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# Aspiration/Reservation-Based Decision Support—a Step Beyond Goal Programming

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ABSTRACT

Real-life decision problems are usually so complex that they cannot be modelled with a single objective function, thus creating a need for clear and efficient techniques for handling multiple criteria to support the decision process. A widely used technique and one commonly taught in general OR/MS courses is goal programming, which is clear and appealing. On the other hand, goal programming is strongly criticized by multiple-criteria optimization specialists for its non-compliance with the efficiency (Pareto-optimality) principle. In this paper we show how the implementation techniques of goal programming can be used to model the reference point method and its extension, aspiration/reservation-based decision support. Thereby we show a congruence between these approaches and suggest how the GP model with relaxation of some traditional assumptions can be extended to an efficient decision support technique meeting the efficiency principle and other standards of multiobjective optimization theory.

KEY WORDS Aspiration/reservation-based DSS Efficiency Goal programming Pareto optimality

## 1. INTRODUCTION

Consider a decision problem defined as an optimization problem with $k$ objective functions. We assume without loss of generality that all the objective functions are to be minimized. The problem can then be formulated as

$$\text{minimize } \mathbf{F}(\mathbf{x}) \tag{1}$$
$$\text{subject to } \mathbf{x} \in Q \tag{2}$$

where $\mathbf{F} = (F_1, \ldots, F_k)$ represents a vector of $k$ objective functions, $Q$ denotes the feasible set of the problem and $\mathbf{x}$ is a vector of decision variables. Consider further an achievement vector

$$\mathbf{q} = \mathbf{F}(\mathbf{x}) \tag{3}$$

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which measures the achievement of decision **x** with respect to the specified set of $k$ objectives $F_1, \ldots, F_k$. Let $Y$ denote the set of all attainable achievement vectors

$$Y = \{\mathbf{q} = \mathbf{F}(\mathbf{x}) \colon \mathbf{x} \in Q\} \tag{4}$$

i.e. all the vectors corresponding to feasible solutions. It is clear that an achievement vector is better than another provided that at least one individual achievement is better whereas no other one is worse. Such a relation is called domination of achievement vectors and is mathematically formalized as

$$\text{if } \mathbf{q}' \neq \mathbf{q}'' \text{ and } q_i' \leqslant q_i'' \text{ for all } i = 1, \ldots, k$$
$$\text{then } \mathbf{q}' \text{ dominates } \mathbf{q}'' \text{ and } \mathbf{q}'' \text{ is dominated by } \mathbf{q}'$$

Unfortunately, there usually does not exist an achievement vector that dominates all others with respect to all the criteria, i.e.

$$\text{does not exist } \mathbf{y} \in Y \text{ such that for any } \mathbf{q} \in Y$$
$$y_i \leqslant q_i \quad \text{for all } i = 1, \ldots, k$$

Thus in terms of strict mathematical relations we cannot distinguish the best achievement vector. Instead we classify each achievement vector **q** as a dominated one or as a non-dominated one. The dominated achievement vectors represent obviously non-optimal decisions. On the other hand, all the non-dominated achievement vectors can be considered as optimal from some point of view. The non-dominated vectors are non-comparable on the basis of the specified set of objective functions.

The feasible solutions (decisions) that generate non-dominated achievement vectors are called efficient or Pareto-optimal solutions to the multiobjective problem. This means that each feasible decision for which one cannot improve any individual achievement without worsening another is an efficient decision.

It seems clear that the solution of multiobjective optimization problems should simply depend on identification of the efficient solutions. There exist such approaches, especially for linear programmes, where the efficient set, despite being infinite, can be described by finite information (vertices of faces). However, the finite characterization of the efficient set for a real-life problem is usually so large that it cannot be considered a solution to the decision problem. Thus the need arises for further analysis, or rather decision support, to help the decision maker (DM) in selecting one efficient solution for implementation. Of course, the original objective functions do not allow one to select any efficient solution as better than any other. Therefore this analysis usually depends on additional information about the DM's preferences gained during an interactive process.

There are various concepts as to how to gain such additional information from the DM (Steuer, 1986). The classical interactive procedures for multiple-criteria decision analysis assume the so-called rational behaviour of decision makers: they know the decision problem and are consistent and coherent in the decision process. In other words the DM is assumed to be a *homo economicus* who has a perfect knowledge of all relevant aspects of the decision problem environment and whose preferences are stable (e.g. Isard, 1969). Usually the existence of some individual or group utility function (Fishburn, 1970) representing these stable preferences is assumed. The interactive decision support process then depends on identification of this utility function (e.g. Zionts and Wallenius, 1976), which, if known, could be easily optimized. However, as stressed by many researchers and practitioners, the DM's understanding of the decision problem usually evolves during the interaction with the decision support system, and there are numerous examples in which people systematically violate consistency and coherence in their preferences (e.g. MacLean, 1985; Tversky and Kahneman, 1985; and references cited therein).

The hypothesis that people seldom maximize a utility function, first analysed in detail by Simon (1957), led to approaches based on the so-called satisficing behaviour. In this approach, depending on recurrent observation, it is assumed that people tend to summarize their learning of the state of the world by forming aspirations of desirable outcomes for their decisions. When the outcomes fail to satisfy their aspirations, people tend to seek ways to improve the outcomes. When their aspirations are satisfied, however, their attention turns to other outcomes.

The satisficing behaviour concept is usually operationalized via goal programming. Goal programming (GP), originally proposed by Charnes and Cooper (1961) and further developed by others (e.g. Ijiri, 1965; Ignizio, 1976; Lee, 1972), requires a transformation of objectives into goals by specification of an aspiration level for each objective. A feasible solution that minimizes deviations from the aspiration levels is then an optimal solution to the GP problem. Various measures of multidimensional deviations have been proposed. Charnes and Cooper (1961) minimized a sum of weighted deviations. Widely used is lexicographic (or pre-emptive priority) GP, where a hierarchy of goals is presumed (Ignizio, 1982). The aspiration levels are considered as part of the data for the GP model. However, the levels can be changed during the decision process if the GP model is used as a basis of some interactive decision support system. One of the most important advantages of the interactive GP approach is that it does not (necessarily) require the DMs to be consistent and coherent in their preferences.

Goal programming, unfortunately, does not satisfy the efficiency (Pareto-optimality) principle. Goal programming only yields decisions that have outcomes closest to the specified aspiration levels, which agrees with the strict satisficing behaviour concept. This has led to the development of the quasi-satisficing approach as a compromise between the strict satisficing methodology and optimization. The quasi-satisficing approach deals with the so-called scalarizing achievement function, which, when optimized, generates efficient decisions relative to the specified aspiration levels. The function is somewhat similar to a utility function and in fact can be used as an approximation to a class of utility functions. It is, however, explicitly dependent on aspiration levels stated and modified by the DM and thereby makes operational the concept of adaptive dependence of utility on learning and context. Completeness, computational robustness and controllability of the interactive scheme are more important here than consistency and coherence of the DM's preferences (Wierzbicki, 1986). An excellent formalization of the quasi-satisficing approach to multiobjective optimization was proposed and developed mainly by Wierzbicki (1982) as the reference point method. The reference point method was later extended to permit additional information from the DM and eventually led to efficient implementations of the so-called aspiration/reservation-based decision support (ARBDS) approach with many successful applications (Lewandowski and Wierzbicki, 1989).

In this paper we show how the implementation techniques of goal programming can be used to model the ARBDS approach. Thereby we show a congruence between these approaches and suggest how the GP model with relaxation of some traditional assumptions can be extended to an efficient decision support technique meeting the efficiency principle and other standards of multiobjective optimization theory. The paper is organized as follows. In Section 2 we briefly review techniques of the GP approach and discuss its failure with respect to the efficiency principle. In Section 3 we show how GP techniques can be used to model the reference point approach. These results are extended in Section 4 to the full ARBDS approach.

## 2. GOAL PROGRAMMING AND EFFICIENCY PRINCIPLE

The GP approach requires the DM to specify *the most desired value* for each objective function as the aspiration level. The objective functions (1) are then transformed into goals:

$$\begin{array}{lll} F_i(\mathbf{x}) + d_i^- - d_i^+ = a_i \quad \text{for } i = 1, \ldots, k & & \\ d_i^- \geqslant 0, & d_i^+ \geqslant 0, & d_i^- d_i^+ = 0 \end{array} \tag{5}$$

where $a_i$ is the aspiration level for the $i$th objective and $d_i^-$ and $d_i^+$ are negative and positive goal deviations respectively, i.e. non-negative state variables which measure deviations of the current value of the $i$th objective function from the corresponding aspiration level. An optimal solution is one that minimizes the deviations from the aspiration levels. Various measures of multidimensional deviations have been proposed. They are expressed as achievement functions. The simplest achievement function was introduced by Charnes and Cooper (1961) as a sum of weighted deviations (weighted GP)

$$g(\mathbf{d}^-, \mathbf{d}^+) = \sum_{i=1}^{k} (w_i^- d_i^- + w_i^+ d_i^+) \tag{6}$$

where $w_i^-$ and $w_i^+$ are weights corresponding to several goal deviations. The weights represent additional information reflecting the relative importance of the various goal deviations to the DM. Therefore they must be considered as additional parameters (data) of the GP model specified by the DM. It is seldom explicitly pointed out, but following GP philosophy, it is understood that all the weights are non-negative.

The achievement function (6) can be recognized mathematically as the weighted $l_1$-norm. Use of other $l_p$-norms to measure multidimensional distances yields other reasonable achievement functions defined as

$$g(\mathbf{d}^-, \mathbf{d}^+) = \left( \sum_{i=1}^{k} (w_i^- d_i^- + w_i^+ d_i^+)^p \right)^{1/p} \tag{7}$$

In particular, for $p = 2$ we get the classical least-squares problem. The $l_2$-norm is rarely used in GP since in the case of LP problems it destroys their linear structure. In fact, Charnes and Cooper (1961) proposed the weighted linear GP model as an approximation to the least-squares problem.

For $p = \infty$ the achievement function (7) takes the form of the weighted Chebychev norm

$$g(\mathbf{d}^-, \mathbf{d}^+) = \max_{1 \leqslant i \leqslant k} (w_i^- d_i^- + w_i^+ d_i^+) \tag{8}$$

The corresponding GP model is referred to as a fuzzy GP model owing to its reflection of the fuzzy approach to mathematical programming (Ignizio, 1982). The fuzzy GP model can be implemented via LP techniques and allows the linear structure of the original multiobjective problem to be maintained. Nevertheless, this model is not frequently used.

Widely used is lexicographic (or pre-emptive priority) GP, where some hierarchy of goals is presumed (Ignizio, 1982). A vector of a few achievement functions is constructed,

$$\mathbf{g}(\mathbf{d}^-, \mathbf{d}^+) = [g_1(\mathbf{d}^-, \mathbf{d}^+), g_2(\mathbf{d}^-, \mathbf{d}^+), \ldots, g_m(\mathbf{d}^-, \mathbf{d}^+)] \tag{9}$$

where $g_j(\mathbf{d}^-, \mathbf{d}^+)$ are achievement functions of type (6), (7) or (8), and minimized according to lexicographic order. This means that the first achievement function is minimized, then within the set of optimal solutions to the first function the second function is minimized, and so on until all the specified functions are minimized.

The aspiration levels and weights are considered part of the data for the GP model and have to be specified by the DM. However, they can be changed during the analysis depending on the DM's evolving perception of the decision problem if a GP model is used as a basis of some interactive decision support system.

The concept of aspiration levels is clear and intuitively appealing. However, the requirement of having some weights set by the DM is frequently criticized. Convincing proposals have been presented by Dyer and Forman (1991) to use GP coupled with the analytic hierarchy process (Saaty, 1980). The latter can be used to derive the weights by making pairwise judgments about the relative importance of criteria. The lexicographic GP model simplifies the problem of weight definition, since the DM needs only to specify weights within the group of goals considered at the same priority level. Albeit, just in this case, usage of weights as control parameters raises theoretical concerns. Namely, lexicographic optimization is essentially unstable (Klepikova, 1985). This means that some arbitrarily small perturbations of the problem coefficients can dramatically change the optimal set as well as the optimal achievement vector. Fortunately, under reasonable assumptions the lexicographic GP is stable with respect to changes in the aspiration levels (Ogryczak, 1988), but it is not stable with respect to changes in the weights.

The most serious weakness of the GP approach to multiobjective optimization is non-compliance with the efficiency principle. Simply, the GP approach does not attempt to use additional information to find an efficient solution. In effect the solution to the GP model is often non-efficient. By specifying an attainable set of aspiration levels, we receive exactly what we ask for even if we could get something better. However, unfortunately, typical GP models using achievement functions (6), (7), (8) or (9) frequently generate non-efficient solutions even when a non-attainable set of aspiration levels is specified. This is shown by the following example.

**Example 1**

Consider a linear problem with two objectives:

$$\begin{aligned} &\text{minimize } (x_1, x_2) \\ &\text{subject to } x_1 + x_2 \geqslant 3 \\ &\qquad x_1 \geqslant 1, \qquad x_2 \geqslant 1 \end{aligned}$$

The efficient set for this problem is

$$x_1 + x_2 = 3, \qquad x_1 \geqslant 1, \qquad x_2 \geqslant 1$$

i.e. the entire line segment between vertices (1,2) and (2,1), including both vertices.

Let us transform the problem into a GP one and specify a non-attainable vector of aspiration levels $a_1 = 0$ and $a_2 = 3$:

$$\begin{aligned} &x_1 + d_1^- - d_1^+ = 0 \\ &x_2 + d_2^- - d_2^+ = 3 \\ &x_1 + x_2 \geqslant 3 \\ &x_1 \geqslant 1, \qquad x_2 \geqslant 1, \qquad d_1^- \geqslant 0, \qquad d_1^+ \geqslant 0 \\ &d_2^- \geqslant 0, \qquad d_2^+ \geqslant 0, \qquad d_1^- d_1^+ = 0, \qquad d_2^- d_2^+ = 0 \end{aligned}$$

It can be verified that the point $\mathbf{x}=(1,3)$ is an optimal solution to the GP problem with any of the achievement functions (6), (7), (8) or (9) provided that only non-negative weights are used. Point (1,3) is a feasible solution but not an efficient solution. □

## 3. GP MODEL OF THE REFERENCE POINT APPROACH

Thus the GP approach has very important advantages but does not satisfy the efficiency principle. The quasi-satisficing approach attempts to use the advantages of GP without its weaknesses. This was formalized and developed mainly by Wierzbicki (1982) as the reference point method.

The reference point method is an interactive technique. The DM specifies requirements, as in GP, in terms of aspiration levels. Depending on the specified aspiration levels, a special scalarizing achievement function is built, which, when minimized generates an efficient solution to the problem. The computed efficient solution is presented to the DM as the current solution, allowing comparison with previous solutions and modifications of the aspiration levels if necessary.

The scalarizing achievement function not only guarantees efficiency of the solution but also reflects the DM's expectation as specified via the aspiration levels. In building the function, the following assumption regarding the DM's expectations is made.

**Assumption 1**

The DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not satisfy one or more of the aspiration levels. □

One of the simplest scalarizing functions takes the form (cf. Steuer, 1986)

$$s(\mathbf{q},\mathbf{a},\boldsymbol{\lambda}) = \max_{1\leqslant i\leqslant k} \{\lambda_i(q_i-a_i)\} + \epsilon \sum_{i=1}^{k} \lambda_i(q_i-a_i) \tag{10}$$

where $\mathbf{a}$ denotes the vector of aspiration levels, $\boldsymbol{\lambda}$ is a scaling vector, $\lambda_i > 0$, and $\epsilon$ is an arbitrarily small positive number. Minimization of the scalarizing achievement function (10) over the feasible set (2), (3) generates an efficient solution. The selection of the solution within the efficient set depends on two vector parameters: an aspiration vector $\mathbf{a}$ and a scaling vector $\boldsymbol{\lambda}$. In practice the former is usually designated as a control tool for use by the DM whereas the latter is automatically calculated on the basis of some prior analysis (cf. Grauer *et al.*, 1984). The small scalar $\epsilon$ is introduced only to guarantee efficiency in the case of a non-unique optimal solution.

The reference point method, although using the same main control parameters (aspiration levels), always generates an efficient solution to the multiobjective problem whereas GP does not. Therefore it is of interest to find a reason for this advantage and determine if it really does not apply in GP models. In this section we will show how the reference point method can be modelled via the GP methodology.

Let us analyse the formula (10) defining the scalarizing achievement function. The scalarizing function is defined there as a sum of the weighted Chebychev norm of the differences between individual achievements $q_i$ and the corresponding aspiration levels $a_i$ and a small regularization term (the sum of the differences). Usage of the Chebychev norm is important in generating efficient solutions for non-convex problems (e.g. discrete ones) and it must always be accompanied by some regularization term.

Let us concentrate on the main term. The Chebychev norm is available in GP modelling via fuzzy goal programming. The differences $q_i - a_i$ can be easily expressed in terms of goal deviations $d_i^-$ and $d_i^+$ defined according to equations (5). Thus nothing prevents modelling the main term of the scalarizing achievement function via the GP methodology. In fact we can form an equivalent GP achievement function

$$g_1(\mathbf{d}^-, \mathbf{d}^+) = \max_{1 \leqslant i \leqslant k} (-w_i^- d_i^- + w_i^+ d_i^+) \tag{11}$$

where weights $w_i^-$ and $w_i^+$ associated with several goal deviations replace the scaling factors used in the scalarizing achievement function, e.g. for an exact model of the function (10) one needs to put $w_i^- = w_i^+ = \lambda_i$. However, there is one specificity in the GP achievement function (11); namely, there is a negative weight coefficient $-w_i^-$ associated with the negative deviation $d_i^-$. This is the reason why the reference point method attempts to reach an efficient solution even if the aspiration levels are attainable. This small change in the coefficient represents, however, a crucial change in the GP philosophy, where all weights are assumed to be non-negative. Provided that we accept negative weight coefficients, we can consider the function (11) as a specific case of GP achievement functions.

Adding a regularization term to the function (11) can destroy its GP form. However, under lexicographic optimization we can avoid the problem of choosing an arbitrarily small positive parameter $\epsilon$ (cf. (10)) and introduce the regularization term using an additional priority level:

$$g_2(\mathbf{d}^-, \mathbf{d}^+) = \sum_{i=1}^{k} (-w_i^- d_i^- + w_i^+ d_i^+) \tag{12}$$

Finally, we can form the following lexicographic GP problem.

**Problem 1**

$$\begin{aligned} &\text{lexmin } \mathbf{g}(\mathbf{d}^-, \mathbf{d}^+) = [g_1(\mathbf{d}^-, \mathbf{d}^+), g_2(\mathbf{d}^-, \mathbf{d}^+)] \\ &\text{subject to } F_i(\mathbf{x}) + d_i^- - d_i^+ = a_i \quad \text{for } i = 1, \ldots, k \\ &\qquad d_i^- \geqslant 0, \qquad d_i^+ \geqslant 0, \qquad d_i^- d_i^+ = 0 \\ &\qquad \mathbf{x} \in Q \end{aligned}$$

□

**Example 2**
Consider again the linear problem from Example 1. By specification of aspiration levels $a_1 = 0$ and $a_2 = 3$ the problem was transformed into the GP problem

$$\begin{aligned} &x_1 + d_1^- - d_1^+ = 0 \\ &x_2 + d_2^- - d_2^+ = 3 \\ &x_1 + x_2 \geqslant 3 \\ &x_1 \geqslant 1, \qquad x_2 \geqslant 1, \qquad d_1^- \geqslant 0, \qquad d_1^+ \geqslant 0 \\ &d_2^- \geqslant 0, \qquad d_2^+ \geqslant 0, \qquad d_1^- d_1^+ = 0, \qquad d_2^- d_2^+ = 0 \end{aligned}$$

It can be verified that for any positive weights $w_1^-$ $w_1^+$, $w_2^-$ and $w_2^+$ lexicographic minimization of the achievement functions

$$\begin{aligned} &g_1(\mathbf{d}^-, \mathbf{d}^+) = \max\{(-w_1^- d_1^- + w_1^+ d_1^+), (-w_2^- d_2^- + w_2^+ d_2^+)\} \\ &g_2(\mathbf{d}^-, \mathbf{d}^+) = -w_1^- d_1^- + w_1^+ d_1^+ - w_2^- d_2^- + w_2^+ d_2^+ \end{aligned}$$

generates point $\mathbf{x} = (1,2)$ as a unique optimal solution. The first-level optimization generates the line segment between points (1,2) and $(1,3 + w_1^+/w_2^+)$, including the ends. Next, the second optimization selects point (1,2) as a unique optimal solution. Thus we get an efficient solution. □

We will show that the above lexicographic GP problem, i.e. Problem 1, always generates an efficient solution to the original multiobjective problem (Proposition 1) satisfying simultaneously the rules of the reference point approach, i.e. Assumption 1 (Proposition 2).

**Proposition 1**

For any aspiration levels $a_i$ and any positive weights $w_i^-$ and $w_i^+$, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ is an optimal solution to Problem 1, then $\bar{\mathbf{x}}$ is an efficient solution to the multiobjective problem (1), (2).

*Proof*

Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ be an optimal solution to Problem 1. Suppose that $\bar{\mathbf{x}}$ is not efficient to the problem (1), (2). This means there exists a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant F_i(\bar{\mathbf{x}}) \quad \text{for all } i = 1, 2, \ldots, k \tag{13}$$

and for some index $j$ $(1 \leqslant j \leqslant k)$

$$F_j(\mathbf{x}) < F_j(\bar{\mathbf{x}})$$

or in other words

$$\sum_{i=1}^{k} F_i(\mathbf{x}) < \sum_{i=1}^{k} F_i(\bar{\mathbf{x}}) \tag{14}$$

The deviations $\bar{d}_i^-$ and $\bar{d}_i^+$ satisfy the relations

$$\bar{d}_i^+ = (F_i(\bar{\mathbf{x}}) - a_i)_+$$
$$\bar{d}_i^- = (a_i - F_i(\bar{\mathbf{x}}))_+$$

where $(\cdot)_+$ denotes the non-negative part of a quantity. Let us define similar deviations for the vector $\mathbf{x}$ as

$$d_i^+ = (F_i(\mathbf{x}) - a_i)_+ \quad \text{for } i = 1, 2, \ldots, k$$
$$d_i^- = (a_i - F_i(\mathbf{x}))_+ \quad \text{for } i = 1, 2, \ldots, k$$

$(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^+)$ is a feasible solution to Problem 1 and, owing to (13) and (14), for any positive weights $w_i^-$ and $w_i^+$ the following inequalities are satisfied:

$$-w_i^- d_i^- + w_i^+ d_i^+ \leqslant -w_i^- \bar{d}_i^- + w_i^+ \bar{d}_i^+ \quad \text{for all } i = 1, 2, \ldots, k$$
$$\sum_{i=1}^{k} (-w_i^- d_i^- + w_i^+ d_i^+) < \sum_{i=1}^{k} (-w_i^- \bar{d}_i^- + w_i^+ \bar{d}_i^+)$$

Hence we get

$$g_1(\mathbf{d}^-, \mathbf{d}^+) \leqslant g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+), \qquad g_2(\mathbf{d}^-, \mathbf{d}^+) < g_2(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ for Problem 1. Thus $\bar{\mathbf{x}}$ must be an efficient solution to the original multiobjective problem (1), (2). □

**Proposition 2**

For any aspiration levels $a_i$ and any positive weights $w_i^-$ and $w_i^+$, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ is an optimal solution to Problem 1, then any deviation $\bar{d}_i^+$ is positive only if there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant a_i \quad \text{for all } i = 1, \ldots, k$$

*Proof*

Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ be an optimal solution to Problem 1. Suppose that for some $j$

$$\bar{d}_j^+ > 0, \qquad \text{i.e. } F_j(\bar{\mathbf{x}}) > a_j$$

and there exists a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant a_i \quad \text{for all } i = 1, \ldots, k$$

Let us define deviations for the vector $\mathbf{x}$ as

$$d_i^+ = (F_i(\mathbf{x}) - a_i)_+ = 0 \quad \text{for } i = 1, 2, \ldots, k$$
$$d_i^- = (a_i - F_i(\mathbf{x}))_+ \geqslant 0 \quad \text{for } i = 1, 2, \ldots, k$$

$(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^+)$ is a feasible solution to Problem 1 and for any positive weights $w_i^-$ and $w_i^+$ the following inequality is satisfied:

$$\max_{1 \leqslant i \leqslant k} (-w_i^- d_i^- + w_i^+ d_i^+) \leqslant 0 < w_j^+ \bar{d}_j^+ \leqslant \max_{1 \leqslant i \leqslant k} (-w_i^- \bar{d}_i^- + w_i^+ \bar{d}_i^+)$$

Hence

$$g_1(\mathbf{d}^-, \mathbf{d}^+) < g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^+)$ for Problem 1. Thus there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant a_i \quad \text{for all } i = 1, \ldots, k$$

and thereby Assumption 1 is satisfied. □

## 4. GP MODEL OF THE ARBDS APPROACH

The reference point method has been extended to allow additional information from the DM, not only through aspiration levels but also through reservation levels, so that the DM can specify desired as well as required values for given objectives. This has led to efficient implementations of the so-called aspiration/reservation-based decision support (ARBDS) approach with many successful applications (Lewandowski and Wierzbicki, 1989).

The scalarizing achievement function for ARBDS works similarly as in the reference point method, but it has to reflect the DM's expectation via two groups of control parameters: aspiration and reservation levels. Namely, while building the function, the following assumptions regarding the DM's expectations are made (extending Assumption 1 from the reference point method).

**Assumption 2**
The DM prefers outcomes that satisfy all the reservation levels to any outcome that does not satisfy one or more of the reservation levels. □

**Assumption 3**
Provided that all the reservation levels are satisfied, the DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not satisfy one or more of the aspiration levels. □

One of the simplest scalarizing functions for ARBDS takes the form

$$s(\mathbf{q}, \mathbf{a}, \mathbf{r}) = \max_{1 \leqslant i \leqslant k} u_1(q_i, a_i, r_i) + (\epsilon/k) \sum_{i=1}^{k} u_i(q_i, a_i, r_i) \tag{15}$$

where **a** and **r** denote vectors of aspiration and reservation levels respectively, $\epsilon$ is an arbitrarily small positive number and $u_i$ is a function which measures the deviation of results from the DM's expectations with respect to the $i$th objective, depending on the given aspiration level $a_i$ and reservation level $r_i$.

The function $u_i(q_i, a_i, r_i)$ is a strictly monotone function of $q_i$ with value $u_i = 0$ if $q_i = a_i$ and $u_i = 1$ if $q_i = r_i$. This function can be interpreted as a measure of the DM's dissatisfaction with the current value of the $i$th objective function. It can be defined, for instance, as a piecewise linear function (Lewandowski and Wierzbicki, 1988)

$$u_i(q_i, a_i, r_i) = \begin{cases} -\beta(q_i - a_i)/(q_i^{\mathrm{b}} - a_i) & \text{if } q_i \leqslant a_i \\ (q_i - a_i)/(r_i - a_i) & \text{if } a_i < q_i < r_i \\ \gamma(q_i - r_i)/(q_i^{\mathrm{w}} - r_i) + 1 & \text{if } q_i \geqslant r_i \end{cases} \tag{16}$$

where $q_i^{\mathrm{b}}$ and $q_i^{\mathrm{w}}$ denote the best and worst possible values of the $i$th objective respectively, which are assumed to be known from the prior analysis, and $\beta$ and $\gamma$ are arbitrarily defined positive parameters. $\beta$ represents additional DM's satisfaction caused by achievement better than the corresponding aspiration level, whereas $\gamma > 1$ represents dissatisfaction connected with achievement worse than the reservation level.

In a successful implementation of the ARBDS system for the multiobjective transshipment problem with facility location (Ogryczak *et al.*, 1989) an even simpler type of function $u_i$ has been used:

$$u_i(q_i, a_i, r_i) = \begin{cases} -\beta(q_i - a_i)/(r_i - a_i) & \text{if } q_i \leqslant a_i \\ (q_i - a_i)/(r_i - a_i) & \text{if } a_i < q_i < r_i \\ \gamma(q_i - r_i)/(r_i - a_i) + 1 & \text{if } q_i \geqslant r_i \end{cases} \tag{17}$$

This is also a piecewise linear function but does not require any estimation of the best and worst values. Under the reasonable assumption that the parameters $\beta$ and $\gamma$ satisfy inequalities $0 < \beta < 1$ and $\gamma > 1$, the achievement functions (17) are convex and thus can be modelled via LP methodology. Consequently, the entire scalarizing achievement function (15) can be modelled with LP methodology.

As shown in the previous section, the reference point method can be implemented with the GP techniques. In this section we will show how the ARBDS approach can be modelled via the GP methodology. The main difference between these two approaches is in the usage of the second vector of control parameters (reservation levels) in the ARBDS approach. The reservation levels can be introduced into the GP model, however. The simplest way is to build two goals for each objective function: one connected with deviations from the aspiration level and the second connected with deviations from the reservation level. However, we can avoid this increase in the problem size using a modelling technique similar to interval GP (cf. Ignizio, 1982; Ogryczak, 1988). We simply transform the objective functions into the goals

$$F_i(\mathbf{x}) + d_i^- - d_i^a - d_i^r = a_i \quad \text{for } i = 1, \ldots, k \tag{18}$$
$$d_i^- \geqslant 0, \qquad 0 \leqslant d_i^a \leqslant r_i - a_i, \qquad d_i^r \geqslant 0, \qquad d_i^- d_i^a = 0, \qquad (r_i - a_i - d_i^a) d_i^r = 0$$

where $a_i$ and $r_i$ denote aspiration and reservation levels for the $i$th objective respectively and $d_i^-$, and $d_i^a$ and $d_i^r$ are non-negative state variables which measure deviations of the current value of the $i$th objective function from the corresponding aspiration and reservation levels: $d_i^-$ is the negative deviation from the aspiration level, $d_i^a$ is the positive deviation from the aspiration level within the interval between the aspiration and reservation level and $d_i^r$ is the positive deviation from the reservation level. The goals (18) differ from the typical ones (5) only through the splitting of the positive deviation $d_i^+$ into a sum of two deviations $d_i^a$ and $d_i^r$, where the first one is limited to the interval between the aspiration and reservation levels and the second can be positive only if $d_i^a = r_i - a_i$.

The most important advantage of the ARBDS approach, as for the reference point method, is in its generation of efficient solutions. The basis for this advantage is concealed in the formulae for the scalarizing achievement functions (15) and (16) or (17). Using three types of deviations defined in (18), one can write both formulae (16) and (17) as

$$u_i(q_i, a_i, r_i) = \begin{cases} -\beta w_i^- d_i^- & \text{if } q_i \leqslant a_i \\ w_i^a d_i^a & \text{if } a_i < q_i < r_i \\ \gamma w_i^r d_i^r + 1 & \text{if } q_i \geqslant r_i \end{cases} \tag{19}$$

where $w_i^-$, $w_i^a$ and $w_i^r$ are positive weights defined depending on the corresponding aspiration and reservation levels, while $\beta$ and $\gamma$ are arbitrarily defined positive parameters. Thus, like the standard GP techniques, the ARBDS approach deals with deviations accompanied by weights, but these weights are now calculated automatically. Provided that $w_i^a = 1/(r_i - a_i)$, as in formulae (16) and (17), the function (19) can be written as

$$u_i(d_i^-, d_i^a, d_i^r) = -\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r \tag{20}$$

which is a weighted sum of the deviations. However, as in the reference point method, there is one specificity in this formula. There is a negative weight coefficient $-\beta w_i^-$ associated with the negative deviation $d_i^-$. As previously, this small change in coefficient represents a crucial change in the GP philosophy.

Now let us analyse formula (15) defining the final scalarizing achievement function. The scalarizing function is built there, as in (10), as a sum of the Chebychev norm of the individual achievements $u_i$ and a small regularization term (the sum of the achievements). Using fuzzy goal programming, we can express the main term as the function

$$g_1(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) = \max_{1 \leqslant i \leqslant k} \{-\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r\} \tag{21}$$

Similarly, as in the previous section, we can use lexicographic optimization here to avoid the problem of choosing an arbitrarily small positive parameter $\epsilon$ (cf. (15)) and introduce the regularization term as an additional priority level:

$$g_2(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) = \sum_{i=1}^{k} (-\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r) \tag{22}$$

Finally, we can form the following lexicographic GP problem.

**Problem 2**

lexmin $\mathbf{g}(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) = [g_1(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r),\ g_2(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r)]$

subject to $F_i(\mathbf{x}) + d_i^- - d_i^a - d_i^r = a_i$ for $i = 1, \ldots, k$

$d_i^- \geqslant 0, \quad 0 \leqslant d_i^a \leqslant r_i - a_i, \quad d_i^r \geqslant 0, \quad d_i^- d_i^a = 0, \quad (r_i - a_i - d_i^a) d_i^r = 0$

$\mathbf{x} \in Q$

where $w_i^-$, $w_i^a$ and $w_i^r$ are positive weights depending on the corresponding aspiration and reservation levels (e.g. to satisfy formula (17), one can put $w_i^- = w_i^a = w_i^r = 1/(r_i - a_i)$), while $\beta$ and $\gamma$ are arbitrarily defined positive parameters. □

We will show that the above lexicographic GP problem, i.e. Problem 2, always generates an efficient solution to the original multiobjective problem (Proposition 3) satisfying simultaneously the rules of the ARBDS approach, i.e. Assumptions 2 (Proposition 4) and 3 (Proposition 5).

**Proposition 3**

For any aspiration and reservation levels $a_i < r_i$, any positive parameters $\beta$ and $\gamma$, any positive weights $w_i^-$ and $w_i^r$, and $w_i^a = 1/(r_i - a_i)$, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ is an optimal solution for Problem 2, then $\bar{\mathbf{x}}$ is an efficient solution to the multiobjective problem (1), (2).

*Proof*

Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ be an optimal solution to Problem 2. Suppose that $\bar{\mathbf{x}}$ is not efficient to the problem (1), (2). This means there exists a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant F_i(\bar{\mathbf{x}}) \quad \text{for all } i = 1, 2, \ldots, k \tag{23}$$

and for some index $j (1 \leqslant j \leqslant k)$

$$F_j(\mathbf{x}) < F_j(\bar{\mathbf{x}})$$

or in other words

$$\sum_{i=1}^{k} F_i(\mathbf{x}) < \sum_{i=1}^{k} F_i(\bar{\mathbf{x}}) \tag{24}$$

The deviations $\bar{d}_i^-$, $\bar{d}_i^a$ and $\bar{d}_i^r$ satisfy the relations

$$\bar{d}_i^r = (F_i(\bar{\mathbf{x}}) - r_i)_+$$
$$\bar{d}_i^a = (F_i(\bar{\mathbf{x}}) - \bar{d}_i^r - a_i)_+$$
$$\bar{d}_i^- = (a_i - F_i(\bar{\mathbf{x}}))_+$$

Let us define similarly deviations for the vector $\mathbf{x}$ as

$$d_i^r = (F_i(\mathbf{x}) - r_i)_+$$
$$d_i^a = (F_i(\mathbf{x}) - d_i^r - a_i)_+$$
$$d_i^- = (a_i - F_i(\mathbf{x}))_+$$

$(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r)$ is a feasible solution to Problem 2 and, owing to (23) and (24), for any positive weights $w_i^-$ and $w_i^r$ the following inequalities are satisfied:

$$-\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r \leqslant -\beta w_i^- \bar{d}_i^- + w_i^a \bar{d}_i^a + \gamma w_i^r \bar{d}_i^r \quad \text{for all } i = 1, 2, \ldots, k$$
$$\sum_{i=1}^{k} (-\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r) < \sum_{i=1}^{k} (-\beta w_i^- \bar{d}_i^- + w_i^a \bar{d}_i^a + \gamma w_i^r \bar{d}_i^r)$$

Hence we get

$$g_1(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) \leqslant g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r) \qquad g_2(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) < g_2(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{d}^r)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ for Problem 2. Thus $\bar{\mathbf{x}}$ must be an efficient solution to the original multiobjective problem (1), (2). □

**Proposition 4**

For any aspiration and reservation levels $a_1 < r_i$, any positive parameters $\beta$ and $\gamma$, any positive weights $w_i^-$ and $w_i^r$, and $w_i^a = 1/(r_i - a_i)$, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ is an optimal solution to Problem 2, then any deviation $\bar{d}_i^r$ is positive only if there does not exist a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant r_i \quad \text{for all } i = 1, \ldots, k$$

*Proof*

Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ be an optimal solution to Problem 2. Suppose that for some $j$

$$\bar{d}_j^r > 0, \qquad \text{i.e. } F_j(\bar{\mathbf{x}}) > r_j$$

and there exists a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant r_i \quad \text{for all } i = 1, \ldots, k$$

We define deviations for the vector $\mathbf{x}$ as

$$d_i^r = (F_i(\mathbf{x}) - r_i)_+ = 0 \qquad \text{for } i = 1, 2, \ldots, k$$
$$d_i^a = (F_i(\mathbf{x}) - d_i^r - a_i)_+ \geqslant 0 \quad \text{for } i = 1, 2, \ldots, k$$
$$d_i^- = (a_i - F_i(\mathbf{x}))_+ \geqslant 0 \qquad \text{for } i = 1, 2, \ldots, k$$

$(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r)$ is a feasible solution to Problem 2 and for any positive weights $w_i^-$ and $w_i^r$ the following inequalities are satisfied:

$$\max_{1 \leqslant i \leqslant k} \{-\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r\} \leqslant \max_{1 \leqslant i \leqslant k} \{-\beta w_i^- d_i^- + w_i^a d_i^a\} \leqslant 1$$

$$\max_{1 \leqslant i \leqslant k} \{-\beta w_i^- \bar{d}_i^- + w_i^a \bar{d}_i^a + \gamma w_i^r \bar{d}_i^r\} \geqslant 1 + \gamma w_j^r \bar{d}_j^r > 1$$

Hence

$$g_1(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) < g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ for Problem 2. Thus there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant r_i \quad \text{for all } i = 1, \ldots, k$$

and thereby Assumption 2 is satisfied. □

**Proposition 5**
For any aspiration and reservation levels $a_i < r_i$, any positive parameters $\beta$ and $\gamma$, any positive weights $w_i^-$ and $w_i^r$, and $w_i^a = 1/(r_i - a_i)$, if $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ with $\bar{\mathbf{d}}^r = \mathbf{0}$ is an optimal solution to Problem 2, then any deviation $\bar{d}_i^a$ is positive only if there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant a_i \quad \text{for all } i = 1, \ldots, k$$

*Proof*
Let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \mathbf{0})$ be an optimal solution to Problem 2. Suppose that for some $j$

$$\bar{d}_j^a > 0, \qquad \text{i.e. } F_j(\bar{\mathbf{x}}) > a_j$$

and there exists a vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant a_i \quad \text{for all } i = 1, \ldots, k$$

We define deviations for the vector $\mathbf{x}$ as

$$\begin{aligned}
d_i^r &= (F_i(\mathbf{x}) - r_i)_+ = 0 && \text{for } i = 1, 2, \ldots, k \\
d_i^a &= (F_i(\mathbf{x}) - d_i^r - a_i)_+ = 0 && \text{for } i = 1, 2, \ldots, k \\
d_i^- &= (a_i - F_i(\mathbf{x}))_+ \geqslant 0 && \text{for } i = 1, 2, \ldots, k
\end{aligned}$$

$(\mathbf{x}, \mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r)$ is a feasible solution to Problem 2 and for any positive weights $w_i^-$ and $w_i^r$ the following inequalities are satisfied:

$$\max_{1 \leqslant i \leqslant k} \{-\beta w_i^- d_i^- + w_i^a d_i^a + \gamma w_i^r d_i^r\} \leqslant \max_{1 \leqslant i \leqslant k} \{-\beta w_i^- d_i^-\} \leqslant 0$$

$$\max_{1 \leqslant i \leqslant k} \{-\beta w_i^- \bar{d}_i^- + w_i^a \bar{d}_i^a + \gamma w_i^r \bar{d}_i^r\} \geqslant w_j^a \bar{d}_j^a > 0$$

Hence

$$g_1(\mathbf{d}^-, \mathbf{d}^a, \mathbf{d}^r) < g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ for Problem 2. Thus there does not exist any vector $\mathbf{x} \in Q$ such that

$$F_i(\mathbf{x}) \leqslant a_i \quad \text{for all } i = 1, \ldots, k$$

and thereby Assumption 3 is satisfied. □

Note that neither proposition assumes any specific relation between weights associated with several deviations. It is not necessary because we directly put into Problem 2 the requirements

$$d_i^- d_i^a = 0, \qquad (r_i - a_i - d_i^a) d_i^r = 0 \tag{25}$$

to guarantee proper calculation of all the deviations. The first requirement $d_i^- d_i^a = 0$ is easy to implement in linear programming. The second one requires special techniques even in the LP case (e.g. special ordered sets). It turns out, however, that the requirements (25) can be simply omitted in the constraints of Problem 2 provided that the weights satisfy some relations natural for the ARBDS philosophy. This is made precise in Proposition 6.

**Proposition 6**

For any aspiration and reservation levels $a_i < r_i$, if the weights satisfy the relations

$$w_i^a = 1/(r_i - a_i), \qquad 0 < \beta w_i^- < w_i^a, \qquad \gamma w_i^r > w_i^a$$

then any $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ optimal solution to Problem 2 with omitted constraints (25) satisfies these requirements, i.e.

$$\bar{d}_i^- \bar{d}_i^a = 0, \qquad (r_i - a_i - \bar{d}_i^a) \bar{d}_i^r = 0$$

*Proof*

Let Problem 2′ denote Problem 2 with omitted constraints (25) and let $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ be an optimal solution for Problem 2′. Suppose that for some $j$

$$\bar{d}_j^- d_j^a > 0$$

Then we can decrease both $\bar{d}_j^-$ and $\bar{d}_j^a$ by the same small positive quantity. This means that for small enough positive $\delta$ the vector $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^- - \delta \mathbf{e}_j, \bar{\mathbf{d}}^a - \delta \mathbf{e}_j, \bar{\mathbf{d}}^r)$ is feasible to Problem 2′. Owing to $0 < \beta w_i^- < w_i^a$, the following inequality is valid:

$$-\beta w_j^- (\bar{d}_j^- - \delta) + w_j^a (\bar{d}_j^a - \delta) + \gamma w_j^r \bar{d}_j^r < -\beta w_j^- \bar{d}_j^- + w_j^a \bar{d}_j^a + \gamma w_j^r \bar{d}_j^r$$

*Hence we get*

$$g_1(\bar{\mathbf{d}}^- - \delta \mathbf{e}_j, \bar{\mathbf{d}}^a - \delta \mathbf{e}_j, \bar{\mathbf{d}}^r) \leqslant g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r), \qquad g_2(\bar{\mathbf{d}}^- - \delta \mathbf{e}_j, \bar{\mathbf{d}}^a - \delta \mathbf{e}_j, \bar{\mathbf{d}}^r) < g_2(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ for Problem 2′. Suppose now that for some $j$

$$(r_i - a_i - \bar{d}_i^a) \bar{d}_i^r > 0$$

Then we can decrease $\bar{d}_j^r$ and simultaneously increase $\bar{d}_j^a$ by the same small positive quantity. This means that for sufficiently small positive $\delta$ the vector $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a + \delta \mathbf{e}_j, \bar{\mathbf{d}}^r - \delta \mathbf{e}_j)$ is feasible to Problem 2′. Owing to $\gamma w_i^r > w_i^a$, the following inequality is valid:

$$-\beta w_j^- \bar{d}_j^- + w_j^a (\bar{d}_j^a + \delta) + \gamma w_j^r (\bar{d}_j^r - \delta) < -\beta w_j^- \bar{d}_j^- + w_j^a \bar{d}_j^a + \gamma w_j^r \bar{d}_j^r$$

Hence we get

$$g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a + \delta \mathbf{e}_j, \bar{\mathbf{d}}^r - \delta \mathbf{e}_j) \leqslant g_1(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r), \qquad g_2(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a + \delta \mathbf{e}_j, \bar{\mathbf{d}}^r - \delta \mathbf{e}_j) < g_2(\bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$$

which contradicts optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ for Problem 2′. Thus $(\bar{\mathbf{x}}, \bar{\mathbf{d}}^-, \bar{\mathbf{d}}^a, \bar{\mathbf{d}}^r)$ satisfies both conditions of (25). □

According to Proposition 6, conditions (25) can be dropped from the formulation of Problem 2 provided that all the individual achievement functions $u_i$ (cf. (15)) keep the rule of relatively high penalty for exceeding the reservation level and relatively small bonus for results better than the corresponding aspiration level. Note that the function $u_i$ defined in formula (17) meets all the assumptions of Proposition 6 under the reasonable assumption that the parameters $\beta$ and $\gamma$ satisfy the inequalities $0<\beta<1$ and $\gamma>1$. In the case of the function $u_i$ defined in formula (16), however, one can encounter some difficulties in finding proper values for the parameters $\beta$ and $\gamma$.

## 5. CONCLUDING REMARKS

Goal programming does not satisfy the efficiency (Pareto-optimality) principle. Simply, the GP approach does not necessarily suggest decisions that optimize the objective functions. It only yields decisions that have outcomes closest to the specified aspiration levels. This weakness of goal programming led to the development of the reference point method, which, though using the same main control parameters as GP, always generates an efficient solution to the multiobjective problem. The reference point method has been extended to permit additional information from the DM and eventually has led to efficient implementations of the ARBDS approach with many successful applications.

In this paper we have shown that the implementation techniques of goal programming can be used to model the reference point method as well as the ARBDS approach. Namely, we have shown that by employing lexicographic and fuzzy GP with properly defined weights, we get a GP achievement function that satisfies all the requirements for the scalarizing achievement function used in the reference point or ARBDS approaches. Usage of negative weights is the reason why the scalarizing achievement function attempts to reach an efficient solution even if the aspiration levels are attainable. This small technical change represents, however, a crucial change in the GP philosophy, where all the weights are assumed to be non-negative. We do not wish to debate whether goal programming with negative weights is still goal programming, but instead we are interested in the practical advantages of the congruence proved in the paper.

From our point of view the most important benefit is the possibility of using efficient GP implementation techniques to model the ARBDS approach. It allows us to simplify and demystify implementations of the ARBDS approach and thereby extend applications of this powerful method. Moreover, it provides an opportunity to build unique decision support systems providing the DM with both GP and ARBDS approaches.

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