

# Asset Prices with Heterogeneity in Preferences and Beliefs\*

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## **Abstract**

In this paper, we study asset prices in a dynamic, general-equilibrium Lucas endowment economy where agents have expected (power) utility and differ with respect to both beliefs and the preference parameters for the subjective rate of time preference and risk aversion. We solve in closed form for the following quantities: the equilibrium consumption allocation and its dynamics; the state price density and its dynamics, which are characterized in terms of the riskless interest rate and the market price of risk; the stock price, the equity risk premium, and the volatility of stock returns; and, the term structure of interest rates along with the term premium. Our solution allows us to identify how heterogeneity in preference parameters and in beliefs is reflected in equilibrium asset returns.

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# 1 Introduction and motivation

Two key characteristics of economic agents are their beliefs and preferences. Our objective in this paper is to study the effect of heterogeneity in *both* of these characteristics on optimal consumption and portfolio policies, and the resulting asset prices, in a general equilibrium stochastic dynamic exchange economy with agents who have expected (power) utility. The main contribution of our work is to solve *in closed form* for consumption policies, portfolio policies, and asset prices in a dynamic general equilibrium economy where agents have heterogeneous beliefs and preferences. In particular, we solve in closed form for the following quantities: the equilibrium consumption allocation across agents and its dynamics over time; the optimal portfolios of individual investors; the state price density and its dynamics, which are characterized in terms of the riskless interest rate and the market price of risk; the stock price, the equity risk premium, and the volatility of stock returns; and, the term structure of interest rates and the term premium. The closed-form results also allow one to identify the conditions under which equilibrium in such an economy will be stationary, in the sense that both agents survive in the long-run.

The paper that is closest to our work is Cvitanić, Jouini, Malamud, and Napp (2009), which also studies asset prices in an economy where agents have expected utility and differ with respect to both beliefs and their preference parameters. This paper provides bounds on asset prices and characterizes their behavior in the limit when only one agent survives. However, it does not provide closed-form solutions for these quantities. In fact, Cvitanić and Malamud (2009b, p. 3) write that: “when risk aversion is heterogeneous, SDF [stochastic discount factor] is the solution to highly non-linear equation (1) [in their paper], and no explicit solution is possible, except for some very special values of risk aversion; see, for example, Wang (1996).” In contrast to Cvitanić, Jouini, Malamud, and Napp (2009), we provide closed-form solutions for optimal consumption, portfolio policies, and asset prices without restricting the risk aversion of the two agents to special values.

Most of the other papers in the existing literature with heterogeneous agents allow for *either* differences in beliefs *or* differences in preferences. We first discuss the literature that considers heterogeneity in preferences and then the literature that considers differences in beliefs. The effect of different time-discount factors on efficient allocation of consumption is studied in Gollier and Zeckhauser (2005). The effect of heterogeneity in risk aversion on asset prices is examined in several papers, most of which assume that investors have expected utility; for example, Dumas (1989) studies the riskfree rate and the risk premium in a production economy, Wang (1996) examines the term structure in an exchange economy, Basak and Cuoco (1998) and Kogan, Makarov, and Uppal

(2007) analyze the effect of borrowing and short-sale constraints in an exchange economy, Benninga and Mayshar (2000) and Weinbaum (2001) examine the effect of heterogeneity in risk aversion on volatility and the valuation of options, Bhamra and Uppal (2009), examine the effect of derivatives on the volatility of stock market returns, Longstaff and Wang (2009) investigate the relation between credit and asset prices, and Cvitanić and Malamud (2009a,b,c) consider equilibrium with multiple heterogeneous traders who maximize utility of only terminal wealth. In contrast to these papers that assume investors have expected utility, Chan and Kogan (2002) and Xiouros and Zapatero (2009) study asset prices in an economy where agents have “catching-up-with-the-Joneses” preferences. And, finally there are papers that work with Epstein and Zin (1989) recursive preferences that allow for a distinction between risk aversion and the elasticity of intertemporal substitution. For example, Guvenen (2005), studies asset pricing in a model with heterogeneity in elasticity of intertemporal substitution, Isaenko (2008), studies the term structure in a model where agents differ in both their risk aversion and elasticity of intertemporal substitution, and Gomes and Michaelides (2008) study portfolio decisions of households and asset prices in a model where agents are heterogeneous not just in terms of preferences but are also exposed to uninsurable income shocks in the presence of borrowing constraints.

When there are multiple agents who differ in their risk aversion, there is rarely a complete characterization of equilibrium that is entirely analytical. For example, for the case of expected utility, Wang (1996) and Longstaff and Wang (2009) provide closed form expressions for only particular parameter values; Kogan and Uppal (2001) characterize the equilibrium in production and exchange economies using perturbation analysis in the neighborhood of log utility; Bhamra and Uppal (2009) compare stock-market-return volatility in the case where agents have access to a derivative security and where they do not, but without solving explicitly for volatility; Dumas (1989) solves numerically for the interest rate in a production economy; for the case of “catching-up-with-the-Joneses” preferences, Chan and Kogan (2002) rely on numerical solutions, the working-paper version of Chan and Kogan (2002) provides approximate analytic results in the neighborhood of log utility using perturbation analysis, and Xiouros and Zapatero (2009) provide an expression for the value function of the central planner assuming a Gamma distribution for the risk tolerances of the investors but asset prices are obtained using numerical methods. The analysis in Guvenen (2005), Isaenko (2008), and Gomes and Michaelides (2008) is also numerical.

We now discuss the literature on the effect of heterogeneous beliefs on asset prices. Essentially, there are two ways to generate heterogeneity in beliefs. In the first approach, agents receive different information. This is the classical approach, adopted in the early noisy-rational-expectations

literature (see, for instance, Grossman and Stiglitz (1980), Hellwig (1980), Wang (1993), and Shleifer and Statman (1994)). In this class of models, one group of (informed) agents receives private signals and then there is a second group of agents (noise-traders), which trades for exogenous reasons and thereby prevents the price from fully revealing the private information of the informed agents. The second approach for generating heterogeneity, which is the one we adopt, is to have agents who “agree to disagree” about some aspect of the underlying economy, and in this class of models it is assumed that agents do not learn from each other’s behavior. Morris (1995) provides a good philosophical discussion of this modeling approach.<sup>1</sup> Excellent reviews of this literature are provided in Basak (2005) and Jouini and Napp (2007).

To summarize, the main contribution of our paper is that in contrast to the existing literature on general equilibrium models of asset pricing that considers either heterogeneous preferences *or* heterogeneous beliefs, we allow for heterogeneity in preferences *and* beliefs, we do *not* restrict the preference parameters of the agents to particular values, and we solve in *closed form* not just for the interest rate and market price of risk, but also for the stock price, equity premium, volatility of stock market returns, term structure of interest rates, and prices of contingent claims. We show explicitly that our results nest the results in the models that consider an exchange economy with agents who have expected utility with different degrees of risk aversion, such as, Wang (1996), Kogan and Uppal (2001), Bhamra and Uppal (2009), and Longstaff and Wang (2009), and that they nest also the results in models where agents have expected utility with heterogeneous beliefs, for instance, Basak (2005). A major advantage of our characterization of equilibrium is that it allows us to identify which features of asset returns can be explained by heterogeneity in preferences and/or beliefs and which features cannot. For instance, we find that heterogeneity in risk aversion can generate volatility in asset returns that is in excess of volatility of fundamentals. Our analysis also allows us to identify the conditions under which the equilibrium in such an economy will be stationary, in the sense that both agents survive in the long-run.

Our paper makes a contribution also on the technical front by demonstrating how one can obtain a closed-form solution to the consumption sharing rule between agents without restricting the risk aversion of the two agents to special values. This consumption-sharing rule is a non-linear algebraic equation, which reduces to a polynomial of degree  $n$  when the ratio of the risk aversion

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<sup>1</sup>Papers using this modeling approach include: Williams (1977) Harrison and Kreps (1978), Varian (1985), Abel (1990), De Long, Shleifer, Summers, and Waldmann (1990a,b), Harris and Raviv (1993), Timmermann (1993, 1996), Detemple and Murthy (1994), Kandel and Pearson (1995), Barberis, Shleifer, and Vishny (1998), Zapatero (1998), Hong and Stein (1999), Veronesi (1999), Basak (2000), Cecchetti, Lam, and Mark (2000), Gallmeyer (2000), Jarrow (1980), David and Veronesi (2002), Duffie, Garleanu, and Pedersen (2002), Scheinkman and Xiong (2003), Berrada (2006) Buraschi and Jiltsov (2006), Kogan, Ross, Wang, and Westerfield (2006), David (2008), Gallmeyer and Hollifield (2008), Yan (2008), Dumas, Kurshev, and Uppal (2009), and Xiong and Yan (2009).

of one agent to that of the other equals  $n$ . This polynomial can of course be solved in closed-form when  $n$  equals two, three or four. We are able to construct a closed-form solution for all other cases; that is, the ratio of risk aversions can be equal to any real number greater than 1. Central to our approach is a theorem due to Lagrange. Given the ubiquity of nonlinear problems in economics and finance, we expect that the approach we use can be applied also in other problems, which previously would have called for numerical methods.<sup>2</sup> One area of research where closed-form solutions are essential is in the study of survival and price impact (see, for example, Kogan, Ross, Wang, and Westerfield (2006)).

The rest of the paper is arranged as follows. In Section 2, we describe our model of an exchange economy with heterogenous agents, and explain how one can solve for the value function of the central planner and the individual agents. The equilibrium consumption allocation is given in Section 3, which also includes a discussion of survival of the agents and stationarity of the equilibrium. A full characterization of asset prices and the properties of asset returns is provided in Section 4. The wealth of individual investors and their optimal portfolio policies are described in Section 5. We conclude in Section 6. Our main results are highlighted in propositions and detailed proofs for all the results are provided in the appendix.

## 2 The model

In this section, we describe the features of the model of the economy we are considering. Below, we explain our assumptions about the information structure, the endowment process, the financial assets in the economy, and the preferences of agents. The equilibrium consumption allocations in this economy are described in the next section.

We consider a continuous-time, pure-exchange economy with an infinite time horizon. There is a single consumption good that serves as the numeraire. It is modeled as an exogenously specified endowment process. There are two types of investors,  $k \in \{1, 2\}$ . Each investor has constant relative risk averse utility (CRRA). The two types of agents are allowed to differ in their rates of time preference and relative risk aversions. Furthermore, the two types of agents have different beliefs about the expected growth rate of the endowment, which they do not update. In summary, our model differs from the standard Lucas (1978) model along two dimensions. First, preferences

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<sup>2</sup>For example, our approach can be used to solve for equilibrium asset prices in closed-form in an exchange economy with two dividend trees and *two* agents with power utility, who *differ in their relative risk aversion*. This is in contrast to Cochrane, Longstaff, and Santa Clara (2008) and Martin (2008), where agents have the same level of relative risk aversion.

are heterogeneous. Second, agents have different beliefs. We adopt the convention of subscripting by  $k$  the quantities related to Agent  $k$ , where  $k \in \{1, 2\}$ .

## 2.1 The information structure and endowment process

The uncertainty in the economy is represented by a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$  on which is defined a one-dimensional Brownian motion  $Z$ . The economy is modeled as being endowed with a single non-storable consumption good. The true evolution of the aggregate endowment,  $Y$ , which in our model is equivalent to both aggregate dividends and aggregate consumption, is:

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dZ_t, Y_0 > 0, \quad (1)$$

in which  $\mu_Y$  and  $\sigma_Y$  are constants.

## 2.2 Financial assets

There are two financial assets in the economy: a risky asset (stock) with one share outstanding and a locally risk-free bond in zero net supply. The stock is a claim on the aggregate endowment. The price of the stock, which can be interpreted as the market portfolio, is denoted  $S_t$ , and its *cumulative* return,  $R_t$ , which consists of capital gains plus dividends, is described by the process:

$$\frac{dS_t + Y_t dt}{S_t} = dR_t = \mu_{R,t} dt + \sigma_{R,t} dZ_t. \quad (2)$$

The price of the locally risk-free bond, is denoted  $B_t$ , and its risk-free return  $r_t$  is described by the process

$$\frac{dB_t}{B_t} = r_t dt. \quad (3)$$

The expected return on the stock,  $\mu_{R,t}$ , the volatility of stock returns,  $\sigma_{R,t}$ , and the locally risk-free rate will be determined endogenously in equilibrium.

## 2.3 Beliefs of the two agents

Agent  $k$  believes that the expected growth rate of the endowment process takes the constant value,  $\mu_{Y,k}$ . Agent  $k$ 's beliefs can be represented by an exponential martingale  $\xi_{k,t}$ , given by

$$\xi_{k,t} = e^{-\frac{1}{2}\sigma_{\xi,k}^2 t + \sigma_{\xi,k} Z_t}, \quad (4)$$



where

$$\sigma_{\xi,k} = \frac{\mu_{Y,k} - \mu_Y}{\sigma_Y}. \quad (5)$$

The exponential martingale,  $\xi_{k,t}$ , defines the probability measure  $\mathbb{P}^k$  on  $(\Omega, \mathcal{F})$ , via

$$\mathbb{P}^k(e_T) = E_t[1_{e_T}\xi_{k,T}], \forall t, T \in [0, \infty), t \leq T, \quad (6)$$

where  $e_T$  is an event which occurs at time  $T$  and  $\mathbb{P}^k(e_T)$  is the probability of its occurrence based on information known at time  $t$ . Hence, by Girsanov's Theorem,

$$\frac{dY_t}{Y_t} = \mu_{Y,k} dt + \sigma_Y dZ_{k,t}, \quad (7)$$

where  $Z_{k,t} = Z_t - \sigma_{\xi,k}t$  is a standard Brownian motion under  $\mathbb{P}^k$ . From the above, we can see that under  $\mathbb{P}^k$ , which represents Agent  $k$ 's beliefs, the expected growth rate of the endowment is  $\mu_{Y,k}$ .<sup>3</sup>

We quantify the *level of disagreement* between the two agents via the process,  $\xi_t$ , defined by

$$\xi_t = \frac{\xi_{2,t}}{\xi_{1,t}}. \quad (8)$$

Hence,

$$\xi_t = e^{-\frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2)t + (\sigma_{\xi,2} - \sigma_{\xi,1})Z_t}, \quad (9)$$

and

$$\frac{d\xi_t}{\xi_t} = \mu_{\xi} dt + \sigma_{\xi} dZ_t, \quad (10)$$

where

$$\mu_{\xi} = -\sigma_{\xi,1}(\sigma_{\xi,2} - \sigma_{\xi,1}), \quad (11)$$

$$\sigma_{\xi} = (\sigma_{\xi,2} - \sigma_{\xi,1}). \quad (12)$$

When the agents agree with each other,  $\xi_t$  is a constant. However, if Agent 2 is more optimistic than Agent 1,  $\mu_{Y,2} > \mu_{Y,1}$ , which implies that  $\sigma_{\xi,2} > \sigma_{\xi,1}$  so that  $\mu_{\xi}$ , the expected growth rate of  $\xi_t$ , is negative. Also, when Agent 2 is more optimistic than Agent 1,  $\sigma_{\xi}$  is positive, and therefore, positive shocks to endowment growth lead to positive shocks to  $\xi_t$ , that is,  $\xi_t$  is procyclical. Observe also that under the measure  $\mathbb{P}^1$ ,  $\xi_t$  is an exponential martingale, given by

$$\xi_t = e^{-\frac{1}{2}\sigma_{\xi}^2 t + \sigma_{\xi} Z_{1,t}}. \quad (13)$$

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<sup>3</sup>Note that the measures  $\mathbb{P}^1$ ,  $\mathbb{P}^2$  and  $\mathbb{P}$  are all equivalent, i.e. they agree on which events are impossible.

## 2.4 Preferences of the two agents

The consumption of Agent  $k$  at instant  $u$  is denoted by  $C_{k,u}$  and the instantaneous utility from consumption is assumed to be time additive and given by a power function:

$$U_k(C_{k,u}) = e^{-\beta_k u} \frac{C_{k,u}^{1-\gamma_k}}{1-\gamma_k}, \quad (14)$$

where  $\beta_k$  is the constant subjective discount rate (that is, the rate of time preference) and  $\gamma_k$  is the degree of relative risk aversion. Without loss of generality, we assume that Agent 1's relative risk aversion is less than that of Agent 2:  $\gamma_1 < \gamma_2$ .

Given her beliefs, represented by the measure  $\mathbb{P}^k$ , the expected utility of Agent  $k$  at time  $t$  from consuming  $C_{k,u}$  is given by

$$V_{k,t} = E_t^k \left[ \int_t^\infty e^{-\beta_k(u-t)} \frac{C_{k,u}^{1-\gamma_k}}{1-\gamma_k} du \right], \quad (15)$$

where  $E_t^k$  denotes the time- $t$  conditional expectation operator with respect to the measure  $\mathbb{P}^k$ .

## 2.5 The optimization problem of each agent

Each agent  $k$  is assumed to have an initial allocation of  $a_k$  shares of the stock, with  $a_1 + a_2 = 1$ . Thus, the value of the initial allocation of agent  $k$  is  $a_k S_0$ .

The problem of agent  $k$  is to maximize lifetime utility, given by  $V_{k,0}$  in (15), subject to a static budget constraint. The budget constraint requires that the present value of all future consumption is no more than the initial wealth with which each agent is endowed:

$$E_0^k \left[ \int_0^\infty \frac{\pi_{k,u}}{\pi_{k,0}} C_{k,u} du \right] \leq a_k S_0, \quad (16)$$

in which  $\pi_{k,u}$  is the marginal utility of investor  $k$  at date  $u$  (referred to by an array of names such as state-price density, stochastic discount factor, and present-value operator):

$$\kappa_k \frac{\pi_{k,u}}{\pi_{k,0}} = \frac{\partial U(C_{k,u})}{\partial C_{k,u}} = e^{-\beta_k u} C_{k,u}^{-\gamma_k}, \quad (17)$$

and  $\kappa_k$  is the Lagrange multiplier on the static budget constraint in (16). The process for  $\pi_{k,u}$  is given by (see Duffie (2001, Section 6.D, p. 106)):

$$\frac{d\pi_{k,t}}{\pi_{k,t}} = -r_t dt - \theta_{k,t} dZ_t, \quad (18)$$

in which  $r_t$  is the risk-free interest rate, which is the same across agents, and  $\theta_{k,t}$  is the agent-specific market price of risk.

Existence of a solution requires that the following condition be satisfied so that the integral in (15) is well defined:

$$\beta_k > (1 - \gamma_k)\mu_Y - \frac{1}{2}\gamma_k(1 - \gamma_k)\sigma_Y^2. \quad (19)$$

## 2.6 The equilibrium

The notion of equilibrium that we use is an extension of the equilibrium in the single-agent model of Lucas (1978). Both agents optimize their expected lifetime utility and all markets must clear. So, in equilibrium, the two individuals consume all of the aggregate endowment. And, in the financial market we require that the two investors together hold all the shares that are a claim on aggregate dividends, and their aggregate holding of the zero supply risk-free bond must net to zero.

## 2.7 The central planner

Given our assumption that investors can trade in a stock and a locally risk-free asset, financial markets are dynamically complete relative to the filtrations of the two agents. When markets are dynamically complete, one can solve for equilibrium consumption policies using a “central-planner,” whose social welfare function is a weighted average of the value functions of individual agents, as shown in Basak (2005). In contrast to the case of identical beliefs, if the agents have heterogeneous beliefs, Basak (2005) shows that the weights used to construct the central planner’s value function are stochastic. The central planner’s problem is given by

$$\sup_{C_1 + C_2 \leq Y} \sum_{k=1}^2 \lambda_{k,t} U_k(C_{k,t}), \quad (20)$$

where  $\lambda_{k,t} = \lambda_{k,0}\xi_{k,t}$ .

## 3 Equilibrium consumption allocations and stationarity

In this section we derive exact closed-form expressions for equilibrium consumption allocations and also characterize the evolution of the consumption-sharing rule.

### 3.1 The consumption-sharing rule and its dynamics

We use the first-order condition for consumption to obtain the equation for the consumption sharing rule, which shows how aggregate consumption is divided between the two agents in equilibrium. The consumption sharing rule is given by

$$\lambda_{1,0}\xi_{1,t} e^{-\beta_1 t} C_{1,t}^{-\gamma_1} = \lambda_{2,0}\xi_{2,t} e^{-\beta_2 t} C_{2,t}^{-\gamma_2}. \quad (21)$$

Equation (21) is a consequence of the first-order condition for consumption, which follows from the central planner's problem (20). In order to solve explicitly for the equilibrium allocations, we write Agent  $k$ 's consumption share as  $\nu_{k,t} = \frac{C_{k,t}}{Y_t}$ , where  $0 \leq \nu_k \leq 1$ , and  $\nu_1 + \nu_2 = 1$ . Then:

$$\lambda_{1,0}\xi_{1,t} e^{-\beta_1 t} \nu_{1,t}^{-\gamma_1} Y_t^{-\gamma_1} = \lambda_{2,0}\xi_{2,t} e^{-\beta_2 t} \nu_{2,t}^{-\gamma_2} Y_t^{-\gamma_2}, \quad (22)$$

which can be rewritten as

$$\nu_{2,t}^\eta A_t = \nu_{1,t}, \quad (23)$$

where

$$A_t = \left( e^{(\beta_2 - \beta_1)t} Y_t^{\gamma_2 - \gamma_1} \frac{\lambda_{1,0}}{\lambda_{2,0}} \xi_t^{-1} \right)^{\frac{1}{\gamma_1}}, \quad (24)$$

and

$$\eta = \gamma_2 / \gamma_1. \quad (25)$$

When  $\eta \in \{1/4, 1/3, 1/2, 1, 2, 3, 4\}$ , the above equation can be written as a polynomial of degree 4 or less, thus allowing us to solve for the equilibrium consumption allocation in closed-form in terms of radicals, using standard results from polynomial theory, as pointed out in Wang (1996).<sup>4</sup>

<sup>4</sup>For example, if  $\eta = 2$ , solving the quadratic equation for  $\nu_{2,t}$  and taking the root that lies between 0 and 1, gives

$$\nu_{2,t} = \frac{1}{2A_t} \left( \sqrt{1 + 4A_t} - 1 \right).$$

Similarly, if  $\eta = 3$ , solving the cubic equation for  $\nu_{2,t}$ , and taking the root that lies between 0 and 1, gives:

$$\nu_{2,t} = - \left( \frac{2}{3D_t} \right)^{1/3} + \frac{1}{3A_t} \left( \frac{2}{3D_t} \right)^{-1/3} \quad \text{with } D_t = 9A_t^2 + \sqrt{3} \sqrt{27A_t^4 + 4A_t^3}.$$

And, when  $\eta = 4$ , solving the quartic equation for  $\nu_{2,t}$ , and taking the root that lies between 0 and 1, gives:

$$\nu_2 = \frac{\vartheta_{2,t}^{1/2}}{2} - \frac{1}{2} \left( -\frac{\vartheta_{1,t}}{2^{1/3} 3^{2/3} A_t} - \frac{2}{A_t \vartheta_{2,t}^{1/2}} + \frac{4 \left( \frac{2}{3} \right)^{1/3}}{\vartheta_{1,t}} \right)^{1/2},$$

$$\vartheta_{1,t} = \left( \sqrt{3} \times \sqrt{256A_t^3 + 27A_t^2 + 9A_t} \right)^{1/3}, \quad \text{and } \vartheta_{2,t} = \frac{\vartheta_{1,t}^{1/3}}{2^{1/3} 3^{2/3} A_t} - \frac{4 \left( \frac{2}{3} \right)^{1/3}}{\vartheta_{1,t}^{1/3}}.$$

Because polynomials of order 5 and above do not admit closed-form solutions in terms of radicals, it would appear that going beyond the results in Wang (1996) by solving for the consumption-sharing rule in closed-form when  $\eta$  is an integer greater than or equal to 5 is not possible. However, when  $\eta$  is an integer greater than or equal to 5, the consumption shares can be obtained in closed-form by using hypergeometric functions. We go further still by showing that when  $\eta$  is *any real number strictly greater than one*, it is possible to derive closed-form, convergent, series solutions for the sharing rule.<sup>5</sup>

**Proposition 1** *For all  $\eta \geq 1$ , Agent 2's, equilibrium share of the aggregate endowment,  $\nu_{k,t} = \frac{C_{1,t}}{Y_t}$ , is given by*

$$\nu_{2,t} = \begin{cases} 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta}{n-1} & , A_t < R \\ -\sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^n}{n} \binom{\frac{n}{\eta}}{n-1} & , A_t > R \end{cases} \quad (26)$$

where  $R = \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$ , and for  $z \in \mathbb{C}$  and  $k \in \mathbb{N}$ , and  $\binom{z}{k} = \prod_{j=1}^k \frac{z-k+j}{j}$  is the generalized binomial coefficient.

We see from the implicit expression for  $\nu_{2,t}$  in (23) that the consumption shares of the two agents will depend on  $A_t$ , which from (24) depends on the difference in the subject discount rates,  $\beta_1$  and  $\beta_2$ , the difference in risk aversions,  $\gamma_1$  and  $\gamma_2$ , and the difference in beliefs,  $\xi_t^{-1} = \xi_{1,t}/\xi_{2,t}$ .

From (24), we also see that  $A_t$  will evolve over time, and that its evolution will have a deterministic component and also a stochastic component, where the stochastic component depends on the stochastic behavior of aggregate endowment and the differences in beliefs. Below, we first define aggregate risk aversion, and then describe the dynamic behavior of the consumption sharing rule.

**Definition 1** *The aggregate relative risk aversion,  $\mathbf{R}_t$ , in the economy is defined as the consumption-share weighted harmonic average of individual agents' relative risk aversions:*

$$\mathbf{R}_t = \left( \frac{1}{\gamma_1} \nu_{1,t} + \frac{1}{\gamma_2} \nu_{2,t} \right)^{-1}. \quad (27)$$

*Equivalently, the aggregate risk tolerance in the economy,  $1/\mathbf{R}_t$ , is the consumption-share weighted average of individual agents' risk tolerances,  $1/\gamma_k$ .*

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<sup>5</sup>Because the derivation of the sharing rule for the general case where  $\eta$  is any real number strictly greater than one are given in full in the Appendix, the derivation showing how the sharing rule can be expressed in terms of hypergeometric functions when  $\eta$  is an integer greater than or equal to 5 is omitted but is available upon request.

**Proposition 2** *The true evolution of the sharing rule is given by*

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_{1,t}} dt + \sigma_{\nu_{1,t}} dZ_t, \quad (28)$$

where

$$\begin{aligned} \mu_{\nu_{1,t}} = & \nu_{2,t} \frac{1}{\gamma_1} \frac{1}{\gamma_2} \mathbf{R}_t \left\{ (\beta_2 - \beta_1) + (\gamma_2 - \gamma_1) \mu_Y - \mu_\xi \right. \\ & + \frac{1}{2} (\gamma_2 - \gamma_1) \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 \\ & + \frac{1}{2} \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \sigma_\xi^2 \\ & \left. - \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t \left[ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right] \sigma_Y \sigma_\xi \right\}, \quad (29) \end{aligned}$$

$$\sigma_{\nu_{1,t}} = \nu_{2,t} \frac{1}{\gamma_1} \frac{1}{\gamma_2} \mathbf{R}_t [(\gamma_2 - \gamma_1) \sigma_Y - \sigma_\xi]. \quad (30)$$

*Agent k believes the evolution of the sharing rule is given by*

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_{1,t}}^{\mathbb{P}^k} dt + \sigma_{\nu_{1,t}} dZ_{k,t}, \quad (31)$$

where

$$\begin{aligned} \mu_{\nu_{1,t}}^{\mathbb{P}^1} = & \nu_{2,t} \left\{ \frac{1}{\gamma_1} \frac{1}{\gamma_2} (\beta_2 - \beta_1) \mathbf{R}_t + \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t \mu_{Y,1} \right. \\ & + \frac{1}{2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 \\ & + \frac{1}{2} \frac{1}{\gamma_1 \gamma_2} \mathbf{R}_t \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \sigma_\xi^2 \\ & \left. - \frac{1}{\gamma_1 \gamma_2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t^2 \left[ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right] \sigma_Y \sigma_\xi \right\}, \quad (32) \end{aligned}$$

and

$$\begin{aligned}
\mu_{\nu_{1,t}}^{\mathbb{P}^2} &= \nu_{2,t} \left\{ \frac{1}{\gamma_1} \frac{1}{\gamma_2} (\beta_2 - \beta_1) \mathbf{R}_t + \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t \mu_{Y,2} - \frac{1}{\gamma_1 \gamma_2} \mathbf{R}_t \sigma_\xi^2 \right. \\
&\quad + \frac{1}{2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 \\
&\quad + \frac{1}{2} \frac{1}{\gamma_1 \gamma_2} \mathbf{R}_t \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \sigma_\xi^2 \\
&\quad \left. - \frac{1}{\gamma_1 \gamma_2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \mathbf{R}_t^2 \left[ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right] \sigma_Y \sigma_\xi \right\}. \quad (33)
\end{aligned}$$

We first discuss the expression for the volatility of the sharing rule. From (30), we see that the volatility of the sharing rule,  $\sigma_{\nu_{1,t}}$ , is driven by differences in risk aversion and differences in beliefs, but not differences in subjective discount rates, which have only a deterministic effect and so appear only in the expression for  $\mu_{\nu_{1,t}}$ . The expression for  $\sigma_{\nu_{1,t}}$  shows that an increase in heterogeneity in risk aversion leads to an increase in the volatility of the consumption share of Agent 1. However, heterogeneity in beliefs leads to an increase in the variance of the consumption share only if  $\sigma_{\xi,k}$  is negative, that is,  $\mu_{Y,2} < \mu_{Y,1}$  implying that Agent 1 is optimistic relative to Agent 2.

Similarly, we see from (29), exactly how  $\mu_{\nu_{1,t}}$  depends on differences in subjective discount rates, risk aversions, and beliefs. We also see how  $\mu_{\nu_{1,t}}$  is affected by the volatility of aggregate endowment growth,  $\sigma_Y$ , the volatility of the disagreement process,  $\sigma_\xi$ , and the covariance between these two processes,  $\sigma_Y \sigma_\xi$ .

### 3.2 Survival of agents and stationarity in the economy

Next, we derive conditions under which both agents survive in the long run. We say that the economy is stationary if both agents survive. To formalize the concept of survival, we introduce two complementary concepts of survival: *almost sure (a.s.) survival* with respect to a particular measure, and *mean survival* with respect to a particular measure. We define *almost sure survival* as follows.

**Definition 2** *Agent  $k$  survives  $\mathbb{P}$ -a.s. if*

$$\lim_{t \rightarrow \infty} \nu_{k,t} > 0, \mathbb{P} - a.s. \quad (34)$$

*Similarly, Agent  $k$  survives  $\mathbb{P}^j$ -a.s. if*

$$\lim_{t \rightarrow \infty} \nu_{k,t} > 0, \mathbb{P}^j - a.s. \quad (35)$$

To understand the above concept of survival, note that if an agent's consumption share is strictly above zero with a probability of less than one, under  $\mathbb{P}$  say, then she does not survive  $\mathbb{P}$ -almost surely. Furthermore, the probability measure is important, because an agent may believe she survives almost surely (with respect to the measure representing her beliefs), when in fact, she almost surely does not survive under the true measure  $\mathbb{P}$ .

We define *mean survival* with respect to a particular measure as follows.

**Definition 3** *Agent  $k$  survives in the mean with respect to  $\mathbb{P}$  if*

$$\lim_{u \rightarrow \infty} E_t \nu_{k,t+u} > 0. \quad (36)$$

*Similarly, Agent  $k$  survives in the mean with respect to  $\mathbb{P}^j$  if*

$$\lim_{u \rightarrow \infty} E_t^j \nu_{k,t+u} > 0. \quad (37)$$

To understand the difference between survival in mean and survival almost surely under say,  $\mathbb{P}$ , note that if an agent's consumption share tends to zero with probability 1/2 and to one with probability 1/2, then the long-run mean of her consumption share will be 1/2. Thus, in the almost surely sense, she does not survive, but in the mean sense, she does.

The economy is stationary if both agents survive. Each concept of survival leads to a corresponding concept of stationary: *almost sure stationarity* under a particular measure, and *mean stationarity* under a particular measure.

One can see immediately from (23) that both agents survive almost surely under the true measure  $\mathbb{P}$  and the economy is hence almost surely stationary under  $\mathbb{P}$ , if agents' relative risk aversions are equal and the exponential decay rates of the deterministic component of the weights in the social planner problem are equal. We can also show the latter two conditions are not only sufficient, but are also necessary. Formally, we have the following result:

**Proposition 3** *1. The economy is almost surely stationary under  $\mathbb{P}$  if and only if*

$$(\beta_1 - \beta_2) - (\gamma_2 - \gamma_1) \left( \mu_Y - \frac{1}{2} \sigma_Y^2 \right) - \frac{1}{2} (\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) = 0, \quad (38)$$

*and*

$$(\mu_{Y,2} - \mu_{Y,1}) = (\gamma_2 - \gamma_1) \sigma_Y^2. \quad (39)$$



2. *Agent 1 believes the economy is almost surely stationary if and only if*

$$(\beta_2 - \beta_1) + \frac{1}{2}\sigma_\xi^2 + (\gamma_2 - \gamma_1)(\mu_{Y,1} - \frac{1}{2}\sigma_Y^2) = 0, \quad (40)$$

and

$$\mu_{Y,2} - \mu_{Y,1} = (\gamma_2 - \gamma_1)\sigma_Y^2. \quad (41)$$

3. *Agent 2 believes the economy is almost surely stationary if and only if*

$$(\beta_2 - \beta_1) - \frac{1}{2}\sigma_\xi^2 + (\gamma_2 - \gamma_1)(\mu_{Y,2} - \frac{1}{2}\sigma_Y^2) = 0, \quad (42)$$

and

$$\mu_{Y,2} - \mu_{Y,1} = (\gamma_2 - \gamma_1)\sigma_Y^2. \quad (43)$$

The conditions for mean stationarity are given below.

**Proposition 4** 1. *The economy is mean stationary under  $\mathbb{P}$  if and only if*

$$(\beta_1 - \beta_2) - (\gamma_2 - \gamma_1) \left( \mu_Y - \frac{1}{2}\sigma_Y^2 \right) - \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) = 0. \quad (44)$$

2. *Agent 1 believes the economy mean stationary if and only if*

$$(\beta_2 - \beta_1) + \frac{1}{2}\sigma_\xi^2 + (\gamma_2 - \gamma_1)(\mu_{Y,1} - \frac{1}{2}\sigma_Y^2) = 0. \quad (45)$$

3. *Agent 2 believes the economy is mean stationary if and only if*

$$(\beta_2 - \beta_1) - \frac{1}{2}\sigma_\xi^2 + (\gamma_2 - \gamma_1)(\mu_{Y,2} - \frac{1}{2}\sigma_Y^2) = 0. \quad (46)$$

## 4 Asset prices and risk premia for stocks and bonds

In this section, we compute asset prices and properties of their returns by using the state-price density. We then use these results to analyze how heterogeneity in beliefs, rates of time preference and in risk aversion impacts equilibrium asset prices.

## 4.1 The equilibrium state-price density

We use the result for the optimal consumption sharing rule to derive an intuitively appealing expression for the equilibrium state-price density, and use this to understand how heterogeneity in preferences and beliefs affects the equilibrium risk-free rate and market price of risk. We then derive an alternative expression for the state-price density, which allows us to compute asset prices in closed form.

Agent  $k$ 's state-price density,  $\pi_{k,t}$ , is given by

$$\pi_{k,t} = \lambda_{k,0} e^{-\beta_k t} Y_t^{-\gamma_k} \nu_{k,t}^{-\gamma_k}. \quad (47)$$

It then follows from the first-order condition for consumption in (21), that

$$\xi_{1,t} \pi_{1,t} = \xi_{2,t} \pi_{2,t}, \quad (48)$$

that is,

$$\pi_{1,t} = \pi_{2,t} \xi_t. \quad (49)$$

Before characterizing the state-price density for each agent, we define the aggregate rate of time preference, the aggregate beliefs, and the aggregate prudence in this economy.

**Definition 4** *The aggregate rate of time preference in the economy,  $\beta_t$ , is given by the weighted arithmetic mean of individual agents' rates of time preference, where the weights are the consumption-share weighted relative risk tolerances of the two investors:*

$$\beta_t = w_{1,t} \beta_1 + w_{2,t} \beta_2, \quad (50)$$

$$w_k = \frac{\frac{1}{\gamma_k} \nu_{k,t}}{\frac{1}{\gamma_1} \nu_{1,t} + \frac{1}{\gamma_2} \nu_{2,t}}, \text{ and } w_1 + w_2 = 1, \quad (51)$$

**Definition 5** *The aggregate beliefs,  $\mu_{Y,t}$ , in this economy are given by the weighted arithmetic mean of the beliefs of individual agents, where the weights are the consumption-share weighted relative risk tolerances of the two investors, as defined in (51)*

$$\mu_{Y,t} = w_{1,t} \mu_{Y,1} + w_{2,t} \mu_{Y,2}. \quad (52)$$

**Definition 6** *The quantity  $\mathbf{P}_t$  is the aggregate prudence in the economy when agents beliefs are identical:*

$$\mathbf{P}_t = (1 + \gamma_1) \left( \frac{\mathbf{R}_t}{\gamma_1} \right)^2 \nu_{1,t} + (1 + \gamma_2) \left( \frac{\mathbf{R}_t}{\gamma_2} \right)^2 \nu_{2,t}, \quad (53)$$

and  $\mathbf{P}_t^{har}$  is the weighted harmonic mean of individual relative agents' prudences, when agents have power utility, where the weights are the consumption shares, i.e.

$$\mathbf{P}_t^{har} = \left( \sum_{k=1}^2 \frac{\nu_{k,t}}{1 + \gamma_k} \right)^{-1}. \quad (54)$$

The following proposition characterizes each agent's equilibrium state-price density in terms of closed-form expressions for perceived market prices of risk and the risk-free rate.

**Proposition 5** *Agent  $k$ 's state-price density,  $\pi_{k,t}$ , is:*

$$\frac{d\pi_{k,t}}{\pi_{k,t}} = -r_t dt - \theta_{k,t} dZ_{k,t}, \quad (55)$$

where the market price of risk perceived by Agents  $k$ ,  $\theta_k$ , is given by

$$\theta_{1,t} = \mathbf{R}_t \left( \sigma_Y - \frac{\nu_{2,t}}{\gamma_2} \sigma_\xi \right), \quad (56)$$

$$\theta_{2,t} = \mathbf{R}_t \left( \sigma_Y + \frac{\nu_{1,t}}{\gamma_1} \sigma_\xi \right), \quad (57)$$

and the locally risk-free rate is given by

$$\begin{aligned} r_t = & \beta_t + \mathbf{R}_t \boldsymbol{\mu}_Y - \frac{1}{2} \mathbf{R}_t \mathbf{P}_t \sigma_Y^2 \\ & + w_{1,t} w_{2,t} \left( 1 - \frac{1}{2} \frac{w_{1,t} w_{2,t}}{v_{1,t} v_{2,t}} \mathbf{R}_t^{-1} \mathbf{P}_t^{har} \right) \sigma_\xi^2 - \mathbf{R}_t w_{1,t} w_{2,t} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \sigma_Y \sigma_\xi, \end{aligned} \quad (58)$$

where the weights,  $v_{k,t}$ , are given by

$$v_{k,t} = \frac{\frac{\nu_{k,t}}{1 + \gamma_k}}{\sum_{k=1}^2 \frac{\nu_{k,t}}{1 + \gamma_k}}, \quad \text{where } v_{1,t} + v_{2,t} = 1. \quad (59)$$

In the corollary below, we consider the special case where agents have identical beliefs that are also the correct ones.

**Corollary 1** *Suppose agents have identical and correct beliefs. Then the equilibrium state-price density,  $\pi_t$ , is:*

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \theta_t dZ_t, \quad (60)$$

where the market price of risk,  $\theta$ , is given by

$$\theta_t = \mathbf{R}_t \sigma_Y, \quad (61)$$

and the locally risk-free rate is given by

$$r_t = \beta_t + \mathbf{R}_t \mu_Y - \frac{1}{2} \mathbf{R}_t \mathbf{P}_t \sigma_Y^2. \quad (62)$$

In the next corollary, we study the case where the two agents have different beliefs but identical preferences.

**Corollary 2** *Suppose agents have identical preferences, i.e.  $\beta_1 = \beta_2 = \beta$ , and  $\gamma_1 = \gamma_2 = \gamma$ , but different beliefs. Then the equilibrium locally risk-free rate is given by*

$$r_t = \beta + \gamma \sum_{k=1}^2 \nu_{k,t} \mu_{Y,k} - \frac{1}{2} \gamma (1 + \gamma) \sigma_Y^2 + \frac{1}{2} \nu_{1,t} \nu_{2,t} \left(1 - \frac{1}{\gamma}\right) \sigma_\xi^2, \quad (63)$$

and the equilibrium market price of risk perceived by Agent  $k$  is given by

$$\theta_{k,t} = \gamma \sigma_Y - (1 - \nu_{k,t}) \sigma_\xi. \quad (64)$$

In the following proposition, we present the solution for each agent's state-price density using convergent series.

**Proposition 6** *Agent 1's state-price density is given by*

$$\pi_{1,t} = \begin{cases} a_2 e^{-\beta_2 t} Y_t^{-\gamma_2} \xi_t \left(1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1}\right) & , A_t < R, \\ a_1 e^{-\beta_1 t} Y_t^{-\gamma_1} \left(1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1}\right) & , A_t > R, \end{cases} \quad (65)$$

and Agent 2's state-price density is given by

$$\pi_{2,t} = \begin{cases} a_2 e^{-\beta_2 t} Y_t^{-\gamma_2} \left(1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1}\right) & , A_t < R, \\ a_1 e^{-\beta_1 t} Y_t^{-\gamma_1} \xi_t^{-1} \left(1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1}\right) & , A_t > R. \end{cases} \quad (66)$$

## 4.2 Valuation of risky assets, the risk premium, and volatility of returns

We now use the expressions for the state-price densities in the proposition above to derive a nonlinear ordinary differential equation for the price-dividend ratio of a generic asset. Then we use the closed-form expression for the state-price density to derive the exact solution to the nonlinear ordinary differential equation.

Consider the cash flow process

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X^{sys} dZ_t + \sigma_X^{id} dZ_t^{id}, \quad (67)$$

where  $Z_t^{id}$  is a standard Brownian motion under  $\mathbb{P}$ , orthogonal to  $Z_t$ . Under measure  $\mathbb{P}^k$ ,  $k \in \{1, 2\}$ , the dynamics of the cash flow process are given by

$$\frac{dX_t}{X_t} = \mu_{X,k} dt + \sigma_X^{sys} dZ_{k,t} + \sigma_X^{id} dZ_t^{id}, \quad (68)$$

where  $\mu_{X,k}$  is given by

$$\frac{\mu_{X,k} - \mu_X}{\sigma_X^{sys}} = \frac{\mu_{Y,k} - \mu_Y}{\sigma_Y}. \quad (69)$$

Observe that the correlation between shocks to the growth rates of  $X_t$  and shocks to the growth rate of  $Y_t$  is given by

$$\rho_{XY} = \frac{\sigma_X^{sys}}{\sigma_X}, \quad (70)$$

where  $\sigma_X = \sqrt{(\sigma_X^{sys})^2 + (\sigma_X^{id})^2}$ . The value of a claim which pays out the cash flow,  $X$ , per unit time, in perpetuity, is given by

$$P_t^X = X_t E_t^1 \int_t^\infty \frac{\pi_{1,u} X_u}{\pi_{1,t} X_t} du, \quad (71)$$

or equivalently by

$$P_t^X = X_t E_t \int_t^\infty \frac{\pi_{1,u} \xi_{1,u} X_u}{\pi_{1,t} \xi_{1,t} X_t} du, \quad (72)$$

or

$$P_t^X = X_t E_t^2 \int_t^\infty \frac{\pi_{2,u} X_u}{\pi_{2,t} X_t} du, \quad (73)$$

Note also that when  $\mu_X = \mu_Y$ ,  $\sigma_X^{sys} = \sigma_Y$ , and  $\sigma_X^{id} = 0$ , then the above price reduces to the value of the stock market,  $P$ , given by

$$P_t = Y_t E_t^1 \int_t^\infty \frac{\pi_{1,u} Y_u}{\pi_{1,t} Y_t} du. \quad (74)$$

The price,  $P_t^X$ , can be written in terms of the price-dividend ratio,  $p_t^X$ , i.e.

$$P_t^X = X_t p_t^X, \quad (75)$$

where

$$p_t^X = E_t^1 \int_t^\infty \frac{\pi_{1,u} X_u}{\pi_{1,t} X_t} du = E_t \int_t^\infty \frac{\pi_{1,u} \xi_{1,u} X_u}{\pi_{1,t} \xi_{1,t} X_t} du = E_t^2 \int_t^\infty \frac{\pi_{2,u} X_u}{\pi_{2,t} X_t} du. \quad (76)$$

The price-dividend ratio,  $p_t^X$ , depends on the distribution of consumption across the two agents in the economy, and would be a constant if agents' consumption shares were constant. Hence, the price-dividend ratio is a function of the consumption share, that is,  $p_t^X = p^X(\nu_{1,t})$ .

Observe that agents agree on prices, which is a consequence of no arbitrage. However, agents do not agree on risk premia. To see this note that the price,  $P_t^X$ , satisfies the basic asset pricing equation

$$E_t^1 \left[ \frac{dP_t^X + X_t dt}{P_t^X} - r_t dt \right] = -E_t^1 \left[ \frac{dP_t^X}{P_t^X} \frac{d\pi_{1,t}}{\pi_{1,t}} \right]. \quad (77)$$

Applying Ito's Lemma to (75) and substituting into the left-hand side of (77) gives the following expression for the expected risk premium on the asset perceived by Agent 1:

$$\mu_{X,1} + \mu_{\nu_{1,t}}^{\mathbb{P}^1} \sigma_X^{sys} \sigma_{\nu_{1,t}} \frac{\nu_{1,t} p_{\nu_{1,t}}^X}{p_t^X} + \frac{1}{2} \nu_{1,t}^2 \frac{p_{\nu_{1,t} \nu_{1,t}}^X}{p_t^X} \sigma_{\nu_{1,t}}^2 + \frac{1}{p_t^X} - r_t. \quad (78)$$

The first term in the above expression comes from Agent's perception of expected cash flow growth and the term,  $\frac{1}{p_t^X}$ , is the contribution of the current cash flow to expected return and  $r$  is of course the risk-free rate. The remaining terms reflect the impact of the time-varying discount rate on the risk premium. The sole reason the discount rate is time-varying is because the consumption shares of the two agents are time varying. Hence, the drift and diffusion terms,  $\mu_{\nu_{1,t}}^{\mathbb{P}^1}$ , and  $\sigma_{\nu_{1,t}}$ , which account for the dynamics of the consumption sharing rule, appear in the expression for the expected risk premium.

Applying Ito's Lemma to (75) and substituting into the right-hand side of (77) gives an alternative expression for the expected risk premium on the asset:

$$\mathbf{R}_t \left( \sigma_Y - \frac{\nu_{2,t}}{\gamma_2} \sigma_\xi \right) \sigma_X^{sys} + \mathbf{R}_t \left( \sigma_Y - \frac{\nu_{2,t}}{\gamma_2} \sigma_\xi \right) \sigma_{\nu_{1,t}} \frac{\nu_{1,t} p_{\nu_{1,t}}^X}{p_t^X}. \quad (79)$$

The first term in the above expression gives the contribution of cash flow risk to the risk premium and the second term prices the risk inherent in time-varying returns.

Equating the alternative expressions for the asset's risk premium gives the nonlinear ordinary differential equation:

$$\begin{aligned}
0 &= \mu_{X,1} - \mathbf{R}_t \left( \sigma_Y - \frac{\nu_{2,t}}{\gamma_2} \sigma_\xi \right) \sigma_X^{sys} \\
&+ \left[ \mu_{\nu_{1,t}}^{\mathbb{P}^1} + \left( \sigma_X^{sys} - \mathbf{R}_t \left( \sigma_Y - \frac{\nu_{2,t}}{\gamma_2} \sigma_\xi \right) \right) \sigma_{\nu_{1,t}} \right] \frac{\nu_{1,t} p_{\nu_{1,t}}^X}{p_t^X} + \frac{1}{2} \nu_{1,t}^2 \frac{p_{\nu_{1,t}\nu_{1,t}}^X}{p_t^X} \sigma_{\nu_{1,t}}^2 + \frac{1}{p_t^X} - r_t.
\end{aligned} \tag{80}$$

The above equation has natural boundary conditions:  $p^X(0) = \frac{1}{\bar{r}_2 + \gamma_2 \sigma_X^{sys} \sigma_Y - \mu_{X,1}}$  and  $p^X(1) = \frac{1}{\bar{r}_1 + \gamma_1 \sigma_X^{sys} \sigma_Y - \mu_{X,1}}$ , which are a consequence of the equation's limiting behavior at  $\nu_{k,t} = 0$ ,  $k \in \{1, 2\}$ .

The nonlinear differential equation for the stock-price can be solved numerically by standard methods. We derive an exact closed-form solution by using the series expression for the state-price density in Proposition 6 to directly evaluate the expectation of the integral in the right-hand side of (76). We state this result as Proposition 7 below:

**Proposition 7** *The price of the claim to the cash flow,  $X_t$ , is given by  $P_t^X = p_t^X X_t$ , where*

$$p_t^X = \nu_{2,t}^{\gamma_2} p_{l,t}^X + \nu_{1,t}^{\gamma_1} p_{r,t}^X, \tag{81}$$

where

$$p_{l,t}^X = \zeta_{0,l,t}^X - \gamma_2 \sum_{n=1}^{\infty} \zeta_{n,l,t}^X \frac{\left( -\frac{\nu_{1,t}}{\nu_{2,t}} \right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1}, \tag{82}$$

$$p_{r,t}^X = \zeta_{0,r,t}^X - \gamma_1 \sum_{n=1}^{\infty} \zeta_{n,r,t}^X \frac{\left( -\frac{\nu_{2,t}}{\nu_{1,t}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1}, \tag{83}$$

where  $\zeta_{n,l,t}^X$ ,  $(\zeta_{n,r,t}^X)$  are the prices of fundamental securities when Agent 2 (Agent 1) is the sole agent in the economy, and which pay out  $A_t^n$ ,  $(A_t^{-n/\eta})$  units of consumption per unit time whenever  $A_t < R$ ,  $(A_t > R)$ . The prices of these fundamental securities are given in closed-form by

$$\zeta_{n,l,t}^X = \begin{cases} -\frac{1}{\frac{1}{2}(\sigma_A)^2(n-a_-^\xi(k_2))(n-a_+(k_2))} + \frac{1}{\frac{1}{2}(\sigma_A)^2(n-a_+^\xi(k_2))(a_+^\xi(k_2)-a_-^\xi(k_2))} \left( \frac{A_t}{R} \right)^{a_+^\xi(k_2)-n}, & A_t < R, \\ \frac{1}{\frac{1}{2}(\sigma_A)^2(n-a_-^\xi(k_2))(a_+^\xi(k_2)-a_-^\xi(k_2))} \left( \frac{A_t}{R} \right)^{a_-^\xi(k_2)-n}, & A_t \geq R, \end{cases} \tag{84}$$

and

$$\zeta_{n,r,t}^X = \begin{cases} \frac{1}{\frac{1}{2}(\sigma_A)^2 \binom{n+a_+(k_1)}{\eta} \binom{a_+(k_1)-a_-(k_1)}{a_+(k_1)+\frac{n}{\eta}}} \left(\frac{A_t}{R}\right)^{a_+(k_1)+\frac{n}{\eta}}, & A_t < R, \\ \frac{1}{\frac{1}{2}(\sigma_A)^2 \binom{n+a_-}{\eta} \binom{a_+(k_1)-a_-(k_1)}{a_+(k_1)+\frac{n}{\eta}}} \left(\frac{A_t}{R}\right)^{a_-(k_1)+\frac{n}{\eta}} - \frac{1}{\frac{1}{2}(\sigma_A)^2 \binom{n+a_+(k_1)}{\eta} \binom{n+a_-(k_1)}{\eta}}, & A_t \geq R, \end{cases} \quad (85)$$

where

$$\hat{\mu}_A^1 = \mu_{A,1} + (\sigma_X^{sys} - \gamma_1 \sigma_Y) \sigma_A, \quad (86)$$

$$\hat{\mu}_A^{2,\xi} = \mu_{A,1} + (\sigma_X^{sys} + \sigma_\xi - \gamma_2 \sigma_Y) \sigma_A, \quad (87)$$

$$\mu_{A,1} = \frac{\beta_2 - \beta_1}{\gamma_1} + (\eta - 1) \left( \mu_{Y,1} - \frac{1}{2} \sigma_Y^2 \right) + \frac{1}{2\gamma_1} \sigma_\xi^2 + \frac{1}{2} \sigma_A^2, \quad (88)$$

$$\sigma_A = (\eta - 1) \sigma_Y - \frac{1}{\gamma_1} \sigma_\xi, \quad (89)$$

$$k_k = \bar{r}_k + \gamma_k \sigma_X^{sys} \sigma_Y - \mu_{X,k}, \quad k \in \{1, 2\}, \quad (90)$$

where

$$a_\pm(k_1) = \frac{-(\hat{\mu}_A^1 - \frac{1}{2}(\sigma_A)^2) \pm \sqrt{(\hat{\mu}_A^1 - \frac{1}{2}(\sigma_A)^2)^2 + 2k_1(\sigma_A)^2}}{(\sigma_A)^2}, \quad (91)$$

$$a_\pm^\xi(k_2) = \frac{-(\hat{\mu}_A^{2,\xi} - \frac{1}{2}(\sigma_A)^2) \pm \sqrt{(\hat{\mu}_A^{2,\xi} - \frac{1}{2}(\sigma_A)^2)^2 + 2k_2(\sigma_A)^2}}{(\sigma_A)^2}. \quad (92)$$

In the next proposition, we give expressions for the volatility of returns on the claim paying  $X_t$  per unit time in perpetuity and its market risk premium, in terms of  $p^X(\nu_{1,t})$  and  $\frac{\partial p_{\nu_{1,t}}^X}{\partial \nu_{1,t}}$ .

**Proposition 8** *The risk premium on the claim paying  $X_t$  per unit time in perpetuity,  $\mu_R^X - r$ , is given by*

$$\mu_{R,t}^X - r_t = \mathbf{R}_t \left( \sigma_Y - \frac{\nu_{2,t}}{\gamma_2} \sigma_\xi \right) \sigma_{R,t}^{X,sys}, \quad (93)$$

and the volatility of the claim's returns,  $\sigma_{R,t}^X$ , is

$$\sigma_{R,t}^X = \sigma_{R,t}^{X,id} + \sigma_{R,t}^{X,sys}, \quad (94)$$

where the idiosyncratic component of the volatility of the claim's returns is given by

$$\sigma_{R,t}^{X,id} = \sigma_X^{id}, \quad (95)$$



and the systematic component of the volatility of the claim's returns is given by

$$\sigma_{R,t}^{X,sys} = \sigma_X^{sys} + \sigma_{\nu_1,t} \frac{\nu_{1,t}}{p_t^X} \frac{\partial p_{\nu_1,t}^X}{\partial \nu_{1,t}}, \quad (96)$$

where

$$\frac{\partial p_t^X}{\partial \nu_{1,t}} = -\gamma_2 \nu_{2,t}^{\gamma_2} \frac{p_{l,t}^X}{\nu_{1,t}} + \gamma_1 \nu_{1,t}^{\gamma_1} \frac{p_{r,t}^X}{\nu_{1,t}} \quad (97)$$

$$+ \nu_{2,t}^{\gamma_2} \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ \frac{\partial \zeta_{0,l,t}^X}{\partial \ln A_t} - \gamma_2 \sum_{n=1}^{\infty} \left( \frac{\partial \zeta_{n,l,t}^X}{\partial \ln A_t} + n \zeta_{n,l,t}^X \right) \frac{\left( \frac{-\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}} \right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right]$$

$$+ \nu_{1,t}^{\gamma_1} \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ \frac{\partial \zeta_{0,r,t}^X}{\partial \ln A_t} - \gamma_1 \sum_{n=1}^{\infty} \left( \frac{\partial \zeta_{n,r,t}^X}{\partial \ln A_t} - \frac{n}{\eta} \zeta_{n,r,t}^X \right) \frac{\left( \frac{-\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right], \quad (98)$$

$$\frac{\partial \zeta_{n,l,t}^X}{\partial \ln A_t} = \begin{cases} -\frac{1}{\frac{1}{2}(\sigma_A)^2(a_+^\xi(k_2) - a_-^\xi(k_2))} \left( \frac{A_t}{R} \right)^{a_+^\xi(k_2) - n}, & A_t < R, \\ -\frac{1}{\frac{1}{2}(\sigma_A)^2(a_+^\xi(k_2) - a_-^\xi(k_2))} \left( \frac{A_t}{R} \right)^{a_-^\xi(k_2) - n}, & A_t \geq R, \end{cases} \quad (98)$$

and

$$\frac{\partial \zeta_{n,r,t}^X}{\partial \ln A_t} = \begin{cases} \frac{1}{\frac{1}{2}(\sigma_A)^2(a_+(k_1) - a_-(k_1))} \left( \frac{A_t}{R} \right)^{a_+(k_1) + \frac{n}{\eta}}, & A_t < R, \\ \frac{1}{\frac{1}{2}(\sigma_A)^2(a_+(k_1) - a_-(k_1))} \left( \frac{A_t}{R} \right)^{a_-(k_1) + \frac{n}{\eta}}, & A_t \geq R. \end{cases} \quad (99)$$

### 4.3 Valuation of bonds and the term premium

We now derive a closed-form expression for time- $t$  price of a zero coupon claim which pays out  $X_T$  units of consumption at time  $T$ . The price of this claim is denoted by  $V_{T-t}^X$ , and is given by

$$V_{T-t}^X = X_T v_{T-t}^X, \quad (100)$$

where

$$v_{T-t}^X = E_t^1 \left[ \frac{\pi_{1,T}}{\pi_{1,t}} \frac{X_T}{X_t} \right] = E_t \left[ \frac{\pi_{1,T}}{\pi_{1,t}} \frac{\xi_{1,T}}{\xi_{1,t}} \frac{X_T}{X_t} \right] = E_t^2 \left[ \frac{\pi_{2,T}}{\pi_{2,t}} \frac{X_T}{X_t} \right]. \quad (101)$$

The price of the above risky zero coupon claim reduces to the price of a risk-free zero coupon bond when  $\mu_{X,1} = \mu_{X,2} = \mu_X = 0$ , and  $\sigma_X^{sys} = \sigma_X^{id} = 0$ .

**Proposition 9** *The time- $t$  price of the claim which pays out  $X_T$  units of consumption at time  $T$  is given by  $V_{T-t}^X = v_{T-t}^X X_t$ , where*

$$v_{T-t}^X = \nu_{2,t}^{\gamma_2} \left( \phi_{0,l,t} - \gamma_2 \sum_{n=1}^{\infty} \phi_{n,l,t} \frac{\left(-\frac{\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}}\right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) + \nu_{1,t}^{\gamma_1} \left( \phi_{0,r,t} - \gamma_1 \sum_{n=1}^{\infty} \phi_{n,r,t} \frac{\left(-\frac{\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}}\right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right), \quad (102)$$

where  $\phi_{n,l,t}$ ,  $(\phi_{n,r,t})$  are the prices of fundamental securities when Agent 2 (Agent 1) is the sole agent in the economy, and which pay out  $A_T^n$ ,  $(A_T^{-n/\eta})$  units of consumption at time  $T$  if  $A_T < R$ ,  $(A_T > R)$ . The prices of these fundamental securities are given in closed-form by

$$\phi_{n,l,T-t} = e^{-[k_2 - n\hat{\mu}_A^2 - \frac{1}{2}n(n-1)\sigma_A^2](T-t)} \Phi \left( \frac{\ln \left( \frac{R}{A_T} \right) - \left( \hat{\mu}_A^{2,\xi} + \frac{1}{2}(2n-1)\sigma_A^2 \right) (T-t)}{\sigma_A(T-t)^{1/2}} \right), \quad (103)$$

and

$$\phi_{n,r,T-t} = e^{-\left[ k_1 + \frac{n}{\eta} \left( \hat{\mu}_A^1 - \frac{1}{2} \left( 1 + \frac{2n}{\eta} \right) \sigma_A^2 \right) \right] (T-t)} \left[ 1 - \Phi \left( \frac{\ln \left( \frac{R}{A_T} \right) - \left( \hat{\mu}_A^1 - \frac{1}{2} \left( 1 + \frac{2n}{\eta} \right) \sigma_A^2 \right) (T-t)}{\sigma_A(T-t)^{1/2}} \right) \right], \quad (104)$$

where  $\Phi(\cdot)$  is the cumulative normal distribution function.

## 5 Wealth and portfolio holdings of each individual agent

**Proposition 10** *Agent 1's wealth at time  $t$  is given by  $W_{1,t} = w_{1,t}^Y Y_t$ , where*

$$w_{1,t}^Y = \nu_{2,t}^{\gamma_2} \sum_{n=1}^{\infty} \frac{\left(-\frac{\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}}\right)^n}{n} \left[ (1 - \gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y + \nu_{1,t}^{\gamma_1} \left[ \zeta_{0,r,t}^Y - (1 - \gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{\left(-\frac{\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}}\right)^n}{n} \zeta_{n,r,t}^Y \right], \quad (105)$$

and Agent 2's wealth is given by

$$W_{2,t} = P_t^Y - W_{1,t}. \quad (106)$$

The proportion of Agent 1's wealth invested in the stock market,  $\Pi_{1,t}$ , is given by

$$\Pi_{1,t} = \frac{\sigma_{W_{1,t}}}{\sigma_{R,t}}, \quad (107)$$

where

$$\sigma_{W_{1,t}} = \sigma_Y + \frac{w_{1,t}^Y}{\nu_{1,t}} \frac{\partial w_{1,t}^Y}{\partial \nu_{1,t}}, \quad (108)$$

and

$$\begin{aligned} w_{1,t}^Y = & \nu_{2,t}^{\gamma_2} \sum_{n=1}^{\infty} \frac{\left(\frac{-\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}}\right)^n}{n} \left[ (1 - \gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y \\ & + \nu_{1,t}^{\gamma_1} \left[ \zeta_{0,r,t}^Y - (1 - \gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{\left(\frac{-\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}}\right)^n}{n} \zeta_{n,r,t}^Y \right]. \end{aligned} \quad (109)$$

The proportion of Agent 2's wealth invested in the stock market,  $\Pi_{2,t}$ , is given by

$$\Pi_{2,t} = \frac{P_t - W_{1,t}\Pi_{1,t}}{P_t}. \quad (110)$$

The proportion of Agent  $k$ 's wealth invested in the locally risk-free bond is  $1 - \Pi_{k,t}$ .

## 6 Conclusion

In this paper, we study an endowment economy where there are two types of agents, each with expected (power) utility. The two agents are heterogeneous with respect to their preference parameters for the subjective rate of time preference and relative risk aversion, and also with respect to their beliefs. The two agents can invest in a stock, which is a claim on endowment, and an instantaneously risk-free asset, which is in zero net supply. We solve for the equilibrium in this economy and identify the optimal consumption sharing rule. We use this to identify the market price of risk, the locally risk-free interest rate, the stock price, the equity market risk premium, and the volatility of stock returns and the prices of bonds and the term premium. We then analyze how heterogeneity in preferences affects prices of stocks and bonds. We find, for instance, that heterogeneity in risk aversion can generate volatility in asset returns that is in excess of volatility of fundamentals.

## A Appendix: Two lemmas

We shall make repeated use of the following two lemmas, the proofs of which are given in the supplementary appendix that is available from the authors.

**Definition A1** *The date- $t$  price of the fundamental financial security which pays out  $E_t^n$  units of consumption per unit time in perpetuity as long as  $E_u < B$ , where  $E_u = E_t e^{(\mu - \frac{1}{2}\sigma^2)(u-t) + \sigma(Z_u - Z_t)}$  and the discount rate is assumed to be  $k_2$ , is given by  $V_{2,n,t} = V_{2,n}(E_t)$ , where*

$$V_{2,n}(E_t) = E_t \int_t^\infty e^{-k_2(u-t)} E_u^n 1_{\{E_u < B\}}. \quad (\text{A1})$$

*The date- $t$  price of the fundamental financial security which pays out  $E_t^{-n/\eta}$  units of consumption per unit time in perpetuity as long as  $E_u > B$ , where  $E_u = E_t e^{(\mu - \frac{1}{2}\sigma^2)(u-t) + \sigma(Z_u - Z_t)}$  and the discount rate is assumed to be  $k_1$ , is given by  $V_{1,n,t} = V_{1,n}(E_t)$ , where*

$$V_{1,n}(E_t) = E_t \int_t^\infty e^{-k_1(u-t)} E_u^{-n/\eta} 1_{\{E_u > B\}}. \quad (\text{A2})$$

**Lemma A1** *The prices of the fundamental securities are given by*

$$V_{2,n}(E) = \begin{cases} -\frac{E^n}{\frac{1}{2}\sigma^2(n-a_-(k_2))(n-a_+(k_2))} + \frac{B^n}{\frac{1}{2}\sigma^2(n-a_+(k_2))(a_+(k_2)-a_-(k_2))} \left(\frac{E}{B}\right)^{a_+(k_2)} & , E < B, \\ \frac{B^n}{\frac{1}{2}\sigma^2(n-a_-(k_2))(a_+(k_2)-a_-(k_2))} \left(\frac{E}{B}\right)^{a_-(k_2)} & , E \geq B. \end{cases}, \quad (\text{A3})$$

and

$$V_{1,n}(E) = \begin{cases} \frac{B^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+(k_1)\right)(a_+(k_1)-a_-(k_1))} \left(\frac{E}{B}\right)^{a_+(k_1)} & , e < b, \\ \frac{B^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_-\right)(a_+(k_1)-a_-(k_1))} \left(\frac{E}{B}\right)^{a_-(k_1)} - \frac{E^{-\frac{n}{\eta}}}{\frac{1}{2}\sigma^2\left(\frac{n}{\eta}+a_+(k_1)\right)\left(\frac{n}{\eta}+a_-(k_1)\right)} & , e \geq b. \end{cases} \quad (\text{A4})$$

where

$$a_\pm(k) = \frac{-(\mu - \frac{1}{2}\sigma^2) \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2k\sigma^2}}{\sigma^2}. \quad (\text{A5})$$

**Definition A2** *The date- $t$  price of the fundamental financial security which pays out  $E_T^n$  units of consumption at time  $T$  if  $E_T < B$ , where  $E_T = E_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(Z_T - Z_t)}$  and the discount rate is assumed to be  $k_2$ , is given by  $L_{2,n,t} = L_{2,n}(E_t)$ , where*

$$L_{2,n}(E_t) = E_t e^{-k_2(T-t)} E_T^n 1_{\{E_T < B\}}. \quad (\text{A6})$$

*The date- $t$  price of the fundamental financial security which pays out  $E_T^{-n/\eta}$  units of consumption at time  $T$  if  $E_T > B$ , where  $E_T = E_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(Z_T - Z_t)}$  and the discount rate is assumed to be  $k_1$ , is given by  $L_{1,n,t} = L_{1,n}(E_t)$ , where*

$$L_{1,n}(E_t) = E_t e^{-k_1(T-t)} E_T^{-n/\eta} 1_{\{E_T > B\}}. \quad (\text{A7})$$

**Lemma A2** *The prices of the zero-coupon fundamental securities are given by*

$$L_{2,n}(E_t) = E_t^n e^{-[k_2 - n\mu - \frac{1}{2}n(n-1)\sigma^2](T-t)} \Phi \left( \frac{\ln \left( \frac{B}{E_t} \right) - \left( \mu + \frac{1}{2}(2n-1)\sigma^2 \right) (T-t)}{\sigma(T-t)^{1/2}} \right), \quad (\text{A8})$$

and

$$L_{1,n}(E_t) = E_t^{-\frac{n}{\eta}} e^{-\left[ k_2 + \frac{n}{\eta} \left( \mu - \frac{1}{2} \left( 1 + \frac{n}{\eta} \right) \sigma^2 \right) \right] (T-t)} \left[ 1 - \Phi \left( \frac{\ln \left( \frac{B}{E_t} \right) - \left( \mu - \frac{1}{2} \left( 1 + 2\frac{n}{\eta} \right) \sigma^2 \right) (T-t)}{\sigma(T-t)^{1/2}} \right) \right]. \quad (\text{A9})$$

## B Appendix: Proofs

### B.1 Proof of Proposition 1: Consumption-sharing rule

We now construct a series solution for

$$A_t \nu_{2,t}^\eta = 1 - \nu_{2,t}, \quad (\text{B1})$$

where  $\eta = \gamma_2/\gamma_1$ . Note that the above equation is equivalent to

$$A_t (1 - \nu_{1,t})^\eta = \nu_{1,t}, \quad (\text{B2})$$

which implicitly defines  $\nu_{1,t}$  in terms of  $A_t$ . To find  $\nu_{1,t}$ , we apply Theorem C2, expanding around the point  $\nu_{1,t} = 0$ , with

$$f(z) = z(1-z)^{-\eta} \quad (\text{B3})$$

$$\varphi(z) = (1-z)^\eta \quad (\text{B4})$$

$$g(z) = z, \quad (\text{B5})$$

after showing that  $f$  is complex analytic in some neighborhood of 0. We know from the binomial series expansion, that for  $z \in \mathbb{C}$ , such that  $|z| < 1$ ,

$$(1-z)^{-\eta} = \sum_{n=0}^{\infty} \binom{-\eta}{k} (-1)^n z^n, \quad (\text{B6})$$

where  $\binom{-\eta}{k} = \prod_{j=1}^k \frac{-\eta - k + j}{j}$  is the generalized binomial coefficient. Therefore,  $(1-z)^{-\eta}$  is analytic in the open ball  $\{z \in \mathbb{C} : |z| < 1\}$ . Since  $z$  is complex analytic for all  $z \in \mathbb{C}$ , it follows that  $f$  as defined in (B3) is analytic in the open ball  $\{z \in \mathbb{C} : |z| < 1\}$ . It therefore follows from Theorem C2 that

$$\nu_{1,t} = \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} [(1-x)^\eta]_{x=0}. \quad (\text{B7})$$

Since

$$\frac{d^{n-1}}{dx^{n-1}} [(1-x)^{\eta n}] = (-)^{n-1} \eta n (\eta n - 1) (\eta n - 2) \dots (\eta n - (n-2)) (1-x)^{\eta n - (n-1)}, \quad (\text{B8})$$

it follows that

$$\nu_{1,t} = - \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{\eta n}{n-1}, \quad (\text{B9})$$

$$\nu_{2,t} = 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{\eta n}{n-1}. \quad (\text{B10})$$

We shall now determine the radius of convergence of the above series. From d'Alembert's ratio test, it follows that the above series converges absolutely for all  $A \in \mathbb{C}$  s.t.  $|A| < R$ , where

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \left| \frac{\binom{\eta n}{n-1}}{\binom{\eta(n+1)}{n}} \right|. \quad (\text{B11})$$

We wish to evaluate the above limit for all  $\eta \in \mathbb{R}$  such that  $\eta > 1$ . Hence,  $\binom{\eta n}{n-1}$  and  $\binom{\eta(n+1)}{n}$  are positive and real, and so

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\binom{\eta n}{n-1}}{\binom{\eta(n+1)}{n}}. \quad (\text{B12})$$

We note that the generalized binomial coefficient,  $\binom{z}{k} = \prod_{j=1}^k \frac{z-k+j}{j}$ , can be written as

$$\binom{z}{k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\Gamma(k+1)}, \quad (\text{B13})$$

where  $\Gamma(z)$  is the Gamma function, which for  $\Re(z) > 0$ , has the integral representation,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \quad (\text{B14})$$

The Euler Beta function,  $B(x, y)$ , defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (\text{B15})$$

can be written in terms of the Gamma function as follows,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (\text{B16})$$

Together with (B13), the above expression implies that the generalized binomial coefficient is given by

$$\binom{z}{k} = \frac{1}{(z+1)B(z-k+1, k+1)}. \quad (\text{B17})$$

Hence,

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{B((\eta-1)(n+1), n+1)}{B((\eta-1)n, n)}. \quad (\text{B18})$$

To evaluate the above limit, we start by recalling Stirling's series for the Gamma function

$$\Gamma(z) = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left( 1 + O\left(\frac{1}{z}\right) \right), \quad (\text{B19})$$

which together with (B16) implies that

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{\frac{((\eta-1)(n+1))^{(\eta-1)(n+1)-\frac{1}{2}} (n+1)^{(n+1)-\frac{1}{2}}}{((\eta-1)(n+1)+(n+1))^{((\eta-1)(n+1)+(n+1)-\frac{1}{2})}}}{\frac{((\eta-1)n)^{((\eta-1)n)-\frac{1}{2}} n^{n-\frac{1}{2}}}{(((\eta-1)n)+n)^{((\eta-1)n)+n-\frac{1}{2}}}}}. \quad (\text{B20})$$

Simplifying the above expression gives

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{\frac{(\eta-1)^{(\eta-1)(n+1)-1/2} (n+1)^{\eta(n+1)-1}}{[\eta(n+1)]^{\eta(n+1)-1/2}}}{\frac{(\eta-1)^{(\eta-1)n-1/2} n^{\eta n-1}}{(\eta n)^{\eta n-1/2}}} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{\frac{(\eta-1)^{(\eta-1)(n+1)-1/2} (n+1)^{-1/2}}{\eta^{\eta(n+1)-1/2}}}{\frac{(\eta-1)^{(\eta-1)n-1/2} n^{-1/2}}{\eta^{\eta n-1/2}}} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1)+1}{\eta n+1} \frac{(\eta-1)^{\eta-1}}{\eta^\eta} \sqrt{\frac{n}{n+1}} \\ &= \frac{(\eta-1)^{\eta-1}}{\eta^\eta}. \end{aligned} \quad (\text{B21})$$

Since  $A_t$  is a geometric Brownian motion, it is positive and real. Hence, the right-hand side of (B10) is absolutely convergent for  $A_t < \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$ .

We now derive a series expansion for  $\nu_{2,t}$  in terms of  $A_t$ , which is absolutely convergent for  $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$ . We start by rearranging (B1) to obtain

$$\nu_{2,t} = A_t^{-1/\eta} (1 - \nu_{2,t})^{1/\eta}. \quad (\text{B22})$$

To find  $\nu_{2,t}$ , we apply Theorem C2, expanding around the point  $\nu_{2,t} = 0$ , with  $f$ ,  $\varphi$  and  $g$ , defined as in

$$f(z) = z(1-z)^{-1/\eta} \quad (\text{B23})$$

$$\varphi(z) = (1-z)^{1/\eta} \quad (\text{B24})$$

$$g(z) = z. \quad (\text{B25})$$

We can show that our newly defined  $f$  is analytic in the open ball,  $\{z \in \mathbb{C} : |z| < 1\}$ , in the same way as for (B3). Hence, Theorem C2 implies that

$$\nu_{2,t} = \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left[ (1-x)^{n/\eta} \right]_{x=0}. \quad (\text{B26})$$

Because

$$\frac{d^{n-1}}{dx^{n-1}} \left[ (1-x)^{n/\eta} \right] = (-)^{n-1} \frac{n}{\eta} \left( \frac{n}{\eta} - 1 \right) \left( \frac{n}{\eta} - 2 \right) \dots \left( \frac{n}{\eta} - (n-2) \right) (1-x)^{\frac{n}{\eta} - (n-1)}, \quad (\text{B27})$$

it follows that

$$\begin{aligned} \nu_{2,t} &= - \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta}}{n-1} \\ &= \sum_{n=1}^{\infty} \frac{(-)^{n-1} (A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta}}{n-1}. \end{aligned} \quad (\text{B28})$$

By comparing the above expression with (B9), we can see that (B28) is absolutely convergent if  $A_t^{-1/\eta} < \frac{(\frac{1}{\eta}-1)^{\frac{1}{\eta}-1}}{\frac{1}{\eta}}$ , that is, if  $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$ .

## B.2 Proof of Proposition 2: Dynamics of consumption-sharing rule

We now derive a stochastic differential equation satisfied by  $\nu_{1,t}$  by treating  $\nu_{1,t}$  as a function of  $t$ ,  $Y$  and  $\xi$ . Differentiating (22) implicitly with respect to  $t$  gives

$$\beta_1 + \gamma_1 \frac{1}{\nu_{1,t}} \frac{\partial \nu_{1,t}}{\partial t} = \beta_2 - \gamma_2 \frac{1}{\nu_{2,t}} \frac{\partial \nu_{1,t}}{\partial t}. \quad (\text{B29})$$

Solving for  $\partial \nu_{1,t} / \partial t$ , we obtain

$$\frac{\partial \nu_{1,t}}{\partial t} = \frac{1}{\gamma_1} \frac{1}{\gamma_2} \nu_{1,t} \nu_{2,t} (\beta_2 - \beta_1) \mathbf{R}_t, \quad (\text{B30})$$

where  $\mathbf{R}_t$  is the average relative risk aversion in the economy, defined by

$$\mathbf{R}_t = \left( \frac{1}{\gamma_1} \nu_{1,t} + \frac{1}{\gamma_2} \nu_{2,t} \right)^{-1}. \quad (\text{B31})$$



Differentiating (22) implicitly with respect  $Y_t$  and solving for  $\partial\nu_{1,t}/\partial Y_t$  gives

$$Y_t \frac{\partial\nu_{1,t}}{\partial Y_t} = \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t. \quad (\text{B32})$$

Partial differentiation of each side of (B32) with respect to  $Y_t$  and solving for  $\partial^2\nu_{1,t}/\partial Y_t^2$  gives

$$Y_t^2 \frac{\partial^2\nu_{1,t}}{\partial Y_t^2} = \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right). \quad (\text{B33})$$

Differentiating (22) implicitly with respect to  $\xi$  gives

$$\begin{aligned} a_1 e^{-\beta_1 t} Y_t^{-\gamma_1} \nu_{1,t}^{-\gamma_1} \left( -\frac{\gamma_1}{\nu_{1,t}} \frac{\partial\nu_{1,t}}{\partial \xi_t} \right) &= a_2 e^{-\beta_2 t} Y_t^{-\gamma_2} \nu_{2,t}^{-\gamma_2} \xi_t \frac{1}{\xi_t} + a_2 e^{-\beta_2 t} Y_t^{-\gamma_1} \nu_{2,t}^{-\gamma_2} \xi_t \left( -\frac{\gamma_2}{\nu_{2,t}} \frac{\partial\nu_{2,t}}{\partial \xi_t} \right) \\ -\frac{\gamma_1}{\nu_{1,t}} \frac{\partial\nu_{1,t}}{\partial \xi_t} &= \frac{1}{\xi_t} + \frac{\gamma_2}{\nu_{2,t}} \frac{\partial\nu_{1,t}}{\partial \xi_t} \\ \frac{\partial\nu_{1,t}}{\partial \xi_t} \left( \frac{\gamma_1}{\nu_{1,t}} + \frac{\gamma_2}{\nu_{2,t}} \right) &= -\xi_t^{-1} \\ \frac{\partial\nu_{1,t}}{\partial \xi_t} &= -\frac{\xi_t^{-1} \nu_{1,t} \nu_{2,t}}{\gamma_1 \gamma_2} \mathbf{R}_t. \end{aligned} \quad (\text{B34})$$

Therefore,

$$\begin{aligned} \frac{\partial^2\nu_{1,t}}{\partial^2 \xi_t} &= -\frac{1}{\gamma_1 \gamma_2} \frac{\partial}{\partial \xi_t} \left[ \xi_t^{-1} \nu_{1,t} \nu_{2,t} \mathbf{R}_t \right] \\ &= -\frac{1}{\gamma_1 \gamma_2} \left[ -\xi_t^{-2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t + \xi_t^{-1} \frac{\partial(\nu_{1,t} \nu_{2,t} \mathbf{R}_t)}{\partial \xi_t} \right]. \end{aligned} \quad (\text{B35})$$

Now note that

$$\begin{aligned} \frac{\partial(\nu_{1,t} \nu_{2,t} \mathbf{R}_t)}{\partial \xi_t} &= \nu_{1,t} \nu_{2,t} \frac{\partial \mathbf{R}_t}{\partial \xi_t} + \mathbf{R}_t \left( \nu_{1,t} \frac{\partial \nu_{2,t}}{\partial \xi_t} + \nu_{2,t} \frac{\partial \nu_{1,t}}{\partial \xi_t} \right) \\ &= \nu_{1,t} \nu_{2,t} \frac{\partial \mathbf{R}_t}{\partial \xi_t} + \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial \xi_t} (\nu_{2,t} - \nu_{1,t}). \end{aligned} \quad (\text{B36})$$

We now compute  $\frac{\partial \mathbf{R}_t}{\partial \xi_t}$ :

$$\begin{aligned} \frac{\partial \mathbf{R}_t}{\partial \xi_t} &= -\mathbf{R}_t^2 \left( \frac{1}{\gamma_1} \frac{\partial \nu_{1,t}}{\partial \xi_t} + \frac{1}{\gamma_2} \frac{\partial \nu_{2,t}}{\partial \xi_t} \right) \\ &= -\mathbf{R}_t^2 \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \frac{\partial \nu_{1,t}}{\partial \xi_t}. \end{aligned} \quad (\text{B37})$$

Therefore

$$\begin{aligned} \frac{\partial(\nu_{1,t} \nu_{2,t} \mathbf{R}_t)}{\partial \xi_t} &= -\nu_{1,t} \nu_{2,t} \mathbf{R}_t^2 \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \frac{\partial \nu_{1,t}}{\partial \xi_t} + \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial \xi_t} (\nu_{2,t} - \nu_{1,t}) \\ &= \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial \xi_t} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \\ &= -\xi_t^{-1} \nu_{1,t} \nu_{2,t} \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right). \end{aligned} \quad (\text{B38})$$

Hence,

$$\begin{aligned}
\frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2} &= -\frac{1}{\gamma_1 \gamma_2} \left[ -\xi_t^{-2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t - \xi_t^{-2} \nu_{1,t} \nu_{2,t} \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \\
&= \frac{1}{\gamma_1 \gamma_2} \left[ \xi_t^{-2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t + \xi_t^{-2} \nu_{1,t} \nu_{2,t} \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \\
&= \frac{1}{\gamma_1 \gamma_2} \xi_t^{-2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right]. \tag{B39}
\end{aligned}$$

The mixed partial derivative,  $\frac{\partial^2 \nu_{1,t}}{\partial Y \partial \xi_t}$ , is given by

$$\begin{aligned}
\frac{\partial^2 \nu_{1,t}}{\partial Y \partial \xi_t} &= -\frac{1}{\gamma_1 \gamma_2} \frac{\partial}{\partial Y_t} \left[ \xi_t^{-1} \nu_{1,t} \nu_{2,t} \mathbf{R}_t \right] \\
&= -\frac{1}{\gamma_1 \gamma_2} \xi_t^{-1} \frac{\partial}{\partial Y_t} \left[ \nu_{1,t} \nu_{2,t} \mathbf{R}_t \right] \\
&= -\frac{1}{\gamma_1 \gamma_2} \xi_t^{-1} \left\{ \mathbf{R}_t \frac{\partial}{\partial Y_t} [\nu_{1,t} \nu_{2,t}] + \nu_{1,t} \nu_{2,t} \frac{\partial \mathbf{R}_t}{\partial Y_t} \right\}. \tag{B40}
\end{aligned}$$

Hence, we compute

$$\begin{aligned}
\frac{\partial}{\partial Y_t} [\nu_{1,t} \nu_{2,t}] &= \frac{\partial \nu_{1,t}}{\partial Y_t} \nu_{2,t} + \frac{\partial \nu_{2,t}}{\partial Y_t} \nu_{1,t} \\
&= \frac{\partial \nu_{1,t}}{\partial Y_t} \nu_{2,t} - \frac{\partial \nu_{1,t}}{\partial Y_t} \nu_{1,t} \\
&= \frac{\partial \nu_{1,t}}{\partial Y_t} (\nu_{2,t} - \nu_{1,t}), \tag{B41}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathbf{R}_t}{\partial Y_t} &= -\mathbf{R}_t^2 \left( \frac{1}{\gamma_1} \frac{\partial \nu_{1,t}}{\partial Y_t} + \frac{1}{\gamma_2} \frac{\partial \nu_{2,t}}{\partial Y_t} \right) \\
&= -\mathbf{R}_t^2 \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \frac{\partial \nu_{1,t}}{\partial Y_t}. \tag{B42}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{\partial^2 \nu_{1,t}}{\partial Y \partial \xi_t} &= -\frac{1}{\gamma_1 \gamma_2} \xi_t^{-1} \left\{ \mathbf{R}_t \frac{\partial}{\partial Y_t} [\nu_{1,t} \nu_{2,t}] + \nu_{1,t} \nu_{2,t} \frac{\partial \mathbf{R}_t}{\partial Y_t} \right\} \\
&= -\frac{1}{\gamma_1 \gamma_2} \xi_t^{-1} \left\{ \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial Y_t} (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t^2 \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \frac{\partial \nu_{1,t}}{\partial Y_t} \right\} \\
&= -\frac{1}{\gamma_1 \gamma_2} \xi_t^{-1} \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial Y_t} \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\} \\
&= -\frac{1}{\gamma_1 \gamma_2} \xi_t^{-1} \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial Y_t} \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\} \\
&= \frac{-1}{\gamma_1 \gamma_2} \xi_t^{-1} \mathbf{R}_t \frac{\partial \nu_{1,t}}{\partial Y_t} \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\} \\
&= \frac{-1}{\gamma_1 \gamma_2} Y_t^{-1} \xi_t^{-1} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t^2 \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\}. \tag{B43}
\end{aligned}$$

From Ito's Lemma

$$\begin{aligned}
d\nu_{1,t} &= \left( \frac{\partial \nu_{1,t}}{\partial t} + Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} \mu_Y + \xi_t \frac{\partial \nu_{1,t}}{\partial \xi_t} \mu_\xi + \frac{1}{2} Y_t^2 \frac{\partial^2 \nu_{1,t}}{\partial Y_t^2} \sigma_Y^2 + \frac{1}{2} \xi_t^2 \frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2} \sigma_\xi^2 + \xi_t Y_t \frac{\partial \nu_{1,t}^2}{\partial \xi_t \partial Y_t} \sigma_Y \sigma_\xi \right) dt \\
&\quad + \left( Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} \sigma_Y + \xi_t \frac{\partial \nu_{1,t}}{\partial \xi_t} \sigma_\xi \right) dZ_t, \tag{B44}
\end{aligned}$$

which under measure  $\mathbb{P}^1$  becomes

$$\begin{aligned}
d\nu_{1,t} &= \left( \frac{\partial \nu_{1,t}}{\partial t} + Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} \mu_{Y,1} + \frac{1}{2} Y_t^2 \frac{\partial^2 \nu_{1,t}}{\partial Y_t^2} \sigma_Y^2 + \frac{1}{2} \xi_t^2 \frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2} \sigma_\xi^2 + \xi_t Y_t \frac{\partial \nu_{1,t}^2}{\partial \xi_t \partial Y_t} \sigma_Y \sigma_\xi \right) dt \\
&\quad + \left( Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} \sigma_Y + \xi_t \frac{\partial \nu_{1,t}}{\partial \xi_t} \sigma_\xi \right) dZ_{1,t}, \tag{B45}
\end{aligned}$$

and under measure  $\mathbb{P}^2$

$$\begin{aligned}
d\nu_{1,t} &= \left( \frac{\partial \nu_{1,t}}{\partial t} + Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} \mu_{Y,2} + \xi_t \frac{\partial \nu_{1,t}}{\partial \xi_t} \sigma_\xi^2 + \frac{1}{2} Y_t^2 \frac{\partial^2 \nu_{1,t}}{\partial Y_t^2} \sigma_Y^2 + \frac{1}{2} \xi_t^2 \frac{\partial^2 \nu_{1,t}}{\partial \xi_t^2} \sigma_\xi^2 + \xi_t Y_t \frac{\partial \nu_{1,t}^2}{\partial \xi_t \partial Y_t} \sigma_Y \sigma_\xi \right) dt \\
&\quad + \left( Y_t \frac{\partial \nu_{1,t}}{\partial Y_t} \sigma_Y + \xi_t \frac{\partial \nu_{1,t}}{\partial \xi_t} \sigma_\xi \right) dZ_{2,t}. \tag{B46}
\end{aligned}$$

Hence, under  $\mathbb{P}$

$$\begin{aligned}
d\nu_{1,t} &= \left( \frac{1}{\gamma_1} \frac{1}{\gamma_2} \nu_{1,t} \nu_{2,t} (\beta_2 - \beta_1) \mathbf{R}_t + \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \mu_Y - \frac{\nu_{1,t} \nu_{2,t}}{\gamma_1 \gamma_2} \mathbf{R}_t \mu_\xi \right. \\
&\quad + \frac{1}{2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 \\
&\quad + \frac{1}{2} \frac{1}{\gamma_1 \gamma_2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \sigma_\xi^2 \\
&\quad - \frac{1}{\gamma_1 \gamma_2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t^2 \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\} \sigma_Y \sigma_\xi \Bigg) dt \\
&\quad + \left( \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \sigma_Y - \frac{\nu_{1,t} \nu_{2,t}}{\gamma_1 \gamma_2} \mathbf{R}_t \sigma_\xi \right) dZ_t, \tag{B47}
\end{aligned}$$

under measure  $\mathbb{P}^1$

$$\begin{aligned}
d\nu_{1,t} &= \left( \frac{1}{\gamma_1} \frac{1}{\gamma_2} \nu_{1,t} \nu_{2,t} (\beta_2 - \beta_1) \mathbf{R}_t + \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \mu_{Y,1} \right. \\
&\quad + \frac{1}{2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 \\
&\quad + \frac{1}{2} \frac{1}{\gamma_1 \gamma_2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \sigma_\xi^2 \\
&\quad - \frac{1}{\gamma_1 \gamma_2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t^2 \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\} \sigma_Y \sigma_\xi \Bigg) dt \\
&\quad + \left( \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \sigma_Y - \frac{\nu_{1,t} \nu_{2,t}}{\gamma_1 \gamma_2} \mathbf{R}_t \sigma_\xi \right) dZ_{1,t}, \tag{B48}
\end{aligned}$$

and under measure  $\mathbb{P}^2$

$$\begin{aligned}
d\nu_{1,t} &= \left( \frac{1}{\gamma_1} \frac{1}{\gamma_2} \nu_{1,t} \nu_{2,t} (\beta_2 - \beta_1) \mathbf{R}_t + \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \mu_{Y,2} - \frac{\nu_{1,t} \nu_{2,t}}{\gamma_1 \gamma_2} \mathbf{R}_t \sigma_\xi^2 \right. \\
&\quad + \frac{1}{2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{\mathbf{R}_t^2}{\gamma_1 \gamma_2} - 2 \right) \sigma_Y^2 \\
&\quad + \frac{1}{2} \frac{1}{\gamma_1 \gamma_2} \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left[ 1 + \frac{\mathbf{R}_t}{\gamma_1 \gamma_2} \left( -\nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) + \nu_{2,t} - \nu_{1,t} \right) \right] \sigma_\xi^2 \\
&\quad - \frac{1}{\gamma_1 \gamma_2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t^2 \left\{ (\nu_{2,t} - \nu_{1,t}) - \nu_{1,t} \nu_{2,t} \mathbf{R}_t \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \right\} \sigma_Y \sigma_\xi \Bigg) dt \\
&\quad + \left( \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \nu_{1,t} \nu_{2,t} \mathbf{R}_t \sigma_Y - \frac{\nu_{1,t} \nu_{2,t}}{\gamma_1 \gamma_2} \mathbf{R}_t \sigma_\xi \right) dZ_{2,t}. \tag{B49}
\end{aligned}$$

### B.3 Proof of Proposition 3: Almost-sure survival

Equation (23) can be rewritten as

$$\nu_{2,t}^{\gamma_2} = Y_0^{-(\gamma_2-\gamma_1)} \frac{\lambda_{2,0}}{\lambda_{1,0}} e^{-(\beta_2-\beta_1)t} e^{-\frac{1}{2}(\sigma_{\xi,2}^2-\sigma_{\xi,1}^2)t+(\sigma_{\xi,2}-\sigma_{\xi,1})Z_t} e^{-(\gamma_2-\gamma_1)[(\mu_Y-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_t]} \nu_{1,t}^{\gamma_1}. \quad (\text{B50})$$

Thus,

$$\nu_{2,t}^{\eta} = \left( Y_0^{-(\gamma_2-\gamma_1)} \frac{\lambda_{2,0}}{\lambda_{1,0}} e^{-(\beta_2-\beta_1)t} e^{-\frac{1}{2}(\sigma_{\xi,2}^2-\sigma_{\xi,1}^2)t+(\sigma_{\xi,2}-\sigma_{\xi,1})Z_t} e^{-(\gamma_2-\gamma_1)[(\mu_Y-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_t]} \right)^{1/\gamma_1} \nu_{1,t}, \quad (\text{B51})$$

which implies that

$$\nu_{2,t}^{\eta} \left( Y_0^{(\gamma_2-\gamma_1)} \frac{\lambda_{1,0}}{\lambda_{2,0}} e^{(\beta_2-\beta_1)t} e^{\frac{1}{2}(\sigma_{\xi,2}^2-\sigma_{\xi,1}^2)t+(\sigma_{\xi,1}-\sigma_{\xi,2})Z_t} e^{(\gamma_2-\gamma_1)[(\mu_Y-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_t]} \right)^{1/\gamma_1} = \nu_{1,t}. \quad (\text{B52})$$

Now recall the standard results that

$$\lim_{t \rightarrow \infty} e^{at+bZ_t} = \begin{cases} \infty, & \mathbb{P} - a.s., & a > 0, \\ 0, & \mathbb{P} - a.s., & a < 0, \end{cases} \quad (\text{B53})$$

and

$$\limsup_{t \rightarrow \infty} e^{bZ_t} = \infty, \quad (\text{B54})$$

$$\liminf_{t \rightarrow \infty} e^{bZ_t} = 0. \quad (\text{B55})$$

Thus both agents will survive  $\mathbb{P}$ -a.s., i.e. the economy will be almost surely stationary under  $\mathbb{P}$  if and only if

$$\beta_1 - \beta_2 - (\gamma_2 - \gamma_1) \left( \mu_Y - \frac{1}{2}\sigma_Y^2 \right) - \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) = 0, \quad (\text{B56})$$

and

$$\sigma_{\xi,2} - \sigma_{\xi,1} = (\gamma_2 - \gamma_1)\sigma_Y. \quad (\text{B57})$$

The above set of conditions is equivalent to

$$\beta_1 + \gamma_1(\mu_Y - \mu_{Y,1}) + \frac{1}{\psi_1}(\mu_{Y,1} - \frac{1}{2}\sigma_Y^2) + \frac{1}{2}\sigma_{\xi,1}^2 = \beta_2 + \gamma_2(\mu_Y - \mu_{Y,2}) + \frac{1}{\psi_2}(\mu_{Y,2} - \frac{1}{2}\sigma_Y^2) + \frac{1}{2}\sigma_{\xi,2}^2, \quad (\text{B58})$$

and

$$\mu_{Y,2} - \mu_{Y,1} = (\gamma_2 - \gamma_1)\sigma_Y^2. \quad (\text{B59})$$

Under  $\mathbb{P}^1$ , (B52) becomes

$$\nu_{2,t}^{\eta} \left( Y_0^{(\gamma_2-\gamma_1)} \frac{\lambda_1}{\lambda_2} e^{(\beta_2-\beta_1)t} e^{\frac{1}{2}\sigma_{\xi}^2 t+(\sigma_{\xi,1}-\sigma_{\xi,2})Z_t} e^{(\gamma_2-\gamma_1)[(\mu_{Y,1}-\frac{1}{2}\sigma_Y^2)t+\sigma_Y Z_{1,t}] } \right)^{1/\gamma_1} = \nu_{1,t}. \quad (\text{B60})$$

It follows that the economy is almost surely stationary under  $\mathbb{P}^1$  if and only if

$$\beta_2 - \beta_1 + \frac{1}{2}\sigma_\xi^2 + (\gamma_2 - \gamma_1)(\mu_{Y,1} - \frac{1}{2}\sigma_Y^2) = 0, \quad (\text{B61})$$

and

$$\mu_{Y,2} - \mu_{Y,1} = (\gamma_2 - \gamma_1)\sigma_Y^2. \quad (\text{B62})$$

Under  $\mathbb{P}^2$ , (B52) becomes

$$\nu_{2,t}^\eta \left( Y_0^{(\gamma_2 - \gamma_1)} \frac{\lambda_{1,0}}{\lambda_{2,0}} e^{(\beta_2 - \beta_1)t} e^{-\frac{1}{2}\sigma_\xi^2 t + (\sigma_{\xi,1} - \sigma_{\xi,2})Z_t} e^{(\gamma_2 - \gamma_1)[(\mu_{Y,2} - \frac{1}{2}\sigma_Y^2)t + \sigma_Y Z_{2,t}]} \right)^{1/\gamma_1} = \nu_{1,t}. \quad (\text{B63})$$

It follows that the economy is almost surely stationary under  $\mathbb{P}^2$  if and only if

$$\beta_2 - \beta_1 - \frac{1}{2}\sigma_\xi^2 + (\gamma_2 - \gamma_1)(\mu_{Y,2} - \frac{1}{2}\sigma_Y^2) = 0, \quad (\text{B64})$$

and

$$\mu_{Y,2} - \mu_{Y,1} = (\gamma_2 - \gamma_1)\sigma_Y^2. \quad (\text{B65})$$

#### B.4 Proof of Proposition 4: Survival in the mean

First we compute  $E_t \nu_{2,t+u}$ ,  $E_t^1 \nu_{2,t+u}$ , and  $E_t^2 \nu_{2,t+u}$ . Then we take limits as  $u \rightarrow \infty$ . Thus,

$$E_t^1 \nu_{2,t+u} = E_t \left[ 1 - \sum_{n=1}^{\infty} \frac{\left(-A_{t+u}^{-\frac{1}{\eta}}\right)^{-n}}{n} \binom{\frac{n}{\eta}}{n-1} A_{t+u}^n \mathbf{1}_{\{A_{t+u} < R\}} \right] \quad (\text{B66})$$

$$- E_t \left[ \sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^{-n}}{n} \binom{\frac{n}{\eta}}{n-1} \mathbf{1}_{\{A_{t+u} > R\}} \right]. \quad (\text{B67})$$

Because the integrand is complex analytic, term-by-term integration is possible. Hence,

$$\begin{aligned} E_t \nu_{2,t+u} &= E_t [\mathbf{1}_{\{A_{t+u} < R\}}] + \sum_{n=1}^{\infty} \frac{(-)^{-n}}{n} \binom{n\eta}{n-1} E_t [A_{t+u}^n \mathbf{1}_{\{A_{t+u} < R\}}] \\ &\quad - \sum_{n=1}^{\infty} \frac{(-)^{-n}}{n} \binom{\frac{n}{\eta}}{n-1} E_t \left[ A_{t+u}^{\frac{n}{\eta}} \mathbf{1}_{\{A_{t+u} > R\}} \right]. \end{aligned} \quad (\text{B68})$$

From Lemma A2 it follows that

$$E_t [A_{t+u}^n \mathbf{1}_{\{A_{t+u} < R\}}] = A_t^n e^{n(\mu_{A,1} - \frac{1}{2}\sigma_A^2)u} e^{n^2\sigma_A^2 u} \Phi \left( \frac{\ln \left( \frac{R}{A_t} \right) - (\mu_A - \frac{1}{2}\sigma_A^2)u}{\sigma\sqrt{u}} - n\sigma_A\sqrt{u} \right), \quad (\text{B69})$$

and

$$E_t \left[ A_{t+u}^{-\frac{n}{\eta}} 1_{\{A_{t+u} > R\}} \right] = A_t^{-\frac{n}{\eta}} e^{-\frac{n}{\eta}(\mu_A - \frac{1}{2}\sigma_A^2)u} e^{\left(\frac{n}{\eta}\right)^2 \sigma_A^2 u} \Phi \left( \frac{-\ln\left(\frac{R}{A_t}\right) + (\mu_A - \frac{1}{2}\sigma_A^2)u}{\sigma\sqrt{u}} - \frac{n}{\eta}\sigma_A\sqrt{u} \right). \quad (\text{B70})$$

Therefore,

$$\begin{aligned} E_t \nu_{2,t+u} &= \Phi \left( \frac{\ln\left(\frac{R}{A_t}\right) - (\mu_A - \frac{1}{2}\sigma_A^2)u}{\sigma\sqrt{u}} \right) \\ &+ \sum_{n=1}^{\infty} \frac{(-)^{-n}}{n} \binom{n\eta}{n-1} A_t^n e^{n(\mu_A - \frac{1}{2}\sigma_{A,1}^2)u} e^{n^2\sigma_A^2 u} \Phi \left( \frac{\ln\left(\frac{R}{A_t}\right) - (\mu_A - \frac{1}{2}\sigma_A^2)u}{\sigma\sqrt{u}} - n\sigma_A\sqrt{u} \right) \\ &- \sum_{n=1}^{\infty} \frac{(-)^{-n}}{n} \binom{\frac{n}{\eta}}{n-1} A_t^{-\frac{n}{\eta}} e^{-\frac{n}{\eta}(\mu_A - \frac{1}{2}\sigma_A^2)u} e^{\left(\frac{n}{\eta}\right)^2 \sigma_A^2 u} \Phi \left( \frac{-\ln\left(\frac{R}{A_t}\right) + (\mu_A - \frac{1}{2}\sigma_A^2)u}{\sigma\sqrt{u}} - \frac{n}{\eta}\sigma_A\sqrt{u} \right). \end{aligned} \quad (\text{B71})$$

In the supplementary appendix, we show that  $\lim_{u \rightarrow \infty}$  and  $\sum_{n=1}^{\infty}$  can be interchanged, which implies that

$$\lim_{u \rightarrow \infty} E_t [\nu_{2,t+u}] = \lim_{u \rightarrow \infty} \Phi \left( \frac{\ln\left(\frac{R}{A_t}\right) - (\mu_A - \frac{1}{2}\sigma_A^2)u}{\sigma\sqrt{u}} \right) = \begin{cases} 0 & , \mu_A - \frac{1}{2}\sigma_A^2 > 0, \\ \frac{1}{2} & , \mu_A - \frac{1}{2}\sigma_A^2 = 0, \\ 1 & , \mu_A - \frac{1}{2}\sigma_A^2 < 0. \end{cases} \quad (\text{B72})$$

Therefore, the economy is mean stationary under  $\mathbb{P}$  if and only if

$$\beta_1 - \beta_2 - (\gamma_2 - \gamma_1) \left( \mu_Y - \frac{1}{2}\sigma_Y^2 \right) - \frac{1}{2}(\sigma_{\xi,2}^2 - \sigma_{\xi,1}^2) = 0. \quad (\text{B73})$$

Similarly, we can evaluate  $E_t^1 \nu_{2,t}$  and  $E_t^2 \nu_{2,t}$ , and obtain necessary and sufficient conditions for mean stationarity under  $\mathbb{P}^1$  and  $\mathbb{P}^2$ , respectively.

## B.5 Proof of Proposition 5: Riskfree rate and market price of risk

Agent 1's state price density,  $\pi_{1,t}$ , is given by

$$\pi_{1,t} = a_1 e^{-\beta_1 t} Y_t^{-\gamma_1} \nu_{1,t}^{-\gamma_1}. \quad (\text{B74})$$

It follows from Ito's Lemma that

$$\begin{aligned} \frac{d\pi_{1,t}}{\pi_{1,t}} &= - \left[ \beta_1 + \gamma_1 \left( \mu_Y + \mu_{\nu_{\gamma_1,t}}^{\mathbb{P}^1} + \sigma_Y \sigma_{\nu_{\gamma_1,t}} \right) - \frac{1}{2} \gamma_1 (1 + \gamma_1) (\sigma_Y + \sigma_{\nu_{\gamma_1,t}})^2 \right] dt - \\ &\quad \gamma_1 (\sigma_Y + \sigma_{\nu_{\gamma_1,t}}) dZ_{1,t}. \end{aligned} \quad (\text{B75})$$

Hence, from the result that

$$\frac{d\pi_{1,t}}{\pi_{1,t}} = -r_t dt - \theta_t dZ_{1,t}, \quad (\text{B76})$$

we have

$$\theta_{1,t} = \gamma_1 (\sigma_Y + \sigma_{\nu_{\gamma_1,t}}), \quad (\text{B77})$$

and

$$r_t = \beta_1 + \gamma_1 (\mu_Y + \mu_{\nu_{\gamma_1,t}} + \sigma_Y \sigma_{\nu_{\gamma_1,t}}) - \frac{1}{2} \gamma_1 (1 + \gamma_1) (\sigma_Y + \sigma_{\nu_{\gamma_1,t}})^2. \quad (\text{B78})$$

Substituting the expression for  $\sigma_{\nu_{\gamma_1,t}}$  from (30) into (B77) and simplifying gives (56). Substituting the expressions for  $\sigma_{\nu_{\gamma_1,t}}$  and  $\mu_{\nu_{\gamma_1,t}}$  from (30) and (29), respectively into (B78) and simplifying gives

$$\begin{aligned} r_t = & \frac{\beta_1 \frac{\nu_1}{\gamma_1} + \beta_2 \frac{\nu_2}{\gamma_2}}{\frac{\nu_1}{\gamma_1} + \frac{\nu_2}{\gamma_2}} + \mathbf{R}_t \sum_{k=1}^2 w_{k,t} \mu_{Y,k} - \frac{1}{2} \mathbf{R}_t \mathbf{P}_t^{\text{power},1} \sigma_Y^2 \\ & + \frac{\nu_{1,t} \mathbf{R}_t}{\gamma_1} \frac{\nu_{2,t} \mathbf{R}_t}{\gamma_2} \sigma_\xi^2 - \frac{1}{2} \left( \frac{\nu_{1,t} \mathbf{R}_t}{\gamma_1} \frac{\nu_{2,t} \mathbf{R}_t}{\gamma_2} \right)^2 \mathbf{R}_t^{-1} \left( \frac{1 + \gamma_1}{\nu_{1,t}} + \frac{1 + \gamma_2}{\nu_{2,t}} \right) \sigma_\xi^2 \\ & - \mathbf{R}_t \left( \frac{\nu_{1,t} \mathbf{R}_t}{\gamma_1} \frac{\nu_{2,t} \mathbf{R}_t}{\gamma_2} \right) \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \sigma_Y \sigma_\xi, \end{aligned} \quad (\text{B79})$$

where  $\mathbf{P}_t$  is the average prudence in the economy when both investors there are no differences in beliefs, i.e.

$$\mathbf{P}_t = (1 + \gamma_1) \left( \frac{\mathbf{R}_t}{\gamma_1} \right)^2 \nu_{1,t} + (1 + \gamma_2) \left( \frac{\mathbf{R}_t}{\gamma_2} \right)^2 \nu_{2,t}. \quad (\text{B80})$$

Further simplification (B79) of gives

$$r_t = \beta_t + \mathbf{R}_t \sum_{k=1}^2 w_{k,t} \mu_{Y,k} - \frac{1}{2} \mathbf{R}_t \mathbf{P}_t \sigma_Y^2 \quad (\text{B81})$$

$$+ w_{1,t} w_{2,t} \sigma_\xi^2 - \frac{1}{2} (w_{1,t} w_{2,t})^2 \mathbf{R}_t^{-1} \left( \frac{1 + \gamma_1}{\nu_{1,t}} + \frac{1 + \gamma_2}{\nu_{2,t}} \right) \sigma_\xi^2 - \mathbf{R}_t w_{1,t} w_{2,t} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \sigma_Y \sigma_\xi$$

$$= \beta_t + \mathbf{R}_t \sum_{k=1}^2 w_{k,t} \mu_{Y,k} - \frac{1}{2} \mathbf{R}_t \mathbf{P}_t \sigma_Y^2 \quad (\text{B82})$$

$$+ w_{1,t} w_{2,t} \left( 1 - \frac{1}{2} \frac{w_{1,t} w_{2,t}}{v_{1,t} v_{2,t}} \mathbf{R}_t^{-1} \mathbf{P}_t^{\text{har}} \right) \sigma_\xi^2 - \mathbf{R}_t w_{1,t} w_{2,t} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \sigma_Y \sigma_\xi$$

where  $\beta_t$  is the average rate of time preference in the economy, defined by a weighted arithmetic mean of individual agents' rates of time preference,

$$\beta_t = \beta_1 w_{1,t} + \beta_2 w_{2,t}, \quad (\text{B83})$$

where

$$w_k = \frac{\frac{1}{\gamma_k} \nu_{k,t}}{\frac{1}{\gamma_1} \nu_{1,t} + \frac{1}{\gamma_2} \nu_{2,t}}, \quad (\text{B84})$$

and

$$w_1 + w_2 = 1. \quad (\text{B85})$$



The weights  $v_{1,t}$  and  $v_{2,t}$  are defined by

$$v_{k,t} = \frac{\frac{\nu_{k,t}}{1+\gamma_k}}{\sum_{k=1}^2 \frac{\nu_{k,t}}{1+\gamma_k}}, \quad (\text{B86})$$

where

$$v_{1,t} + v_{2,t} = 1 \quad (\text{B87})$$

and  $\mathbf{P}_t^{har}$  is the weighted harmonic mean of individual relative agents' prudences, when agents have power utility, where the weights are the consumption shares, i.e.

$$\mathbf{P}_t^{har} = \left( \sum_{k=1}^2 \frac{\nu_{k,t}}{1+\gamma_k} \right)^{-1} \quad (\text{B88})$$

## B.6 Proof of Corollary 1: Riskfree rate and market price of risk under correct beliefs

Equations (61) and (62) follow from (56) and (58), respectively, after setting  $\mu_{Y,1} = \mu_{Y,2} = \mu_Y$ , and simplifying.

## B.7 Proof of Corollary 2: Riskfree rate and market price of risk under identical preferences

Equations (64) and (63) follow from (56) and (58), respectively, after setting  $\beta_1 = \beta_2 = \beta$ ,  $\gamma_1 = \gamma_2 = \gamma$ ,  $\psi_1 = \psi_2 = \psi$ , and simplifying.

## B.8 Proof of Proposition 6: State-price density

Agent  $k$ 's state-price density is given by (47).

To find a closed-form expression for Agent  $k$ 's state-price density, we find series expansions for  $\nu_{k,t}^{-\gamma_k}$ ,  $k \in \{1, 2\}$ . To find a series expansion for  $\nu_{2,t}^{-\gamma_2}$ , note that

$$\nu_{2,t}^{-\gamma_2} = (1 - \nu_{1,t})^{-\gamma_2}, \quad (\text{B89})$$

and use Theorem C2 to expand around the point  $\nu_{1,t} = 0$ . To do this we define

$$g(z) = (1 - z)^{-\gamma_2}, \quad (\text{B90})$$

which is complex analytic in the open ball  $\{z \in \mathbb{C} : |z| < 1\}$ . Hence, with  $f$  and  $\varphi$  defined as in (B3) and (B4), respectively, Theorem C2 implies that

$$\begin{aligned}
g(\nu_{1,t}) &= (1 - \nu_{1,t})^{-\gamma_2} \\
&= g(0) + \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} [g'(x)\varphi(x)^n]_{x=0} \\
&= 1 + \sum_{n=1}^{\infty} \frac{A_t^n}{n!} \frac{d^{n-1}}{dx^{n-1}} [\gamma_2(1-x)^{n\eta-\gamma_2-1}]_{x=0}.
\end{aligned} \tag{B91}$$

Since,

$$\begin{aligned}
&\frac{d^{n-1}}{dx^{n-1}} \gamma_2(1-x)^{n\eta-\gamma_2-1} \\
&= \gamma_2(-)^{n-1} (n\eta - \gamma_2 - 1)(n\eta - \gamma_2 - 2) \dots (n\eta - \gamma_2 - (n-1)) (1-x)^{n\eta-\gamma_2-(n-1)},
\end{aligned} \tag{B92}$$

it follows that

$$\nu_{2,t}^{-\gamma_2} = 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1}. \tag{B93}$$

D'Alembert's ratio test implies that the above series converges absolutely for all  $A \in \mathbb{C}$  such that  $|A| < R$ , where

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\binom{n\eta - \gamma_2 - 1}{n-1}}{\binom{\eta(n+1) - \gamma_2 - 1}{n}}. \tag{B94}$$

Using (B17), we rewrite the above expression as

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1) - \gamma_2}{\eta n - \gamma_2} \frac{B((\eta-1)(n+1) - \gamma_2 - 1, n+1)}{B((\eta-1)n - \gamma_2 - 1, n)}. \tag{B95}$$

Hence, using (B16) and (B19), we obtain

$$R = \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1) - \gamma_2}{\eta n - \gamma_2} \frac{\frac{[(\eta-1)(n+1) - (1+\gamma_2)]^{(\eta-1)(n+1) - (1+\gamma_2) - 1/2} (n+1)^{n+1-1/2}}{[\eta(n+1) - (1+\gamma_2)]^{\eta(n+1) - (1+\gamma_2) - 1/2}}}{\frac{[(\eta-1)n - (1+\gamma_2)]^{(\eta-1)n - (1+\gamma_2) - 1/2} n^{n-1/2}}{[\eta n - (1+\gamma_2)]^{\eta n - (1+\gamma_2) - 1/2}}}. \tag{B96}$$

We now simplify the expression

$$\frac{\frac{[(\eta-1)(n+1) - (1+\gamma_2)]^{(\eta-1)(n+1) - (1+\gamma_2) - 1/2} (n+1)^{n+1-1/2}}{[\eta(n+1) - (1+\gamma_2)]^{\eta(n+1) - (1+\gamma_2) - 1/2}}}{\frac{[(\eta-1)n - (1+\gamma_2)]^{(\eta-1)n - (1+\gamma_2) - 1/2} n^{n-1/2}}{[\eta n - (1+\gamma_2)]^{\eta n - (1+\gamma_2) - 1/2}}}. \tag{B97}$$

Simplifying the numerator of the above expression gives

$$\begin{aligned}
& \frac{[(\eta - 1)(n + 1) - (1 + \gamma_2)]^{(\eta-1)(n+1)-(1+\gamma_2)-\frac{1}{2}} (n + 1)^{n+1-1/2}}{[\eta(n + 1) - (1 + \gamma_2)]^{\eta(n+1)-(1+\gamma_2)-\frac{1}{2}}} \\
&= \frac{(\eta - 1)(n + 1)^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}} (n + 1)^{n+1-\frac{1}{2}} \left[1 - \frac{1+\gamma_2}{(\eta-1)(n+1)}\right]^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}}}{[\eta(n + 1)]^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}} \left[1 - \frac{1+\gamma_2}{\eta(n+1)}\right]^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}}} \\
&= \frac{(\eta - 1)^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}} (n + 1)^{\eta(n+1)-(\gamma_2+1)-1} \left[1 - \frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}}}{\eta^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}} (n + 1)^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}} \left[1 - \frac{\gamma_2+1}{\eta(n+1)}\right]^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}}} \\
&= \frac{(\eta - 1)^{(\eta-1)(n+1)-(\gamma_2+1)-\frac{1}{2}} \left(\left[1 - \frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{(n+1)}\right)^{(\eta-1)} \left[1 - \frac{\gamma_2+1}{(\eta-1)(n+1)}\right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n+1} \eta^{\eta(n+1)-(\gamma_2+1)-\frac{1}{2}} \left(\left[1 - \frac{\gamma_2+1}{\eta(n+1)}\right]^{(n+1)}\right)^\eta \left[1 - \frac{\gamma_2+1}{\eta(n+1)}\right]^{-(1+\gamma_2+\frac{1}{2})}}. \quad (\text{B98})
\end{aligned}$$

Similarly for the denominator of (B97)

$$\begin{aligned}
& \frac{[(\eta - 1)n - (1 + \gamma_2)]^{(\eta-1)n-(1+\gamma_2)-1/2} n^{n-1/2}}{[\eta n - (1 + \gamma_2)]^{\eta n-(1+\gamma_2)-1/2}} \\
&= \frac{(\eta - 1)^{(\eta-1)n-(\gamma_2+1)-\frac{1}{2}} \left(\left[1 - \frac{\gamma_2+1}{(\eta-1)n}\right]^n\right)^{(\eta-1)} \left[1 - \frac{\gamma_2+1}{(\eta-1)n}\right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n} \eta^{\eta n-(\gamma_2+1)-\frac{1}{2}} \left(\left[1 - \frac{\gamma_2+1}{\eta n}\right]^n\right)^\eta \left[1 - \frac{\gamma_2+1}{\eta n}\right]^{-(1+\gamma_2+\frac{1}{2})}}. \quad (\text{B99})
\end{aligned}$$

Therefore,

$$\begin{aligned}
R &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \frac{\eta(n+1) - \gamma_2}{\eta n - \gamma_2} \frac{\frac{(\eta-1)^{(\eta-1)(n+1) - (\gamma_2+1) - \frac{1}{2}} \left( \left[ 1 - \frac{\gamma_2+1}{(\eta-1)(n+1)} \right]^{(n+1)} \right)^{(\eta-1)} \left[ 1 - \frac{\gamma_2+1}{(\eta-1)(n+1)} \right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n+1} \eta^{n(n+1) - (\gamma_2+1) - \frac{1}{2}} \left( \left[ 1 - \frac{\gamma_2+1}{\eta(n+1)} \right]^{(n+1)} \right)^\eta \left[ 1 - \frac{\gamma_2+1}{\eta(n+1)} \right]^{-(1+\gamma_2+\frac{1}{2})}}}{\frac{(\eta-1)^{(\eta-1)n - (\gamma_2+1) - \frac{1}{2}} \left( \left[ 1 - \frac{\gamma_2+1}{(\eta-1)n} \right]^n \right)^{(\eta-1)} \left[ 1 - \frac{\gamma_2+1}{(\eta-1)n} \right]^{-(1+\gamma_2+\frac{1}{2})}}{\sqrt{n} \eta^{n^2 - (\gamma_2+1) - \frac{1}{2}} \left( \left[ 1 - \frac{\gamma_2+1}{\eta n} \right]^n \right)^\eta \left[ 1 - \frac{\gamma_2+1}{\eta n} \right]^{-(1+\gamma_2+\frac{1}{2})}}} \\
&= \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} \frac{\eta(n+1) - \gamma_2}{\eta n - \gamma_2} \frac{\frac{(\eta-1)^{(\eta-1)} \left( \left[ 1 - \frac{\gamma_2+1}{(\eta-1)(n+1)} \right]^{(n+1)} \right)^{(\eta-1)} \left[ 1 - \frac{\gamma_2+1}{(\eta-1)(n+1)} \right]^{-(1+\gamma_2+\frac{1}{2})}}{\eta^\eta \left( \left[ 1 - \frac{\gamma_2+1}{\eta(n+1)} \right]^{(n+1)} \right)^\eta \left[ 1 - \frac{\gamma_2+1}{\eta(n+1)} \right]^{-(1+\gamma_2+\frac{1}{2})}}}{\frac{\left( \left[ 1 - \frac{\gamma_2+1}{(\eta-1)n} \right]^n \right)^{(\eta-1)} \left[ 1 - \frac{\gamma_2+1}{(\eta-1)n} \right]^{-(1+\gamma_2+\frac{1}{2})}}{\left( \left[ 1 - \frac{\gamma_2+1}{\eta n} \right]^n \right)^\eta \left[ 1 - \frac{\gamma_2+1}{\eta n} \right]^{-(1+\gamma_2+\frac{1}{2})}}} \\
&= \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta} \lim_{n \rightarrow \infty} \frac{\frac{\left( \left[ 1 - \frac{\gamma_2+1}{(\eta-1)(n+1)} \right]^{(n+1)} \right)^{(\eta-1)}}{\left( \left[ 1 - \frac{\gamma_2+1}{\eta(n+1)} \right]^{(n+1)} \right)^\eta}}{\frac{\left( \left[ 1 - \frac{\gamma_2+1}{(\eta-1)n} \right]^n \right)^{(\eta-1)}}{\left( \left[ 1 - \frac{\gamma_2+1}{\eta n} \right]^n \right)^\eta}} \\
&= \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta} \frac{e^{-(\gamma_2+1)}}{e^{-(\gamma_2+1)}}, \tag{B100}
\end{aligned}$$

since  $e^x = \lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n$ . Hence,

$$R = \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}. \tag{B101}$$

Since  $A_t$  is a geometric Brownian motion,  $A_t$  is real and positive, and so the right-hand side of (B93) is absolutely convergent if  $A_t < \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$ . Hence, Agent 2's state-price density is given by

$$\pi_{2,t} = a_2 e^{-\beta_2 t} Y_t^{-\gamma_2} \left( 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right), \quad A_t < \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}. \tag{B102}$$

To find an expression for the state-price density when  $A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$ , we find a series expansion for  $\nu_{1,t}^{-\gamma_1}$ , which is absolutely convergent for  $A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$ . Note that

$$\nu_{1,t}^{-\gamma_1} = (1 - \nu_{2,t})^{-\gamma_1}, \tag{B103}$$

and use Theorem C2 to expand around the point  $\nu_{2,t} = 0$ . To do this, we define

$$g(z) = (1 - z)^{-\gamma_1}, \tag{B104}$$

which is complex analytic in the open ball  $\{z \in \mathbb{C} : |z| < 1\}$ . Hence, with  $f$  and  $\varphi$  defined as in (B23) and (B24), respectively, Theorem C2 implies that

$$\begin{aligned}
g(\nu_{2,t}) &= (1 - \nu_{2,t})^{-\gamma_1} \\
&= g(0) + \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} [g'(x)\varphi(x)^n]_{x=0} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(A_t^{-1/\eta})^n}{n!} \frac{d^{n-1}}{dx^{n-1}} [\gamma_1(1-x)^{\frac{n}{\eta}-\gamma_1-1}]_{x=0}.
\end{aligned} \tag{B105}$$

Because,

$$\gamma_1(1-x)^{\frac{n}{\eta}-\gamma_1-1} = \gamma_1(-)^{n-1} \left(\frac{n}{\eta} - \gamma_1 - 1\right) \left(\frac{n}{\eta} - \gamma_1 - 2\right) \dots \left(\frac{n}{\eta} - \gamma_1 - (n-1)\right) (1-x)^{\frac{n}{\eta}-\gamma_1-(n-1)}, \tag{B106}$$

it follows that

$$\nu_{1,t}^{-\gamma_1} = 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1}. \tag{B107}$$

By comparing the above expression with (B93), we can see that (B107) is absolutely convergent if  $A_t^{-1/\eta} < \frac{(\frac{1}{\eta}-1)^{\frac{1}{\eta}-1}}{\frac{1}{\eta}}$ , i.e. if  $A_t > \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$ . Hence, Agent 1's state-price density is given by

$$\pi_{1,t} = a_1 e^{-\beta_1 t} Y_t^{-\gamma_1} \left( 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right), \quad A_t > \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}. \tag{B108}$$

For ease of notation, we define  $R = \frac{(\eta-1)^{(\eta-1)}}{\eta^\eta}$ . Equations (65) and (66) follow from (B102), (B108) and (49).

## B.9 Proof of Proposition 7: Prices of risky assets

We now derive a closed-form expression for (76). We use (66) and (66) to write Agent 1's state-price density as

$$\begin{aligned}
\pi_{1,t} &= \lambda_{2,0} e^{-\beta_2 t} Y_t^{-\gamma_2} \xi_t \left( 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) 1_{\{A_t < R\}} \\
&+ \lambda_{1,0} e^{-\beta_1 t} Y_t^{-\gamma_1} \left( 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right) 1_{\{A_t > R\}},
\end{aligned} \tag{B109}$$

which is complex analytic for all  $A \in \mathbb{C}$ , such that  $|A| \neq R$ . Since the event  $\{A_u = R\}$  is of measure zero, it follows from (76) that

$$p_t^X = (\pi_{1,t} X_t)^{-1} j_t, \tag{B110}$$

where

$$\begin{aligned}
j_t &= E_t^1 \int_t^\infty \left[ a_2 e^{-\beta_2 u} \xi_u X_u Y_u^{-\gamma_2} \left( 1 - \gamma_2 \sum_{n=1}^\infty \frac{(-A_u)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) 1_{\{A_u < R\}} \right. \\
&\quad \left. + a_1 e^{-\beta_1 u} X_u Y_u^{-\gamma_1} \left( 1 - \gamma_1 \sum_{n=1}^\infty \frac{(-A_u^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right) 1_{\{A_u > R\}} \right] du. \quad (\text{B111})
\end{aligned}$$

Since the integrand in the above expression is complex analytic, term-by-term integration is valid and the resulting expression will also be complex analytic. Hence,

$$j_t = \lambda_{2,0} j_{l,t} + \lambda_{1,0} j_{r,t}, \quad (\text{B112})$$

where

$$j_{l,t} = e^{-\beta_2 t} X_t Y_t^{-\gamma_2} \xi_t \left( \zeta_{0,l,t} - \gamma_2 \sum_{n=1}^\infty \zeta_{n,l,t} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right), \quad (\text{B113})$$

$$j_{r,t} = e^{-\beta_1 t} X_t Y_t^{-\gamma_1} \left( \zeta_{0,r,t} - \gamma_1 \sum_{n=1}^\infty \zeta_{n,r,t} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right), \quad (\text{B114})$$

and

$$\zeta_{n,l,t} = E_t^1 \int_t^\infty e^{-\beta_2(u-t)} \frac{X_u}{X_t} \left( \frac{Y_u}{Y_t} \right)^{-\gamma_2} \frac{\xi_u}{\xi_t} \left( \frac{A_u}{A_t} \right)^n 1_{\{A_u < R\}} du, \quad n \in \mathbb{N}_0, \quad (\text{B115})$$

$$\zeta_{n,r,t} = E_t^1 \int_t^\infty e^{-\beta_1(u-t)} \frac{X_u}{X_t} \left( \frac{Y_u}{Y_t} \right)^{-\gamma_1} \left( \frac{A_u}{A_t} \right)^{-n/\eta} 1_{\{A_u > R\}} du, \quad n \in \mathbb{N}_0, \quad (\text{B116})$$

where  $\mathbb{N}_0$  is the set of natural numbers including zero. Note that

$$\frac{X_u}{X_t} \left( \frac{Y_u}{Y_t} \right)^{-\gamma_2} \frac{\xi_u}{\xi_t} = e^{[\beta_2 - (\bar{r}_2 + \gamma_2 \sigma_X^{sys} \sigma_Y - \mu_{X,2})](u-t)} \frac{M_{2,u}^\xi}{M_{2,t}^\xi}, \quad (\text{B117})$$

where

$$\bar{r}_k = \beta_k + \gamma_k \mu_{Y,k} - \frac{1}{2} \gamma_k (1 + \gamma_k) \sigma_Y^2, \quad (\text{B118})$$

is the risk-free rate when only Agent  $k$  is present in the economy, and  $M_{2,t}^\xi$  is the following exponential martingale under  $\mathbb{P}^1$ :

$$\frac{dM_{2,t}^\xi}{M_{2,t}^\xi} = \sigma_X^{id} dZ_t^{id} + (\sigma_X^{sys} + \sigma_\xi - \gamma_2 \sigma_Y) dZ_{1,t}, \quad M_{2,t}^\xi = 1. \quad (\text{B119})$$

Also

$$\frac{X_u}{X_t} \left( \frac{Y_u}{Y_t} \right)^{-\gamma_1} = e^{[\beta_1 - (\bar{r}_1 + \gamma_1 \sigma_X^{sys} \sigma_Y - \mu_{X,1})](u-t)} \frac{M_{1,u}}{M_{1,t}}, \quad (\text{B120})$$

where  $M_{1,t}$  is the following exponential martingale under  $\mathbb{P}^1$ :

$$\frac{dM_{1,t}}{M_{1,t}} = \sigma_X^{id} dZ_t^{id} + (\sigma_X^{sys} - \gamma_1 \sigma_Y) dZ_{1,t}, \quad M_{1,t} = 1. \quad (\text{B121})$$

We can thus define the new probability measures  $\hat{\mathbb{P}}^{2,\xi}$  and  $\hat{\mathbb{P}}^1$  on  $(\Omega, \mathcal{F})$  via

$$\hat{\mathbb{P}}^{2,\xi}(A) = E(1_A M_{2,T}^\xi), \quad A \in \mathcal{F}_T, \quad (\text{B122})$$

and

$$\hat{\mathbb{P}}^1(A) = E(1_A M_{1,T}), \quad A \in \mathcal{F}_T, \quad (\text{B123})$$

respectively. It follows that

$$\zeta_{n,l,t} = \hat{E}_t^{2,\xi} \int_t^\infty e^{-k_2(u-t)} \left( \frac{A_u}{A_t} \right)^n 1_{\{A_u < R\}} du, \quad n \in \mathbb{N}_0, \quad (\text{B124})$$

$$\zeta_{n,r,t} = \hat{E}_t^1 \int_t^\infty e^{-k_1(u-t)} \left( \frac{A_u}{A_t} \right)^{-n/\eta} 1_{\{A_u > R\}} du, \quad n \in \mathbb{N}_0, \quad (\text{B125})$$

where  $\hat{E}_t^{2,\xi}[\cdot]$  and  $\hat{E}_t^1[\cdot]$  are the time- $t$  conditional expectation operator under  $\hat{\mathbb{P}}^{2,\xi}$  and  $\hat{\mathbb{P}}^1$ , respectively, and

$$k_k = \bar{r}_k + \gamma_k \sigma_X^{sys} \sigma_Y - \mu_{X,k}, \quad (\text{B126})$$

is the discount rate used to value a security paying  $X$  units of consumption per unit time in perpetuity, when Agent  $k$  is the sole agent in the economy.

From Lemma A1, it follows that

$$\zeta_{n,l,t} = \begin{cases} -\frac{1}{\frac{1}{2}(\sigma_A)^2(n-a_-^\xi(k_2))(n-a_+(k_2))} + \frac{1}{\frac{1}{2}(\sigma_A)^2(n-a_+^\xi(k_2))(a_+^\xi(k_2)-a_-^\xi(k_2))} \left( \frac{A_t}{R} \right)^{a_+^\xi(k_2)-n}, & A_t < R, \\ \frac{1}{\frac{1}{2}(\sigma_A)^2(n-a_-^\xi(k_2))(a_+^\xi(k_2)-a_-^\xi(k_2))} \left( \frac{A_t}{R} \right)^{a_-^\xi(k_2)-n}, & A_t \geq R. \end{cases}, \quad (\text{B127})$$

and

$$\zeta_{n,r,t} = \begin{cases} \frac{1}{\frac{1}{2}(\sigma_A)^2\left(\frac{n}{\eta}+a_+(k_1)\right)(a_+(k_1)-a_-(k_1))} \left( \frac{A_t}{R} \right)^{a_+(k_1)+\frac{n}{\eta}}, & A_t < R, \\ \frac{1}{\frac{1}{2}(\sigma_A)^2\left(\frac{n}{\eta}+a_-(k_1)\right)(a_+(k_1)-a_-(k_1))} \left( \frac{A_t}{R} \right)^{a_-(k_1)+\frac{n}{\eta}} - \frac{1}{\frac{1}{2}(\sigma_A)^2\left(\frac{n}{\eta}+a_+(k_1)\right)\left(\frac{n}{\eta}+a_-(k_1)\right)}, & A_t \geq R. \end{cases} \quad (\text{B128})$$

where

$$\hat{\mu}_A^1 = \mu_{A,1} + (\sigma_X^{sys} - \gamma_1 \sigma_Y) \sigma_A, \quad (\text{B129})$$

$$\hat{\mu}_A^{2,\xi} = \mu_{A,1} + (\sigma_X^{sys} + \sigma_\xi - \gamma_2 \sigma_Y) \sigma_A, \quad (\text{B130})$$

$$\mu_{A,1} = \frac{\beta_2 - \beta_1}{\gamma_1} + (\eta - 1) \left( \mu_{Y,1} - \frac{1}{2} \sigma_Y^2 \right) - \frac{1}{2\gamma_1} \sigma_\xi^2 + \frac{1}{2} \sigma_A^2, \quad (\text{B131})$$

$$\sigma_A = (\eta - 1) \sigma_Y + \frac{1}{\gamma_1} \sigma_\xi, \quad (\text{B132})$$

where

$$a_{\pm}(k_1) = \frac{-(\hat{\mu}_A^1 - \frac{1}{2}(\sigma_A)^2) \pm \sqrt{(\hat{\mu}_A^1 - \frac{1}{2}(\sigma_A)^2)^2 + 2k_1(\sigma_A)^2}}{(\sigma_A)^2}, \quad (\text{B133})$$

$$a_{\pm}^{\xi}(k_2) = \frac{-(\hat{\mu}_A^{2,\xi} - \frac{1}{2}(\sigma_A)^2) \pm \sqrt{(\hat{\mu}_A^{2,\xi} - \frac{1}{2}(\sigma_A)^2)^2 + 2k_2(\sigma_A)^2}}{(\sigma_A)^2}. \quad (\text{B134})$$

Hence, we have closed-form expressions for the functions  $\zeta_{n,l,t}$  and  $\zeta_{n,r,t}$  which appear in  $j_t$ :

$$\begin{aligned} j_t &= \lambda_{2,0} e^{-\beta_2 t} X_t Y_t^{-\gamma_2} \xi_t \left( \zeta_{0,l,t} - \gamma_2 \sum_{n=1}^{\infty} \zeta_{n,l,t} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \\ &+ \lambda_{1,0} e^{-\beta_1 t} X_t Y_t^{-\gamma_1} \left( \zeta_{0,r,t} - \gamma_1 \sum_{n=1}^{\infty} \zeta_{n,r,t} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right). \end{aligned} \quad (\text{B135})$$

From (B2) it follows that

$$\pi_{1,t} X_t \nu_{1,t}^{\gamma_1} = \lambda_{1,0} e^{-\beta_1 t} X_t Y_t^{-\gamma_1} \quad (\text{B136})$$

$$\pi_{1,t} X_t \nu_{2,t}^{\gamma_2} = \lambda_{2,0} e^{-\beta_2 t} X_t Y_t^{-\gamma_2} \xi_t. \quad (\text{B137})$$

Hence,

$$\begin{aligned} j_t &= \pi_{1,t} X_t \left[ \nu_{2,t}^{\gamma_2} \left( \zeta_{0,l,t} - \gamma_2 \sum_{n=1}^{\infty} \zeta_{n,l,t} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \right. \\ &\left. + \nu_{1,t}^{\gamma_1} \left( \zeta_{0,r,t} - \gamma_1 \sum_{n=1}^{\infty} \zeta_{n,r,t} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right) \right], \end{aligned} \quad (\text{B138})$$

which implies that

$$\begin{aligned} p_t^X &= \nu_{2,t}^{\gamma_2} \left( \zeta_{0,l,t} - \gamma_2 \sum_{n=1}^{\infty} \zeta_{n,l,t} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \\ &+ \nu_{1,t}^{\gamma_1} \left( \zeta_{0,r,t} - \gamma_1 \sum_{n=1}^{\infty} \zeta_{n,r,t} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right). \end{aligned} \quad (\text{B139})$$

It will now be useful to define  $\zeta_{n,l}(A_t) = \zeta_{n,l,t}$  and  $\zeta_{n,r}(A_t) = \zeta_{n,r,t}$ . Using (23), we can write the price-dividend ratio,  $p_t$ , purely in terms of  $\nu_{1,t}$  and  $\nu_{2,t} = 1 - \nu_{1,t}$ , i.e.

$$\begin{aligned} p_t^X &= \nu_{2,t}^{\gamma_2} \left( \zeta_{0,l} \left( \frac{\nu_{1,t}}{\nu_{2,t}} \right) - \gamma_2 \sum_{n=1}^{\infty} \zeta_{n,l} \left( \frac{\nu_{1,t}}{\nu_{2,t}} \right) \frac{\left( -\frac{\nu_{1,t}}{\nu_{2,t}} \right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \\ &+ \nu_{1,t}^{\gamma_1} \left( \zeta_{0,r} \left( \frac{\nu_{1,t}}{\nu_{2,t}} \right) - \gamma_1 \sum_{n=1}^{\infty} \zeta_{n,r} \left( \frac{\nu_{1,t}}{\nu_{2,t}} \right) \frac{\left( -\frac{\nu_{2,t}}{\nu_{1,t}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right). \end{aligned} \quad (\text{B140})$$



We can also write

$$p_t^X = \nu_{2,t}^{\gamma_2} p_{l,t}^X + \nu_{1,t}^{\gamma_1} p_{r,t}^X, \quad (\text{B141})$$

where

$$p_{l,t}^X = \zeta_{0,l,t}^X - \gamma_2 \sum_{n=1}^{\infty} \zeta_{n,l,t}^X \frac{\left(-\frac{\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}}\right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1}, \quad (\text{B142})$$

$$p_{r,t}^X = \zeta_{0,r,t}^X - \gamma_1 \sum_{n=1}^{\infty} \zeta_{n,r,t}^X \frac{\left(-\frac{\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}}\right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1}. \quad (\text{B143})$$

## B.10 Proof of Proposition 8: Risk premium and volatility of risky assets

Note that

$$\frac{\partial p_t^X}{\partial \nu_{1,t}} = -\gamma_2 \nu_{2,t}^{\gamma_2} \frac{p_{l,t}^X}{\nu_{2,t}} + \gamma_1 \nu_{1,t}^{\gamma_1} \frac{p_{r,t}^X}{\nu_{1,t}} + \nu_{2,t}^{\gamma_2} \frac{\partial p_{l,t}^X}{\partial \nu_{1,t}} + \nu_{1,t}^{\gamma_1} \frac{\partial p_{r,t}^X}{\partial \nu_{1,t}}, \quad (\text{B144})$$

where

$$\frac{\partial p_t^X}{\partial \nu_{1,t}} = -\gamma_2 \nu_{2,t}^{\gamma_2} \frac{p_{l,t}^X}{\nu_{2,t}} + \gamma_1 \nu_{1,t}^{\gamma_1} \frac{p_{r,t}^X}{\nu_{1,t}} + \nu_{2,t}^{\gamma_2} \frac{\partial p_{l,t}^X}{\partial \nu_{1,t}} + \nu_{1,t}^{\gamma_1} \frac{\partial p_{r,t}^X}{\partial \nu_{1,t}}, \quad (\text{B145})$$

and

$$\frac{\partial p_{l,t}^X}{\partial \nu_{1,t}} = \frac{\partial A_t}{\partial \nu_{1,t}} \left[ \frac{\partial \zeta_{0,l,t}^X}{\partial A_t} - \gamma_2 \sum_{n=1}^{\infty} \left( \frac{\partial \zeta_{n,l,t}^X}{\partial A_t} + n \frac{\zeta_{n,l,t}^X}{A_t} \right) \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right]. \quad (\text{B146})$$

Now

$$A = \nu_2^{-\eta} - \nu_2^{1-\eta} \quad (\text{B147})$$

$$\frac{\partial A}{\partial \nu_2} = -\gamma_2 \nu_2^{-(\eta+1)} \mathbf{R}_t^{-1} \quad (\text{B148})$$

$$\frac{\partial A}{\partial \nu_1} = \gamma_2 \nu_2^{-(\eta+1)} \mathbf{R}_t^{-1} \quad (\text{B149})$$

$$= \gamma_2 \frac{A_t}{\nu_1 \nu_2} \mathbf{R}_t^{-1}. \quad (\text{B150})$$

Therefore,

$$\frac{\partial p_{l,t}^X}{\partial \nu_{1,t}} = \frac{\gamma_2}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ A_t \frac{\partial \zeta_{0,l,t}^X}{\partial A_t} - \gamma_2 \sum_{n=1}^{\infty} \left( A_t \frac{\partial \zeta_{n,l,t}^X}{\partial A_t} + n \zeta_{n,l,t}^X \right) \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right]. \quad (\text{B151})$$

Also,

$$\frac{\partial p_{r,t}^X}{\partial \nu_{1,t}} = \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ A_t \frac{\partial \zeta_{0,r,t}^X}{\partial A_t} - \gamma_1 \sum_{n=1}^{\infty} \left( \frac{A_t \partial \zeta_{n,r,t}^X}{\partial A_t} - \frac{n}{\eta} \zeta_{n,r,t}^X \right) \frac{\left( -A_t^{-\frac{1}{\eta}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right]. \quad (\text{B152})$$

Therefore,

$$\frac{\partial p_t^X}{\partial \nu_{1,t}} = -\gamma_2 \nu_{2,t}^{\gamma_2} \frac{p_{l,t}^X}{\nu_{2,t}} + \gamma_1 \nu_{1,t}^{\gamma_1} \frac{p_{r,t}^X}{\nu_{1,t}} + \nu_{2,t}^{\gamma_2} \frac{\partial p_{l,t}^X}{\partial \nu_{1,t}} + \nu_{1,t}^{\gamma_1} \frac{\partial p_{r,t}^X}{\partial \nu_{1,t}} \quad (\text{B153})$$

$$= -\gamma_2 \nu_{2,t}^{\gamma_2} \frac{p_{l,t}^X}{\nu_{1,t}} + \gamma_1 \nu_{1,t}^{\gamma_1} \frac{p_{r,t}^X}{\nu_{1,t}} \quad (\text{B154})$$

$$+ \nu_{2,t}^{\gamma_2} \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ A_t \frac{\partial \zeta_{0,l,t}^X}{\partial A_t} - \gamma_2 \sum_{n=1}^{\infty} \left( A_t \frac{\partial \zeta_{n,l,t}^X}{\partial A_t} + n \zeta_{n,l,t}^X \right) \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right]$$

$$+ \nu_{1,t}^{\gamma_1} \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ A_t \frac{\partial \zeta_{0,r,t}^X}{\partial A_t} - \gamma_1 \sum_{n=1}^{\infty} \left( A_t \frac{\partial \zeta_{n,r,t}^X}{\partial A_t} - \frac{n}{\eta} \zeta_{n,r,t}^X \right) \frac{\left( -A_t^{-\frac{1}{\eta}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right].$$

Hence,

$$\frac{\partial p_t^X}{\partial \nu_{1,t}} = -\gamma_2 \nu_{2,t}^{\gamma_2} \frac{p_{l,t}^X}{\nu_{1,t}} + \gamma_1 \nu_{1,t}^{\gamma_1} \frac{p_{r,t}^X}{\nu_{1,t}} \quad (\text{B155})$$

$$+ \nu_{2,t}^{\gamma_2} \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ \frac{\partial \zeta_{0,l,t}^X}{\partial \ln A_t} - \gamma_2 \sum_{n=1}^{\infty} \left( \frac{\partial \zeta_{n,l,t}^X}{\partial \ln A_t} + n \zeta_{n,l,t}^X \right) \frac{\left( -\frac{\nu_{1,t}}{\nu_{2,t}} \right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right]$$

$$+ \nu_{1,t}^{\gamma_1} \gamma_2 \frac{1}{\nu_{1,t} \nu_{2,t}} \mathbf{R}_t^{-1} \left[ \frac{\partial \zeta_{0,r,t}^X}{\partial \ln A_t} - \gamma_1 \sum_{n=1}^{\infty} \left( \frac{\partial \zeta_{n,r,t}^X}{\partial \ln A_t} - \frac{n}{\eta} \zeta_{n,r,t}^X \right) \frac{\left( -\frac{\nu_{2,t}}{\nu_{1,t}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right].$$

## B.11 Proof of Proposition 9: Prices of bonds

Consider the time- $t$  price of a claim which pays off  $X_T$  units of consumption at date  $T$ . Hence,

$$V_{T-t}^X = X_t v_{T-t}^X, \quad (\text{B156})$$

where

$$v_{T-t}^X = E_t^1 \left[ \frac{\pi_{1,T} X_T}{\pi_{1,t} X_t} \right]. \quad (\text{B157})$$

From the expression for the state-price density in (65), it follows that

$$\begin{aligned} v_{T-t}^X &= (\pi_{1,t} X_t)^{-1} E_t \left[ \lambda_{2,0} e^{-\beta_2 T} X_T Y_T^{-\gamma_2} \xi_T \left( 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_T)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) 1_{\{A_T < R\}} \right. \\ &\quad \left. + \lambda_{1,0} e^{-\beta_1 T} X_T Y_T^{-\gamma_1} \left( 1 - \gamma_1 \sum_{n=1}^{\infty} \frac{(-A_T^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right) 1_{\{A_T > R\}} \right] du. \end{aligned} \quad (\text{B158})$$

Hence,

$$v_{T-t}^X = (\pi_{1,t} X_t)^{-1} [\lambda_{2,0} j_{l,T-t}^\phi + \lambda_{1,0} j_{r,T-t}^\phi], \quad (\text{B159})$$

where

$$j_{l,T-t}^\phi = e^{-\beta_2 t} X_t Y_t^{-\gamma_2} \xi_t \left( \phi_{0,l,t} - \gamma_2 \sum_{n=1}^{\infty} \phi_{n,l,T-t} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right), \quad (\text{B160})$$

$$j_{r,T-t}^\phi = e^{-\beta_1 t} X_t Y_t^{-\gamma_1} \left( \phi_{0,r,T-t} - \gamma_1 \sum_{n=1}^{\infty} \phi_{n,r,T-t} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right), \quad (\text{B161})$$

and

$$\phi_{n,l,T-t} = E_t e^{-\beta_2(T-t)} \frac{X_T}{X_t} \left( \frac{Y_T}{Y_t} \right)^{-\gamma_2} \frac{\xi_T}{\xi_t} \left( \frac{A_T}{A_t} \right)^n 1_{\{A_T < R\}}, \quad n \in \mathbb{N}_0, \quad (\text{B162})$$

$$\phi_{n,r,T-t} = E_t e^{-\beta_1(T-t)} \frac{X_T}{X_t} \left( \frac{Y_T}{Y_t} \right)^{-\gamma_1} \left( \frac{A_T}{A_t} \right)^{-n/\eta} 1_{\{A_T > R\}}, \quad n \in \mathbb{N}_0, \quad (\text{B163})$$

where  $\mathbb{N}_0$  is the set of natural numbers including zero. Changing measure from  $\mathbb{P}^1$  to  $\hat{\mathbb{P}}^{2,\xi}$  and  $\hat{\mathbb{P}}^1$ , respectively gives

$$\phi_{n,l,T-t} = \hat{E}_t^{2,\xi} e^{-k_2(T-t)} \left( \frac{A_T}{A_t} \right)^n 1_{\{A_T < R\}}, \quad n \in \mathbb{N}_0, \quad (\text{B164})$$

$$\phi_{n,r,T-t} = \hat{E}_t^1 e^{-k_1(T-t)} \left( \frac{A_T}{A_t} \right)^{-n/\eta} 1_{\{A_T > R\}}, \quad n \in \mathbb{N}_0. \quad (\text{B165})$$

From Lemma A2 it follows that

$$\phi_{n,l,T-t}(A_t) = e^{-[k_2 - n\hat{\mu}_A^2 - \frac{1}{2}n(n-1)\sigma_A^2](T-t)} \Phi \left( \frac{\ln \left( \frac{R}{A_t} \right) - \left( \hat{\mu}_A^{2,\xi} + \frac{1}{2}(2n-1)\sigma_A^2 \right) (T-t)}{\sigma_A(T-t)^{1/2}} \right), \quad (\text{B166})$$

and

$$\phi_{n,r,T-t}(A_t) = e^{-\left[ k_1 + \frac{n}{\eta} \left( \hat{\mu}_A^1 - \frac{1}{2} \left( 1 + \frac{2n}{\eta} \right) \sigma_A^2 \right) \right] (T-t)} \left[ 1 - \Phi \left( \frac{\ln \left( \frac{R}{A_t} \right) - \left( \hat{\mu}_A^1 - \frac{1}{2} \left( 1 + \frac{2n}{\eta} \right) \sigma_A^2 \right) (T-t)}{\sigma_A(T-t)^{1/2}} \right) \right]. \quad (\text{B167})$$

Hence,

$$\begin{aligned}
v_{T-t}^X &= \nu_{2,t}^{\gamma_2} \left( \phi_{0,l,T-t} - \gamma_2 \sum_{n=1}^{\infty} \phi_{n,l,T-t} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \\
&\quad + \nu_{1,t}^{\gamma_1} \left( \phi_{0,r,T-t} - \gamma_1 \sum_{n=1}^{\infty} \phi_{n,r,T-t} \frac{(-A_t)^{\frac{1}{\eta}}}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right) \\
&= \nu_{2,t}^{\gamma_2} \left( \phi_{0,l,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right) - \gamma_2 \sum_{n=1}^{\infty} \phi_{n,l,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right) \frac{\left( -\frac{\nu_{2,t}}{\nu_{2,t}^{\eta}} \right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \\
&\quad + \nu_{1,t}^{\gamma_1} \left( \phi_{0,r,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right) - \gamma_1 \sum_{n=1}^{\infty} \phi_{n,r,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right) \frac{\left( -\frac{\nu_{2,t}}{\nu_{1,t}^{1/\eta}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right) \\
&= \nu_{2,t}^{\gamma_2} \left( \phi_{0,l,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right) - \gamma_2 \sum_{n=1}^{\infty} \phi_{n,l,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right) \frac{\left( -\frac{\nu_{2,t}}{\nu_{2,t}^{\eta}} \right)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \right) \\
&\quad + \nu_{1,t}^{\gamma_1} \left( \phi_{0,r,T-t} - \gamma_1 \sum_{n=1}^{\infty} \phi_{n,r,T-t} \frac{\left( -\frac{\nu_{2,t}}{\nu_{1,t}^{1/\eta}} \right)^n}{n} \binom{\frac{n}{\eta} - \gamma_1 - 1}{n-1} \right), \tag{B168}
\end{aligned}$$

where for ease of notation, we omit the arguments in  $\phi_{n,l,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right)$  and  $\phi_{n,r,T-t} \left( \frac{\nu_{1,t}}{\nu_{2,t}^{\eta}} \right)$

## B.12 Proof of Proposition 10: Wealth and portfolio weights

We start by deriving expressions for each agent's financial wealth at date  $t$ , denoted by  $W_{k,t}$  for Agent  $k \in \{1, 2\}$ . Since  $W_{1,t} + W_{2,t} = P_t$ , we need only derive an expression for  $W_{1,t}$ . We know that

$$W_{1,t} = E_t \left[ \int_t^{\infty} \frac{\pi_{1,u}}{\pi_{1,t}} C_{1,u} du \right]. \tag{B169}$$

Hence,

$$\begin{aligned}
W_{1,t} &= \pi_{1,t}^{-1} \left\{ \lambda_{2,0} E_t \left[ \int_t^{\infty} e^{-\beta_2 u} Y_u^{-\gamma_2} \nu_{2,u}^{-\gamma_2} \xi_u C_{1,u} 1_{\{A_u < R\}} du \right] \right. \\
&\quad \left. + \lambda_{1,0} E_t \left[ \int_t^{\infty} e^{-\beta_1 u} Y_u^{-\gamma_1} \nu_{1,u}^{-\gamma_1} C_{1,u} 1_{\{A_u > R\}} du \right] \right\}. \tag{B170}
\end{aligned}$$

It follows that

$$\begin{aligned}
W_{1,t} &= \pi_{1,t}^{-1} \left\{ \lambda_{2,0} E_t^1 \left[ \int_t^\infty e^{-\beta_2 u} Y_u^{1-\gamma_2} \nu_{2,u}^{-\gamma_2} \nu_{1,u} \xi_u 1_{\{A_u < R\}} du \right] \right. \\
&\quad \left. + \lambda_{1,0} E_t^1 \left[ \int_t^\infty e^{-\beta_1 u} Y_u^{1-\gamma_1} \nu_{1,u}^{1-\gamma_1} 1_{\{A_u > R\}} du \right] \right\}. \tag{B171}
\end{aligned}$$

Thus,

$$\begin{aligned}
W_{1,t} &= \pi_{1,t}^{-1} \left\{ \lambda_{2,0} E_t^1 \left[ \int_t^\infty e^{-\beta_2 u} Y_u^{1-\gamma_2} (\nu_{2,u}^{-\gamma_2} - \nu_{2,u}^{1-\gamma_2}) \xi_u 1_{\{A_u < R\}} du \right] \right. \\
&\quad \left. + \lambda_{1,0} E_t^1 \left[ \int_t^\infty e^{-\beta_1 u} Y_u^{1-\gamma_1} \nu_{1,u}^{1-\gamma_1} 1_{\{A_u > R\}} du \right] \right\}. \tag{B172}
\end{aligned}$$

Since the series expression in (B107) is valid for all real  $\gamma_1$ , it follows that

$$\nu_{1,t}^{1-\gamma_1} = 1 - (1 - \gamma_1) \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1}{n-1}, \quad |A_t| > R. \tag{B173}$$

We already know that (B93) provides a convergent series expansion for  $|A_t| < R$  for all real  $\gamma_2$ . Hence,

$$\nu_{2,t}^{1-\gamma_2} = 1 - (1 - \gamma_2) \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2}{n-1}. \tag{B174}$$

Therefore,

$$\begin{aligned}
\nu_{2,u}^{-\gamma_2} - \nu_{2,u}^{1-\gamma_2} &= 1 - \gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} \\
&\quad - \left[ 1 - (1 - \gamma_2) \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2}{n-1} \right] \\
&= -\gamma_2 \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2 - 1}{n-1} + (1 - \gamma_2) \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \binom{n\eta - \gamma_2}{n-1} \\
&= \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1 - \gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right], \tag{B175}
\end{aligned}$$

which converges for  $|A_t| < R$ . Therefore,

$$\begin{aligned}
W_{1,t} &= \pi_{1,t}^{-1} \left\{ \lambda_{2,0} E_t^1 \left[ \int_t^\infty e^{-\beta_2 u} Y_u^{1-\gamma_2} \times \right. \right. \\
&\quad \left. \left( \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1 - \gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \right) \xi_u 1_{\{A_u < R\}} du \right] \right. \\
&\quad \left. + \lambda_{1,0} E_t^1 \left[ \int_t^\infty e^{-\beta_1 u} Y_u^{1-\gamma_1} \left[ 1 - (1 - \gamma_1) \sum_{n=1}^{\infty} \frac{(-A_t^{-1/\eta})^n}{n} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \right] 1_{\{A_u > R\}} du \right] \right\}. \tag{B176}
\end{aligned}$$

Since the integrand is complex analytic, term-by-term integration is valid. Hence,

$$\begin{aligned}
\pi_{1,t}W_{1,t} &= \lambda_{2,0} \sum_{n=1}^{\infty} \frac{(-)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \times \\
& E_t^1 \left[ \int_t^{\infty} e^{-\beta_2 u} Y_u^{1-\gamma_2} A_t^n \xi_u 1_{\{A_u < R\}} du \right] \\
& + \lambda_{1,0} E_t^1 \left[ \int_t^{\infty} e^{-\beta_1 u} Y_u^{1-\gamma_1} 1_{\{A_u > R\}} du \right] \\
& - (1-\gamma_1) \lambda_{1,0} \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{(-)^n}{n} E_t^1 \left[ \int_t^{\infty} e^{-\beta_1 u} Y_u^{1-\gamma_1} A_t^{-n/\eta} 1_{\{A_u > R\}} du \right] \\
& = \lambda_{2,0} e^{-\beta_2 t} Y_t^{1-\gamma_2} \xi_t \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \times \\
& E_t^1 \left[ \int_t^{\infty} e^{-\beta_2(u-t)} \left( \frac{Y_u}{Y_t} \right)^{1-\gamma_2} \frac{\xi_u}{\xi_t} \left( \frac{A_u}{A_t} \right)^n \xi_u 1_{\{A_u < R\}} du \right] \times \\
& + \lambda_{1,0} e^{-\beta_1 t} Y_t^{1-\gamma_1} E_t^1 \left[ \int_t^{\infty} e^{-\beta_1(u-t)} \left( \frac{Y_u}{Y_t} \right)^{1-\gamma_1} 1_{\{A_u > R\}} du \right] \\
& - (1-\gamma_1) \lambda_{1,0} e^{-\beta_1 t} Y_t^{1-\gamma_1} \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \times \\
& E_t^1 \left[ \int_t^{\infty} e^{-\beta_1(u-t)} \left( \frac{Y_u}{Y_t} \right)^{1-\gamma_1} \left( \frac{A_u}{A_t} \right)^{-n/\eta} 1_{\{A_u > R\}} du \right] \\
& = \lambda_{2,0} e^{-\beta_2 t} Y_t^{1-\gamma_2} \xi_t \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y \\
& + \lambda_{1,0} e^{-\beta_1 t} Y_t^{1-\gamma_1} \left[ \zeta_{0,r,t}^Y - (1-\gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \zeta_{n,r,t}^Y \right]. \tag{B177}
\end{aligned}$$

Thus,  $w_{1,t}^Y = \frac{W_{1,t}}{Y_t}$ , is given by

$$\begin{aligned}
w_{1,t}^Y &= \nu_{2,t}^{\gamma_2} \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y \\
& + \nu_{1,t}^{\gamma_1} \left[ \zeta_{0,r,t}^Y - (1-\gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \zeta_{n,r,t}^Y \right]. \tag{B178}
\end{aligned}$$

Hence,

$$\begin{aligned}
w_{1,t}^Y &= \nu_{2,t}^{\gamma_2} \sum_{n=1}^{\infty} \frac{\left(-\frac{\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}}\right)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y \\
&\quad + \nu_{1,t}^{\gamma_1} \left[ \zeta_{0,r,t}^Y - (1-\gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{\left(-\frac{\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}}\right)^n}{n} \zeta_{n,r,t}^Y \right]. \tag{B179}
\end{aligned}$$

Thus,

$$\begin{aligned}
w_{1,t}^Y &= \nu_{2,t}^{\gamma_2} \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y \\
&\quad + \nu_{1,t}^{\gamma_1} \left[ \zeta_{0,r,t}^Y - (1-\gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \zeta_{n,r,t}^Y \right]. \tag{B180}
\end{aligned}$$

Hence,

$$w_{1,t}^Y = \nu_{2,t}^{\gamma_2} w_{1,l,t}^Y + \nu_{1,t}^{\gamma_1} w_{1,r,t}^Y, \tag{B181}$$

where

$$w_{1,l,t}^Y = \sum_{n=1}^{\infty} \frac{\left(-\frac{\nu_{1,t}}{\nu_{2,t}^{\frac{1}{\eta}}}\right)^n}{n} \left[ (1-\gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right] \zeta_{l,n,t}^Y \tag{B182}$$

$$w_{1,r,t}^Y = \left[ \zeta_{0,r,t}^Y - (1-\gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{\left(-\frac{\nu_{2,t}}{\nu_{1,t}^{\frac{1}{\eta}}}\right)^n}{n} \zeta_{n,r,t}^Y \right]. \tag{B183}$$

To find the optimal portfolio policies note that

$$W_{k,t} = N_{k,t}^B B_t + N_{k,t}^P P_t, \tag{B184}$$

where  $N_{k,t}^B$  and  $N_{k,t}^P$  are the number of bonds and units of stock, respectively, held by Agent  $k$ . Market clearing implies that

$$0 = \sum_{k=1}^2 N_{k,t}^B, \tag{B185}$$

$$1 = \sum_{k=1}^2 N_{k,t}^P. \tag{B186}$$

Thus, we need only determine  $N_{1,t}^P$ , and given this, it follows that

$$N_{2,t}^P = 1 - N_{1,t}^P \quad (\text{B187})$$

$$N_{1,t}^B = -N_{2,t}^B = \frac{W_{1,t} - N_{1,t}^P P_t}{B_t}. \quad (\text{B188})$$

Applying Ito's Lemma to (B184) when  $k = 1$ , gives

$$dW_{1,t} = B_t dN_{1,t}^B + P_t dN_{k,t}^P + N_{1,t}^B dB_t + N_{1,t}^P dP_t. \quad (\text{B189})$$

The self-financing condition

$$B_t dN_{1,t}^B + P_t dN_{k,t}^P + N_{1,t}^B dB_t = 0, \quad (\text{B190})$$

implies that

$$dW_{1,t} = N_{1,t}^P dP_t, \quad (\text{B191})$$

and hence

$$\frac{dW_{1,t}}{W_{1,t}} = \Pi_{1,t} \frac{dP_t}{P_t} \quad (\text{B192})$$

where

$$\Pi_{k,t} = \frac{N_{k,t}^P P_t}{W_{k,t}} \quad (\text{B193})$$

is the proportion of Agent  $k$ 's wealth held in the stock market. Hence,

$$\Pi_{1,t} = \frac{\sigma_{W_{1,t}}}{\sigma_{R,t}}, \quad (\text{B194})$$

where  $\sigma_{W_{1,t}}$  is given by

$$\frac{dW_{1,t}}{W_{1,t}} = \mu_{W_{1,t}} dt + \sigma_{W_{1,t}} dZ_t, \quad (\text{B195})$$

and

$$\sigma_{W_{1,t}} = \sigma_Y + \frac{\nu_{1,t}}{w_{1,t}^Y} \frac{\partial w_{1,t}^Y}{\partial \nu_{1,t}}, \quad (\text{B196})$$

where

$$\frac{\partial w_{1,t}^Y}{\partial \nu_{1,t}} = -\gamma_2 \nu_{2,t}^{\gamma_2 - 1} w_{1,l,t}^Y + \gamma_1 \nu_{1,t}^{\gamma_1 - 1} w_{1,r,t}^Y + \nu_{2,t}^{\gamma_2} \frac{\partial w_{1,l,t}^Y}{\partial \nu_{1,t}} + \nu_{1,t}^{\gamma_1} \frac{\partial w_{1,r,t}^Y}{\partial \nu_{1,t}} \quad (\text{B197})$$

and

$$\begin{aligned} \frac{\partial w_{1,l,t}^Y}{\partial \nu_{1,t}} &= \frac{\partial A_t}{\partial \nu_{1,t}} \sum_{n=1}^{\infty} \left[ \frac{\left(-\frac{\nu_{1,t}}{\nu_{2,t}^\eta}\right)^n}{n} \frac{\partial \zeta_{l,n,t}^Y}{\partial A_t} + \left(-\frac{\nu_{1,t}}{\nu_{2,t}^\eta}\right)^{n-1} \zeta_{l,n,t}^Y \right] \times \\ &\quad \left[ (1 - \gamma_2) \binom{n\eta - \gamma_2}{n-1} - \gamma_2 \binom{n\eta - \gamma_2 - 1}{n-1} \right], \end{aligned} \quad (\text{B198})$$

$$w_{1,r,t} = \zeta_{0,r,t}^Y - (1 - \gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta} - \gamma_1}{n-1} \frac{\left(-\frac{\nu_{2,t}}{\nu_{1,t}^\eta}\right)^n}{n} \zeta_{n,r,t}^Y. \quad (\text{B199})$$



$$\begin{aligned}
\frac{\partial w_{1,t}^Y}{\partial \nu_{1,t}} &= \gamma_2 \nu_{2,t}^{\gamma_2-1} \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1-\gamma_2) \binom{n\eta-\gamma_2}{n-1} - \gamma_2 \binom{n\eta-\gamma_2-1}{n-1} \right] \zeta_{l,n,t}^Y \\
&\quad + \gamma_2 \nu_{2,t}^{\gamma_2-1} \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \left[ (1-\gamma_2) \binom{n\eta-\gamma_2}{n-1} - \gamma_2 \binom{n\eta-\gamma_2-1}{n-1} \right] \zeta_{l,n,t}^Y \\
&\quad + \nu_{1,t}^{\gamma_1} \left[ \frac{\partial \zeta_{0,r,t}^Y}{\partial \nu_{1,t}} - (1-\gamma_1) \sum_{n=1}^{\infty} \binom{\frac{n}{\eta}-\gamma_1}{n-1} \frac{\partial}{\partial \nu_{1,t}} \frac{(-A_t^{-\frac{1}{\eta}})^n}{n} \zeta_{n,r,t}^Y \right]. \tag{B200}
\end{aligned}$$

Expressed in words, the proportion of Agent 1's wealth held in the stock market equals the ratio of the volatility of her total portfolio return to the volatility of the stock market.

## C Some results from complex analysis

In this section, we state a number of definitions and theorems from complex analysis that are used in the Appendix of the paper.

**Definition C1** *If  $U$  is an open subset of  $\mathbb{C}$  and  $f : U \rightarrow \mathbb{C}$  is a complex function on  $U$ , we say that  $f$  is complex differentiable at a point  $z_0$  of  $U$  if the limit*

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{C1}$$

*exists. The limit here is taken over all sequences of complex numbers approaching  $z_0$ , and for all such sequences the difference quotient has to approach the same number  $f'(z_0)$ .*

**Definition C2** *If  $f$  is complex differentiable at every point  $z_0$  in  $U$ , we say that  $f$  is holomorphic on  $U$ . We say that  $f$  is holomorphic at the point  $z_0$  if it is holomorphic on some neighborhood of  $z_0$ . We say that  $f$  is holomorphic on some non-open set  $A$  if it is holomorphic in an open set containing  $A$ .*

**Definition C3** *A function  $f$  is complex analytic on an open set  $D$  in the complex plane if for any  $z_0$  in  $D$  one can write*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{C2}$$

*in which the coefficients  $a_0, a_1, \dots$  are complex numbers and the series is convergent for  $z$  in a neighborhood of  $z_0$ .*

**Theorem C1** *A function  $f$  is complex analytic on an open set  $D$  in the complex plane if and only if it is holomorphic in  $D$ .*

We are now ready to state the theorem that allows us to find closed-form series expansions for the sharing rule and complex analytic functions of the sharing rule.

**Theorem C2 (Lagrange)** *Suppose the dependence between the variables  $w$  and  $z$  is implicitly defined by an equation of the form*

$$w = f(z), \tag{C3}$$

*where  $f$  is complex analytic in a neighborhood of 0 and  $f'(0) \neq 0$ . Then for any function  $g$  which is complex analytic in a neighborhood of 0,*

$$g(z) = g(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} g'(x) [\varphi(x)^n] \right]_{x=0}, \tag{C4}$$

*where  $\varphi(z) = \frac{z}{f(z)}$ .*

Note that the above theorem does not provide a radius of convergence for the series (C4). While the original proof of Theorem C2 due to Lagrange is not very straightforward, a relatively easier proof can be obtained by using Cauchy's Integral Formula.

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