Asset Prices in An Exchange Economy

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SUFE

Question

- Risk-free assets and risky assets
  - bank deposit, government debt

$$R_{t+1}$$

  - stock

$$\text{pay } p_t, \text{return } p_{t+1} + d_{t+1}$$

- How those risky assets are priced?

Objective

Theoretical examination of stochastic behavior of equilibrium asset prices

- Returns of an asset is stochastic
- Existence of equilibrium asset prices
- Behavior of equilibrium asset prices

Model Environment

- Identical consumers
- A single consumption good
- A number of different productive units
- Nature determines output - no input
  - pure exchange
- An assest is a claim to all or part of the output of one of these units
- Shock
  - productivity in each unit fluctuates stochastically through time

Consumption and Utility

- Representative consumer:

$$E\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\}$$

  c_t stochastic process representing consumption of a single good
- n distinct productive units, output is perishable

$$0 \leq c_t \leq \sum_{i=1}^{n} y_{it}$$

Output

- Production is entirely "exogenous": no labor or capital inputs
- Output y_t follows a Markov process
  - transition funciton

$$F(y', y) = \Pr\left\{y_{t+1} \leq y' | y_t = y\right\}$$

Asset holdings

- Ownership in this productive units is determined each period in a competitive stock market
  - Each unit has outstanding one perfectly divisible equity share
  - A share entitles its owner as of the beginning of t to all of the unit's output in period t
  - Shares are traded, after payment of real dividends, at a competitively determined price vector $p_t = (p_{1t}, ...p_{nt})$
  - Let $z_t = (z_{1t}, ..., z_{nt})$ denote a consumer's beginning-of-period share holdings

Quantities of consumption and asset holdings

- All output will be consumed $(c_t = \sum_{i=0}^{n} y_{it})$
- All shares will be held $(z_t = (1, ..., 1) = \underline{1})$

States and price function

- Physical state of the economy:
  - The current output vector y_t summarizes all relevant information on current physical state
  - Knowledge of the transition function $F(y', y)$
- Equilibrium should be expressible as some fixed function $p(\cdot)$ of the state of the economy, or $p_t = p(y_t)$
  - the ith coordinate $p_i(y_t)$ is the price of a share of unit i when the economy is in the state y_t.
- Knowledge of the transition function $F(y', y)$ and this function $p(y)$ will suffice to determine the stochastic character of the price process $\{p_t\}$

Policy funcitons

- Decision rules $c_t = c(z_{t,}y_t, p_t)$ and $z_{t+1} = z(z_{t,}y_t, p_t)$
- A consumer's current consumption and portfolio decisions, c_t and z_{t+1}, depend on his beginning of period portfolio, z_t, the prices he faces, p_t, and the relevant information he possesses on current and future states of the economy, y_t

Definition of equilibrium

Definition

An equilibrium is a continuous function $p(y) : E^{n+} \rightarrow E^{n+}$ and a continuous, bounded function $v(z, y) : E^{n+} \times E^{n+} \rightarrow R^{+}$ such that
(i)

$$v(z, y) = \max_{c,\ x} \left\{ U(c) + \beta \int v(x, y') dF(y', y) \right\}$$

subject to

$$c + p(y) \cdot x \leq y \cdot z + p(y) \cdot z,\ \ c \geq 0,\ \ 0 \leq x \leq \bar{z},$$

where $\bar{z}$ is a vector with components exceeding one;
(ii) for each y, $v(\underline{1}, y)$ is attained by $c = \sum_i y_i$ and $x = \underline{1}$.

Definition of equilibrium

- Condition (i) says that, given the behavior of prices, a consumer allocates his resources $y \cdot z + p(y) \cdot z$ optimally among current consumption c and end-of-period share holdings x
- Condition (ii) requires that these consumption and portfolio decisions be market clearing

Lucas's Tree

- Every one is endowed with a tree that can produce fruit

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- Our interest:
  The movement of the asset prices

Construction of the equilibrium - propositions

Proposition 1: *For each continuous price function $p(\cdot)$ there is a unique, bounded, continuous, nonnegative function $v(z,\ y : p)$ satisfying (i). For each y, $v(z,\ y : p)$ is an increasing, concave function of z.*

Proposition 2 (Envelope theorem)

Proposition 2: If $v(z, y; p)$ is attained at (c, x) with $c > 0$, then v is differentiable with respect to z at (z, y) and

$$\frac{\partial v(z, y; p)}{\partial z_i} = U'(c)\left[y_i + p_i(y)\right], \quad i = 1, ..., n$$

Solution of the price function

- First order condition

$$U'(c)p_i(y) = \beta \int \frac{\partial v(x, y')}{\partial x_i} dF(y', y)$$

$$c + p(y)x = yz + p(y)z$$

provided $c, x > 0$

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- If next period's optimum consumption $c\prime$ is also positive, Proposition 2 implies

$$\frac{\partial v(x, y')}{\partial x_i} = U'(c') \left[y_i' + p_i(y') \right]$$

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- One may think this equation, loosely, as equating the marginal rate of substitution of current for future consumption to the market rate of transformation, as given in the market rate of return on security i
- Mathematically, it is a stochastic Euler equation

Solution of the price function

- Define

$$g_i(y) = \beta \int U'\left(\sum_j y_j'\right) y_i' dF(y', y)$$

$$f_i(y) = U'\left(\sum_j y_j\right) p_i(y)$$

We have n independent functional equations

$$f_i(y) = g_i(y) + \beta \int f_i(y') dF(y', y)$$

Solution of the price function

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- the price functions
$$p_i(y) = \frac{f_i(y)}{U'\left(\sum_j y_j\right)}$$
will solve (*), and $p(y) = (p_1(y), ..., p_n(y))$ will be the equilibrium price function

Solution by contraction mapping

- If f is any continuous, bounded, nonnegative function on E^{n+}, the function $T_i f : E^{n+} \to R^+$ given by

$$(T_i f)(y) = g_i(y) + \beta \int f(y')dF(y', y)$$

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  is well-defined and continuous in y
- Since U is concave and bounded (by B, say) we have for any c :

$$0 = U(0) \leq U(c) + U'(c)(-c) \leq B - cU'(c)$$

  So that $cU'(c) < B$ for all c.

Solution by contraction mapping

- $cU'(c) < B$, it follows that the functions $g_i(y)$ are bounded,

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- Evidently, solutions to $T_i f = f$ are solutions to $f(y) = g_i(y) + \beta \int f(y') dF(y', y)$

Characterize the price function – one asset

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- The crucial issues are the information content of the current state y (that is, the way $F(y', y)$ varies with y) and the degree of "risk aversion" (the curvature of U)
- Suppose, as first case, that $\{y_t\}$ is a sequence of independent random variables: $F(y', y) = \phi(y')$
- Then $g(y)$ is the constant

$$\bar{g} = \beta \int y' U'(y') d\phi(y') = \beta E\left[yU'(y)\right]$$

Characterize the price function – one asset

- Calculating f from

$$(Tf)(y) = \bar{g} + \beta \int f(y')d\phi(y')$$

$$(T^2 f)(y) = \bar{g} + \beta \left[\bar{g} + \beta \int f(y')d\phi(y') \right]$$

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- We get

$$f(y) = \frac{\bar{g}}{1-\beta}, \ f'(y) = 0$$

Characterize the price function – one asset

- Differentiating

$$p(y) = \frac{f(y)}{U'(y)}$$

gives

$$p'(y) = -\frac{\beta E\left[yU'(y)\right]U''(y)}{(1-\beta)\left[U'(y)\right]^2} = p(y)\frac{-U''(y)}{U'(y)} > 0$$

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- In a period of high transitory income, then, agents attempt to distribute part of the windfall over future periods (marginal utility decreases), via securities purchases. This attempt is frustrated (since storage is precluded) by an increase in asset prices

Autocorrelated production disturbances

- Restrict the stochastic difference equation governing y_t to have its root between zero and one

$$y_{t+1} = \rho y_t + \varepsilon_{t+1} \qquad \rho \in (0, 1)$$

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- CDF $F(y', y) = \Pr\{y_{t+1} \leq y' | y_t = y\}$
  $F_1 > 0$
  $F_2 < 0$: the higher the y_t, the more likely the higher y_{t+1}

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  $F_1 > 0$
  $F_2 < 0$: the higher the y_t, the more likely the higher y_{t+1}
- Use the change of variable $u = F(y', y)$, and invert to get $y' = G(u, y)$, $G_2 = \partial y' / \partial y$
  By substitution we take into account that y affects $y\prime$,
  $u = F(G(u, y), y)$, completely differentiation gives

$$F_1 G_2 + F_2 = 0, \quad G_2 = -F_2 / F_1$$

Lemma 1

Lemma

Let F satisfy $0 < -F_2 < F_1$, and let $h(y)$ have a derivative bounded between 0 and $h'_M > 0$. Then

$$0 \leq \frac{d}{dy} \int h(y')dF(y', y) \leq h'_M$$

Proof.

$\frac{d}{dy} \int_0^1 h(y')dF(y', y) = \frac{d}{dy} \int_0^1 h(G(u, y))du = \int_0^1 h'(G)G_2(u, y)du$, the result follows. □

Bounds using lemma 1

- $$g'(y) = \beta \frac{d}{dy} \int U'(y')y' dF(y', y)$$
- the derivative of $U'(y)y$
  - $U''(y)y + U'(y) = U'\left(1 - \left(\frac{-yU''(y)}{U'(y)}\right)\right)$

take 0 and $\bar{a}$ as lower and upper bounds on $U''(y)y + U'(y)$, then apply Lemma 1

$$0 \leq g'(y) \leq \beta \bar{a}$$

Bound - 0

- Let $f(y)$ be the solution to $f(y) = g(y) + \beta \int f(y')dF(y', y)$

$$f'(y) = g'(y) + \beta \int f'(y')G_2(u, y)dF(y', y)$$

substitute

$$f'(y') = g'(y') + \beta \frac{d}{dy'} \int f(y'')dF(y'', y')$$

$$\begin{aligned} f'(y') &= g'(y') + \beta \int g'(y')G_2(u, y)dF(y', y) \\ &\quad + \beta^2 \int \left[\frac{d}{dy'} \int f(y'')dF(y'', y') \right] G_2(u, y)dF(y', y) \\ &\geq 0 \end{aligned}$$

Upper bound

$$
\begin{aligned}
f'(y) &= g'(y) + \beta \frac{d}{dy} \int f(y') dF(y', y) \\
&= g'(y) + \beta \int f'(y') G_2(u, y) dF(y', y) \\
&\leq g'(y) + \beta \int f'(y') dF(y', y) \qquad \text{given } G_2 = \frac{-F_2}{F_1} < 1 \\
&\leq g'(y) + \beta \int \left[g'(y') + \beta \frac{d}{dy'} \int f(y'') dF(y'', y') \right] dF(y', y) \\
&\leq g'(y) + \beta \int g'(y') dF(y', y) \\
&\quad + \beta^2 \int \int g'(y'') dF(y'', y') dF(y', y) + \ldots \\
&\leq \bar{a} + \beta \bar{a} + \beta^2 \bar{a} + \ldots \\
&\leq \frac{\beta \bar{a}}{1 - \beta}
\end{aligned}
$$

The elasticity of the equilibrium price function

- $$\begin{aligned} p(y) &= \frac{f(y)}{U'(y)} \\ p'(y) &= \frac{U'(y)f'(y) - f(y)U''(y)}{[U'(y)]^2} \end{aligned}$$

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- $$\frac{yp'(y)}{p(y)} = \underbrace{\frac{yf'(y)}{f(y)}}_{\substack{\text{"information} \\ \text{effect"} \\ \text{sign of } f'(y)}} - \underbrace{\frac{yU''(y)}{U'(y)}}_{\text{income effect } (+)}$$

$f(y)$ information about future dividends

The elasticity of the equilibrium price function

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  It shows how to translate such knowledge into knowledge about asset prices

The elasticity of the equilibrium price function

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- Depends on our knowledge of the curvature of U
  It shows how to translate such knowledge into knowledge about asset prices
- $f'(y) > 0$, so that the information effect is positive
  Thus, new optimistic information on future dividends leads to increased asset prices