

$$\begin{aligned}
\log E(e^{i\theta I_\lambda}) &= \int_{\mathbb{R}^d} \int_0^\infty (e^{i\theta a(x)s} - 1)\, n(dx, ds) \\
&= \lambda |B(0,1)| \int_0^\infty E(e^{i\theta S/r^\beta} - 1) r^{d-1}\, dr \\
&= \lambda |B(0,1)| \int_0^\infty E(e^{i\theta St} - 1) \beta^{-1} t^{-d/\beta - 1}\, dt \\
&= \lambda |B(0,1)| E(S^{d/\beta}) \int_0^\infty (e^{i\theta t} - 1) \beta^{-1} t^{-d/\beta - 1}\, dt \\
&= \lambda |B(0,1)| E(S^{d/\beta})\, C(\operatorname{sign} \theta)\, |\theta|^{d/\beta}, \qquad (1)
\end{aligned}$$

where the constant $C(\cdot)$ depends on the sign of the real number θ. This shows that I_λ has an α-stable distribution with stable index $\alpha = d/\beta < 1$. Within this context, such links between Poisson shot noise models and α-stable distributions have been utilized in e.g. [S], [XP] for similar models where the signal S has a symmetric distribution and the resulting interference process is symmetric α-stable. The stable distribution for I_λ is highly variable. For instance, the only finite moments are those of order $\gamma < d/\beta$. This is again an artifact of the singular shape of a_0.

It follows by inspecting the above characteristic function that the interference field admits the stable type scaling relation

$$I_\lambda(y) \stackrel{d}{=} \lambda^{\beta/d}\, I_1(y)$$

Moreover, for $\theta > 0$,

$$\begin{aligned}
-\log E(e^{-\theta I_\lambda}) &= \lambda |B(0,1)| E(S^{d/\beta}) \int_0^\infty (1 - e^{-\theta t}) \beta^{-1} t^{-d/\beta - 1}\, dr \\
&= \lambda C_{d,\beta} E(S^{d/\beta})\, \theta^{d/\beta},
\end{aligned}$$

where

$$C_{d,\beta} = |B(0,1)|\Gamma(1-d/\beta)/d.$$

Next, position a source of signal power S at $x \in \mathbb{R}^d$. The emitted signal is received at the origin uncorrupted by interference if the signal to interference and noise ratio exceeds a threshold value T in the sense

$$\mathrm{SINR} = \frac{S\,a(x)}{W + I_\lambda} > T.$$

The probability of a successful transmission is therefore given by $P(S\,a(x) > T(W + I_\lambda)$. If we ignore the background noise by putting $W = 0$ and use interference scaling it is seen that transmission over distance r with attenuation parameter β that requires signal to interference ratio T has a success probability which scales according to

$$p_r(\lambda) = P(Sa_0(x) > TI_\lambda) = P(S > (\lambda^{1/d}r)^\beta TI_1) = p_{\lambda^{1/d}r}(1)$$

(c.f. [BBM], Lemma 3.3). For the special case of Rayleigh fading with S exponential of mean $1/\mu$, the success probability equals

$$P(S\,a_0(x) > TI_\lambda) = E(e^{-\mu TI_\lambda/a_0(x)})$$

By (1),

$$\begin{aligned} E(e^{-\mu TI_\lambda/a_0(x)}) &= \exp\Big\{-\lambda C_{d,\beta}E(S^{d/\beta})\,(\mu T/a_0(x))^{d/\beta}\Big\} \\ &= \exp\Big\{-\lambda C_{d,\beta}\Gamma(1+d/\beta)\,T^{d/\beta}|x|^d\Big\} \\ &= \exp\Big\{-\lambda|B(0,1)|\frac{\pi/\beta}{\sin(d\pi/\beta)}\,T^{d/\beta}|x|^d\Big\} \end{aligned}$$

Thus, with Rayleigh fading and without external noise

$$p_r(\lambda) = \exp\Big\{-\lambda K_{d,\beta}\,T^{d/\beta}r^d\Big\}, \tag{2}$$

where

$$K_{d,\beta} = \frac{\pi^{d/2}}{\Gamma(1+d/2)}\frac{\pi/\beta}{\sin(d\pi/\beta)}.$$

Returning to the model with external noise W we have

$$P(S\,a_0(x) > T(W + I_\lambda)) = E(e^{-\mu T(W+I_\lambda)/a_0(x)}) = E(e^{-\mu TW|x|^\beta})\,p_{|x|}(\lambda).$$

Finite mean interference field We consider in this section the case of Rayleigh fading with mean signal strength $1/\mu$ and signal attenuation function $a_\kappa(x) = 1/(\kappa + |x|^\beta)$, $x \in \mathbb{R}^d$, for $\beta > d$ and $\kappa > 0$. The interference field $I_\lambda(y)$ at a point y in $\mathbb{R}^d$ is given by

$$I_\lambda(y) = \int_{\mathbb{R}^d}\int_0^\infty s a_\kappa(x-y)\,N(dx,ds), \quad y \in \mathbb{R}^d,$$

which is translation invariant in $\mathbb{R}^d$ with expected value

$$EI_\lambda(y) = \int_{\mathbb{R}^d}\int_0^\infty \frac{s}{\kappa + |x|^\beta}\,\lambda dx F(ds) = \lambda E(S)|B(0,1)|\int_0^\infty \frac{r^{d-1}}{\kappa + r^\beta}\,dr$$

$$= \frac{\lambda}{\mu}\frac{\pi^{d/2}}{\Gamma(1+d/2)}\frac{1}{\kappa^{1-d/\beta}}\frac{\pi/\beta}{\sin(d\pi/\beta)} < \infty.$$

The cumulant-generating function is well-defined for $\theta > -\kappa\mu$ as

$$\log E(e^{\theta I_\lambda(y)}) = \int_{\mathbb{R}^d}\int_0^\infty (e^{\theta a_\kappa(x-y)s} - 1)\, n(dx, ds)$$

$$= \lambda \int_{\mathbb{R}^d} E(e^{\theta S a_\kappa(x)} - 1)\, dx$$

$$= \lambda \int_{\mathbb{R}^d} \frac{\theta a_\kappa(x)}{\mu - \theta a_\kappa(x)}\, dx.$$

Thus, for $\theta > 0$,

$$-\log E(e^{-\theta I_\lambda(y)}) = \lambda|B(0,1)|\int_0^\infty \frac{\theta a_\kappa(r)}{\mu + \theta a_\kappa(r)}\, r^{d-1} dr$$

$$= \lambda|B(0,1)|\int_0^\infty \frac{\theta/\mu}{\kappa + \theta/\mu + r^\beta}\, r^{d-1} dr$$

$$= \lambda K_{d,\beta}\,\frac{\theta/\mu}{(\kappa + \theta/\mu)^{1-d/\beta}}.$$

Suppose that a signal of exponential power S with mean $1/\mu$ is emitted at $x \in \mathbb{R}^d$ independently of the Poisson interference field $(I_\lambda(y))_{y\in\mathbb{R}^d}$ introduced above. Assume the source is at distance $|x| = R$ from the origin and is attenuated by the same function a_κ as the interfering noise. The signal is received successfully at the origin if the signal to interference ratio exceeds a threshold T, which occurs with probability

$$P(Sa_\kappa(x) > TI_\lambda(0)) = Ee^{-\mu T I_\lambda(0)(\kappa + R^\beta)} = \exp\left\{-\lambda K_{d,\beta}\frac{T(\kappa + R^\beta)}{(\kappa + T(\kappa + R^\beta))^{1-d/\beta}}\right\}$$

For $\kappa = 0$, as in (2) we obtain

$$P(Sa_0(x) > TI_\lambda(0)) = \exp\left\{-\lambda K_{d,\beta}\, T^{d/\beta} R^d\right\}.$$

2.5 Node density versus interference

Medium access control protocols aim at preventing users close to each other to access the network and emit signals simultaneously over a shared channel. With many active users per unit of space the interference field increases in strength. This has the counter effect that the signal to noise and interference ratio, and hence the probability of successful transmission between a given user-destination pair, decreases.

A recent study of spatial reuse Aloha, [BBM], uses a similar framework as the one described here to optimize the medium access probability p that maximizes the expected number of successful transmissions in a region. In the rest of this section we give essentially an account of the results of Propositions 4.1 and 4.3 in [BBM]. Given the node intensity λ, if we assume that the stations are potentially dormant and attempt to send signals independently with probability p this amounts to thinning the Poisson point measure resulting in the new intensity λp. Assume that each station which access the medium expects to transmit over a fixed distance r with threshold T, to some destination user which is not considered part of the network. If the accessing station is the Poisson point (X_j, S_j) and the user is located at Y_j with $|X_j - Y_j| = r$, then the transmission is successful if $S_j\, a(X_j - Y_j) > TI_{\lambda p}(Y_j)$. Hence the expected number of successful users in a region $\mathbb{S} \subset \mathbb{R}^d$ in which there is no external noise equals

$$\begin{aligned} E \sum_{X_j \in \mathbb{S}} \mathrm{I}\{S_j a(r) > TI_{\lambda p}(Y_j)\} &= \int_{\mathbb{S}} P(Sa(r) > TI_{\lambda p})\, \lambda p dx \\ &= \lambda p |\mathbb{S}| P(Sa(r) > TI_{\lambda p}) \\ &= \lambda' p_r(\lambda')\, |\mathbb{S}|, \quad \lambda' = \lambda p. \end{aligned}$$

This shows that we can find the optimal medium access probability p for transmission over fixed range r, by maximizing $\lambda p_r(\lambda)$ over $\lambda > 0$. For S not identically vanishing, $p_r(\lambda) > 0$ for some $\lambda > 0$. By Chebyshevs inequality, for $p > 0$

$$p_r(\lambda) = p_{\lambda^{1/d} r}(1) = P(S > T\lambda^{\beta/d} r^{\beta} I_1) \leq E(S^p) E(I_1^{-p}) \frac{1}{T^p \lambda^{p\beta/d} r^{p\beta}} \tag{3}$$

Here,

$$\begin{aligned} E(I_1^{-p}) &= \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} E(e^{-I_1 s})\, ds \\ &= \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} \exp\{-C_{d,\beta} E(S^{d/\beta}) s^{d/\beta}\}\, ds \\ &= \frac{\beta/d}{\Gamma(p)} \int_0^\infty s^{p\beta/d-1} \exp\{-C_{d,\beta} E(S^{d/\beta}) s\}\, ds \\ &= \frac{(\beta/d)\Gamma(p\beta/d)}{\Gamma(p)(C_{d,\beta} E(S^{d/\beta}))^{p\beta/d}} < \infty \end{aligned} \tag{4}$$

Thus, if $E(S^p) < \infty$ for some $p > d/\beta$, then

$$\lambda p_r(\lambda) \leq \mathrm{const} \frac{1}{\lambda^{p\beta/d-1}} \to 0, \quad \lambda \to \infty,$$

and so there exist an optimal node intensity $\lambda_{\max}$ which maximizes the performance of the network, under the given conditions.

A crude bound for $p_r(\lambda)$ is obtained by taking $p = d/\beta$ in (3) and (4). This yields

$$\begin{aligned} p_r(\lambda) &\leq \frac{1}{T^{d/\beta} \lambda r^d} E(S^{d/\beta}) E(I_1^{-d/\beta}) \\ &= \frac{1}{T^{d/\beta} \lambda r^d} \frac{\beta}{\Gamma(d/\beta) |B(0,1)| \Gamma(1 - d/\beta)} \\ &= \frac{1}{K_{d,\beta} T^{d/\beta} \lambda r^d} \end{aligned}$$

For the special case of Rayleigh fading the exact formula (2) shows already that

$$\lambda_{\max}^{-1} = K_{d,\beta} T^{d/\beta} r^d \quad \text{and} \quad \lambda_{\max} p_r(\lambda_{\max}) = \frac{e^{-1}}{K_{d,\beta} T^{d/\beta} r^d}.$$

3 Traffic session modeling

3.1 Rayleigh fading over fixed length session

It is shown in [VTY], based on results in [PY], that it is possible to construct a two-parameter real-valued stochastic process $\{\Gamma_v(t), v \geq 0, t \geq 0\}$ which is a gamma subordinator process in t and a squared Bessel diffusion in v. It is suggested in [K] that this yields an appropriate model for Rayleigh fading developing over time. For fixed v, $\{\Gamma_v(t), t \geq 0\}$ with $\Gamma_v(0) = 0$ is the Lévy process with Lévy measure $\nu(dy) = y^{-1} e^{-y/v}\, dy$. This is a stochastic process with independent increments known as the gamma subordinator. Our interpretation is that the subordinator increments yield the cumulative increase of energy pulses from a given emitter over time. For fixed t, the process $\Gamma_v(t), v \geq 0$, $\Gamma_0(t) = 0$, is a squared Bessel diffusion with fractal dimension $2t$ and variance parameter $v/2$. This means in particular that we have the representation $\Gamma_v(k) = \sum_{j=1}^k (X_j^2 + Y_j^2)$ in terms of Gaussian random variables $X_j, Y_j,\ j \geq 1$, with zero mean and variance $v/2$. Again this provides an interpretation of Rayleigh fading stemming from the variations in the squared amplitude of a complex Gaussian wave.

Taking $t = 1$ and $v = 1/\mu$ we recover the Rayleigh fading model discussed above. To put this in the Poisson point process framework, let $Q_0^v(d\gamma)$ be the distribution for paths $\{\gamma(t), t \geq 0\}$ of the subordinator $\Gamma_v(t)$. We write $\mathcal{D}$ for the state space of increasing pure jump trajectories. With each poisson point in $\mathbb{R}^d$ we associate a gamma subordinator with distribution $Q_0^{a(x)}$ and let $N_\lambda(dx, d\gamma)$ be the Poisson point process in $R^d \times \mathcal{D}$ with intensity measure $\lambda dx\, Q_0^{a(x)}(d\gamma)$. The cumulative interference observed at a point y at time t is given by

$$I_\lambda(t, y) = \int_{\mathbb{R}^d} \int_{\mathcal{D}} \gamma(t) a(x - y)\, N_\lambda(dx, d\gamma),$$

which is stationary in y and has independent increments in t. The functional $I_\lambda(t) = I_\lambda(0, t)$ picks out the terminal value of the path at time t observed at the origin. As in (1), using $a = a_0$,

$$\begin{aligned} \log E(e^{i\theta I_\lambda(t)}) &= \int_{\mathbb{R}^d} \int_{\mathcal{D}} (e^{i\theta\gamma(t)} - 1)\, \lambda dx\, Q_0^{a(x)}(d\gamma) \\ &= \lambda |B(0,1)| \int_0^\infty E(e^{i\theta\Gamma_1(t)/r^\beta} - 1) r^{d-1}\, dr \\ &= \lambda |B(0,1)| E(\Gamma_1(t)^{d/\beta})\, C(\operatorname{sign}\theta)\, |\theta|^{d/\beta}. \end{aligned}$$

Modified multicast model

If we apply the gamma subordinator model to the multicast situation with one active emitter at the origin and receivers placed in $\mathbb{R}^d$ with Poisson intensity λdx, then for a fixed t we have

$$M_3(t) = \int_{\mathbb{R}^d} \mathrm{I}\{\Gamma_{a(x)}(t) > TW\}\, N(dx),$$

where the integrand is a function of the squared Bessel process $\{\Gamma_v(t), v \geq 0\}$, after "time change" $v = a(x)$. With $a = a_0$ this yields

$$\begin{aligned} E(e^{i\theta M_3(t)}) &= E \exp\Big\{\lambda(e^{i\theta}-1)\int_{\mathbb{R}^d} \mathrm{I}\{\Gamma_{a_0(x)}(t) > TW\}\, dx\Big\} \\ &= E \exp\Big\{\lambda(e^{i\theta}-1)|B(0,1)|\int_0^\infty \mathrm{I}\{\Gamma_{a_0(r)}(t) > TW\}\, r^{d-1}\, dr\Big\} \\ &= E \exp\Big\{\lambda(e^{i\theta}-1)|B(0,1)|\int_0^\infty \mathrm{I}\{\Gamma_v(t) > TW\}\, \beta^{-1} v^{-d/\beta-1}\, dv\Big\}. \end{aligned}$$

This is a mixed Poisson representation with a random intensity expressed as a weighted occupation time functional of the Bessel difffusion process.

3.2 Lognormal fading

Lognormal fading is considered to be a multiplicative effect of wave shadowing. Small multiplicative variations of the wave energy subject to a Gaussian approximation on the logarithmic scale, lead to the assumption that the signal power has a lognormal distribution. The changes in shadowing effect occur on a relatively slow time scale. It is natural therefore to assume that the lognormal distribution is fixed throughout a traffic session. Additional random variation due to Rayleigh fading is superimposed, conditional on the lognormal average.

More precisely, let X represent the received power measured in decibel emitted from a station at x. We assume that X is normally distributed $\mathrm{N}(\mu(x), \sigma_L^2)$, where $\mu(x) = -\log(\kappa + |x|^\beta)$ and the variance $\sigma_L^2 > 0$ is a given constant. Put $V_x = e^X$. Then V_x has a lognormal distribution, such that for $s > 0$

$$EV_x^s = \exp\{s\mu(x) + s^2\sigma_L^2/2\} = a_\kappa(x)^s \exp\{s^2\sigma_L^2/2\}. \qquad (5)$$

Conditional on $V_x = v$, recall that the increment $\Gamma_v(k+1) - \Gamma_v(k)$ represents the power of a symbol emitted from x and recieved at the origin during slot k. In this way the evolution of the process $\Gamma_{V_x}(t)$, $t \geq 0$, yields the Rayleigh variation of an emitter in x during a traffic session with lognormal power fading and attenuation a_κ.

Write $F_x(v) = P(V_x \leq v)$ for the distribution function of V_x. The relevant Poisson measure $N_{\lambda,y}(dx, dv, d\gamma)$ for this situation has intensity $\lambda dx\, F_{x-y}(dv)\, Q_0^v(d\gamma)$. The interference field

$$I_\lambda(t, y) = \int_{\mathbb{R}^d}\int_0^\infty\int_{\mathcal{D}} \gamma(t)\, N_{\lambda,y}(dx, dv, d\gamma)$$

at $y=0$ has

$$\begin{aligned}\log E(e^{i\theta I_\lambda(t)}) &= \int_{\mathbb{R}^d}\int_0^\infty\int_{\mathcal{D}}(e^{i\theta\gamma(t)}-1)\,\lambda dx\,F_x(dv)Q_0^v(d\gamma)\\ &= \lambda\int_{\mathbb{R}^d}\int_0^\infty E(e^{i\theta\Gamma_v(t)}-1)\,F_x(dv)dx\\ &= \lambda\int_{\mathbb{R}^d} E\Big[\Big(\frac{V_x}{1-i\theta V_x}\Big)^t-1\Big]\,dx.\end{aligned}$$

SINR success probability

We consider a signal of length t time slots which is emitted from a source in $x\in\mathbb{R}^d$ with $|x|=R$. The signal is successfully received in the origin if the signal to interference ratio exceeds a threshold T in each slot, that is, if

$$\frac{S_k a_\kappa(x)}{I_\lambda(k)-I_\lambda(k-1)}\geq T,\quad k=1,\dots,t,$$

where (S_k) is a sequence of i.i.d. exponential variables with mean $1/\mu$, independent of the interfering field. Hence the success probability is given by

$$p_\lambda(t,x)=E(e^{-\mu T I_\lambda(1)/a_\kappa(x)}\cdots e^{-\mu T(I_\lambda(t)-I_\lambda(t-1))/a_\kappa(x)})=E(e^{-\mu T I_\lambda(t)/a_\kappa(x)}).$$

Now,

$$\begin{aligned}-\log E(e^{-\theta I_\lambda(t)}) &= \int_{\mathbb{R}^d\times\mathbb{R}^+}E(1-e^{-\theta\Gamma_v(t)a(x)})\,\lambda dxF(dv)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^+}\Big\{1-\Big(\frac{\kappa+|x|^\beta}{\theta v+\kappa+|x|^\beta}\Big)^t\Big\}\,\lambda dxF(dv)\\ &= \sum_{k=1}^t\binom{t}{k}\int_{\mathbb{R}^d\times\mathbb{R}^+}\frac{(\theta v)^k(\kappa+|x|^\beta)^{t-k}}{(\theta v+\kappa+|x|^\beta)^t}\,\lambda dxF(dv)\\ &= \sum_{k=1}^t\sum_{\ell=0}^{t-k}\binom{t}{k}\binom{t-k}{\ell}\kappa^\ell\int_{\mathbb{R}^d\times\mathbb{R}^+}\frac{(\theta v)^k|x|^{\beta(t-k-\ell)}}{(\theta v+\kappa+|x|^\beta)^t}\,\lambda dxF(dv)\end{aligned}$$

Here,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|x|^{\beta(t-k-\ell)}}{(\theta v+\kappa+|x|^\beta)^t}\,\lambda dx &= \frac{\lambda\pi^{d/2}}{\Gamma(1+d/2)}\int_0^\infty\frac{r^{\beta(t-k-\ell)}}{(\theta v+\kappa+r^\beta)^t}\,r^{d-1}dr\\ &= \frac{\lambda\pi^{d/2}}{\Gamma(1+d/2)}\frac{(\pi/\beta)}{\sin(d\pi/\beta)}\frac{(-1)^{k+\ell+1}\Gamma(d/\beta+t-k-\ell)}{\Gamma(t)\Gamma(d/\beta-k-\ell+1)}(\theta v+\kappa)^{d/\beta-k-\ell}\end{aligned}$$

Hence

$$-\log E(e^{-\theta I_\lambda(t)}) = \lambda K_{d,\beta} \sum_{k=1}^{t} \sum_{\ell=0}^{t-k} \binom{t}{k} \binom{t-k}{\ell} (-1)^{k+\ell+1} \binom{d/\beta+t-1-k-\ell}{t-1} \kappa^{\ell} \times \int_{\mathbb{R}^+} (\theta v)^k (\theta v+\kappa)^{d/\beta-k-\ell} F(dv)$$

$$= \lambda K_{d,\beta} \sum_{k=1}^{t} \sum_{\ell=k}^{t} \binom{t}{k} \binom{t-k}{\ell-k} (-1)^{\ell+1} \binom{d/\beta+t-1-\ell}{t-1} \kappa^{\ell-k} \times \int_{\mathbb{R}^+} (\theta v)^k (\theta v+\kappa)^{d/\beta-\ell} F(dv)$$

$$= \lambda K_{d,\beta} \sum_{\ell=1}^{t} (-1)^{\ell+1} \binom{d/\beta+t-1-\ell}{t-1} \binom{t}{\ell} \sum_{k=1}^{\ell} \binom{\ell}{k} \kappa^{\ell-k} \times \int_{\mathbb{R}^+} (\theta v)^k (\theta v+\kappa)^{d/\beta-\ell} F(dv)$$

$$= \lambda K_{d,\beta} \sum_{\ell=1}^{t} (-1)^{\ell+1} \binom{d/\beta+t-1-\ell}{t-1} \binom{t}{\ell} \times \int_{\mathbb{R}^+} \Big((\theta v+\kappa)^{\ell} - \kappa^{\ell} \Big) (\theta v+\kappa)^{d/\beta-\ell} F(dv)$$

Moreover,

$$\sum_{\ell=1}^{t} (-1)^{\ell+1} \binom{d/\beta+t-1-\ell}{t-1} \binom{t}{\ell} = \frac{t-1+d/\beta}{d/\beta} \binom{d/\beta+t-2}{t-1}$$

Put

$$H_\ell(\theta,\kappa) = \int_{\mathbb{R}^+} \Big((\theta v+\kappa)^{\ell} - \kappa^{\ell} \Big) (\theta v+\kappa)^{d/\beta-\ell} F(dv) = \int_{\mathbb{R}^+} (\theta v+\kappa)^{d/\beta} \Big(1 - (\frac{\kappa}{\kappa+\theta v})^{\ell} \Big) F(dv)$$

For any ℓ, $1 \le \ell \le t$, we have $H_1(\theta,\kappa) \le H_\ell(\theta,\kappa) \le H_t(\theta,\kappa)$, that is

$$\int_{\mathbb{R}^+} \frac{\theta v}{(\theta v+\kappa)^{1-d/\beta}} F(dv) \le H_\ell(\theta,\kappa) \le \int_{\mathbb{R}^+} (\theta v+\kappa)^{d/\beta} \Big(1 - (\frac{\kappa}{\kappa+\theta v})^{t} \Big) F(dv)$$

The simplest case $V = 1/\mu$ yields, taking $\theta = \mu T / a_\kappa(r)$,

$$p_\lambda(r) = \lambda K_{d,\beta} \frac{t-1+d/\beta}{d/\beta} \binom{d/\beta+t-2}{t-1} H(r),$$

where

$$(\kappa + T(\kappa+r^\beta))^{d/\beta} \frac{T(\kappa+r^\beta)}{\kappa+T(\kappa+r^\beta)} \le H(r) \le (\kappa + T(\kappa+r^\beta))^{d/\beta}.$$

For the case $\kappa = 0$ this is

$$-\log E(e^{-\theta I_\lambda(t)}) = \lambda K_{d,\beta} \frac{t-1+d/\beta}{d/\beta} \binom{d/\beta+t-2}{t-1} \theta^{d/\beta} E(V^{d/\beta})$$

and so, taking $\theta = Tr^\beta/E(V)$,

$$p_\lambda(t,r) = \exp\Big\{ -\lambda K_{d,\beta}\, \frac{t-1+d/\beta}{d/\beta} \binom{d/\beta+t-2}{t-1} \frac{E(V^{d/\beta})}{E(V)^{d/\beta}} T^{d/\beta} r^d \Big\}$$

By (5),

$$\frac{E(V^{d/\beta})}{E(V)^{d/\beta}} = e^{-(d/\beta)(1-d/\beta)\sigma_L^2/2}$$

In conclusion,

$$p_\lambda(t,r) \approx \exp\Big\{ -\lambda \frac{K_{d,\beta}\Gamma(d/\beta)}{d/\beta} (t-1)^{d/\beta}\, e^{-(d/\beta)(1-d/\beta)\sigma_L^2/2}\, T^{d/\beta} r^d \Big\}$$

[Write $\alpha = d/\beta$?]

3.3 Temporal-spatial traffic sessions

We develop the model further by imagining signal emitters at random locations in space that transmit calls with random initial times and random call holding times.

The initial time s, the location x, and the call holding time u of each transmission is given by a point (s,x,u) of a Poisson point measure on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+$ with intensity $\lambda dsdx\, G(du)$, where $G(u) = P(U \le u)$ is the distribution function of the lifelength U of a typical traffic session. With each session (s,x,u) we associate a further mark (v,γ) picked with intensity $F_x(dv)\, Q_0^v(d\gamma)$, which yields independently of the temporal-spatial location (s,x) and the call duration u a lognormal fading level v and a gamma subordinator γ that generate pulses during the time interval $(s,s+u)$. To record the cumulative interference field during an observation interval $[0,t]$, we introduce

$$K_t(s,u) = \int_0^t I\{s < y < s+u\}\, dy = |(s,s+u)\cap(0,t)|,$$

which measures the fraction of the time interval $[0,t]$ during which a session that starts at time s and has duration u is active. Letting $N_\lambda(dsdx,du,dv,d\gamma)$ be the Poisson measure with intensity $\lambda dsdx\, G(du)\, F_x(dv)\, Q_0^v(d\gamma)$, it is seen that the contribution to the interference field at $y=0$ in the interval $[0,t]$ is given by

$$I_\lambda(t) = \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty\int_{\mathcal{D}} \gamma(K_t(s,u))\, N_\lambda(dsdx,du,dv,d\gamma).$$

The expected value is

$$\begin{aligned} EI_\lambda(t) &= \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty E\Gamma_v(K_t(s,u))\, \lambda dsdx\, F_x(dv)\, G(du) \\ &= \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty E(V_x)K_t(s,u)\, \lambda dsdx\, F(du) \\ &= \lambda\, e^{\sigma_L^2/2} \int_{\mathbb{R}^d} a_1(x)\, dx \int_{-\infty}^\infty\int_0^\infty K_t(s,u)\, ds\, F(du) \\ &= \lambda\, e^{\sigma_L^2/2}\, K_{d,\beta}\, \kappa^{-(1-d/\beta)}\, EU\ t \end{aligned}$$

since

$$\int_{-\infty}^{\infty} K_t(s,u)\,ds = \int_{-\infty}^{\infty}\int_0^t I\{y-u<s<y\}\,dy\,ds = ut.$$

For the variance we obtain similarly

$$\begin{aligned}
\mathrm{Var} I_\lambda(t) &= \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty E(\Gamma_v(K_t(s,u))^2)\,\lambda ds dx\, F_x(dv)\, G(du)\\
&= \lambda \int_{\mathbb{R}^d} E(V_x^2)\int_{-\infty}^{\infty}\int_0^\infty K_t(s,u)(K_t(s,u)+1)\,ds\, G(du)\\
&= \lambda\, e^{2\sigma_L^2}\int_{\mathbb{R}^d} a_\kappa^2(x)\,dx\,\Big(tEU + \int_{-\infty}^{\infty}\int_0^\infty K_t(s,u)^2\,ds\,G(du)\Big).
\end{aligned}$$

The remaining double integral may be recast in the form

$$\int_{-\infty}^{\infty}\int_0^\infty K_t(s,u)^2\,ds\,G(du) = \int_0^t\int_0^t dy dy' \int_{|y-y'|}^\infty (1-|y-y'|/u)\,G(du). \tag{6}$$

We remark that the variance of the interference functional may exist finitely even if the call duration distribution is heavy-tailed in such a way that u has finite mean but infinite variance.

3.4 Fluctuations in the interference field

The fluctuations of the Poisson interferers around the mean level are described by the compensated Poisson integral

$$J_\lambda(t) = I_\lambda(t) - EI_\lambda(t) = \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty\int_{\mathcal{D}} \gamma(K_t(s,u))\,\widetilde{N}_\lambda(dsdx,du,dv,d\gamma),$$

where

$$\widetilde{N}_\lambda(dsdx,du,dv,d\gamma) = N_\lambda(dsdx,du,dv,d\gamma) - \lambda dsdx\, G(du)\, F_x(dv)\, Q_0^v(d\gamma)$$

is the compensated Poisson measure. We have

$$\begin{aligned}
&\log E(e^{i\theta J_\lambda(t)}) \qquad (7)\\
&\quad = \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty E(e^{i\theta\Gamma_v(K_t(s,u))} - 1 - i\theta\Gamma_v(K_t(s,u)))\,\lambda dsdx\, G(du)F_x(dv)\\
&\quad = \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty (e^{-K_t(s,u)\log(1-iv\theta)} - 1 - iv\theta K_t(s,u))\,\lambda dsdx\, G(du)F_x(dv).
\end{aligned}$$

It can be shown that

$$\begin{aligned}
\log E(e^{i\theta J_\lambda(t)}) &= R_\lambda(t,\theta)\\
&\quad + \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty (e^{iv\theta K_t(s,u)} - 1 - iv\theta K_t(s,u))\,\lambda dsdx\, G(du)F_x(dv),
\end{aligned} \tag{8}$$

where

$$|R_\lambda(t,\theta)| \le \frac{1}{2}e^{2\sigma_L^2}\int_{\mathbb{R}^d} a(x)^2\,\lambda dx\, E(U)\, t\,\theta^2. \tag{9}$$

Scaling analysis

We investigate the regime of high density networks under time rescaling. This is to say, we let $\lambda \to \infty$ while scanning the interference fluctuations over the time interval $[0, at]$, where $a = a_\lambda \to \infty$ is an additional scaling parameter. Is there a normalizing sequence $b = b_\lambda$ such that $b^{-1} J_\lambda(a)$ has a limit in distribution?

Suppose first that the call holding time has finite second moment, $EU^2 < \infty$. In this case, take any sequence $a \to \infty$ and put $b = \sqrt{\lambda a}$. The finite-dimensional distributions of

$$b^{-1} J_\lambda(at) = \frac{1}{b} \int_{\mathbb{R}\times\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{\mathcal{D}} \gamma(K_{at}(s,u))\, \widetilde{N}_\lambda(dsdx, du, dv, d\gamma)$$

converge to those of a Brownian motion as $\lambda, a \to \infty$. The proof of convergence of the marginal distribution can be carried out by verifying the following approximative steps. By (7),

$$\begin{aligned}
&\log E(e^{i\theta b^{-1} J_\lambda(at)}) \\
&\sim \int_{\mathbb{R}\times\mathbb{R}^d} \int_0^\infty \int_0^\infty (e^{-K_{at}(s,u)\log(1-iv\theta/b)} - 1 - iv\theta K_{at}(s,u)/b)\, \lambda dsdx\, G(du) F_x(dv) \\
&\sim -\frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}^d} \int_0^\infty \int_0^\infty \Big(\frac{v\theta}{b}\Big)^2 (K_{at}(s,u)^2 + K_{at}(s,u))\, \lambda dsdx\, G(du) F_x(dv) \\
&\sim -\frac{1}{2}\theta^2 \int_{\mathbb{R}^d} E(V_x^2)\, dx \int_{-\infty}^\infty \int_0^\infty (K_{at}(as,u)^2 + K_{at}(as,u))\, ds\, G(du) \\
&\sim -\frac{1}{2}\theta^2 \int_{\mathbb{R}^d} E(V_x^2)\, dx\, (E(U^2) + E(U))\, t,
\end{aligned}$$

since

$$K_{at}(as,u) \to u\,\mathrm{I}\{0 < s < t\}, \quad a \to \infty.$$

These approximations can be verified and extended to the finite-dimensional distributions. Thus, the distributional limit of $J_\lambda(at)/\sqrt{\lambda a}$ is Brownian motion with variance $\int_{\mathbb{R}^d} E(V_x^2)\, dx\, E(U^2 + U)$.

3.4.1 Heavy-tailed call holding times

Assume that the distribution for the call durations has a regularly varying tail at infinity, $1 - G(u) = L(u)u^{-\gamma}$, for a slowly varying function L and with index of regular variation γ, $1 < \gamma < 2$. Under this assumption U has finite mean but infinite variance. In this situation there are three possible scaling regimes given by the relative speed at which λ and a tend to infinity. We consider

- Fast connection rate: $\lambda/a^{\gamma-1} \to \infty$, $b^2 = \lambda a^{3-\gamma}$
- Intermediate connection rate: $\lambda/a^{\gamma-1} \to 1$, $b = a$
- Slow connection rate: $\lambda/a^{\gamma-1} \to 0$, $b^\gamma = \lambda a$

Note that in each of these cases we have for the remainder term $R_\lambda(t,\theta)$ introduced in (9),

$$R_\lambda(at,\theta/b)\to 0, \quad \lambda,a\to\infty.$$

Hence for each case we want to find the limit of

$$\begin{aligned}&\log E(e^{i\theta J_\lambda(at)/b})\\ &\quad=\int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty (e^{iv\theta K_{at}(s,u)/b}-1-iv\theta K_{at}(s,u)/b)\,\lambda ds dx\, G(du)F_x(dv).\end{aligned}$$

For fast connection rate

$$\begin{aligned}&\log E(e^{i\theta J_\lambda(at)/b})\\ &\quad\sim -\frac{1}{2}\int_{\mathbb{R}^d} EV_x^2\,dx\int_{-\infty}^\infty\int_0^\infty (a\theta K_t(s,u)/b)^2\,\lambda ads\, G(adu)\\ &\quad\sim -\frac{1}{2}\theta^2\int_{\mathbb{R}^d} EV_x^2\,dx\int_{-\infty}^\infty\int_0^\infty K_t(s,u)^2\,ds\, u^{-\gamma-1}du.\end{aligned}$$

By (6),

$$\begin{aligned}\int_{-\infty}^\infty\int_0^\infty K_t(s,u)^2\,ds\,u^{-\gamma-1}du &= \int_0^t\int_0^t dydy'\int_{|y-y'|}^\infty (1-|y-y'|/u)\,u^{-\gamma-1}\\ &= \text{const}\int_0^t\int_0^t dydy'|y-y'|^{-(\gamma-1)}=\text{const}\,t^{3-\gamma}.\end{aligned}$$

In general, we obtain the finite-dimensional distribution of fractional Brownian motion with Hurst index $H=(3-\gamma)/2$.

The intermediate scaling regime yields the limit

$$\begin{aligned}&\log E(e^{i\theta J_\lambda(at)/a})\\ &\quad\to\int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty (e^{iv\theta K_t(s,u)}-1-iv\theta K_t(s,u))\,dsdx\,u^{-\gamma-1}F_x(dv),\end{aligned}$$

which is the characteristic function of

$$\int_{\mathbb{R}\times\mathbb{R}^d}\int_0^\infty\int_0^\infty K_t(s,u)\,\widetilde{N}(dsdx,du,dv)$$

where $\widetilde{N}$ is a compensated Poisson measure with intensity measure $dsdx\,u^{-\gamma-1}F_x(dv)$. The covariance structure of this process is the same as that of fractional Brownian motion with Hurst index $3-\gamma$.

Finally, the limit in the case of slow connection rate is a stable Lévy process with stable index $1/\gamma$.

References

[BBM] F. Baccelli, B. Blaszczyszyn, and P. Mühlethaler, An Aloha protocol for multihop mobile wireless networks. IEEE Trans. Inf. Theory 52, 421-436, 2006.

[DVJ] D.J. Daley and D. Vere-Jones, *An introduction to the theory of point processes.* Springer-Verlag, New York, NY, 1988.

[G] R. Gaigalas, A Poisson bridge between fractional Brownian motion and the stable Lévy motion. Stoch. Proc. Applications 116, 447-462, 2006.

[GaK] R. Gaigalas and I. Kaj, Convergence of scaled renewal processes and a packet arrival model. Bernoulli 9, 671-703, 2003.

[IHV] J. Ilow, D. Hatzinakos, and A.N. Venetsanopoulos, Performance of FH SS radio networks with interference modeled as a mixture of Gaussian and α-stable noise. IEEE Trans. on Communications 46, 509-520, 1998.

[K] I. Kaj, *Stochastic modeling in broadband communications systems.* SIAM Monographs in Mathematical Modeling and Computation 8. SIAM, Philadelphia, PH, 2002.

[K2] I. Kaj, Limiting fractal random processes in heavy-tailed systems. In Fractals in Engineering, New Trends in Theory and Applications, p 199-218. Eds. J. Levy-Lehel, E. Lutton, Springer-Verlag London 2005.

[KLNS] I. Kaj, L. Leskelä, I. Norros and V. Schmidt, Scaling limits for random fields with long range dependence. To appear: Ann. Probab. 2006.

[KT] I. Kaj and M.S. Taqqu, Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In an Out of Equilibrium 2, Eds. M.E. Vares, V. Sidoravicius. Progress in Probability, Vol 60, 383-427. Birkhauser 2008.

[Ka] O. Kallenberg, *Foundations of Modern Probability.* 2nd ed Springer-Verlag, New York, 2002.

[Ki] J. F. C. Kingman, *Poisson Processes.* Oxford Univ. Press, Oxford, 1993.

[PY] J. Pitman and M. Yor, A decomposition of Bessel bridges. Z. Wahrsch. Verw. Gebiete 59, 425–457, 1982.

[S] E.S. Sousa, Performance of a spread spectrum packet radio network link in a Poisson field of interferers. IEEE Trans. Inf. Theory 38, 1743-1754, 1992.

[VTY] A. M. Vershik, N. V. Tsilevich, and M. Yor, M., Distinguished properties of the gamma process, Preprint 2004.

[YP] X. Yang and A.P. Petropulu, Co-channel interference modeling and analysis in a Poisson field of interferers in wireless communications. IEEE Trans. Signal Proc. 51, 64-76, 2003.