# Asset/Liability Management for Pension Funds Using CVaR Constraints

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This article develops a formal framework for asset/liability management (ALM) for a pension fund. There is an extensive literature on modeling and optimization of portfolio allocation strategies for asset/liability management problems; see, for instance, Ziemba and Mulvey [1998].

A formal study of ALM problems usually focuses on modeling plausible sample paths for liabilities of the fund and returns of instruments in the portfolio. These scenarios are then used to test the performance of various decision rules, usually using an ad hoc approach. A typical decision rule is the so-called constant proportion rule, that rebalances the portfolio at each decision moment in order to maintain a constant allocation of resources across different asset classes. Cairns, Blake, and Dowd [2000] use simulation to compare the choices available to a pension plan member at the time of retirement.

Traditionally, in a multiperiod setting, stochastic differential equations are used to solve asset allocation problems; see, for instance, Merton [1971, 1993], Milevsky [1998], and Cairns, Blake, and Dowd [2000]. These approaches, which lead to analytical solutions, usually study relatively simple strategies (e.g., constant proportion) under restrictive assumptions, such as that the dynamics obey geometric Browning motion.

Much more advanced strategies use stochastic programming approaches. Examples include Carino et al. [1994], Dert [1995, 1998], Boender, Van Aalst, and Heemskerk, [1998], and Kouwenberg [2001]. Implementation of such models has expanded greatly; see Ziemba and Mulvey [1998]. Censor and Zenios [1997] and Zenios [1999] survey large-scale asset/liability applications to portfolio management.

In a stochastic programming approach, one either transforms the set of sample paths into a scenario tree, or generates a scenario tree directly using appropriately chosen (conditional) distributions for each exogenous random variable in the problem. The problem is then formulated as a stochastic programming problem. A stochastic programming approach can find a strategy that is superior to a strategy found using the ad hoc approach, since the latter allows for far fewer possible decisions at each decision moment.

Although stochastic programming techniques have been used successfully in several ALM applications, they are not widely used in financial practice. There are several reasons why. First, sophisticated statistical techniques need to be applied to transform a set of simulated sample paths into a scenario tree. Second, a scenario tree exhibits exponential growth of the number of nodes as the number of decision moments increases, which can produce a very large decision problem. Finally, when the extent of the scenario tree is limited, the resulting tree may not be stochastic enough to describe a realistic and diverse range of uncertain behavior of the system.

We consider an intermediate setting between the ad hoc approach and stochastic programming. This setting was first proposed by Hibiki [1999, 2000] for portfolio optimization problems. Here we extend this approach to ALM problems and combine it with a risk management technique using conditional value at risk (CVaR) constraints (see Rockafellar and Uryasev [2000, 2001]). Optimization is performed using a set of sample paths, thereby eliminating the need to construct a scenario tree.

To allow for a broader decision space than that traditionally used, we allow different decisions to be made for different *bundles* of sample paths, where sample paths are bundled together according to some criteria. Then the same optimal decisions are made for bundles (or groups) of sample paths that exhibit similar performance characteristics.

Sample paths are bundled to avoid solutions that can be anticipated (as in the scenario tree approach), and at the same time to dramatically limit the number of decision variables.[1] The dimension of the problem thus increases linearly with the number of bundles and the number of time periods. This is an improvement over exponential growth of problem size under the stochastic programming approach.

The proposed approach can be viewed as a simplification of stochastic programming that imposes the same decisions for many scenarios. This simplification makes the models easier to solve, and in a way is more intuitive.

The idea of using a decision variable independently of scenarios is not new in financial modeling. Financial models based on string (linear) scenario trees instead of event trees have been suggested by Hiller and Eckstein [1993] and Zenios [1993]. Our advance is that we consider decisions on bundles of sample paths.

We show that, with such a structure and suitable risk measures, it is possible to formulate and solve the problem using linear programming techniques. We use the fact that incorporating CVaR constraints does not destroy the linear structure of a model (see Rockafellar and Uryasev [2000, 2001] ). This quality is an important advantage of CVaR as a risk measure in optimization settings, but there are several other reasons to use CVaR as a risk measure.

CVaR is a subadditive measure of risk (see Pflug [2000] and Rockafellar and Uryasev [2000, 2001]) That is, diversification of a portfolio reduces CVaR. Moreover, CVaR is a *coherent* measure of risk in the sense of Artzner et al. [1999]. The coherency of CVaR was first proved by Pflug [2000]; see also Acerbi, Nordio, and Sirtori [2001], Acerbi and Tasche [2001], and Rockafellar and Uryasev [2001].

Our numerical experiments are conducted using a set of sample paths for liabilities and asset returns for a pension fund in The Netherlands. This set of sample paths was generated at ORTEC Consultants, B.V., using its simulation-based decision support system for asset/liability management for pension funds.

The research is designed as a feasibility study to test new optimization techniques with CVaR risk constraints in a dynamic setting. Practical recommendations on investment decisions are beyond its scope.

We have shown that the technique is stable and robust, allowing for the solution of large-scale problems with a long investment horizon. It can be implemented relatively easily in a realistic investment environment.

## THE PENSION FUND PROBLEM

### Basic Framework

We consider a pension fund that conducts activities as follows: 1) collection of premiums from the sponsor and/or the active employees; 2) investment of available funds; and 3) payment of pensions to retired employees. The fund uses an asset management strategy (a set of investment rules) so that, at each decision moment, the total value of all assets exceeds the liabilities of the fund (which is actually a measure of the fund's future stream of liabilities) with high certainty, and at the same time tries to minimize the contribution rate by the sponsor and active employees of the fund. The problem consists of setting, at each decision moment, a suitable contribution rate and a suitable investment strategy for the funds available to the pension fund.[2]

We denote the time horizon by $T$, and denote the set of decision moments by $t = 0, \ldots, T$. At each time $t$ a decision is made on the value of contributions to the fund and on portfolio allocations, both based on the state of the pension fund at that particular time. We simplify the notation for the moment by suppressing randomness:

$A_t$ = value of all assets owned by the fund at time $t$ (random variable);
$W_t$ = wages earned by active members at time $t$ (random variable);
$y_t$ = contribution rate, i.e., premium paid by the sponsor and/or active employee as a fraction of (a suitable part of) their wages at time $t$ (decision variable);[3]
$\ell_t$ = payments made by the fund to retirees at time $t$ (random variable);

$x_{n,t}$ = money invested in asset $n$ at time $t$ (decision variable);
$r_{n,t}$ = return on investment in asset $n$ at period $t$ (random variable); and
$L_t$ = liabilities (i.e., a measure of the stream of future liabilities) of the fund at time $t$ (random variable).

In addition, let $h(y_1, \dots, y_T)$ denote a measure of the costs of the pension fund. This could be the average of contribution rates, or the present value of all contributions $y_t W_t$. Furthermore, we assume that $h(y_1, \dots, y_T)$ is linear in $y_t$ and non-decreasing in $y_t$. At each decision moment, the balance equation holds:

$$\sum_{n=0}^{N} x_{n,t} = A_t + W_t y_t - l_t \qquad t = 0, \dots, T-1 \tag{1}$$

which equates the sum of all investments, $\Sigma_{n=0}^{N} x_{n,t}$, to assets, $A_t$, plus contributions, $W_t y_t$, minus liabilities, $\ell_t$. The sum $\Sigma_{n=0}^{N} x_{n,t-1}$ invested at time $t-1$ results in the value of all assets at time $t$:

$$A_t = \sum_{n=0}^{N} x_{n,t-1}(1 + r_{n,t}) \cdot$$

At each time period $t = 1, \dots, T$, we want with high certainty to satisfy the liability constraints:

$$A_t = \sum_{n=0}^{N} x_{n,t-1}(1 + r_{n,t}) \geq L_t \text{ with high certainty.} \tag{2}$$

We consider the problem of minimizing costs of the fund:

$$\textbf{minimize } h(y_1, \dots, y_T) \tag{3}$$

subject to 1) *balance* and 2) *liability* constraints.

The ratio of assets to liabilities, $A_t/L_t$, is usually referred to as the *funding ratio* of the pension fund. A target funding ratio of $\psi$ can easily be incorporated by replacing constraint (2) by

$$A_t = \sum_{n=0}^{N} x_{n,t-1}(1 + r_{n,t}) \geq \psi L_t \text{ with high certainty.} \tag{4}$$

Values of $\psi > 1$ often are used to add some extra safety margin to the constraint. For example, a value of $\psi = 1.2$ would give an extra margin of 20% of the value of liabilities. In principle, one could imagine that the constraint (y) should be satisfied with probability 1.0, but it will often be impossible, or at the very least be expensive, to ensure that the liability constraints are met for all possible future outcomes. Therefore, this constraint is relaxed, and we would like to find a solution with a sufficiently high probability of meeting the liability constraints while keeping costs at a reasonable level.

### Conditional Value at Risk

When Equation (4) is violated, we say that we have a *loss*, or that the pension fund is *underfunded*. As a measure of this loss, we use the difference between the right-hand side and left-hand side in (4):[4]

$$f_\psi(x; r, L) = \psi L - \sum_{n=0}^{N} (1 + r_n) x_n \tag{5}$$

Hence, (4) could be replaced with:

$$f_\psi(x; r, L) \leq 0 \text{ with high certainty.} \tag{6}$$

Let $P$ be the joint probability measure of the vector $(r, L)$, and denote by $\Phi_\psi(\zeta; x)$ the cumulative probability distribution of the loss, given $x$:

$$\begin{aligned} \Phi_\psi(x, \zeta) &= \Pr(f_\psi(x; r, L) \leq \zeta) \\ &= \int_{f_\psi(x;r,L) \leq \zeta} P(dr, dL) \end{aligned}$$

which by definition is the probability that the loss $f_\psi(x; r, L)$ does not exceed a threshold value $\zeta$. Now, if $\alpha$ is a confidence level that (6) is not violated, the inequality in (6) can be expressed as follows:

$$\zeta_{\alpha,\psi}(x) \leq 0 \tag{7}$$

where

$$\zeta_{\alpha,\psi}(x) = \min\{\zeta \in \mathbb{R} : \Phi_\psi(x, \zeta) \geq \alpha\}.$$

The value $\zeta_{\alpha,\psi}(x)$ is called the $\alpha$-value at risk ($\alpha$-VaR) and constraint (7) means that the loss in at least

100α% of outcomes must be below or equal to 0 (note that, in general, this threshold level may be chosen to be different from 0). VaR is a widely used risk measure, but it has several notable drawbacks, including:

1. It does not take into account losses exceeding VaR.
2. It may provide an inconsistent picture for various confidence levels.[5]
3. It is not subadditive, i.e., diversification of the portfolio may increase the risk.
4. It is non-convex, which results in computationally difficult, for risk management.

These disadvantages are not shared by the closely related *conditional value at risk* (CVaR), which is the weighted average of VaR and the losses exceeding VaR. Denote the conditional expectation of all losses strictly exceeding VaR by $\phi_{\alpha,\psi}(x)^+$ (where it is supposed that there are losses strictly exceeding VaR). CVaR is then defined as follows:

$$\phi_{\alpha,\psi}(x) = \lambda\,\zeta_{\alpha,\psi}(x) + (1-\lambda)\,\phi_{\alpha,\psi}(x)^+ \,.$$

That is, it is a weighted average of VaR and the conditional expectation of losses strictly exceeding VaR, where the weight equals

$$\lambda = (1-\alpha)^{-1}(\Phi_\psi(x, \zeta_{\alpha,\psi}(x)) - \alpha) \in [0,1]\,.$$

Note that, when the distribution of the losses has continuous density, $\lambda = 0$, and we have $\phi_{\alpha,\psi}(x)^+ = \phi_{\alpha,\psi}(x)$. This is in general not the case when the distribution is discrete (or is approximated by a discrete distribution using sampling of scenarios). CVaR is convex, which makes it possible to construct efficient algorithms for controlling CVaR.

It is easy to see that CVaR always exceeds or equals VaR (i.e., CVaR ≥ VaR). Therefore, we could replace (7) by the CVaR constraint

$$\phi_{\alpha,\psi}(x) \le w \tag{8}$$

for some $w$. With $w = 0$, we have a risk constraint that dominates (i.e., is stronger than) the α-VaR constraint (7). Using a negative $w$ would tighten the constraint further, while a positive $w$ would relax it.

For $i = 1, \ldots, I$, let us denote by $r^i$, $L^i$, a sample of realizations of $(r, L)$ from the probability distribution function $P$. As is shown by Rockafellar and Uryasev [2000, 2001], constraint (8) can be replaced by the system of linear constraints

$$\zeta + \frac{1}{I(1-\alpha)} \sum_{i=1}^{I} z^i \;\le\; w \tag{9}$$

$$\sum_{n=0}^{N} (1 + r_n^i) x_n - \zeta \;\le\; z^i \tag{10}$$

$$z^i \;\ge\; 0 \quad \text{for } i = 1, \ldots, I$$

where $z^i$, $i = 1, \ldots, I$, are dummy variables. If constraint (9) is active at an optimal solution, the corresponding optimal value of $\zeta$ if it is unique will be equal to VaR. If there are many optimal values of $\zeta$, then VaR is the left endpoint of the optimal interval. The left-hand side of inequality (9) will be equal to CVaR.

### Objective Function and Optimization Problem

The asset/liability model is defined by the balance constraints (1) and the risk constraints (9) and (10) that are imposed at each time $t$. The risk constraints are imposed for groups of sample paths.

The objective function of the model is defined as the expected present value of the contributions to the pension fund, i.e., we want to minimize the total cost of funding the pension fund, subject to balance and safety constraints. The problem is solved using formal optimization algorithms.

We develop a more elaborate version of this model as well, where we define approaches for modeling of uncertainties and the structure of solution rules. A formal description of the problem formulation is included in the appendix.

## MODELING OF UNDERLYING STOCHASTIC VARIABLES AND DECISION RULES

The text describes in a non-quantitative way our approach for modeling uncertainties and decision rules. A formal description of the mathematical model is in the appendix.

There are several ways to model dynamics and uncertainty. The most popular approach is to use simulation and to generate many sample paths (scenarios) that represent possible future states of the system. The financial industry mostly uses the simulation approach, where decision rules are specified for choosing the contribution rate and investment strategy. Then, the parameters of the decision rules are adjusted, using trial and error, to provide a satisfactory solution to the problem.

The most typical decision rule is the "fixed-mix strategy." This rule leads to non-convex optimization problems, so only a few researchers apply optimization techniques to this problem; see, e.g., Mulvey et al. [1995].

Alternatively, the set of sample paths can be converted into a scenario tree. Such a conversion lets us use the well-developed stochastic optimization theory to make optimal decisions (for a description of the basic stochastic programming technology, see Ermoliev and Wets [1988], Prekopa [1995], and Birge and Louveaux [1997]). For multistage models, conversion of the set of sample paths to a scenario tree can lead to significant methodological and computational difficulties. The scenario tree increases very rapidly in size for multistage problems, easily exceeding the capacities of even the most advanced computational resources. This forces multistage models to limit the number of scenarios, which in turn severely limits the possibility of reflecting the rich randomness of the actual dynamic stochastic processes.

Neither of these solutions is very satisfying. We use an alternative approach laid out by Hibiki [1999, 2000] for portfolio allocation problems. This setup uses simulation sample paths, yet leads to linear models and a rich decision space, unlike other sample path-based approaches that lead to non-convex multiextremal problems (like the constant proportion rule). We combine this approach with CVaR risk management techniques.

We consider a $T$-period model where time ranges from $t = 0, \ldots, T$, and decisions are taken at times $t = 0, \ldots, T-1$. Randomness in the model is expressed by $I$ sample paths spanning the entire horizon from $t = 0$ until $t = T$. Each path reflects a sequence of possible outcomes for all random parameters in the model. The collection of equally probable sample paths gives a discrete approximation of the probability measure of the random variables ($r$, $W$, $L$, $\ell$).

Ideally, one would like to make different decisions for every path at every time $t = 1, \ldots, T-1$, but this would lead to undesirable anticipativity in the model. This is caused by the fact that, once we start following a specific path, we have full knowledge of the future until time $T$. The simplest way to avoid anticipativity is to make one single decision at each time $t$ for all paths. This means that the values of the decision variables are independent for all path realizations at a given time. This is the basis of current sample path-based approaches.

We relax this approach, without introducing anticipativity, by making the same decision at some time for all sample paths *in a particular bundle of sample paths*. We allow, however, for the presence of multiple bundles of sample paths at each time. This approach is inspired by ideas developed for portfolio optimization by Hibiki [1999, 2000].

We consider two models that will lead to linear optimization problems: the *fixed-value* and *fixed-quantity* models. In the fixed-value model, the dollar value of the positions held in each of the assets will be the same for all scenarios in a given bundle. In the fixed-quantity model, the number of shares of each asset will be the same for all scenarios in a given bundle. Hibiki [1999, 2000] refers to the fixed-quantity case as "fixed-amount"; we have renamed this case to avoid confusion, because amount can refer to both quantity (of shares) and a dollar amount.

Like Hibiki [1999, 2000] in the case of portfolio allocation problems, we find that the fixed-quantity model leads to solutions for the ALM problem for pension funds that are superior to the solutions produced by the fixed-value model. We thus present the mathematical model for the fixed-quantity case, and discuss its performance in several settings. For the fixed-value case, we limit ourselves to a discussion of the numerical results.

The fixed-quantity approach can be implemented in various frameworks. Using a stochastic programming approach, Consiglio, Cocco, and Zenios [2001] consider the fixed-quantity models for the insurance industry.

## Structure of Decision Rules: Grouping of Sample Paths

To avoid anticipativity of solutions, the same decisions are made simultaneously for many sample paths. For reasons of comparison, we examine the extreme case where the same decisions are made for all sample paths at a given time; i.e., at each time there is only a single bundle of sample paths.

Such an approach may lead to very conservative solutions. After a high observed return on investments, we would need to contribute more than necessary to cover liabilities, since the same contributions would also have to cover the liabilities for the case of low returns.

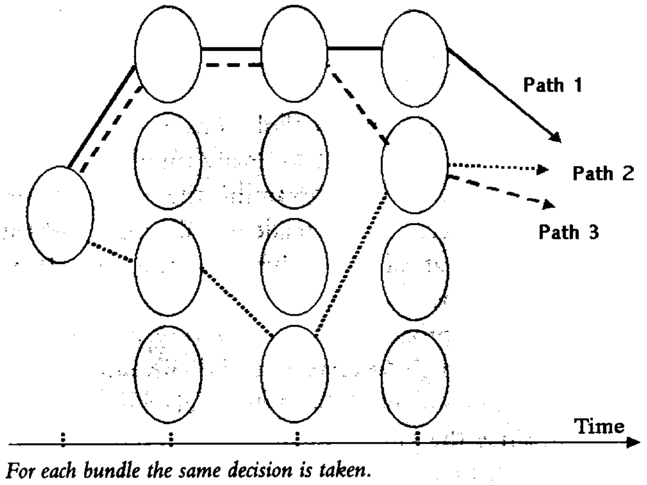
For a more flexible control strategy, we group sample paths into different bundles at each time, and make the same decision for all paths in each node. At time $t$, we group the paths in $K_t$ groups, or bundles, for $t = 1, \ldots, T - 1$. Paths that are in two different groups at time $t$ can pass through the same group at either earlier or later times (or both). Also, paths that pass through one node at time $t$ do not necessarily pass through the same node at any other time. Exhibit 1 illustrates this setup.

There is a lot of flexibility in grouping the paths. Besides having to decide how many groups to have at each time, we should also decide how to allocate paths to these groups. The idea is to group paths into bundles that require similar decisions. Since the funding ratio is a single quantity that characterizes the health of the pension fund, this seems a reasonable measure to use for grouping. A high funding ratio may allow one to invest in risky instruments or to demand a lower contribution from the fund's sponsor and active members. A low funding ratio may mean that the fund is in danger of violating its liability constraints. This means that an increased contribution rate probably is needed.

It seems intuitive that scenarios with similar funding ratios call for similar optimal decisions, and can therefore be grouped together. Unfortunately, the funding ratio is a result of a particular strategy, and cannot therefore be computed a priori. We have chosen to first solve the problem using only a single bundle at each time period. Further, we use the obtained solution to calculate the funding ratio (for each path at each time), and group paths in bundles.

**EXHIBIT 1**

**Grouping of Paths into Bundles at Each Time $t$**



*For each bundle the same decision is taken.*

This approach could be employed in an iterative fashion—namely, by using the optimal strategy for a given grouping to reestimate the funding ratios, which can then in turn be used to regroup the sample paths. Application of this method is beyond the scope of this study, and is left for future research. Our numerical experiments show that this method for grouping according to the funding ratio leads to reasonable results.

### Fixed-Quantity Model

Under the fixed-quantity rule, a position in an asset is represented by the number of shares (i.e., the quantity). The model considers $I$ different sample paths (realizations) for times $t = 0, \ldots, T$. At the beginning of each time period $t = 0, \ldots, T - 1$, an investment decision is made, and at the end of every time period this investment must cover the value of future liabilities in the $\alpha$-CVaR sense. Payments are made to retirees, and a contribution to the fund is made.

In this model, we fix the number of shares in each instrument for all paths in a single group (bundle) at each moment. In this case, the total value of all shares to be purchased might not be equal to the total available wealth, i.e., there is a balancing problem.

For example, consider two different paths, say, $i$ and $j$, belonging to the same group at time $t$. In both paths, the portfolio should be adjusted to hold the same number of shares in two assets, for instance, in asset 1 and asset 2. The total value of these shares may be different in both paths because each path has its own history of asset returns up to time $t$.

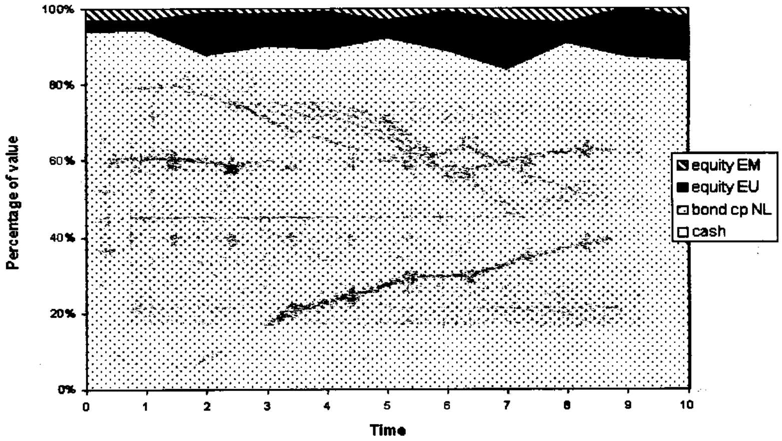
This clearly is not a problem, but the total value of the shares to be purchased may not coincide with the total amount available for investment in one or both paths. We have chosen to address this by allowing the difference to be made up by additional (positive or negative) investments in cash.

Hibiki [1999] chooses the cash position as the path-dependent variable. The idea of using cash as a path-dependent variable has been considered by Hiller and Eckstein [1993] for bonds and Zenios [1993] for mortgage cash flows. Here, we include two different cash positions in the model. Strategic investments in cash, as one of the asset categories, are made using a number of shares that is constant for all paths in a single group at a single time. We also have include a path-dependent deviation from this fixed quantity, to account for excess or shortage wealth.

The path-dependent variable can take positive or negative values. This is consistent with other ALM models. In Kouwenberg [2001], for example, the borrowing of cash is allowed, but not the shorting of other assets. To avoid using this corrective cash to cover liabilities, we require that the average path-dependent cash position be zero over all paths in a group at a given time. The strategic cash position, like the other asset positions, is restricted to be non-negative.

EXHIBIT 2

**Average Monetary Proportions of Instruments**



*One decision is made for all sample paths at each time (the same number of shares for all sample paths at each time). The monetary portfolio composition differs between paths.*

## COMPUTATIONAL RESULTS

### Data and Computational Resources

The data set of sample paths was generated at ORTEC Consultants, B.V., for a pension fund in The Netherlands. It includes 5,000 sample paths for a time horizon of 10 years. Each sample path consists of returns for 12 financial assets, wages for the active members, payments to be made each year, and future liabilities.

The asset categories are: 1) cash Netherlands, 2) commercial paper (CP) Netherlands, 3) CP U.S., 4) CP Japan, 5) CP U.K., 6) bonds emerging markets, 7) equity Europe, 8) equity U.S., 9) equity Japan, 10) equity U.K., 11) equity emerging markets, and 12) private equity. To simplify interpretation of the results, we have mostly limited ourselves to a subset of four asset categories: cash Netherlands, CP Netherlands, equity Europe, and equity emerging markets. The objective is to provide a qualitative assessment of our model and approach, and the simplified portfolio makes it easier to grasp the basic characteristics of the modeling approach.

Some of the model parameters are: 1) a target funding ratio of $\psi = 1.2$; see Equation (5) and Equation (A-4) in the appendix; 2) a confidence level in CVaR constraint $\alpha = 0.95$; see Equation (8) and Equation (A-5) in the appendix; 3) an upper bound in the CVaR constraints set to $w_t = 0$, for all moments and all nodes; see Equation (8) and Equation (A-5) in the appendix. This means that we limit the expected value of the worst 5% underperformances to be 20% over the future value of the liabilities (i.e., the outcome of $L$). All monetary values are scaled so that $A_0 = 1$; see balance Equations (1) and (A-2).

A full list of variables is in Exhibit A in the appendix.

We implement the model using the CPLEX callable library. All computations are made on an IBM Power 3 node with two processors and 512MB of RAM (CPLEX uses one processor, up to 256MB of RAM). The computation time for the problem with $I = 5{,}000$ (number of scenarios), $N = 3$ (number of instruments excluding cash), and $T = 10$ (number of periods) is about three hours.

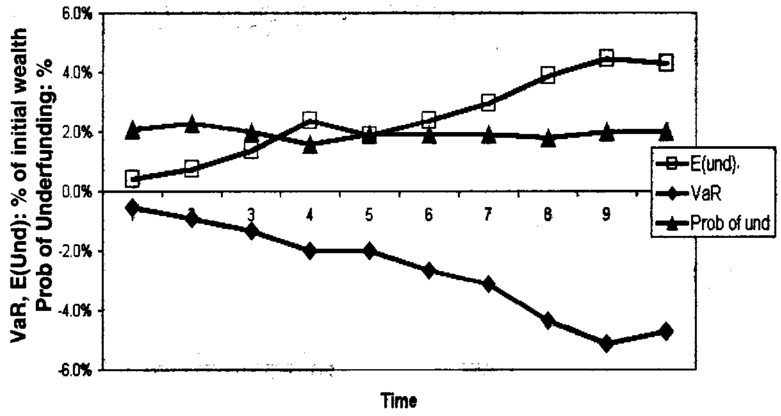
To limit the computation time, we use a subset of 2,000 scenarios. Although in our numerical example we mostly considered $N = 3$, the model can handle quite a high number of assets, as the number of constraints, the number of variables, and the computation time remain relatively constant as long as $N \ll I$.

### One Decision at Each Moment

First, we consider that the contribution rate and portfolio allocation (number of shares) are the same for all sample paths at each time $t = 1, \ldots, T - 1$. The contribution rate and (strategic) portfolio allocation thus depend on time, but do not depend on the sample paths. Although the number of shares invested in each instrument does not depend on the particular sample path that occurs, the actual monetary portfolio composition may differ for various paths because of different prices of the instruments along different paths, and because of the

## EXHIBIT 3

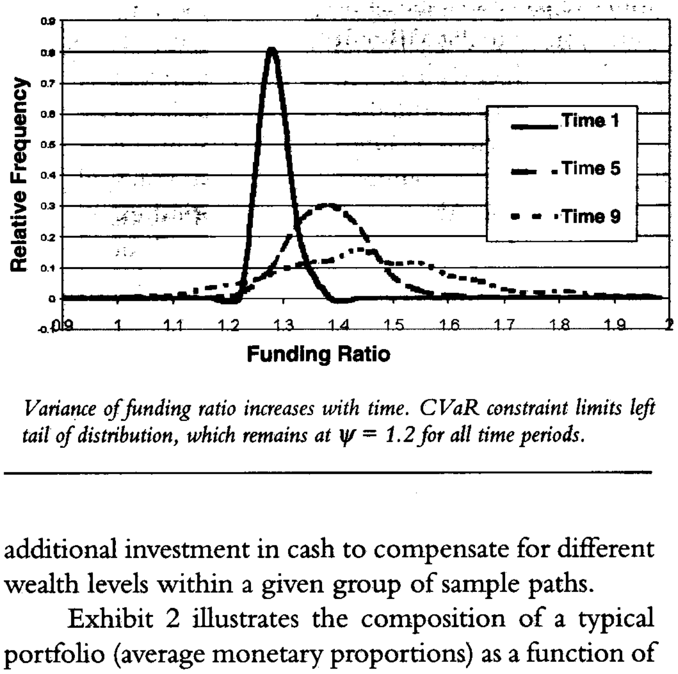
### Evolution of Three Risk Measures with One Decision at Each Moment



*VaR reduces with time. Probability of underfunding stays approximately constant. Expected value of the underfundings increases with time.*

## EXHIBIT 4

### Distribution of Funding Ratio at Different Times with Fixed Allocations



*Variance of funding ratio increases with time. CVaR constraint limits left tail of distribution, which remains at $\psi = 1.2$ for all time periods.*

additional investment in cash to compensate for different wealth levels within a given group of sample paths.

Exhibit 2 illustrates the composition of a typical portfolio (average monetary proportions) as a function of time. On average, more than 84% of the strategic portfolio funds is allocated to bonds (CP Netherlands), while the remaining funds are allocated to equities (equity Europe, and equity emerging markets). The allocation to strategic cash (cash Netherlands) is close to zero. There is a tendency to increase the allocation in equities over time (from 5% to 14%; the maximum value of equities is 16%).

It is interesting to compare the dynamics of portfolio allocation with the dynamics of risk levels. Since the bound on CVaR is set at zero ($w_t = 0, \forall t$), VaR should be negative (CVaR is always greater than or equal to VaR). In addition, Exhibit 3 shows that VaR (the variable $\zeta$ in the model) moves away from zero over time, while the number of underfundings (or probability of underfunding) stays fairly constant. At the same time, the expected value of the underfundings increases.[6]

Exhibit 4 illustrates that the differences in funding ratios between the different paths increase over time. It compares the distribution of the funding ratio at different times. In year 1 (i.e., after one period), the distribution of funding ratios is concentrated around 1.3. Almost 90% are found in the interval 1.20 to 1.25; the lowest value is 1.18, and the highest 1.42. By year 5, the average funding ratio is just below 1.4, but the lowest value found is 1.13, and the highest is 1.69. This pattern continues, and in year 9 we actually observe sample paths where the funding ratio is below 1 (0.3% of the 2,000 paths). So, while the average funding ratio increases over time, the probability that very bad cases will occur increases as well.

In a sensitivity analysis of the model with respect to the funding ratio parameter $\psi$, we vary the parameter from 1.0 to 1.4. Exhibit 5 shows the effect of changing $\psi$ on the contribution rate. The contribution rates for different $\psi$ seem to be very similar. As expected, a higher value of the funding ratio parameter $\psi$ leads to higher contribution rates. Calculations show that parameter $\psi$ can be effectively used to adjust the wealth of the fund.

### Grouping of Sample Paths

The strategy does not depend on the realized sample path, although the actual decisions do change with time. To obtain strategies that adapt to particular situations faced by a fund as measured by the funding ratio, we group sample paths in eight groups at each time.

Grouping significantly improves the performance of the algorithm. For the fixed quantity model with grouping, we reduce costs by about 50% from the cost with one decision at each time; see Exhibit 6. The second set of bars in this graph corresponds to the fixed-quantity model without grouping, and the third set represents the fixed-quantity model with grouping. Three cost values are presented in Exhibit 6: net present value of contributions to the fund, average premium, and contribution rate as a fraction of wages.

The first set of bars in Exhibit 6 represents the fixed-value model. This model is very similar to the fixed-quantity model, but instead of fixing the number of shares, we fix the monetary values invested in each instrument. Because the fixed-value model significantly underperforms the fixed-quantity model, we omit the model details, but we include data on this model for the purposes of comparison in Exhibits 6, 7, and 9.

The dynamics of the contribution rates for the different models are presented in Exhibit 7. Again, we note that the fixed-value model considerably underperforms the fixed-quantity model. The contribution rate in the fixed-quantity model with grouping is lower than the contribution rate in the fixed-quantity model without grouping. In the case of grouping, the contribution rate is calculated as the weighted average of the contribution rates over the eight groups at each time. The contribution rate of the model with grouping exhibits some oscillations that may be suppressed using additional constraints, disallowing significant changes in the contribution rate at each time.

The contribution rate for various groups in the fixed-quantity model with grouping is presented in Exhibit 8. The contribution rate differs for each group of paths. The solid line curve is the weighted average of contribution rates over eight groups. The vertical lines display the range of contribution rates for each time period. The group with the highest funding ratio is represented by diamonds and the group with the next-to-the-highest funding ratio by squares. The two highest funding groups exhibit the highest changes in the contribution rate. While the oscillations can be reduced by modifying the feasible set of contribution rates, the oscillations may not be a severe drawback of this model (if kept under control). Clearly, it will be undesirable to make dramatically different decisions between time periods (contribution rate oscillating from –0.2 to 0.3), but the oscillations do provide a cheaper solution than the solution of the fixed-quantity model without grouping.

Usually, one is interested in the decision at time 0. This is because the program would most likely be implemented in a rolling horizon fashion, i.e., one would rerun the program at each time period to obtain a new optimal initial decision.

**EXHIBIT 5**

**Contribution Rate for Different Values of $\psi$**

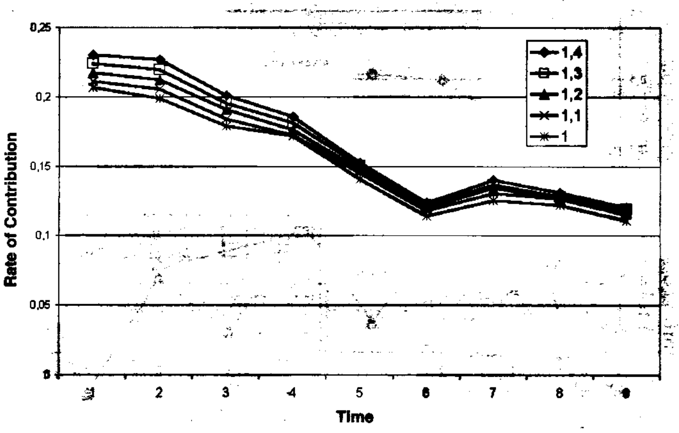
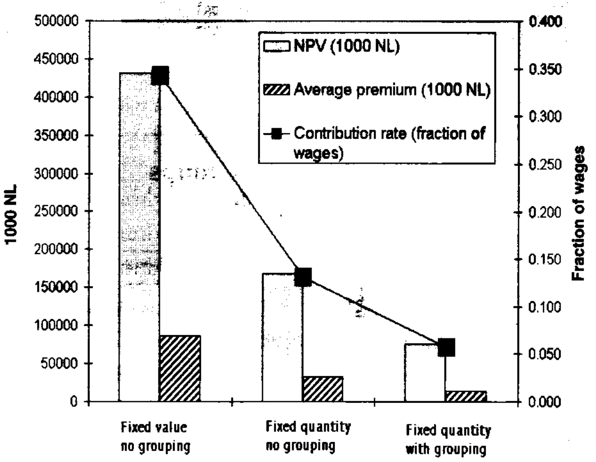


**EXHIBIT 6**

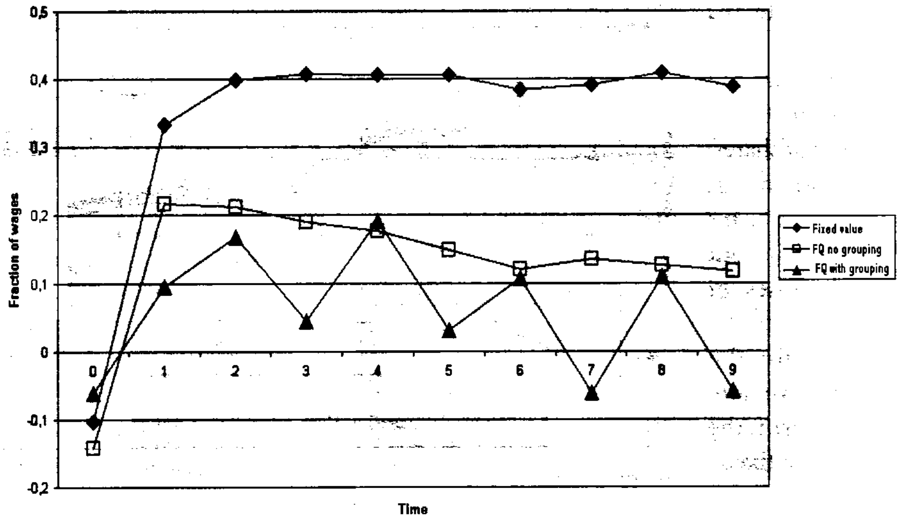
**Comparison of Cost**



We also compare the initial portfolios for the three different models. As you can see in Exhibit 9, there are few differences among the three initial portfolios. The fixed-quantity model with grouping has the highest exposure to equities (highest percentage of risky instruments). The fixed-value model without grouping has the lowest exposure to equities. The CVaR risk level for the three models is the same, however. This illustrates that, with a dynamic control strategy that responds to the situation of the fund, risk can be accounted for in a more cost-effective manner.

**EXHIBIT 7**

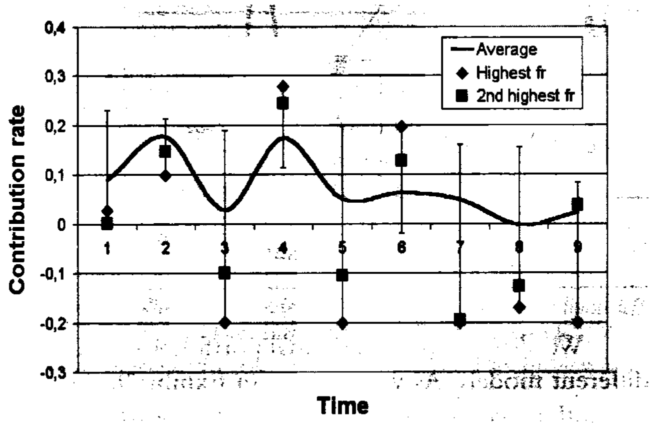
**Comparison of Contribution Rates**



*Contribution rate in the case of grouping calculated as the weighted average of the contribution rates over the eight groups at each time.*

**EXHIBIT 8**

**Contribution Rates for Different Groups**



## CONCLUSIONS

We have explored a new approach to modeling asset/liability management problems for pension funds. We combine CVaR risk management with a new framework of optimal decisions using sample paths. We have formulated and solved several multiple-period optimization models. Our findings can be summarized as follows.

A model for finding optimal contribution rates and portfolio allocations takes into account the funding situation of the fund. Using the CVaR risk measure, the model can be solved with linear programming techniques.

Our approach adds flexibility to the decisions while using only a sample path representation of uncertainties, allowing us to avoid the explosion of the problem dimension required by a stochastic programming approach. The flexibility comes from grouping at each time a set of sample paths that correspond to similar characteristics of the pension fund, and from restricting decisions to vary among different groups of sample paths.

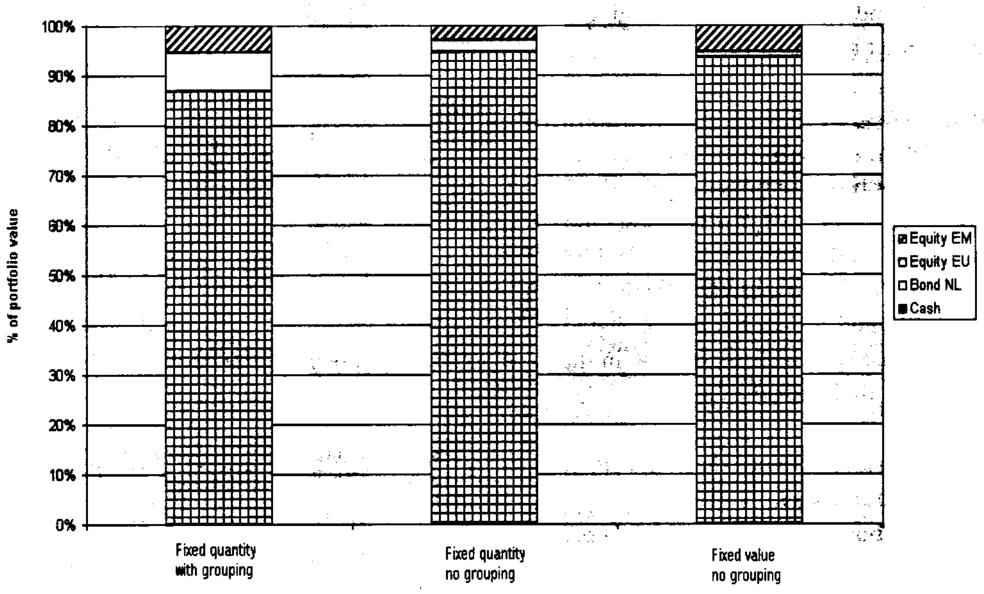
We obtain truly dynamic decisions at moderate computational expense, while allowing for extensive uncertainty through the use of paths. In fact, from a computational point of view, the problem size and solution times are on the same order of magnitude, with or without grouping.

Our main focus is the so-called fixed-quantity model, where for each group of sample paths we optimize the number of shares invested in each asset category. We compare our grouping strategy to a strategy that makes the same decision, regardless of the state of the fund and a so-called fixed-value model without grouping, where instead of the number of shares, we optimize the monetary value invested in each asset category.

Our experiments indicate that dynamic decision-making through the use of paths results in much lower costs to the fund than the alternative without grouping. Clearly, how one groups the sample paths affects the solution. For instance, the solutions obtained show an oscillating behavior over time, which could be a consequence of the particular grouping method.

Future research should address the issue of grouping the sample paths, perhaps by applying the algorithm in an iterative fashion. We also should analyze end-of-study effects by associating a value with the state of the pension fund at the end of the horizon, as well as the effect of choosing different objectives, such as incorporating a measure of the rate of return on the investment portfolio.

## EXHIBIT 9
### Initial Portfolio for Three Models



*Fixed-quantity model with grouping has the highest exposure to equities, about 13%. All portfolios have more than 90% of investments in bonds.*

## APPENDIX
### Mathematical Model

This appendix formally presents the fixed-quantity model with CVaR constraints. We consider the model with grouping of sample paths. A special case of this model is a model without grouping, i.e., the same decision is taken for all paths at each time.

### Notation

***Parameters*** (typical parameter values are included in Exhibit A):

$A_0$ = total initial value of all assets;
$W_0$ = total initial amount of wages;
$l_0$ = initial payments made by the fund;
$p_{n,0}$ = initial market price of asset $n$ (scaled to 1 for all assets);
$\rho_0$ = scale factor that translates the initial cash position into monetary value (set to 1, for simplicity);
$\psi$ = lower bound for funding ratio; typical value for $\psi$ is around 1 or higher;
$\psi_{end}$ = lower bound for funding ratio at the last time period;
$I$ = number of paths;
$N$ = number of assets;
$T$ = number of time intervals;
$\gamma$ = discount factor for contributions in the future;
$w_t$ = CVaR constraint level at time $t$;
$\alpha$ = confidence level in CVaR;
$k(i, t)$ = function returning the node (i.e., group number) through which, at time $t$ passes the $i$-th sample path;
$K_t$ = set of all nodes at time $t$;
$V_t^k$ = set of paths $i$ that pass node $k$ at time $t$;
$\tau^{low}$ = lower bound on cash positions;
$v$ = upper bound on the relative position of an asset in the portfolio ($0 \leq v \leq 1$);
$\xi_{n,t}^{low}$ = lower bound on the position of instrument $n$ at time $t$;
$\xi_{n,t}^{up}$ = upper bound on the position of instrument $n$ at time $t$;
$\underline{y}$ = lower bound on contribution rate; and
$\overline{y}$ = upper bound on contribution rate.

***Random data:***

$p_{n,t}^i$ = market price of asset $n$ for period $t-1$ to $t$ in path $i$;
$\rho_t^i$ = equivalent of a market price for cash at time $t$ in path $i$. This is a conversion factor that converts the

cash position variables $\xi_0$ and $\tau$ into a monetary value;

$L_t^i$ = liability measure that should be met or exceeded by the total value of all assets in the fund at time $t$ in path $i$;

$l_t^i$ = payments of the fund at time $t$ in path $i$; and

$W_t^i$ = total wages at time $t$ in path $i$.

***Decision variables:***

$y_t^k$ = contribution rate at time $t$ in node $k$;

$\xi_{n,t}^k$ = total quantity (i.e., number of shares) of asset $n$ ($n$ = 0, ..., $N$) at time $t$ in node $k$;

$\tau_0$ = additional amount of cash owned at time 0;

$\tau_t^i$ = amount of cash owned at time $t$ in path $i$;

$\zeta_t^k$ = dummy variable that approximates α-VaR in the optimal solution at time $t$ for the decision made at node $k$;

$z_t^i$ = dummy variables associated with the α-CVaR constraint at time $t$ and in path $i$;

$q^i$ = amount of money borrowed at time $T-1$ in path $i$; and

$B^i$ = size of underfundings at time $T$ in path $i$.

## Formulation of the Optimization Problem

First, we formulate the problem. Then, we comment each expression.

$$\textbf{minimize}\quad W_0 y_0 + \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T-1}\frac{W_t^i y_t^{k(i,t)}}{(1+\gamma)^t} + \tag{A-1}$$

$$\lambda_1 \frac{1}{I}\sum_{i=1}^{I}\frac{q^i}{(1+\gamma)^T} + \lambda_2 \frac{1}{I}\sum_{i=1}^{I}\frac{B^i}{(1+\gamma)^T}$$

subject to

$$\sum_{n=0}^{N} p_{n,0}\xi_{n,0} + \rho_{0,0}\tau_0 \;=\; A_0 - l_0 + W_0 y_0 \tag{A-2}$$

$$\sum_{n=0}^{N} p_{n,t}^i\left(\xi_{n,t}^{k(i,t)} - \xi_{n,t-1}^{k(i,t-1)}\right) + \rho_t^i\left(\tau_t^i - \tau_{t-1}^i\right) \tag{A-3}$$

$$= \; W_t^i y_t^{k(i,t)} - l_t^i,\; t = 1,\ldots,T-1, \forall i$$

$$\psi L_t^i - \sum_{n=0}^{N} p_{n,t}^i \xi_{i,t-1}^{k(i,t-1)} - \;\rho_t^i \tau_{t-1}^i - \zeta_t^{k(i,t-1)} \;\le \tag{A-4}$$

$$z_t^i,\; \forall i, t = 1,\ldots,T$$

$$\zeta_t^k + \frac{1}{(1-\alpha)|V_{t-1}^k|}\sum_{i\in V_{t-1}^k} z_t^i \;\le \tag{A-5}$$

$$w_t,\; t = 1,\ldots,T,\; k \in K_{t-1}$$

$$\rho_T^i \tau_{T-1}^i + q^i \;\ge\; 0,\; \forall i \tag{A-6}$$

$$\sum_{n=0}^{N} p_{n,T}^i \xi_{n,T-1}^{k(i,T-1)} + \rho_T^i \tau_{T-1}^i + \; B^i \;\ge\; \psi_{end} L_T^i,\; \forall i \tag{A-7}$$

$$\frac{1}{|V_t^k|}\sum_{i\in V_t^k} \rho_t^i \tau_t^i \;\ge\; 0,\; k \in K_t,\; t = 1,\ldots,T-1 \tag{A-8}$$

$$\frac{1}{v} p_{n,0}\xi_{n,0} \;\le\; \sum_{m=0}^{N} p_{m,0}\xi_{m,0}, \forall n \tag{A-9}$$

$$\frac{1}{v} p_{n,t}^i \xi_{n,t}^{k(i,t)} \;\le\; \sum_{m=0}^{N} p_{m,t}^i \xi_{m,t}^{k(i,t)} \tag{A-10}$$

$$\forall i,\; \forall n, t = 1,\ldots,T-1$$

$$\xi_{n,t}^{low} \;\le\; \xi_{n,t}^k \;\le\; \xi_{n,t}^{up} \qquad t = 0,\ldots,T-1 \tag{A-11}$$

$$\forall n, k \in K_t$$

$$0 \;\le\; z_t^i,\; \forall i,\; t = 1,\ldots,T \tag{A-12}$$

$$0 \;\le\; \tau_0 \tag{A-13}$$

$$\tau^{low} \;\le\; \tau_t^i,\; \forall i, t = 1,\ldots,T-1 \tag{A-14}$$

$$0 \leq q^i, \forall i \qquad \text{(A-15)}$$

$$0 \leq B^i, \forall i \qquad \text{(A-16)}$$

$$y_0 \text{ is free} \qquad \text{(A-17)}$$

$$\underline{y} \leq y_t^k \leq \overline{y} \quad t = 1, \ldots, T-1 \quad \forall i, k \in K_t \qquad \text{(A-18)}$$

$$\zeta_t^k \text{ is free}, t = 1, \ldots, T, \; k \in K_t \qquad \text{(A-19)}$$

***Objective.*** The objective function (A-1) consists of four terms. The first term equals the initial contribution, and the second term equals the expected present value of the contributions to the pension fund. The third term is the expected present value of the loans at the end of the investment horizon. This term is included to ensure that all borrowed money is repaid at the end of the investment horizon.

The final term is the expected present value of the underfundings at time $T$, i.e., the model penalizes cases where we have less wealth than the required level $\psi_{end} L_T^i$. This term is included to prevent end-of-horizon effects.

***Investment Balance.*** Constraints (A-2) and (A-3) are balance equations constraining the amount of money available at the end of a period, plus additional contributions and minus payments, to be equal to the money available at the beginning of the next period.

***CVaR Constraints on Wealth.*** We have one CVaR constraint for each node, and accordingly one auxiliary variable $\zeta_t^k$ for each node $k$ at each time $t$. If the CVaR constraint is active, the optimal value of the variable $\zeta_t^k$ will be equal to $\alpha$–VaR for the appropriate node.[7] In each node $k$ and at each time $t$, the value of the investments must meet the required level $\psi L_t^i$ in an $\alpha$–CVaR sense; see (A-4), and (A-5). The conditional expected value of losses exceeding the VaR level $\zeta_t^k$ is constrained by the CVaR bound $w_t$ in (A-5).

***Terminal Loans.*** At the end of the investment horizon (i.e., when $t = T$), all borrowed money must be repaid. Constraint (A-6) ensures that $q^i$ represents the value of the borrowed cash position at time $t = T$.

***End-of-Horizon Effects.*** $B^i$ in (A-7) measures a shortage of wealth in path $i$ at time $T$ when the wealth is below the required level $\psi_{end} L_T^i$. The value $B^i$ is included in the objective function to prevent undesirable end-of-horizon effects.

***Constraints on Positions.*** The model allows for constraining both the absolute and relative position sizes. The relative position in each asset can be limited to $\nu (0 \leq \nu \leq 1)$ of the total portfolio value with the constraints (A-9) and (A-10). The value $\nu = 1$ implies that we have no bounds on the relative positions. The absolute positions are handled by the constraint (A-11).

***Constraints on Cash Positions.*** Similar to the other assets, the size of the cash position could also be limited. In particular, we do not allow any borrowing at time 0 [constraint (A13)], since borrowing should only be used to compensate for the differences among the sample paths in the same group. Imposing a lower bound on cash positions could reflect the impact of regulations [constraint (A-4)]. We allow borrowing in all paths, but limit the expected value of the cash positions to be no less than zero [constraint (A-8)]. This means that on average we do not borrow.

***Constraints on Contribution Rate.*** We also place limits on the rates of contributions from wages of active members; see, (A-18). The upper bound is denoted by $\overline{y}$, and the lower bound is denoted by $\underline{y}$.

## ENDNOTES

The authors are grateful to ORTEC Consultants, B.V., for providing a data set of sample paths for a pension fund used in this study.

[1] A sample path completely describes future events; the optimal solution for a particular sample path should not "anticipate" (use information about) future events.

## EXHIBIT A
### Typical Parameter Values

| | Parameter | Value |
|---|---|---|
| # of paths | I | 2000 |
| # of asset categories (except cash) | N | 3 |
| length of investment horizon | T | 10 |
| confidence level | $\alpha$ | 0.95 |
| lower bound on funding ratio | $\psi$ | 1.2 |
| lower bound on funding ratio at horizon | $\psi_{end}$ | 1.3 |
| present value discount factor | $\gamma$ | 0.15 |
| penalty coefficient for loans at horizon | $\lambda_1$ | 1 |
| penalty coefficient for underfunding at horizon | $\lambda_2$ | 1 |
| CVaR constraint level | $w_t, \forall t$ | 0 |
| lower bound on cash positions | $\tau^{low}$ | 0 |
| upper bound on the relative position in an asset | $v$ | 0.2 |
| lower bound on position in an asset | $\xi_{n,t}^{low}, \forall n, \forall t$ | 0 |
| upper bound on position in an asset | $\xi_{n,t}^{up}, \forall n, \forall t$ | $\infty$ |
| lower bound on contribution rate | $\underline{y}$ | −0.2 |
| upper bound on contribution rate | $\overline{y}$ | 0.3 |
| number of nodes | $\lvert K_t \rvert, \forall t$ | 8 |

[2] We consider the defined-benefit pension scheme.

[3] A decision variable can be a constant or a stochastic function, depending upon the underlying stochastic variables.

[4] The notation suppresses the dependence of the variables on time for simplicity, but the function and most of the variables in fact depend on $t$.

[5] For example, for one confidence level, an equity may be a dominant contributor to portfolio risk, and for another confidence level, the dominant contributor may be a bond (an interest rate or credit risk).

[6] Recall that underfunding means that the value of the fund is lower than $\psi$ times the liabilities. If $\psi > 1$, this can still mean that we are not truly underfunded, but only that we are not meeting our target funding ratio.

[7] α–VaR equals the smallest optimal value if the solution $\zeta_t^k$ is not unique.

## REFERENCES

Acerbi, C., C. Nordio, and C. Sirtori. "Expected Shortfall as a Tool for Financial Risk Management." Working paper, 2001 (download from http://www.gloriamundi.org).

Acerbi, C., and D. Tasche. "On the Coherence of Expected Shortfall." Working paper, 2001 (download from http://www.gloriamundi.org).

Artzner, P., P. Delbaen, J.-M. Eber, and D. Heath. "Coherent Measures of Risk." *Mathematical Finance*, 9 (1999), pp. 203-228.

Birge, J.R., and F. Louveaux. *Introduction to Stochastic Programming*. Springer Series in Operations Research, 1997.

Boender, C.G.E., P.C. Van Aalst, and F. Heemskerk. "Modeling and Management of Assets and Liabilities of Pension Plans in The Netherlands." In W.T Ziemba and J.M. Mulvey, eds., *Worldwide Asset and Liability Modeling*. Cambridge: Cambridge, University Press, 1998.

Cairns, A.J.G., D. Blake, and K. Dowd. "Optimal Dynamic Asset Allocation for Defined-Contribution Pension Plans." Proceedings of the 10th AFIR Colloquium, Tromsoe, June 2000, pp. 131-154.

Carino, D.R., et al. "The Russel-Yasuda Kasai Model: An Asset/Liability Model for a Japanese Insurance Company Using Multistage Stochastic Programming." *Interfaces*, Vol. 24, No. 1 (1994).

Censor, Y., and S.A. Zenios. "Parallel Optimization." In *Numerical Mathematics and Scientific Computation*. New York: Oxford University Press, 1997.

Consiglio, A., F. Cocco, and S.A. Zenios. "The Value of Integrative Risk Management for Insurance Products with Guarantees." *The Journal of Risk Finance*, Spring 2001, pp. 1-11.

Dert, C. "Asset Liability Management for Pension Funds: A Multistage Chance Constrained Programming Approach." Ph.D. Thesis. Erasmus University, Rotterdam, The Netherlands, 1995.

——. "A Dynamic Model for Asset Liability Management for Defined Benefit Pension Funds." In W.T Ziemba and, J.M. Mulvey, eds., *Worldwide Asset and Liability Modeling*. Cambridge: Cambridge University Press, 1998.

Ermoliev, Yu., and R.J.-B. Wets, eds. *Numerical Techniques for Stochastic Optimization*. Springer Series in Computational Mathematics, 10 (1988).

Hibiki, N. "Multi-Period Stochastic Programming Model Using Simulated Paths for Strategic Asset Allocation." Presented at the 5th Conference of the Association of Asian-Pacific Operations Research Societies within IFORS, Singapore.

——. "Multi-Period Stochastic Programming Models for Dynamic Asset Allocation." In *Stochastic Systems Theory and Its Applications*, Yokohama, 1999, pp. 11-12.

Hiller, R.S., and J. Eckstein. "Stochastic Dedication: Designing Fixed Income Portfolios Using Massively Parallel Benders Decomposition." *Management Science*, 39 (1993), pp. 1422-1438.

Kouwenberg, R. "Scenario Generation and Stochastic Programming Models for Asset Liability Management." *European Journal of Operational Research*, 134 (2001), pp. 279-292.

Merton, R.C. *Continuous-Time Finance*. Cambridge MA, and Oxford UK: Blackwell, revised edition, 1993.

——. "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." *The Journal of Economic Theory*, 3 (1971), pp. 373-413.

Milevsky, M.A. "Optimal Asset Allocation Towards the End of the Life Cycle: To Annuitize or Not to Annuitize?" *The Journal of Risk and Insurance*, 65, (3) (September 1998).

Mulvey, J.M., D.P. Rosenbaum, and B. Shetty. "Strategic Financial Risk Management and Operations Research." *European Journal of Operational Research*, 97 (1995), pp. 1-16.

Pflug, G. "Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk." In S. Uryasev, ed., *Probabilistic Constrained Optimization: Methodology and Applications*. Amsterdam: Kluwer Academic Publishers, 2000.

Prekopa, A. *Stochastic Programming*. Amsterdam: Kluwer Academic Publishers, 1995.

Rockafellar, R.T., and S. Uryasev. "Optimization of Conditional Value-at-Risk." *Journal of Risk*, 2 (2000), pp. 21-41.

——. "Conditional Value-at-Risk for General Loss Distributions." Research report 2001-5. University of Florida, April, 2001 (download from: http://www.ise.ufl.edu/uryasev/cvar2.pdf).

Zenios, S.A. "High Performance Computing for Financial Planning: The Last Ten Years and the Next." *Parallel Computing*, 25 (1999), pp. 2149-2175.

——. "A Model for Portfolio Management with Mortgage-Backed Securities." *Annals of Operations Research*, 43 (1993), pp. 337-356.

Ziemba, T.W., and M.J. Mulvey, eds., *Worldwide Asset and Liability Modeling*. Cambridge: Cambridge University Press, 1998.

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