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ASSET PRICING IN
MULTIPERIOD SECURITIES
MARKETS

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ABSTRACT

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The paper provides an intertemporal version of the capital asset pricing model (CAPM) of Sharpe and Lintner. Although we allow for general changes in the investment opportunity set and for general risk-averse preferences, there are conditions under which two mutual funds are sufficient to generate all optimal portfolios. In particular, we require that the Riesz claim, which represents the date 0 pricing functional for the marketed claims, should lie in a scalar Brownian information set. Then we obtain an instantaneous counterpart to the CAPM pricing formula: a linear relationship between the conditional mean returns on the securities and conditional covariances with the return on the market portfolio. Our use of option pricing techniques requires continuous trading but does not require continuous consumption.

In addition, we consider a large economy with a factor structure, as in Ross' arbitrage pricing theory. The dividends are assumed to have an approximate factor structure, with the factor components lying in the information set generated by an N-dimensional Brownian motion, and with the covariance matrices of the idiosyncratic components having uniformly bounded eigenvalues. We obtain an N-factor version of the pricing formula and relate the factors to the gains processes (price change plus accumulated dividends) for well-diversified portfolios. An approximate factor structure for dividends implies an approximate factor structure for the gains processes of the securities. Furthermore, the assumption that per capita supply is well diversified can motivate our condition that the Riesz claim lies in an N-dimensional Brownian information set.

1. INTRODUCTION AND SUMMARY

The paper provides an intertemporal version of the capital asset pricing model (CAPM) of Sharpe [38] and Lintner [26]. In addition, we consider a large economy with a factor structure, as in Ross' [34, 35] arbitrage pricing theory (APT). This enriches the interpretation of our model and in turn suggests a multifactor extension of it.

To set up the static CAPM, suppose that there is a single consumption good and that agents are interested in certain consumption at date 0 and state contingent consumption at date T. There are K+1 securities representing contingent claims to the good at date T. These claims lie in a Hilbert space H of random variables with finite variance. A share of the kth security pays $d_k \varepsilon H$ units of the good. The 0th security is a riskless asset that pays one unit of the good in all states of nature. The price of a share of the kth security at date 0 is Z_{k0}. We use the 0th security as numeraire, so that $Z_{00} = 1$, and we set $Z_{kT} = d_k$. Under this numeraire, the riskless interest rate is zero.

Markets are frictionless, with no transactions costs and no restrictions on short sales. All agents share the same probability assessments. The jth agent chooses a claim $x_j^* \varepsilon H$ from the linear span of $d_0, \ldots, d_K$; the value of the market portfolio at T is $W = \sum_{j=1}^{J} x_j^*$. The CAPM asserts that the expected change in the price of a security is proportional to its covariance with the value of the market portfolio:

$$(1.1) \qquad E(Z_{kT} - Z_{k0}) = \phi \operatorname{Cov}(Z_{kT} - Z_{k0}, W) \qquad (k=1,\ldots,K).$$

The argument runs as follows: suppose that each agent chooses a mean-variance efficient portfolio, so that the market portfolio is also mean-variance efficient. One can show that all mean-variance efficient claims in M are linear functions of a single claim ρ (mutual fund separation); furthermore, ρ represents the price system in that $Z_{k0} = E(\rho d_k)$. Then (1.1) directly follows.

Now consider an intertemporal model in which security trading can occur at intermediate dates $0 = t_0 < t_1 < \ldots < t_N = T$. In a dynamic programming approach, one chooses a portfolio at t_i to maximize the conditional expectation of a value function defined over wealth at t_{i+1}. The problem is that this value function depends also upon the information available at t_{i+1}. Additional state variables must be introduced to summarize changes in the investment opportunity set, as in Merton [31]. In Merton's continuous time model, the pricing formula contains covariances of price changes with changes in the state variables, so that one does not generally obtain a simple relationship like (1.1).[1/] Cox, Ingersoll and Ross [10] have provided a general equilibrium setting for these additional covariance terms. Equilibrium models in a discrete time framework have been provided by Lucas [29], Brock [4], and Prescott and Mehra [32].[2/]

We shall adopt the continuous trading framework of Merton [30, 31] and Cox, Ingersoll and Ross [10], but we shall not use dynamic programming techniques. There is an information structure (given by an increasing sequence of σ-fields) $\{F_t, 0 \leq t \leq T\}$, and a stochastic process $\underline{Z}_t = (Z_{0t}, \ldots, Z_{Kt})$ giving the prices at date t of the K+1 securities, as a function of the information available at that date. This information is common to all of the agents. As before, agents are interested in certain consumption at t=0 and state contingent consumption at $t=T$. Their endowments consist solely of the consumption good and the securities at $t=0$; there are no nontraded assets. A trading strategy $\underline{\theta}_t = (\theta_{0t},\ldots,\theta_{Kt})$ is a stochastic process in which θ_{kt} specifies how many shares of the kth security to hold at date t, as a function of the information available at that date. An admissible trading strategy must be self-financing in that the value of the portfolio at t equals the initial value plus the accumulated gains (and losses) from trading prior to t.

A contingent claim $x \varepsilon H$ is marketed at $t=0$, denoted by $x \varepsilon M$, if there is an admissible trading strategy $\underline{\theta}$ such that $\underline{\theta}_T \cdot \underline{Z}_T \equiv \sum_{k=0}^{K} \theta_{kT} Z_{kT} = x$. Then (if there are no free lunches) we can follow Harrison and Kreps [19] in defining the implicit price of x at $t=0$ by $\pi(x) = \underline{\theta}_0 \cdot \underline{Z}_0$. We assume that π can be extended to a continuous linear functional ψ on H; so, by a theorem of Riesz, ψ can be represented as $\psi(x) = E(\rho x)$, where $\rho \varepsilon H$.

Now an agent's problem is to choose a claim $x \varepsilon M$ subject to $\pi(x)$ $(= E(\rho x))$ satisfying his budget constraint. The agents are risk-averse in the following sense: if $x = \hat{x} + e$, where $E(e|\hat{x}) = 0$, then they prefer $\hat{x}$ to x. Preferences may differ across the agents, but they all use the same probability measure in making this calculation. Then we are able to show that every optimal claim is a measurable function of ρ, provided that these claims are marketed.

With continuous trading, this restriction on optimal claims can lead to a mutual fund result which, as in the CAPM, leads to a pricing formula. The key to the mutual fund result is the martingale representation theory used by Harrison and Kreps [19] to provide a foundation for the Black-Scholes [1] option pricing formula. Suppose that ρ is in the information set generated by a Brownian motion B (i.e., $\rho \epsilon F^B$). We shall discuss the motivation for this assumption below. Let $\underline{\theta}^*_{jt}$ be the trading strategy chosen by the jth agent. Then there are (nonanticipating) stochastic processes α_j and γ such that

$$(1.2) \qquad \underline{\theta}^*_{jt} \cdot \underline{Z}_t = \underline{\theta}^*_{j0} \cdot \underline{Z}_0 + \int_0^t \alpha_{js} dB_s + \int_0^t \alpha_{js} \gamma_s ds \quad (j=1,\ldots,J;\ 0 \le t \le T)$$

--the value process for any optimal portfolio can be represented as a stochastic integral over a single process $Y_t \equiv B_t + \int_0^t \gamma_s ds$. Hence the value of the market portfolio, $W_t = \sum_{j=1}^J \underline{\theta}^*_{jt} \cdot \underline{Z}_t$, also has such a representation.

From here we use a martingale projection argument to obtain our pricing formula:

$$(1.3) \qquad W_t = W_0 + \int_0^t \alpha_s dB_s + \int_0^t \alpha_s \gamma_s ds,$$

$$(1.4) \qquad Z_{kt} = Z_{k0} + \int_0^t \beta_{ks} dB_s + \int_0^t \beta_{ks} \gamma_s ds + V_{kt} \qquad (k=1,\ldots,K;\ 0 \le t \le T),$$

where α, γ, and β_k are (nonanticipating) stochastic processes and V_k is a martingale that is uncorrelated with B; i.e., $Cov(B_t, V_{kt} | F_s) = 0$ for $0 \le s \le t \le T$. In differential form (if $\alpha_t \neq 0$),

$$(1.5) \qquad dZ_{kt} = \beta_{kt} \alpha_t^{-1} dW_t + dV_{kt}$$

and

$$(1.6) \qquad E(dZ_{kt} | F_t) = \phi_t \, Cov(dZ_{kt}, dW_t | F_t),$$

where $\phi_t = \gamma_t/\alpha_t$. Equation (1.6) is our intertemporal counterpart to the static CAPM equation (1.1).

In order to assess the assumption that $\rho \varepsilon F^B$, suppose the information structure is generated by a vector $\underline{B}_t = (B_{1t},\ldots,B_{Lt})$ of independent Brownian motions. We show that it is not necessarily the case that one can construct a scalar Brownian motion B such that $\rho \varepsilon F^B$. So this condition must be regarded as restrictive. Such a construction is possible if there is an invertible function $g\colon R \to R$ such that $(g(\rho), \underline{B}_{t_1},\ldots,\underline{B}_{t_n})$ has a multivariate normal distribution for any finite set of points $t_j \varepsilon [0, T]$.

An alternative justification, which I prefer, considers a large economy with a countable set of securities. Assume that the security payoffs have an approximate one-factor structure generated by a Brownian motion B; i.e., $d_k = f_k + e_k$, where the factor components f_k are in the information set generated by B, and the covariance matrix of the idiosyncratic components $(e_1,\ldots,e_n)$ has uniformly bounded eigenvalues as $n \to \infty$.[3/] Then we can follow the Pareto-efficiency argument in Connor [8] to motivate the assumption that $\rho \varepsilon F^B$. Furthermore, the role of the market portfolio can be played by any well-diversified portfolio.

The plan of the paper is as follows. Section 2 continues the Introduction by deriving the CAPM equation (1.1) in a way that mimics our treatment of the multiperiod case. Section 3 sets up the information structure and the price system, and follows Harrison and Kreps [19] in defining a new probability measure under which the security prices are martingales. Section 3 also summarizes some martingale theory, in particular the key notion of the martingale covariance process. Section 4 derives the restriction in (1.2) on the value of an optimal portfolio, and Section 5 derives our pricing formulas (1.3 - 1.6).

Section 6 shows how the restriction in (1.2) gives a mutual fund result. The two mutual funds consist of the riskless asset and a $\underline{\lambda}$-fund that holds λ_{kt} shares of the kth security at date t. For any optimal claim, there is a scalar process α such that by holding α_t units of the $\underline{\lambda}$-fund at t, and adjusting the holding of the riskless asset to keep the strategy self-financing, we generate that claim at T. Furthermore, the risky $\underline{\lambda}$-fund is chosen to be instantaneously mean-variance efficient.

Section 7 presents an N-factor version of the pricing formula for a countable set of securities, and it allows for consumption and dividends at intermediate dates. We relate the factors to the gains processes (price change plus accumulated dividends) for well-diversified portfolios, and we show that an approximate factor structure for dividends implies an approximate factor structure for the gains processes of the securities.

2. THE STATIC CAPM

I shall begin by reviewing the Sharpe-Lintner model.[4/] The treatment of the dynamic case will follow it quite closely. Also some of our notation will be set up in this section.

There is a complete probability space (Ω, F, P) and a space H of F-measurable random variables that are square integrable:

$$H = \{x\varepsilon F : E(x^2) = \int x^2(\omega)dP(\omega) < \infty\}$$

($x\varepsilon F$ denotes x is F-measurable). H is a Hilbert space under the mean-square inner product $(x,y) = E(xy)$.

There is a single consumption good and agents are interested in certain consumption at date 0 and state contingent consumption at date T. There are K+1 securities representing claims to the consumption good at T. A share of the kth security pays d_k units of the consumption good, where $d_k \varepsilon H$. The 0th security is a riskless asset with $d_0(\omega) = 1$ for all states $\omega\varepsilon\Omega$, which we denote by $d_0=1_\Omega$. The (nonstochastic) price of the good at $t=0$ is $q\varepsilon R$, where R is the real line. The price of a share of the kth security at $t=0$ is $Z_{k0}\varepsilon R$. We use the 0th security as numeraire, so that $Z_{00} = 1$, and we set $Z_{kT} = d_k$ $(k=0,1,\ldots,K)$. Under this numeraire the riskless interest rate is zero: $Z_{00} = Z_{0T} = 1$.

The set M of marketed claims is the linear span of $d_0,\ldots,d_K$:

$$M = [d_0,\ldots,d_K]$$

$$= \{x = \sum_{k=0}^{K} \theta_k d_k : (\theta_0,\ldots,\theta_K)\varepsilon R^{K+1}\}.$$

If $x = \sum\theta_k d_k$, the (implicit) price of x at t=0 is $\pi(x) = \sum\theta_k Z_{k0}$. We can regard π as defined on M instead of on portfolio vectors in R^{K+1} since if $x = \sum\theta_k d_k = \sum\theta'_k d_k$, an arbitrage argument implies that $\sum\theta_k Z_{k0} = \sum\theta'_k Z_{k0}$; otherwise, a claim to 0 could be sold at a positive price, so that agents could costlessly increase their consumption at t=0. So $\pi:M\to R$ is a linear functional; it is continuous since M is a finite dimensional subspace. Hence, by Riesz's theorem, there is a $\rho\varepsilon M$ such that $\pi(x) = E(\rho x)$.

The jth agent has preferences over R×H represented by a utility function v_j (j=1,...,J). He is risk-averse in the following sense: if $x = \hat{x} + e$, where $E(e) = E(e\hat{x}) = 0$, then $v_j(c,\hat{x}) \geq v_j(c,x)$ for all $c\varepsilon R$, with strict inequality unless e = 0 a.s.[5/] This definition of risk-aversion is objectionable, as argued by Rothschild and Stiglitz [37]; one would like to replace $E(e\hat{x}) = 0$ by $E(e|\hat{x}) = 0$. The continuous trading model will allow us to do that.

Agents have endowments at t=0 consisting of the consumption good and shares in the securities. Consumption at T is provided for by holding a portfolio of securities; there are no nonmarketed endowments. The jth agent solves the following problem: max $v_j(c,x)$ subject to $(c,x)\varepsilon R\times M$ and $qc + \pi(x) \leq a_j$, where a_j is the value at t=0 of his endowment.

Now there are two basic steps, which will be repeated when we consider continuous trading. First we determine the space of efficient portfolios, or, more directly, of efficient claims. This space is generated by ρ. Then we project security prices onto this space.

Given any $x\varepsilon M$, consider its projection $\hat{x}$ onto the linear space $[1_\Omega,\rho]$: $x = \hat{x} + e$, where $E(e) = E(\rho e) = 0$. Hence $v_j(c,\hat{x}) > v_j(c,x)$ for any $c\varepsilon R$ unless $x = \hat{x}$ a.s. Note that $\hat{x}\varepsilon M$ since $1_\Omega = d_0\varepsilon M$ and $\rho\varepsilon M$; furthermore, $\pi(\hat{x}) = \pi(x)$ since $\pi(e) = E(\rho e) = 0$. So if (c_j^*,x_j^*) is chosen by the jth agent, then $x_j^* \varepsilon [1_\Omega, \rho]$.6/ This key mutual fund property implies that the market claim, $W = \sum_{j=1}^{J} x_j^*$, is also in $[1_\Omega, \rho]$: $W = \tau + \alpha\rho$, where $\tau, \alpha \varepsilon R$. Note that W is the value at T of the market portfolio.

Now consider the projection of Z_{kT} onto $[1_\Omega, \rho]$:

$$Z_{kT} = \tau_k + \beta_k \rho + V_k,$$

where $E(V_k) = E(\rho V_k) = 0$ and τ_k, $\beta_k \varepsilon R$ $(k=1,\ldots,K)$. Since $Z_{k0} = E(\rho Z_{kT})$ and $Z_{00} = E\rho = 1$, we have

$$Z_{k0} = \tau_k + \beta_k\ E(\rho^2).$$

Hence

$$Z_{kT} = Z_{k0} + \beta_k(\rho - E\rho) + \beta_k \gamma + V_k,$$

where $\gamma \varepsilon R$. Since

$$W = \tau + \alpha\rho,$$

it follows that, if $\mathrm{Var}(W) \neq 0$,

$$(2.1) \qquad E(Z_{kT} - Z_{k0}) = \phi\ \mathrm{Cov}(Z_{kT} - Z_{k0}, W),$$

where $\phi = \alpha\gamma/\mathrm{Var}(W)$. With the numeraire chosen to give a zero interest rate, the expected change in the price of a security is proportional to its covariance with the value of the market portfolio.

3. CONTINUOUS TRADING: MARTINGALES AND THE PRICE SYSTEM

3.1. The Price System

Suppose that trading can take place at any date in $[0,T]$. We need to extend the static model so that there is a price system and an information structure at each date. The information structure is given by a filtration: $F = \{F_t,\ 0 \le t \le T\}$ is a family of sub-σ- fields with $F_s \subset F_t$ for $0 \le s \le t \le T$ and $F_T = F$. Date 0 events are certain in that $P(A) = 0$ or 1 for $A \varepsilon F_0$. In addition, F is a standard filtration: $F_s = F_{s+} \equiv \cap_{t>s} F_t$ (right-continuity), and F_0 contains all the P-null sets (completion).[7]

The (K+1)-dimensional stochastic process $\underline{Z}_t = (Z_{0t},\ldots,Z_{Kt})$ gives the prices at t of the securities; $\underline{Z}$ is adapted to F in that $\underline{Z}_t$ is F_t-measurable ($\underline{Z}_t \varepsilon F_t$). As before, the kth security pays $d_k \varepsilon H$ units of the consumption good at T and $d_0 = 1_\Omega$. We use the 0th security as numeraire so that $Z_{0t} = 1_\Omega$, and we set $Z_{kT} = d_k$. Assume that $E(Z_{kt}^2) < \infty$ $(k=0,\ldots,K;\ 0 \le t \le T)$.

We also need to specify the admissible trading strategies. Define a simple trading strategy as a (K+1)-dimensional stochastic process $\underline{\theta} = \{\underline{\theta}_t,\ 0 \le t \le T\}$ that satisfies three conditions: (1) $\underline{\theta}_t \varepsilon F_t$; (2) $\theta_{0t} \varepsilon H$, $\sup_{t,\omega} |\theta_{kt}(\omega)| < \infty$ $(k=1,\ldots,K)$; (3) there is a finite integer N and a sequence of dates $0 = t_0 < t_1 < \ldots < t_N = T$ such that $\underline{\theta}_t(\omega)$ is constant over the interval $t_{n-1} \le t < t_n$ for every state $\omega (n=1,\ldots,N)$.[8] Then $\underline{\theta}_t \cdot \underline{Z}_t$ $(= \sum_{k=0}^{K} \theta_{kt} Z_{kt})$ represents the value of the portfolio at t. Define $\underline{\theta}$ to be a self-financing simple strategy if

$$\underline{\theta}_{t_{n-1}} \cdot \underline{Z}_{t_n} = \underline{\theta}_{t_n} \cdot \underline{Z}_{t_n} ;$$

i.e., the value of the portfolio before trading at t_n equals the value after trading. This self-financing requirement can also be expressed as follows: if $t_n < t \leq t_{n+1}$,

$$(3.1) \quad \underline{\theta}_t \cdot \underline{Z}_t = \underline{\theta}_0 \cdot \underline{Z}_0 + \sum_{k=0}^{n-1} \underline{\theta}_{t_k} \cdot (\underline{Z}_{t_{k+1}} - \underline{Z}_{t_k}) + \underline{\theta}_{t_n} \cdot (\underline{Z}_t - \underline{Z}_{t_n})$$

--the value of the portfolio at t is the initial value plus the accumulated capital gains and losses.

We shall assume that the space Θ of admissible trading strategies is linear ($a\underline{\theta} + a'\underline{\theta}' \varepsilon \Theta$ if $\underline{\theta}, \underline{\theta}' \varepsilon \Theta$ and $a, a' \varepsilon R$) and includes the simple, self-financing ones. We shall say more about Θ after we have set up the necessary martingale machinery.

A claim $x \varepsilon H$ is said to be marketed at t=0, which we denote by $x \varepsilon M$, if there is a trading strategy $\theta \varepsilon \Theta$ such that $\underline{\theta}_T \cdot \underline{Z}_T = x$ a.s. The cost of that strategy is $\underline{\theta}_0 \cdot \underline{Z}_0$. We can identify $\underline{\theta}_0 \cdot \underline{Z}_0$ with the price of x if $\underline{\theta}_0 \cdot \underline{Z}_0 = \underline{\theta}_0' \cdot \underline{Z}_0$ for any $\underline{\theta}' \varepsilon \Theta$ with $\underline{\theta}_T' \cdot \underline{Z}_T = x$ a.s. As in Section 2, this follows from an arbitrage argument: if $\underline{\theta}_0 \cdot \underline{Z}_0 > \underline{\theta}_0' \cdot \underline{Z}_0$, consider the portfolio strategy

$$\underline{\gamma}_t = \underline{\theta}_t' - \underline{\theta}_t + [(\underline{\theta}_0 - \underline{\theta}_0') \cdot \underline{Z}_0, 0, \ldots, 0].$$

This is admissible ($\underline{\gamma} \varepsilon \Theta$), requires no initial investment ($\underline{\gamma}_0 \cdot \underline{Z}_0 = 0$), and generates positive consumption at T ($\underline{\gamma}_T \cdot \underline{Z}_T = (\underline{\theta}_0 - \underline{\theta}_0') \cdot \underline{Z}_0 > 0$).

We shall say that a portfolio stategy $\underline{\theta} \varepsilon \Theta$ is a free lunch if $\underline{\theta}_0 \cdot \underline{Z}_0 \leq 0$ and $\underline{\theta}_T \cdot \underline{Z}_T \varepsilon H_+$, where H_+ consists of the claims $x \varepsilon H$ with $P\{x \geq 0\} = 1$ and $P\{x > 0\} > 0$. Assume there are no free lunches and define the price of claims in M by $\pi(x) = \underline{\theta}_0 \cdot \underline{Z}_0$, where $\underline{\theta} \in \Theta$ and $\underline{\theta}_T \cdot \underline{Z}_T = x$ a.s. We shall assume that π admits an extension ψ to all of H, where $\psi: H \to R$ is a continuous, strictly positive, linear functional: $\psi(x) = \pi(x)$ for $x \in M$ and $\psi(x) > 0$ if $x \in H_+$. Harrison and Kreps [19] provide general conditions on preferences under which there must be such an extension in order for optimal net trades to exist, and hence in order for (M,π) to be viable as an equilibrium price system.

Then by Riesz's theorem there is a $\rho \in H$ such that $\psi(x) = E(\rho x)$ for all $x \in H$; $\rho > 0$ a.s. since ψ is strictly positive. We shall assume that ρ is uniformly bounded above and away from zero: there is a $\delta \in R$ such that $P\{0 < \delta \leq \rho \leq \delta^{-1}\} = 1$. From our choice of numeraire, $\psi(1_\Omega) = E(\rho) = 1$. So following Harrison and Kreps [19] we can define a new probability measure P^*, with $P^*(A) = \int_A \rho dP$; let $E^*(x) = \int x dP^*$ for $x \in H$. Then $\underline{Z}$ is a (P^*,F)-martingale:

LEMMA 1. $E^*(\underline{Z}_t | F_s) = \underline{Z}_s$, $\qquad 0 \leq s \leq t \leq T$.

Proof. See Harrison and Kreps [19, Theorem 2].[9/]

3.2. The Martingale Covariance Process

With an eye to applications, we are mainly interested in martingales generated by stochastic integrals over Brownian motion. However, the structure of our arguments is somewhat clearer if we work with general continuous martingales. In particular, this helps to underline the key role of the covariance process and the associated notion of martingale projection.[10/]

We shall assume that all martingales adapted to F are continuous; more precisely, if X is a martingale (under P or P*) adapted to F, then the function $t \mapsto X_t(\omega)$ is continuous on [0,T] for $\omega\varepsilon\Omega'$, where $P(\Omega') = 1$. An example is the filtration generated by a vector of independent Brownian motions.

We shall say that a (P,F)-martingale X is square-integrable, or $X\varepsilon M^2$, if $\|X\|_2 \equiv (EX_T^2)^{\frac{1}{2}} < \infty$. (Note that, by Jensen's inequality, $EX_t^2 \leq EX_T^2$, $0 \leq t \leq T$). If $X\varepsilon M^2$, there is a unique decomposition of X^2 as the sum of a continuous martingale and a continuous, increasing, integrable process with initial value 0. This latter process is known as the quadratic variation or variance process and will be denoted by $\langle X \rangle$.[11/] It can be obtained by partitioning the interval [0,T] into $0 = t_{0n} < t_{1n} < \ldots < t_{k(n),n} = t$ and forming the quadratic variation

$$S_t^n = \sum_{j=0}^{k(n)-1} (X_{t_{j+1,n}} - X_{t_{jn}})^2;$$

if $\max \{|t_{j+1,n} - t_{jn}|,\ j=0,\ldots,k(n)-1\} \to 0$ as $n \to \infty$, then

$$E|S_t^n - \langle X\rangle_t| \to 0. \tag{3.2}$$

Given $X\varepsilon M^2$, define $\Pi_2(X)$ to be the set of predictable processes such that $\|\alpha\|_X \equiv (E\int_0^T \alpha_s^2\, d\langle X\rangle_s)^{\frac{1}{2}} < \infty$.[12/] Then the stochastic integral $Y_t = \int_0^t \alpha_s\, dX_s$ is well-defined for $\alpha\varepsilon\Pi_2(X)$, and Y is a square-integrable (P,F)-martingale ($Y\varepsilon M^2$).

By analogy with

$$Cov(x,y) = \tfrac{1}{4}[Var(x+y) - Var(x-y)] \quad (x,y\varepsilon H),$$

define the covariance process for the martingales X and Y in M^2 by

$$\langle X,Y\rangle_t = \tfrac{1}{4}[\langle X+Y\rangle_t - \langle X-Y\rangle_t].$$

Note that $\langle X,X\rangle = \langle X\rangle$. Then $X_tY_t - \langle X,Y\rangle_t$ is a martingale, and $\langle X,Y\rangle$ is the unique continuous, bounded variation, integrable process that has this property and initial value 0. We can obtain $\langle X,Y\rangle$ by forming a partition as in (3.2) and then

$$(3.3)\qquad E\left|\sum_j (X_{t_{j+1,n}} - X_{t_{jn}})(Y_{t_{j+1,n}} - Y_{t_{jn}}) - \langle X,Y\rangle_t\right| \to 0$$

as $n \to \infty$.

If $\langle X,Y\rangle_t = 0$ for $0 \le t \le T$, then X_tY_t is a martingale and so

$$E(X_tY_t|F_s) = X_sY_s = E(X_t|F_s)\ E(Y_t|F_s)$$

for $0 \le s \le t \le T$. Hence X_t and Y_t are uncorrelated conditional on F_s, an extremely useful property. We shall say that the martingales X and Y are uncorrelated.

If $Y_t = \int_0^t \alpha_s dX_s$ and $V_t = \int_0^t \beta_s dW_s$, where X, $W \epsilon M^2$, $\alpha \epsilon \Pi_2(X)$, and $\beta \epsilon \Pi_2(W)$, then $\langle Y,V\rangle_t = \int_0^t \alpha_s\beta_s d\langle X,W\rangle_s$. If B is a Brownian motion, then $\langle B\rangle_t = t$ and so the covariance process for $\int_0^t \alpha_s dB_s$ and $\int_0^t \beta_s dB_s$ is $\int_0^t \alpha_s\beta_s ds$.

These definitions apply equally well under P*. So if X and Y are (P*,F)-martingales, $X \epsilon M^2$ denotes $\|X\|_2 \equiv (E^*X_T^2)^{\frac{1}{2}} < \infty$, $\langle X\rangle$ is the variance process (so $X_t^2 - \langle X\rangle_t$ is a (P*,F)-martingale), $\langle X,Y\rangle$ is the covariance process, and $\Pi_2(X)$ is the set of predictable processes α such that

$\|\alpha\|_X \equiv (E^* \int_0^T \alpha_s^2 d\langle X\rangle_s)^{1/2} < \infty$. If we are not explicit, the measure that the martingale property refers to should be clear from the context. The filtration is always F unless we say otherwise. Note that $P(A) = 0$ if and only if $P^*(A) = 0$, so there is no ambiguity in the use of "almost surely."

Since $\underline{Z}$ is a P^*-martingale, if $\underline{\theta}$ is a simple, self-financing, portfolio strategy, then (3.1) implies that $\underline{\theta}_t \cdot \underline{Z}_t$ is a square-integrable martingale under P^*. That $\underline{\theta}_t \cdot \underline{Z}_t$ is a P^*-martingale is the appropriate generalization of "self-financing"; see Harrison and Pliska [20]. The square-integrability is convenient, and we shall assume that $\underline{\theta}_t \cdot \underline{Z}_t \varepsilon M^2$ under P^* for all $\underline{\theta} \varepsilon \Theta$. This is all we need to say about Θ to obtain our asset pricing formula. After deriving the formula, we shall give a definition for Θ in Section 6 which allows us to exhibit portfolio strategies that can serve as mutual funds.

4. PREFERENCES AND THE REPRESENTATION OF OPTIMAL CLAIMS

4.1. The Structure of Optimal Claims

As in Section 2, the preferences of the jth agent are given by a utility function $v_j: R \times H \to R$ $(j=1,\ldots,J)$. Now, however, we adopt a more appealing definition of risk aversion: If $x, \hat{x} \varepsilon H$ and $x = \hat{x} + e$, where $E(e|\hat{x}) = 0$, then $v_j(c,\hat{x}) \geq v_j(c,x)$ for all $c \varepsilon R$, with strict inequality unless $e = 0$ a.s. This holds, for example, if $v_j(c,x) = E[u_j(c,x)]$, where $u(c,\cdot)$ is strictly concave in its second argument (and supposing that the expectation exists).

The jth agent has an endowment consisting of $\bar{c}_j$ units of the consumption good and $\bar{\theta}_{jk}$ units of the kth security at t=0. So his budget constraint at t=0 is determined by $a_j = q\bar{c}_j + \underline{\bar{\theta}}_j \cdot \underline{Z}_0$. He solves the following problem:

$$\max_{c,x} v_j(c,x)$$

$$\text{subject to } c \varepsilon R,\ x \varepsilon M,\ qc + \pi(x) \leq a_j.$$

We shall assume that this problem has a solution, which we denote by c_j^*, x_j^*.

Now we show how risk-aversion restricts the set of optimal claims. Define $H(\rho)$ as the set of claims in H that are measureable with respect to ρ:

$$H(\rho) = \{x \varepsilon H: x = g(\rho) \text{ a.s. for some measurable function } g: R \to R\}.$$

LEMMA 2. If $H(\rho) \subset M$, then $x_j^* \varepsilon H(\rho)$ $(j=1,\ldots,J)$.

Proof. Let $\hat{x} = E(x^*|\rho)$, so $x^* = \hat{x} + e$ with $E(e|\rho) = 0$. (We have dropped the j subscripts to simplify notation.) Then $\hat{x} \varepsilon H(\rho) \subset M$ (by Jensen's inequality), and $(c^*,\hat{x})$ satisfies the budget constraint: $\pi(\hat{x}) = \pi(x^*) - \pi(e) = \pi(x^*)$ since $\pi(e) = E(\rho e) = 0$. $E(e|\hat{x}) = 0$ implies that $(c^*,\hat{x})$ is strictly preferred to (c^*,x^*) unless $e=0$ a.s. Q.E.D.

A Representative Agent

Lemma 2 shows that an optimal claim is a function of ρ if the set of marketed claims is sufficiently large. With some additional structure, we can sum over the agents and then invert to express ρ as a function of aggregate consumption. This result is not needed to obtain our pricing formula, but it does enrich the interpretation of ρ.

Suppose that the preferences of the jth agent are given by

$$v_j(c,x) = w_j(c) + E[u_j(x)],$$

where $u_j:R \to R$ is increasing and strictly concave with derivative u_j'. If $y \varepsilon M$ and $\pi(y) = E(\rho y) = 0$, then a necessary condition for x_j^* to be an optimal claim is that $E[u_j(x_j^* + \alpha y)]$ is maximized over $\alpha \varepsilon R$ at $\alpha^* = 0$. Under suitable conditions, this requires $E[y u_j'(x_j^*)] = 0$. If $M = H$ (complete markets), then $E[y u_j'(x_j^*)] = 0$ for all y in H with $E(\rho y) = 0$, and so $u_j'(x_j^*) = \lambda_j \rho$ a.s., where $\lambda_j \varepsilon R$ is positive. Since u_j' is strictly decreasing, $x_j^* = g_j(\rho)$ a.s., where $g_j:A_j \to R$ is strictly decreasing on its domain $A_j \subset R$. Hence $\bar{x} \equiv \sum_{j=1}^{J} x_j^* = g(\rho)$ a.s., where $g \equiv \sum_{j=1}^{J} g_j$ is strictly decreasing, and so $\rho = h(\bar{x})$ a.s., where h is strictly decreasing.

From here we could construct a representative agent, with time-additive, strictly concave, von Neumann-Morgenstern preferences, whose optimal claim is $\bar{x}$. Such constructions have been used in [16, 17, 9, 22]. For our purposes, however, all that matters is that ρ is a measurable function of aggregate consumption: $\rho(\omega) = h[\bar{x}(\omega)]$ for (almost) all states ω. Note that this implies, given Lemma 2, that optimal claims are functions of aggregate consumption, as in Breeden and Litzenberger [3].

4.2. Martingale Representation

In Section 2 we used mean-variance preferences to restrict the optimal claims to linear functions of ρ. Lemma 2 is weaker, but with continuous trading it can lead to a sharp restriction on the values of optimal portfolios. In order to make this connection, we need the following representation for $H(\rho)$:

CONDITION (R). There is a P*-martingale $Y \varepsilon M^2$ with $Y_0 = 0$ and $E^*(\langle Y\rangle_T^2) < \infty$ such that if $x \varepsilon H(\rho)$, then

$$x = E^*(x) + \int_0^T \alpha_s dY_s$$

for some $\alpha \varepsilon \Pi_2(Y)$.

The motivation for Condition (R) comes from representation results for functionals of Brownian motion. Suppose that $\underline{B} = (B_1, \ldots, B_L)$ is a vector of independent Brownian motions under P; we shall follow the convention that a Brownian motion has initial value equal to 0. Let $F_t^{\underline{B}}$ denote the

σ-field generated by $\{\underline{B}_s,\ 0 \le s \le t\}$, let $\mathcal{F}^{\underline{B}} = \mathcal{F}^{\underline{B}}_T$, and let $F^{\underline{B}}$ denote the corresponding filtration.[13/] The martingale representation theorem of Kunita and Watanabe [25; 15, p. 88] implies that if X is a square-integrable $(P, F^{\underline{B}})$-martingale, then it can be represented as:

$$X_t = X_0 + \sum_{n=1}^{L} \int_0^t \alpha_{ns} dB_{ns},$$

where $\alpha_n \varepsilon \Pi_2(B_n)$. Harrison and Kreps [19] and Kreps [24] have shown how martingale representations provide the foundation for the Black-Scholes option pricing formula. Duffy and Huang [14] have shown how these representations relate to the number of continuously traded securities needed to implement an Arrow-Debreu equilibrium.

Suppose that $Y \varepsilon M^2$ is a P*-Brownian motion and let $X_t = E^*(x|F_t^Y)$, where $x \varepsilon H(\rho)$. If $\rho \varepsilon F^Y$ then $x \varepsilon F^Y$, and the martingale representation theorem gives $x = X_T = E^*(x) + \int_0^T \alpha_s dY_s$. The following lemma provides a corresponding result in terms of the original measure P. The proof follows Duffy and Huang [14, Proposition 6.3].

LEMMA 3. If there is a P-Brownian motion $B \varepsilon M^2$ with $\rho \varepsilon F^B$, then Condition (R) holds.

<u>Proof</u>. Let $V_t = E(\rho|F_t^B)$. The Kunita-Watanabe result implies that

$$V_t = 1 + \int_0^t \beta_s dB_s,$$

where $\beta \varepsilon \Pi_2(B)$ and $\beta_t \varepsilon F_t^B$. Girsanov's theorem [27, Theorem 6.2] implies that

$$(4.1) \qquad Y_t \equiv B_t - \int_0^t V_s^{-1} \beta_s ds$$

is a P*-Brownian motion in M^2. If $x \varepsilon H$, then Lipster and Shiryayev [27, Theorem 5.20] implies that the (P^*, F^B)-martingale $S_t = E^*(x | F_t^B)$ can be represented as $S_t = E^*(x) + \int_0^t \alpha_s dY_s$, where $\alpha \varepsilon \Pi_2(Y)$.14/ The result follows since $S_T = x$ if $x \varepsilon H(\rho)$. Q.E.D.

Suppose that F is generated by an L-dimensional P-Brownian motion: $F = F^{\underline{B}}$. There is a counterexample in Appendix B to show that given $y \varepsilon H$, it need not be true that there is a scalar Brownian motion $B \varepsilon M^2$ such that $y \varepsilon F^B$. Hence the Lemma 3 condition that $\rho \varepsilon F^B$ must be regarded as restrictive.

I shall indicate a special case in which a scalar Brownian motion B can be constructed such that $\rho \varepsilon F^B$. Suppose that $F = F^{\underline{B}}$, and that there is some measurable function $g: R \to R$ with a measurable inverse such that $g(\rho)$ together with $\underline{B}$ form a Gaussian system; that is, the distribution of $(g(\rho), \underline{B}_{t_1}, \ldots, \underline{B}_{t_n})$ is multivariate normal for any finite set of points $t_i \varepsilon [0, T]$. Then we have the following representation for $g(\rho)$:

$$g(\rho) = E[g(\rho)] + \sum_{n=1}^{L} \int_0^T \alpha_{ns} \, dB_{ns},$$

where α_n: $[0, T] \to R$ is a deterministic function. (This follows from the L-dimensional version of [27, Theorem 5.6; 28, p. 13].) Define $\gamma_t = (\sum_{n=1}^L \alpha_{nt}^2)^{\frac{1}{2}}$ and, to avoid some complications, suppose that $\gamma_t > 0$ $(0 \leq t \leq T)$. Then

$$B_t = \sum_{n=1}^{L} \int_0^t \gamma_s^{-1} \alpha_{ns} \, dB_{ns}$$

is a scalar Brownian motion in M^2, and

$$g(\rho) = E[g(\rho)] + \int_0^T \gamma_s \, dB_s$$

[27, Lemma 6.9]. Since γ_t is deterministic, it follows that $g(\rho) \varepsilon F^B$ and so $\rho \varepsilon F^B$. Note that, given a representative agent, this argument applies if some invertible function of aggregate consumption together with $\underline{B}$ form a Gaussian system.

We shall return to the representation problem in Section 7, when we consider an N-factor version of Condition (R). There we relate the factors to well-diversified portfolios and show how Condition (R) can be motivated from the assumption that aggregate per capita consumption is well diversified.

Now we shall show how Condition (R) combines with Lemma 2 to restrict the values of optimal portfolios.

LEMMA 4. Suppose that Condition (R) holds. If $x \varepsilon H(\rho)$ and there is a portfolio strategy $\underline{\theta} \varepsilon \Theta$ such that $\underline{\theta}_T \cdot \underline{Z}_T = x$ a.s., then there is an $\alpha \varepsilon \Pi_2(Y)$ such that

$$\underline{\theta}_t \cdot \underline{Z}_t = \underline{\theta}_0 \cdot \underline{Z}_0 + \int_0^t \alpha_s dY_s \qquad (0 \leq t \leq T).$$

<u>Proof</u>. (R) implies that there is an $\alpha \varepsilon \Pi_2(Y)$ such that

$$x = E^*(x) + \int_0^T \alpha_s dY_s .$$

Since $\underline{\theta}_t \cdot \underline{Z}_t$ and $\int_0^t \alpha_s dY_s$ are P*-martingales,

$$\underline{\theta}_t \cdot \underline{Z}_t = E^*(\underline{\theta}_T \cdot \underline{Z}_T | F_t) = E^*(x | F_t)$$

$$= E^*(x) + \int_0^t \alpha_s dY_s.$$

Then the result follows from $E^*(x) = \psi(x) = \pi(x) = \underline{\theta}_0 \cdot \underline{Z}_0$. Q.E.D.

Lemmas 2 and 4 establish the key mutual fund property. If $H(\rho)$ is marketed, then the value of an optimal portfolio is restricted to be a stochastic integral over a single martingale Y, which is common to all optimal portfolios. A portfolio strategy that, together with the riskless asset, plays the role of a mutual fund is displayed in Section 6. The form of that strategy is not needed for the derivation of our asset pricing formula in the next section.

5. THE ASSET PRICING FORMULA

Following the development of the static CAPM, we need a counterpart for the projection of Z_k on ρ. It is given by the martingale projection theorem, which uses the notion of the stable subspace generated by a square-integrable martingale X with $X_0 = 0$. This stable subspace $S(X)$ consists of the stochastic integrals over X:

$$S(X) \;=\; \{V \varepsilon M^2 : V_t \;=\; \int_0^t \alpha_s dX_s,\; \alpha \varepsilon \Pi_2(X)\}.$$

LEMMA 5. Suppose that Condition (R) holds. Then

$$(5.1) \quad Z_{kt} \;=\; Z_{k0} + \int_0^t \beta_{ks} dY_s + V_{kt} \qquad (k=1,\ldots,K;\ 0 \le t \le T),$$

where $\beta_k \varepsilon \Pi_2(Y)$ and V_k is a square-integrable P*-martingale that is uncorrelated with $S(Y)$: $\langle X, V_k \rangle_t = 0$ if $X \varepsilon S(Y)$.

Proof. This is a direct application of the martingale projection theorem: M^2 is a Hilbert space under the inner product $(X_1, X_2) = E^*(X_{1T} X_{2T})$; $S(Y)$ is a closed subspace of M^2 under the M^2 norm, and so the Hilbert space projection theorem gives $Z_{kt} - Z_{k0} = \hat{Z}_{kt} + V_{kt}$, where $\hat{Z}_k \varepsilon S(Y)$ and $(X_T, V_{kT}) = 0$ for all $X \varepsilon S(Y)$; this implies that V_k is uncorrelated with $S(Y)$ [15, pp. 87, 88]. Q.E.D.

Let $\underline{\theta}_j^* \varepsilon \Theta$ be the portfolio strategy chosen by the jth agent and define W as the value of the market portfolio:

$$W_t = \sum_{j=1}^{J} \underline{\theta}^*_{jt} \cdot \underline{Z}_t \qquad (0 \leq t \leq T).$$

The projection result in Lemma 5 is not by itself restrictive; we could get such a representation by projecting Z_k onto any square-integrable P*-martingale. The restrictions come from the fact that $S(Y)$ contains ρ, the claim that represents the price system. So the mutual fund result in Lemmas 2 and 4 implies that $W_t - W_0 \varepsilon S(Y)$. In addition, since ρ is the density of P* with respect to P, we can show that V_k in (5.1) is also a martingale under the original measure P. These are the key points in the following result.

THEOREM 1. Suppose that Condition (R) holds. If $H(\rho) \subset M$, then

(5.2) $$W_t = W_0 + \int_0^t \alpha_s dX_s + \int_0^t \alpha_s \gamma_s d\langle X\rangle_s,$$

(5.3) $$Z_{kt} = Z_{k0} + \int_0^t \beta_{ks} dX_s + \int_0^t \beta_{ks}\gamma_s d\langle X\rangle_s + V_{kt} \qquad (k=1,\ldots,K;\ 0 \leq t \leq T),$$

where X and V_k are square-integrable martingales under P; α, γ, and β_k are in $\Pi_2(X)$; and V_k is uncorrelated with $S(X)$.

Proof. (i) Lemma 2 implies that $x^*_j \varepsilon H(\rho)$ and then Lemma 4 implies that an optimal portfolio strategy satisfies

(5.4) $$\underline{\theta}^*_{jt} \cdot \underline{Z}_t = \underline{\theta}^*_{j0} \cdot \underline{Z}_0 + \int_0^t \alpha_{js} dY_s,$$

where $\alpha_j \varepsilon \Pi_2(Y)$. Hence $\alpha_t \equiv \sum_{j=1}^{J} \alpha_{jt}$ is in $\Pi_2(Y)$ and summing (5.4) over the J agents gives

(5.5) $$W_t = W_0 + \int_0^t \alpha_s dY_s.$$

(ii) Note that $\lambda \equiv \rho^{-1}$ is dP/dP^*. Since $\lambda \varepsilon H(\rho)$ (recall that there is a $\delta \varepsilon R$ such that $0 < \delta \leq \rho \leq \delta^{-1}$ a.s.), Condition (R) implies that

$$\lambda = E^*(\lambda) + \int_0^T \eta_s dY_s,$$

where $\eta \varepsilon \Pi_2(Y)$, and so

(5.6) $$\lambda_t \equiv E^*(\lambda | F_t) = E^*(\lambda) + \int_0^t \eta_s dY_s.$$

V_{kt} in (5.1) is a P-martingale if $\lambda_t V_{kt}$ is a P*-martingale [15, p. 83]; this follows since $\lambda_t - \lambda_0 \varepsilon S(Y)$ and V_k is uncorrelated with $S(Y)$ under P*.

(iii) From Girsanov's theorem [15, p. 83],

$$X_t \equiv Y_t - \int_0^t \lambda_s^{-1} \, d\langle\lambda, Y\rangle_s$$

is a local P-martingale. So there is a sequence $\{\tau(k), k=1,2,\ldots\}$ of stopping times such that $\tau(k)\uparrow T$ a.s. and $X_t^k \equiv X_{t\wedge\tau(k)}$ is a P-martingale ($a \wedge b \equiv \min\{a,b\}$). X is a P-martingale if for each $t\varepsilon[0,T]$

(5.7) $$\{|X_{t\wedge\tau(k)}|, k=1,2,\ldots\}$$

is uniformly integrable [7, Proposition 1.8]. The Kunita-Watanabe inequality gives

$$|X_t| \leq \sup_{0\leq s\leq T} |Y_s| + \delta^{-1}\langle\lambda\rangle_T^{1/2} \langle Y\rangle_T^{1/2} \equiv z$$

[7, eq. (5.17)]. Since λ_t is a bounded P*-martingale, $E^*\langle\lambda\rangle_T^2 < \infty$ [7, Theorem 4.1(i)], and $E^*\langle Y\rangle_T^2 < \infty$ by Condition (R). Hence $E^*(z^2) < \infty$ by Doob's inequality [7, Theorem 1.4], and so $E(z^2) \leq \delta^{-1} E^*(z^2)$ implies that uniform integrability holds in (5.7), and X is a square-integrable martingale under P.

From (5.6), $\langle \lambda, Y\rangle_t = \int_0^t \eta_s d\langle Y\rangle_s$. Note that $P\{\langle X\rangle_t = \langle Y\rangle_t, 0 \leq t \leq T\} = 1$ since $X - Y$ is a continuous process with bounded variation. Hence $\Pi_2(X) = \Pi_2(Y)$ and

$$(5.8) \qquad Y_t = X_t + \int_0^t \lambda_s^{-1} \eta_s d\langle X\rangle_s.$$

Substituting (5.8) into (5.5) and (5.1) gives (5.2) and (5.3) with $\gamma_t = \lambda_t^{-1} \eta_t$. Finally, for any $\zeta \varepsilon \Pi_2(X)$ we have

$$\langle \int \zeta_s dX_s, V_k\rangle_t = \int_0^t \zeta_s d\langle X, V_k\rangle_s = 0,$$

since $\langle X, V_k\rangle_t = \langle Y, V_k\rangle_t = 0$. Q.E.D.

COROLLARY 1. Suppose that there is a P-Brownian motion $B \varepsilon \mathcal{M}^2$ with $\rho \varepsilon F^B$. If $H(\rho) \subset M$, then

$$(5.9) \qquad W_t = W_0 + \int_0^t \alpha_s dB_s + \int_0^t \alpha_s \gamma_s ds,$$

$$(5.10) \qquad Z_{kt} = Z_{k0} + \int_0^t \beta_{ks} dB_s + \int_0^t \beta_{ks}\gamma_s ds + V_{kt} \qquad (k=1,\ldots,K;\ 0 \leq t \leq T),$$

where α, γ, and β_k are in $\Pi_2(B)$, and V_k is a square-integrable martingale under P that is uncorrelated with $\mathcal{S}(B)$.

<u>Proof</u>. From Lemma 3, (4.1) and (5.8), $B - X$ has bounded variation. Since $B - X$ is a continuous P-martingale, $P\{B_t = X_t, 0 \leq t \leq T\} = 1$ [15, p. 54]. Now the result follows from the Theorem since $\langle B\rangle_t = t$. Q.E.D.

Under the assumptions of Corollary 1, let Y be the P*-Brownian motion constructed in Lemma 3. Then (5.1) and (5.5) imply that

$$\langle Z_k, W\rangle_t = \int_0^t \beta_{ks}\alpha_s \, ds,$$

$$\langle W\rangle_t = \int_0^t \alpha_s^2 \, ds,$$

so, if $\alpha_t \neq 0$,

$$\frac{d\langle Z_k, W\rangle_t/dt}{d\langle W\rangle_t/dt} = \beta_{kt}\alpha_t^{-1}.$$

We can obtain consistent estimates of the variance and covariance processes $\langle W\rangle$ and $\langle Z_k, W\rangle$ from a single realization of W and Z_k, by forming the quadratic variation and covariation as in (3.2) and (3.3). Hence a consistent estimator of $\beta_{kt}\alpha_t^{-1}$ may be available as the sampling interval between observations shrinks to zero. The restriction from our pricing formula is that

$$Z_{kt} - \int_0^t \beta_{ks}\alpha_s^{-1} \, dW_s$$

is a P-martingale.

Expressing (5.9) and (5.10) in differential form gives

$$dW_t = \alpha_t dB_t + \alpha_t \gamma_t dt,$$

$$dZ_{kt} = \beta_{kt} dB_t + \beta_{kt}\gamma_t dt + dV_{kt}.$$

The formalism corresponding to the martingale property of B and V_k is $E(dB_t|F_t) = E(dV_{kt}|F_t) = 0$, and corresponding to $\langle B\rangle_t = t$ and $\langle B,V_k\rangle_t = 0$ we have $\mathrm{Var}(dB_t|F_t) = dt$ and $\mathrm{Cov}(dB_t, dV_{kt}|F_t) = 0$. Hence

$$E(dZ_{kt}|F_t) = \beta_{kt}\gamma_t dt,$$

$$\mathrm{Cov}(dZ_{kt}, dW_t|F_t) = \beta_{kt}\alpha_t dt,$$

and, if $\alpha_t \neq 0$,

$$(5.11) \quad E(dZ_{kt}|F_t) = \phi_t \, \mathrm{Cov}(dZ_{kt}, dW_t|F_t),$$

where $\phi_t = \gamma_t/\alpha_t$. This is perhaps the closest counterpart to the static CAPM equation (2.1) that one can hope to obtain in a general intertemporal setting.

Now consider changing the numeraire so that the price of the kth security at t is $\tilde{Z}_{kt} = p_t Z_{kt}$ $(k=1,\ldots,K)$, $\tilde{Z}_{0t} = p_t$, and $p_t \varepsilon F_t$ is positive a.s.[15/] If $dp_t/dt = p_t r_t$, corresponding to a locally riskless interest rate of r_t, then Ito's formula [7, Theorem 5.10] gives

$$d\tilde{Z}_{kt} = p_t dZ_{kt} + Z_{kt} dp_t;$$

using rates of return, we have

$$\tilde{Z}_{kt}^{-1}\, d\tilde{Z}_{kt} = Z_{kt}^{-1}\, dZ_{kt} + r_t dt.$$

Then with $\tilde{W}_t = p_t W_t$, (5.11) becomes

$$\text{(5.12)} \quad E(\tilde{Z}_{kt}^{-1}\, d\tilde{Z}_{kt} | F_t) = r_t dt + \tau_t \operatorname{Cov}(\tilde{Z}_{kt}^{-1}\, d\tilde{Z}_{kt}, \tilde{W}_t^{-1}\, d\tilde{W}_t | F_t),$$

where $\tau_t = \phi_t W_t$.

6. THE MUTUAL FUND PORTFOLIO

Let $\underline{\theta} = (\theta_0,\ldots,\theta_K)\varepsilon\Pi_2(\underline{Z})$ denote $\theta_k\varepsilon\Pi_2(Z_k)$, k=0,...,K, and let $\int_0^t \underline{\theta}_s d\underline{Z}_s$ denote $\sum_{k=0}^K \int_0^t \theta_{ks}dZ_{ks}$. We shall specify Θ, the set of admissible portfolio strategies, to consist of all $\underline{\theta}\varepsilon\Pi_2(\underline{Z})$ such that

$$(6.1) \qquad \underline{\theta}_t\cdot\underline{Z}_t = \underline{\theta}_0\cdot\underline{Z}_0 + \int_0^t \underline{\theta}_s d\underline{Z}_s, \qquad 0 \leq t \leq T.$$

The stochastic integral in (6.1) replaces the summation in the definition of self-financing simple strategies in (3.1). Then Θ is a linear space that contains the simple, self-financing strategies, and (6.1) ensures that $\underline{\theta}_t\cdot\underline{Z}_t$ is a square-integrable P*-martingale.16/

Define $\underline{S}_t = (Z_{1t},\ldots,Z_{Kt})$, and suppose that $\underline{S}$ is an Ito process:

$$(6.2) \qquad \underline{S}_t = \underline{S}_0 + \int_0^t \Phi_s d\underline{B}_s + \int_0^t \underline{\mu}_s ds,$$

where $\underline{B} = (B_1,\ldots,B_K)$ is a vector of independent P-Brownian motions in M^2. The jth element of $\int_0^t \Phi_s d\underline{B}_s$ is $\sum_k \int\phi_{jks}dB_{ks}$, where $\phi_{jk}\varepsilon\Pi_2(B_k)$ (j,k=1,...,K). The $K\times K$ matrix $\Psi_t \equiv \Phi_t\Phi_t'$ is the local covariance matrix for $\underline{S}_t$ (A' denotes the transpose of the matrix A), and it is convenient to assume that its eigenvalues are a.s. uniformly bounded above and away from zero: there is a $\delta\varepsilon R$ such that a.s. we have $0<\delta\leq\underline{a}'\ \Psi_t\ \underline{a}\leq\delta^{-1}$ $(0\leq t\leq T)$ for any $\underline{a}\varepsilon R^K$ with $\underline{a}\cdot\underline{a} = 1$. The $K\times 1$ vector $\underline{\mu}$ is a predictable process with $E(\int_0^T \underline{\mu}_s\cdot\underline{\mu}_s ds)<\infty$. Recall that $Z_{0t} = 1$. If F is generated by $\underline{B}$, we shall write $F = F^{\underline{B}}$.

PROPOSITION 1. Suppose that Condition (R) holds. (i) If $Y_T \varepsilon M$, then $H(\rho) \subset M$. (ii) If $F = F^{\underline{B}}$, then $M = H$. (iii) If $F = F^{\underline{B}}$ and $P\{\underline{\mu}_t \cdot \underline{\mu}_t > 0, 0 \leq t \leq T\} = 1$, then the riskless asset and the mutual fund $\underline{\lambda}_t = \Psi_t^{-1}\, \underline{\mu}_t$ generate $H(\rho)$; i.e., if $x \varepsilon H(\rho)$, there are processes θ_0 and α such that $\underline{\theta}_t = (\theta_{0t}, \alpha_t\lambda_{1t}, \ldots, \alpha_t\lambda_{Kt}) \varepsilon \Theta$ and $\underline{\theta}_T \cdot \underline{Z}_T = x$ a.s.

Proof. See Appendix A.

If Y_T is marketed, there is a portfolio process $\underline{\gamma}$ that generates it. The two mutual funds consist of the riskless asset and the fund that holds γ_{kt} shares of the kth security at t (k=1,...,K). For any optimal claim (i.e., $x \varepsilon H(\rho)$), there is a scalar process α such that by holding α_t units of the $\underline{\gamma}$-fund at t, and adjusting the holding of the riskless asset to keep the strategy self-financing, we generate that claim at T. If $\underline{\mu}_t \neq \underline{0}$, we can use $\Psi_t^{-1}\, \underline{\mu}_t$ for the risky fund. This corresponds to the mean-variance efficient portfolio in the static case.

7. LARGE ECONOMY FACTOR MODELS

7.1. Extension of the Pricing Formula

I shall present an N-factor version of the pricing formula and relate the factors to the gains processes for well-diversified portfolios. This requires a countable set of securities. I shall also extend the consumption space and allow securities to pay dividends before the terminal date.

Suppose that there is a fixed set of dates $0 < \tau_1 < \tau_2 < \ldots < \tau_L = T$ at which consumption and dividend payments may occur. (We allow consumption at t=0 but not dividend payments.) Let $H_i = \{x \varepsilon H: x \varepsilon F_{\tau_i}\}$. A share of the kth security pays $d_{ki} \varepsilon H_i$ units of the consumption good at τ_i (k=1,2,...; i=1,...,L). The 0th security pays no dividends until the terminal date, when it pays one unit of the consumption good: $d_{0i} = 0$ (i=1,...,L-1) and $d_{0L} = 1_\Omega$. We let the 0th security be numeraire and set $Z_{0t} = 1$ $(0 \leq t \leq T)$. In terms of this numeraire, the price of the consumption good at τ_i is $q_i \varepsilon H_i$ (i=1,...,L-1), the price at t=0 is $q_0 \varepsilon R$, and we set $q_L = 1$. We assume there is a $\delta \varepsilon R$ such that $P\{0 < \delta \leq q_i \leq \delta^{-1}, i=1,\ldots,L-1\} = 1$.

Define

$$D_{kt} = \sum_{i:\tau_i \leq t} q_i d_{ki}, \qquad \underline{D}_t = (D_{0t},\ldots,D_{nt}),$$

and define the gains process for the first n+1 securities as

$$\underline{G}_t = \underline{Z}_t - \underline{Z}_0 + \underline{D}_t.$$

Here n is a finite integer which may be arbitrarily large; a more explicit but more cumbersome notation uses $\underline{D}_{nt}$, $\underline{G}_{nt}$, etc. Suppose that securities are sold ex-dividend and set $\underline{Z}_T = (0,\ldots,0)$. Let $\underline{\theta} = (\theta_0,\ldots,\theta_n)$ be a simple trading strategy, as defined in Section 3. Define $H^\tau = H_1 \times \ldots \times H_L$ and consider a claim $\underline{x} = (x_1,\ldots,x_L)\varepsilon H^\tau$. The analog of the self-financing requirement in (3.1) is

$$(7.1) \qquad \underline{\theta}_t\cdot\underline{Z}_t = \underline{\theta}_0\cdot\underline{Z}_0 + \sum_{j=0}^{m-1} \underline{\theta}_{t_j}\cdot(\underline{G}_{t_{j+1}} - \underline{G}_{t_j}) + \underline{\theta}_{t_m}\cdot(\underline{G}_t - \underline{G}_{t_m}) - \sum_{i:\tau_i\leq t} q_i x_i$$

for $t_m < t \leq t_{m+1}$.

A claim $\underline{x}\varepsilon H^\tau$ is said to be marketed at t=0, which we denote by $\underline{x}\varepsilon M$, if it is generated by an admissible trading strategy. The set Θ of these admissible strategies is assumed to be linear and to contain at least the simple strategies that satisfy (7.1) for some $\underline{x}\varepsilon H^\tau$, which is then the $\underline{x}$ generated by that $\underline{\theta}$. An admissible strategy employs only a finite number of securities. If $\underline{\theta}_i\varepsilon\Theta$ generates $\underline{x}_i$, we assume that $\sum_{i=1}^2 a_i\underline{\theta}_i$ generates $\sum_{i=1}^2 a_i\underline{x}_i$ $(a_i\varepsilon R)$.

A trading strategy $\underline{\theta}\varepsilon\Theta$ is a free lunch if $\underline{\theta}_0\cdot\underline{Z}_0 = 0$ and $\underline{\theta}$ generates $\underline{x}\varepsilon H_+^\tau$ (i.e., $x_i \geq 0$, i=1,...,L, a.s. and $\sum_{i=1}^L x_i^2 > 0$ with positive probability). We assume there are no free lunches, which allows us to define the price of $\underline{x}\varepsilon M$ by $\pi(\underline{x}) = \underline{\theta}_0\cdot\underline{Z}_0$, where $\underline{\theta}\varepsilon\Theta$ generates $\underline{x}$.

H^τ is a Hilbert space under the inner product $(\underline{x},\underline{y}) = E(\sum_{i=1}^L x_i y_i)$. We assume that π admits an extension ψ to all of H^τ, where ψ is a continuous, strictly positive linear functional: $\psi(\underline{x}) = \pi(\underline{x})$ if $\underline{x}\varepsilon M$ and $\psi(\underline{x}) > 0$ if $\underline{x}\varepsilon H_+^\tau$. By Riesz's theorem there is a $\underline{\rho}\varepsilon H^\tau$ such that $\psi(\underline{x}) = E(\sum_{i=1}^L \rho_i x_i)$. We shall assume there is a $\delta\varepsilon R$ such that $P\{0 < \delta \leq \rho_i \leq \delta^{-1}, i=1,\ldots,L\} = 1$. Given our choice of numeraire, $\psi[(0,\ldots,0,1_\Omega)] = E(\rho_L) = 1$.

As in Section 3, we define a new probability measure P*, with $P^*(A) = \int_A \rho_L dP$; let $E^*(x) = \int x dP^*$ for $x\varepsilon H$. Then the gains process is a P*-martingale:

LEMMA 6. $E^*(G_{kt}|F_s) = G_{ks}$ $(k=0,1,\ldots;\ 0 \le s \le t \le T)$.

Proof. We follow the proof in Harrison and Kreps [19, Theorem 2].[17/] Fix $k>0$, s, t, and $A\varepsilon F_s$. Consider the case in which $s<\tau_1 \le t$ and $d_{ki} = 0$ for $i\neq 1$. (The argument in the other cases is similar.) Consider the simple trading strategy $\underline{\theta}$ defined by

$$\begin{aligned} \theta_{ku}(\omega) &= 1 \text{ for } u\varepsilon[s,t) \text{ and } \omega\varepsilon A, \\ &= 0 \text{ otherwise;} \end{aligned}$$

$$\begin{aligned} \theta_{0u}(\omega) &= -Z_{ks}(\omega) \text{ for } u\varepsilon[s,\tau_1) \text{ and } \omega\varepsilon A, \\ &= -Z_{ks}(\omega) + q_1(\omega)d_{k1}(\omega) \text{ for } u\varepsilon[\tau_1,t) \text{ and } \omega\varepsilon A, \\ &= -Z_{ks}(\omega) + q_1(\omega)d_{k1}(\omega) + Z_{kt}(\omega) \text{ for } u\varepsilon[t,T] \text{ and } \omega\varepsilon A, \\ &= 0 \text{ otherwise;} \end{aligned}$$

$$\theta_{ju}(\omega) = 0 \text{ for all } j\neq 0,k.$$

This strategy satisfies (7.1) and generates the claim $\underline{x} = (0,\ldots,0,x_L)$ with $x_L = (G_{kt} - G_{ks})1_A$. Since the initial cost of the strategy is zero,

$$0 = \psi(\underline{x}) = E^*[(G_{kt} - G_{ks})1_A].$$

Since this holds for all $A\varepsilon F_s$, we have $G_{ks} = E^*(G_{kt}|F_s)$. Q.E.D.

Since $\underline{G} = (G_0,\ldots,G_n)$ is a square-integrable P*-martingale, we can adopt the following specification for the admissible trading strategies: let $\underline{\theta}\varepsilon\Pi_2(\underline{G})$ denote $\theta_k\varepsilon\Pi_2(G_k)$ $(k=0,\ldots,n)$, and let $\int_0^t \underline{\theta}_s d\underline{G}_s$ denote $\sum_{k=0}^n \int_0^t \theta_{ks} dG_{ks}$; then Θ consists of all $\underline{\theta}\varepsilon\Pi_2(\underline{G})$ (for n=0,1,...) such that

$$(7.2) \qquad \underline{\theta}_t\cdot\underline{Z}_t = \underline{\theta}_0\cdot\underline{Z}_0 + \int_0^t \underline{\theta}_s d\underline{G}_s - \sum_{i:\tau_i\le t} q_i x_i \qquad (0 \le t \le T)$$

for some $\underline{x}\varepsilon H^{\tau}$; we say that $\underline{\theta}$ generates $\underline{x}$ and that $\int_0^t \underline{\theta}_s d\underline{G}_s$ is the gains process corresponding to $\underline{x}$.

In order to use diversification arguments, we shall allow agents to choose claims from $\bar{M}$, the closure of M in H^{τ}. If $\underline{x}\varepsilon\bar{M}$, define the corresponding gains process as

$$(7.3) \qquad G_t = E^*\left(\sum_{i=1}^{L} q_i x_i \mid F_t\right) - \psi(\underline{x}).$$

This corresponds to the following limits: suppose that $\underline{x}_n\varepsilon M$ is generated by the trading strategy $\underline{\theta}_n\varepsilon\Theta$, and that $\underline{x}_n\to\underline{x}$ as $n\to\infty$. Then $\pi(\underline{x}_n) = \psi(\underline{x}_n)\to\psi(\underline{x})$ and $G_t^n \equiv \int_0^t \underline{\theta}_{ns} d\underline{G}_s$ converges in M^2 to the P*-martingale G.

Preferences of the jth agent are represented by a utility function $v_j : R\times H^{\tau} \to R$ and are risk-averse in the following sense: if $\underline{x}$, $\hat{\underline{x}}\varepsilon H^{\tau}$ and $x_i = \hat{x}_i + e_i$ $(i=1,\ldots,L)$, where $E(e_i|\hat{x}_1,\ldots,\hat{x}_i) = 0$, then $v_j(c,\underline{x}) \le v_j(c,\hat{\underline{x}})$ with strict inequality unless $\underline{e} = \underline{0}$ a.s. This holds if $v(c,\underline{x}) = u_0(c) + \sum_{i=1}^L E[u_i(x_i)]$, where u_i is strictly concave.

The jth agent solves the following problem:

$$\max v_j(c,\underline{x})$$

$$\text{subject to } c\varepsilon R,\ \underline{x}\varepsilon\bar{M},\ q_0 c + \psi(\underline{x}) \leq a_j,$$

where $a_j \varepsilon R$. Assume this problem has a solution and denote it by c_j^*, $\underline{x}_j^*$ $(j=1,\ldots,J)$.

Define

$$H(\underline{\rho}) = \{\underline{x}\varepsilon H^T\colon x_i = g_i(\rho_1,\ldots,\rho_i) \text{ a.s. for some measurable function } g_i\colon R^i \to R,\ i=1,\ldots,L\}$$

and

$$H'(\underline{\rho}) = \{x\varepsilon H\colon x = g(\rho_1,\ldots,\rho_L) \text{ a.s. for some measurable function } g\colon R^L \to R\}.$$

LEMMA 7. If $H(\underline{\rho}) \subset \bar{M}$, then $\underline{x}_j^* \varepsilon H(\underline{\rho})$ $(j=1,\ldots,J)$.

Proof. As in Lemma 2.

CONDITION (RN). There is a vector of uncorrelated P*-martingales $\underline{Y} = (Y_1,\ldots,Y_N)$, with $Y_m \varepsilon M^2$, $Y_{m0} = 0$, and $E^*(\langle Y_m\rangle_T^2) < \infty$ $(m=1,\ldots,N)$, such that if $x\varepsilon H'(\underline{\rho})$ then

$$(7.4) \qquad x = E^*(x) + \int_0^T \underline{\alpha}_s d\underline{Y}_s$$

for some $\underline{\alpha}\varepsilon\Pi_2(\underline{Y})$.

LEMMA 8. If there is an N-dimensional vector $\underline{B}$ of independent P-Brownian motions in M^2, and if $\rho_i \varepsilon F^{\underline{B}}$ (i=1,...,L), then Condition (RN) holds.

Proof. As in Lemma 3, we have

$$V_t \equiv E(\rho_L | F_t^{\underline{B}}) = 1 + \int_0^t \underline{\beta}_s d\underline{B}_s,$$

where $\underline{\beta}\varepsilon\Pi_2(\underline{B})$, and Girsanov's theorem [27, Theorem 6.4] implies that

$$(7.5) \qquad \underline{Y}_t \equiv \underline{B}_t - \int_0^t V_s^{-1} \underline{\beta}_s ds$$

is a vector of independent P*-Brownian motions in M^2. Then the result follows since any $y\varepsilon H$ that is measurable with respect to $F^{\underline{B}}$ can be represented as a stochastic integral over $\underline{Y}$ [27, Theorem 5.20; or 23, Theorem 8.3.1]. Q.E.D.

The following result places an (N+1)-mutual fund restriction on the optimal gains processes.

LEMMA 9. Suppose that Condition (RN) holds. If G is the gains process for $\underline{x}\varepsilon\bar{M}$, and if $\underline{x}\varepsilon H(\underline{\rho})$, then there is an $\underline{\alpha}\varepsilon\Pi_2(\underline{Y})$ such that

$$(7.6) \qquad G_t = \int_0^t \underline{\alpha}_s d\underline{Y}_s \qquad (0 \leq t \leq T).$$

Proof. First we need to show that $q_i = \rho_i E^*(\rho_L^{-1} | F_{\tau_i})$, i=1,...,L-1. Consider the claim $\underline{y} = (y_1,\ldots,y_L)$ with $y_i = z\varepsilon H_i$, $y_L = -q_i z$, and $y_j = 0$ for all $j \neq i,L$. Then $\underline{y}\varepsilon M$ and $\psi(\underline{y}) = 0$ implies that $E(\rho_L q_i z) = E(\rho_i z)$, or $E^*(q_i z) = E^*(\rho_L^{-1}\rho_i z) = E^*(\eta\rho_i z)$, where $\eta = E^*(\rho_L^{-1} | F_{\tau_i})$. Since this holds for all $z\varepsilon H_i$, we have $q_i = \eta\rho_i$ a.s.

From (7.3),

$$G_T = \sum_{i=1}^{L} q_i x_i - \psi(\underline{x}).$$

Note that $q_i x_i = E^*(\rho_L^{-1} \rho_i x_i | F_{\tau_i})$. Since $\rho_L^{-1} \rho_i x_i \varepsilon H'(\underline{\rho})$, Condition (RN) implies that

$$\rho_L^{-1} \rho_i x_i = E^*(\rho_L^{-1} \rho_i x_i) + \int_0^T \underline{\alpha}_{is} d\underline{Y}_s,$$

where $\underline{\alpha}_i \varepsilon \Pi_2(\underline{Y})$. Hence

$$q_i x_i = E^*(\rho_L^{-1} \rho_i x_i) + \int_0^{\tau_i} \underline{\alpha}_{is} d\underline{Y}_s,$$

and so

$$G_T = \int_0^T \underline{\alpha}_s d\underline{Y}_s, \tag{7.7}$$

where $\alpha_{mt} = \sum_{i=1}^{L} \tilde{\alpha}_{imt}$ and $\tilde{\alpha}_{imt} = \alpha_{imt}$ if $t \leq \tau_i$; $\tilde{\alpha}_{imt} = 0$ otherwise. Then the result follows by taking expectations conditional on F_t in (7.7). Q.E.D.

Lemmas 7 and 9 show that if $H(\underline{\rho})$ is marketed, then the gains process for any optimal trading strategy can be represented as a stochastic integral over an N-dimensional martingale, which is common to all optimal gains processes.

Let G^{j*} be the gains process chosen by the jth agent and define

$$\bar{G}_t = \sum_{j=1}^{J} G_t^{j*}.$$

If there are a finite number of securities (i.e., $d_{ki} = 0$ for $k > K$), then we interpret $\bar{G}_t$ as the gains process for the market portfolio. If the number of securities is infinite, it would be well to allow for an infinite number of agents, or to interpret our J agents as types, each of which stands for an infinite number of identical agents. We shall not pursue this formally, but rather interpret $\bar{G}_t$ as the aggregate gains process for what may be a proper subset of the agents in the economy.

Define the stable subspace generated by $\underline{Y}$ as

$$S(\underline{Y}) = \{U \varepsilon M^2 \colon U_t = \int_0^t \underline{\alpha}_s d\underline{Y}_s,\ \underline{\alpha} \varepsilon \Pi_2(\underline{Y})\}.$$

LEMMA 10. Suppose that Condition (RN) holds. Then

$$(7.8) \qquad G_{kt} = \int_0^t \underline{\beta}_{ks} d\underline{Y}_s + V_{kt} \qquad (k=1,2,\ldots;\ 0 \leq t \leq T),$$

where $\underline{\beta}_k \varepsilon \Pi_2(\underline{Y})$ and V_k is a square-integrable P*-martingale that is uncorrelated with $S(\underline{Y})$.

Proof. As in Lemma 5.[18/]

THEOREM 2. Suppose that Condition (RN) holds. If $H(\underline{\rho}) \subset \bar{M}$, then

$$(7.9) \qquad \bar{G}_t = \sum_{m=1}^{N} \{\int_0^t \alpha_{ms} dX_{ms} + \int_0^t \alpha_{ms}\gamma_{ms} d\langle X_m\rangle_s\},$$

$$(7.10) \qquad G_{kt} = \sum_{m=1}^{N} \{\int_0^t \beta_{kms} dX_{ms} + \int_0^t \beta_{kms}\gamma_{ms} d\langle X_m\rangle_s\} + V_{kt} \quad (k=1,2,\ldots;\ 0 \leq t \leq T),$$

where $\underline{X} = (X_1, \ldots, X_N)$ is a vector of uncorrelated, square-integrable martingales under P; $\underline{\alpha}$, $\underline{\gamma}$, and $\underline{\beta}_k$ are in $\Pi_2(\underline{X})$; and V_k is a square-integrable martingale under P that is uncorrelated with $S(\underline{X})$.

<u>Proof</u>. As in Theorem 1.

COROLLARY 2. Suppose that there is an N-dimensional vector $\underline{B}$ of independent P-Brownian motions in M^2 and that $\rho_i \varepsilon F^{\underline{B}}$ (i=1,...,L). If $H(\underline{\rho}) \subset \bar{M}$, then

$$(7.11) \qquad \bar{G}_t = \int_0^t \underline{\alpha}_s d\underline{B}_s + \int_0^t \underline{\alpha}_s \cdot \underline{\gamma}_s ds,$$

$$(7.12) \qquad G_{kt} = \int_0^t \underline{\beta}_{ks} d\underline{B}_s + \int_0^t \underline{\beta}_{ks} \cdot \underline{\gamma}_s ds + V_{kt} \qquad (k=1,2,\ldots;\ 0 \le t \le T),$$

where $\underline{\alpha}$, $\underline{\gamma}$, and $\underline{\beta}_k$ are in $\Pi_2(\underline{B})$, and V_k is a square-integrable martingale under P that is uncorrelated with $S(\underline{B})$.

<u>Proof</u>. As in Corollary 1.

These results ought to extend to more general spaces for consumption and dividends. Suppose that cumulative consumption and cumulative dividends are (integrable) bounded variation processes as in Huang [21], and that the price at t=0 of such a claim is given by $\psi(C) = E(\int_0^T \rho_s dC_s)$. Suppose that the optimal claims have the agents consuming in rates, $C_t = \int_0^t c_s ds$ with $E(\int_0^T c_s^2 ds) < \infty$, as in Duffy [13]. If the jth agent has preferences over consumption rates given by $v_j(C) = E[\int_0^T u_{js}(c_s)ds]$, where $u_{js}: R \to R$ is strictly concave, then as before we shall have $c^*_{jt} \varepsilon F^{\rho}_t$ if the set of marketed claims is sufficiently large; i.e., the optimal consumption processes will be adapted to the filtration generated by the ρ process. If there is an N-dimensional vector $\underline{B}$ of independent P-Brownian motions in M^2 with $\rho_t \varepsilon F^{\underline{B}}$ $(0 \leq t \leq T)$, then the pricing formulas in (7.11) and (7.12) will follow.

With continuous consumption, it becomes possible to derive "consumption-β" pricing formulas, as in Breeden [2], but we shall not pursue that here. His model, with the extensions of Grossman and Shiller [18], addresses many of the limitations of the CAPM. Nevertheless, aggregate consumption data have serious limitations if one's objective is to work with daily (or more frequent) data on security price fluctuations.

7.2. Factor Structure and Diversification

Next I want to relate the $\underline{B}$ process in Corollary 2 to the gains process for a well-diversified portfolio.[19/] Let

$$I = \{k(j),\ j=1,2,\ldots\}$$

be a subsequence of the nonnegative integers that indexes the securities in positive net supply.

DEFINITION 1. Suppose that $\underline{B}$ is an N-dimensional vector of independent P-Brownian motions in M^2. The dividend process for the securities in positive net supply has an approximate N-factor structure generated by $\underline{B}$ if

$$(7.13) \qquad d_{ki} = f_{ki} + e_{ki} \qquad (k \varepsilon I;\ i=1,\ldots,L),$$

where $f_{ki} \varepsilon F^{\underline{B}}_{\tau_i}$, $E(e_{ki}) = 0$, and the covariance matrix of $[e_{k(1),i},\ldots,e_{k(n),i}]$ has uniformly bounded eigenvalues as $n \to \infty$.

We shall refer to f_{ki} as the factor component and to v_{ki} as the idiosyncratic component. The bounded eigenvalue condition on the idiosyncratic components is suggested by the analysis in Chamberlain and Rothschild [6]. We could replace the $f_{ki} \varepsilon F^{\underline{B}}$ condition by a restriction that square-integrable functions of the factor components are representable as stochastic integrals over a vector $\underline{Y}$ of uncorrelated P*-martingales.

DEFINITION 2. $\underline{x}\varepsilon\bar{M}$ is a <u>well-diversified claim</u> if there is a sequence of trading strategies $\underline{\theta}_n\varepsilon\Theta$ such that (i) $\underline{\theta}_n$ generates $\underline{x}_n\varepsilon M$ and $\underline{x}_n\to\underline{x}$ in H^τ as $n\to\infty$; (ii) $\underline{\theta}_{nt} = \underline{\theta}_{n0}\varepsilon R^n$ $(0 \leq t \leq T)$ and $\underline{\theta}_{n0}\cdot\underline{\theta}_{n0}\to 0$ as $n\to\infty$; (iii) $\theta_{nk0} = 0$ if $k\notin I$; i.e., $\underline{\theta}_n$ uses only the securities in positive net supply.

For an example of a well-diversified claim, let a_n denote how many of the first n securities are in positive net supply, and set $\theta_{nk0} = 1/a_n$ if $k\varepsilon I$, $\theta_{nk0} = 0$ otherwise $(k=0,1,\ldots,n-1)$. Then $\underline{\theta}_{n0}\cdot\underline{\theta}_{n0} = 1/a_n\to 0$ if there are an infinite number of securities in positive net supply. Since $\underline{\theta}_n$ generates $\underline{x}_n = \sum_{j=1}^{a_n}\underline{d}_{k(j)}/a_n$, $\lim\underline{x}_n$ is a well-diversified claim if this series converges in H^τ as $n\to\infty$.

THEOREM 3. Suppose that $\underline{B}$ is a vector of independent P-Brownian motions in M^2 with $\rho_i\varepsilon F^{\underline{B}}$ $(i=1,\ldots,L)$. If the dividend process for the securities in positive net supply has an approximate N-factor structure generated by $\underline{B}$, and if G is the gains process for a well-diversified claim, then

$$(7.14)\qquad G_t = \int_0^t \underline{\alpha}_s d\underline{B}_s + \int_0^t \underline{\alpha}_s\cdot\underline{\gamma}_s\, ds$$

$$(7.15)\qquad G_{kt} = \int_0^t \underline{\beta}_{ks} d\underline{B}_s + \int_0^t \underline{\beta}_{ks}\cdot\underline{\gamma}_s\, ds + V_{kt} \qquad (k=1,2,\ldots;\ 0\leq t\leq T),$$

where $\underline{\alpha}$, $\underline{\gamma}$, and $\underline{\beta}_k$ are in $\Pi_2(\underline{B})$ ($\underline{\alpha}$ depends upon the well-diversified claim; $\underline{\gamma}$ and $\underline{\beta}_k$ do not), and V_k is a square-integrable martingale under P that is uncorrelated with $S(\underline{B})$. In addition, if $\underline{\phi}_n\varepsilon R^n$ satisfies $\underline{\phi}_n\cdot\underline{\phi}_n\to 0$ as $n\to\infty$ and $\phi_{nk} = 0$ if $k\notin I$, then

(7.16) $$\sup_{0\le t\le T} E[\sum_{k=1}^{n} \phi_{nk}V_{kt}]^2 \to 0$$

as $n\to\infty$; i.e., $\sum_{k=1}^{n} \phi_{nk} V_k \to 0$ in M^2.

<u>Proof</u>. Let $\underline{x}\varepsilon\bar{M}$ denote the well-diversified claim that has G for its gains process, so that

$$G_T = \sum_{i=1}^{L} q_i x_i - \psi(\underline{x}),$$

and let $\{\underline{\theta}_n\}$ and $\{\underline{x}_n\}$ be the sequences specified in Definition 2. Then

$$x_{ni} = \sum_{k=0}^{n-1} \theta_{nk0}(f_{ki} + e_{ki})$$

and $\sum_k \theta_{nk0} e_{ki}$ converges in H to 0 as $n\to\infty$ due to the bounded eigenvalue condition. Hence $\sum_k \theta_{nk0} f_{ki} \varepsilon F^{\underline{B}}$ converges in H to x_i, and so $x_i \varepsilon F^{\underline{B}}$.

Construct the vector

$$\underline{Y}_t = \underline{B}_t + \int_0^t \underline{\gamma}_s ds$$

of independent P*-Brownian motions in (7.5); any $y\varepsilon H$ that is measurable with respect to $F^{\underline{B}}$ can be represented as a stochastic integral over $\underline{Y}$. Following the proof of Lemma 9,

$$\rho_L^{-1}\rho_i x_i = E^*(\rho_L^{-1}\rho_i x_i) + \int_0^T \underline{\alpha}_{is} d\underline{Y}_s,$$

$$q_i x_i = E^*(\rho_L^{-1}\rho_i x_i) + \int_0^{\tau_i} \underline{\alpha}_{is} d\underline{Y}_s,$$

and $G_T = \int_0^T \underline{\alpha}_s d\underline{Y}_s$. Hence

$$G_t = \int_0^t \underline{\alpha}_s d\underline{B}_s + \int_0^t \underline{\alpha}_s \cdot \underline{\gamma}_s ds.$$

The proof of (7.15), like that of (7.12), follows the proof of (5.3) and (5.10) in Theorem 1 and its Corollary; the assumption that $H(\underline{\rho}) \subset \bar{M}$ and the structure of optimal claims are not needed for that argument.

To prove (7.16), note first that $\sum_{k=1}^n \phi_{nk} V_k$ is the residual from the projection of $\sum_{k=1}^n \phi_{nk} G_k$ onto $S(\underline{Y})$ under P*. In addition,

$$\sum_{k=1}^n \phi_{nk} G_k = U_n' + U_n'',$$

where U_n' and U_n'' are P*-martingales with

$$U_{nT}' = \sum_{k=1}^n \phi_{nk} [\sum_{i=1}^L q_i f_{ki} - E^*(\sum_{i=1}^L q_i f_{ki})]$$

and

$$U_{nT}'' = y_n - E^*(y_n), \quad y_n = \sum_{k=1}^n \phi_{nk} (\sum_{i=1}^L q_i e_{ki}).$$

Since $U_n' \varepsilon S(\underline{Y})$, we have

$$U_n'' = Q_n + \sum_{k=1}^n \phi_{nk} V_k$$

for some $Q_n \varepsilon S(\underline{Y})$. Hence

$$\|U_n''\|_2^2 = \|Q_n\|_2^2 + \|\sum_{k=1}^n \phi_{nk} V_k\|_2^2 \to 0$$

as $n \to \infty$, since the bounded eigenvalue condition together with the boundedness of q_i imply that $Ey_n^2 \to 0$, and so $E^*y_n^2 \to 0$. Q.E.D.

We can regard $\int_0^t \underline{\beta}_{ks} d\underline{B}_s + \int_0^t \underline{\beta}_{ks} \cdot \underline{\gamma}_s ds$ as the factor component and V_{kt} as the idiosyncratic component in (7.15). It follows from (7.16) that the covariance matrix of $[V_{k(1),t}, \ldots, V_{k(n),t}]$ has uniformly bounded eigenvalues as $n \to \infty$. So if the dividend process for the securities in positive net supply has an approximate factor structure, then the corresponding gains process also has an approximate factor structure.

In the one-factor case, the differential form of (7.14) and (7.15) is

$$dG_t = \alpha_t dB_t + \alpha_t \gamma_t dt$$

$$dG_{kt} = \beta_{kt} dB_t + \beta_{kt} \gamma_t dt + dV_{kt},$$

or, if $\alpha_t \neq 0$,

$$dG_{kt} = \beta_{kt} \alpha_t^{-1} dG_t + dV_{kt}.$$

Then $E(dV_{kt} | F_t) = 0$ and $\mathrm{Cov}(dG_t, dV_{kt} | F_t) = 0$ imply that

$$E(dG_{kt} | F_t) = \phi_t \,\mathrm{Cov}(dG_{kt}, dG_t | F_t),$$

where $\phi_t = \gamma_t / \alpha_t$. So, under an approximate one-factor structure, the gains process for any well-diversified claim can play the role of the gains process for the market portfolio.

In the N-factor case, let G_t^m be the gains process for a well-diversified claim $\underline{x}^m$, so that

$$dG_t^m = \underline{\alpha}_t^m d\underline{B}_t + \underline{\alpha}_t^m \cdot \underline{\gamma}_t dt \qquad (m=1,\ldots,N),$$

and let A_t be the $N \times N$ matrix with $\underline{\alpha}_t^m$ as its mth column. If A_t is nonsingular, we have

$$dG_{kt} = \sum_{m=1}^{N} \tilde{\beta}_{kmt} dG_t^m + dV_{kt},$$

where $\underline{\tilde{\beta}}_{kt} = A_t^{-1}\,\underline{\beta}_{kt}$, and

$$E(dG_{kt}|F_t) = \sum_{m=1}^{N} \zeta_{mt}\,\mathrm{Cov}(dG_{kt},\, dG_t^m|F_t),$$

where $\underline{\zeta}_t = A_t^{-1}\,\underline{\gamma}_t$. I plan to return in subsequent work to the problem of ensuring that A_t is nonsingular.

Well-Diversified Supply

Connor [8] introduced the condition that per capita supply be well-diversified, and he used it to obtain an exact pricing formula in a static factor model. I want to sketch how his analysis can be applied to our model.

Consider a sequence of economies in which the nth economy has $n\phi_{nj}$ identical agents of type j, where $\sum_{j=1}^{J} \phi_{nj} = 1$ and ϕ_{nj} converges in R to ϕ_j as $n \to \infty$. Only the first n+1 securities are available in the nth economy. Simplify notation by excluding consumption at t=0 and by assuming that the net supply of each security is one share. Then per capita supply of the single perishable good at $t=\tau_i$ is $\sum_{k=0}^{n} d_{ki}/n$ $(i=1,\ldots,L)$. Assume that

$\sum_{k=0}^{n} d_{ki}/n$ converges in H to $\bar{d}_i$ as $n \to \infty$. If the dividend process has an approximate N-factor structure generated by $\underline{B}$, then $\sum_{k=0}^{n} e_{ki}/n$ converges in H to 0, and so $\bar{d}_i \varepsilon F_{\tau_i}^{\underline{B}}$. This restriction on $\bar{d}_i$ is our counterpart to Connor's condition that per capita supply be well diversified.

The preferences of type j agents are given by the utility function $v_j(\underline{x}) = \sum_{i=1}^{L} E[u_{ji}(x_i)]$, where u_{ji}: $R \to R$ is increasing and strictly concave with derivative $u_{ji}^{'}$. Let $\underline{x}^{j*} \varepsilon H^{\tau}$ be the solution to max $v_j(\underline{x})$ subject to $\underline{x} \varepsilon \bar{M}$, $\psi(\underline{x}) \leq a_j \varepsilon R$, and assume that $\sum_{j=1}^{J} \phi_j x_i^{j*} = \bar{d}_i$ a.s. If $\bar{M} = H^{\tau}$ (complete markets), then $\{\underline{x}^{j*}, j=1,\ldots,J\}$ is a Pareto-efficient allocation for the limit economy; i.e., there is no allocation $\{\underline{x}^{j} \varepsilon H^{\tau}\}$ that (i) satisfies the resource constraint for the limit economy: $\sum_{j=1}^{J} \phi_j x_i^{j} \leq \bar{d}_i$ a.s. $(i=1,\ldots,L)$; and (ii) dominates $\{\underline{x}^{j*}\}$: $v_j(\underline{x}^{j}) > v_j(\underline{x}^{j*})$ $(j=1,\ldots,J)$.

Consider the allocation $\{\hat{\underline{x}}^{j}\}$, where $\hat{x}_i^{j} = E(x_i^{j*}|F_{\tau_i}^{\underline{B}})$. This satisfies the resource constraint since

$$\sum_{j=1}^{J} \phi_j \hat{x}_i^{j} = E(\sum_{j=1}^{J} \phi_j x_i^{j*}|F_{\tau_i}^{\underline{B}}) = E(\bar{d}_i|F_{\tau_i}^{\underline{B}}) = \bar{d}_i .$$

Furthermore, $v_j(\hat{\underline{x}}^{j}) > v_j(\underline{x}^{j*})$ unless $\hat{\underline{x}}^{j} = \underline{x}^{j*}$ a.s. Hence Pareto-efficiency of $\{\underline{x}^{j*}\}$ implies that $\underline{x}^{j*} \varepsilon F^{\underline{B}}$. Since $u_{ji}^{'}(x_i^{j*}) = \lambda_j \rho_i$, where $\lambda_j \varepsilon R$ is positive, we have $\rho_i \varepsilon F^{\underline{B}}$. So this argument can motivate the hypothesis of Theorem 3.

APPENDIX A

Proof of Proposition 1. (i). If $Y_T \varepsilon M$, there is a $\underline{\gamma} \varepsilon \Pi_2(\underline{Z})$ such that

$$(A.1) \qquad Y_T \;=\; \underline{\gamma}_T \cdot \underline{Z}_T \;=\; \underline{\gamma}_0 \cdot \underline{Z}_0 + \int_0^T \underline{\gamma}_s d\underline{Z}_s \quad \text{a.s.}$$

If $x \varepsilon H(\rho)$, then Condition (R) implies there is an $\alpha \varepsilon \Pi_2(Y)$ such that

$$x \;=\; E^*(x) + \int_0^T \alpha_s dY_s .$$

From (A.1),

$$Y_t \;=\; E^*(Y_T | F_t) \;=\; \underline{\gamma}_0 \cdot \underline{Z}_0 + \int_0^t \underline{\gamma}_s d\underline{Z}_s .$$

The bounds on the eigenvalues of Ψ imply that $\alpha\underline{\gamma} \varepsilon \Pi_2(\underline{Z})$ and the associative law [15, p. 62] gives

$$x \;=\; E^*(x) + \int_0^T \alpha_s \underline{\gamma}_s d\underline{Z}_s .$$

Define the portfolio strategy $\underline{\theta} = (\theta_0, \underline{\theta}_1)$ as follows: $\underline{\theta}_{1t} = (\alpha_t \gamma_{1t}, \ldots, \alpha_t \gamma_{Kt})$ and

$$(A.2) \qquad \theta_{0t} \;=\; E^*(x) + \int_0^t \alpha_s \underline{\gamma}_s d\underline{Z}_s - \underline{\theta}_{1t} \cdot \underline{S}_t .$$

Then $\underline{\theta} \varepsilon \Theta$ and $\underline{\theta}_T \cdot \underline{Z}_T = x$ a.s.

(ii) and (iii). By the martingale representation theorem [25],

$$\rho_t \equiv E(\rho|F_t) = 1 + \int_0^t \underline{\tau}_s d\underline{B}_s,$$

where $\underline{\tau} \varepsilon \Pi_2(\underline{B})$. By Girsanov's theorem [27, Theorem 6.4],

(A.3) $$\underline{B}^*_t \equiv \underline{B}_t - \int_0^t \rho_s^{-1} \underline{\tau}_s ds$$

is a vector of independent P*-Brownian motions in M^2. From (6.2) and (A.3)

(A.4) $$\underline{S}_t = \underline{S}_0 + \int_0^t \Phi_s d\underline{B}^*_s + \int_0^t \underline{\zeta}_s ds,$$

where $\underline{\zeta}_t = \underline{\mu}_t + \rho_t^{-1} \Phi_t \underline{\tau}_t$. Since $\underline{S}_t$ and $\int_0^t \Phi_s d\underline{B}^*_s$ are P*-martingales, the absolutely continuous component of (A.4) is a.s. identically 0 [15, p. 54] and so $\underline{\zeta}_t = 0$ for almost every t. (This result, when $\underline{S}$ is a Markov diffusion, is in Harrison and Kreps [19, p. 398].)

By the martingale representation theorem in [27, Theorem 5.20; or 23, Theorem 8.3.1], we have

(A.5) $$\eta_t \equiv E^*(\rho^{-1}|F_t) = 1 + \int_0^t \underline{\beta}_s d\underline{B}^*_s,$$

where $\underline{\beta} \varepsilon \Pi_2(\underline{B}^*)$. Hence $\eta_T = \rho^{-1} = dP/dP^*$, and by Girsanov's theorem,

(A.6) $$\underline{\tilde{B}}_t \equiv \underline{B}^*_t - \int_0^t \eta_s^{-1} \underline{\beta}_s ds$$

is a P-Brownian motion in M^2. From (A.3) and (A.6), $\underline{B}_t - \underline{\tilde{B}}_t$ is an absolutely continuous P-martingale, and so $P\{\underline{B}_t = \underline{\tilde{B}}_t, 0 \leq t \leq T\} = 1$. Hence

$$\text{(A.7)} \qquad \underline{S}_t = \underline{S}_0 + \int_0^t \Phi_s d\underline{B}_s^*$$

$$= \underline{S}_0 + \int_0^t \Phi_s d\underline{B}_s + \int_0^t \eta_s^{-1} \Phi_s \underline{\beta}_s ds.$$

Comparing (6.2) and (A.7), we have a.s. $\underline{\mu}_t = \eta_t^{-1} \Phi_t \underline{\beta}_t$ for almost every t, and so (A.5) and (A.7) give

$$\text{(A.8)} \qquad \rho^{-1} = 1 + \int_0^T \eta_s \underline{\lambda}_s d\underline{S}_s,$$

where $\underline{\lambda}_t = \Psi_t^{-1} \underline{\mu}_t$.

Since Y is a square-integrable P*-martingale, it can be represented as

$$Y_t = Y_0 + \int_0^t \underline{\kappa}_s d\underline{B}_s^* = Y_0 + \int_0^t \underline{\gamma}_{1s} d\underline{S}_s,$$

where $\underline{\kappa} \varepsilon \Pi_2(\underline{B}^*)$ [27, Theorem 5.20; or 23, Theorem 8.3.1], and $\underline{\gamma}_{1t} \equiv \Phi_t'^{-1} \underline{\kappa}_t \varepsilon \Pi_2(\underline{S})$. We can set $\underline{\theta}_1 = (\theta_1, \ldots, \theta_K) = \underline{\gamma}_1$ and choose θ_0 so that $\underline{\theta} = (\theta_0, \underline{\theta}_1) \varepsilon \Theta$ and $\underline{\theta}_T \cdot \underline{Z}_T = Y_T$ a.s.; hence $Y_T \varepsilon M$. If $x \varepsilon H$, applying this argument to $E^*(x|F_t)$ shows that $x \varepsilon M$. As in (A.2) the riskless asset and $\underline{\gamma}_1$ serve as mutual funds that generate $H(\rho)$. Since $\rho^{-1} \varepsilon H(\rho)$, condition (R) gives

$$\text{(A.9)} \qquad \rho^{-1} = 1 + \int_0^T \upsilon_s dY_s = 1 + \int_0^T \upsilon_s \underline{\gamma}_{1s} d\underline{S}_s,$$

where $\upsilon \varepsilon \Pi_2(Y)$. From (A.8) and (A.9), $\eta_t \underline{\lambda}_t = \upsilon_t \underline{\gamma}_{1t}$ for almost all t a.s. Since $P\{\underline{\mu}_t \cdot \underline{\mu}_t > 0, 0 \leq t \leq T\} = 1$, we have $\upsilon_t^2 > 0$ and $\underline{\gamma}_{1t} = \upsilon_t^{-1} \eta_t \Psi_t^{-1} \underline{\mu}_t$ for almost all t a.s. Q.E.D.

APPENDIX B

Suppose that $F = F^{\underline{B}}$, where $\underline{B} = (B_1, B_2)$ is a vector of independent Brownian motions. Define $y = \int_0^T B_{1s}\, dB_{2s}$. We shall assume that there is a Brownian motion $B \epsilon M^2$ such that $y \epsilon F^B$ and obtain a contradiction. Our counterexample was suggested by an example in [23, p. 204].

By the martingale representation theorem,

$$y = \int_0^T \alpha_s\, dB_s,$$

where $\alpha \epsilon \Pi_2(B)$ and $\alpha_t \epsilon F_t^B$, and

$$B_t = \int_0^t \lambda_{1s}\, dB_{1s} + \int_0^t \lambda_{2s}\, dB_{2s} \qquad (0 \leq t \leq T),$$

where $\lambda_1 \epsilon \Pi_2(B_1)$ and $\lambda_2 \epsilon \Pi_2(B_2)$. Hence

$$y_t \equiv E(y|F_t) = \int_0^t B_{1s}\, dB_{2s} = \int_0^t \alpha_s\, dB_s$$

$$= \int_0^t \alpha_s \lambda_{1s}\, dB_{1s} + \int_0^t \alpha_s \lambda_{2s}\, dB_{2s} \qquad (0 \leq t \leq T),$$

and so

$$\text{(B.1)} \qquad 0 = E\Big[\int_0^T \alpha_s \lambda_{1s}\, dB_{1s} + \int_0^T (\alpha_s \lambda_{2s} - B_{1s}) dB_{2s}\Big]^2$$

$$= E \int_0^T \alpha_s^2 \lambda_{1s}^2\, ds + E \int_0^T (\alpha_s \lambda_{2s} - B_{1s})^2\, ds.$$

Note that $\langle y \rangle_t = \int_0^t B_{1s}^2 \, ds$, and so $B_{1t}^2 \varepsilon F^B$ $(0 \leq t \leq T)$. Hence there is a $\gamma \varepsilon \Pi_2(B)$ such that

$$B_{1t}^2 - t = 2\int_0^t B_{1s} \, dB_{1s} = \int_0^t \gamma_s \, dB_s$$

$$= \int_0^t \gamma_s \, \lambda_{1s} \, dB_{1s} + \int_0^t \gamma_s \, \lambda_{2s} \, dB_{2s} \qquad (0 \leq t \leq T),$$

and so

(B.2) $$0 = E\left[\int_0^T (\gamma_s \, \lambda_{1s} - 2B_{1s}) dB_{1s} + \int_0^T \gamma_s \, \lambda_{2s} \, dB_{2s}\right]^2$$

$$= E \int_0^T (\gamma_s \, \lambda_{1s} - 2B_{1s})^2 ds + E \int_0^T \gamma_s^2 \, \lambda_{2s}^2 \, ds.$$

Let μ denote Lebesgue measure on $[0, T]$. Since $\lambda_{1t}^2 + \lambda_{2t}^2 = 1$ for almost every (t, ω) under the product measure $\mu \times P$ ($\mu \times P$ - a.e.), (B.1) and (B.2) imply that

$$0 = \alpha_t^2 \, \lambda_{1t}^2 = B_{1t}^2 \, \lambda_{1t}^2 = \gamma_t^2 \, \lambda_{1t}^2/4$$

$$= B_{1t}^2 \qquad (\mu \times P \text{ - a.e.}),$$

a contradiction.

FOOTNOTES

1/ Simplifications can be obtained by using special functional forms for preferences, as in Cox, Ingersoll, and Ross [11]. See Stapleton and Subrahmanyam [39] for the case of exponential utility; they also give references to earlier work using functional form restrictions.

2/ Constantinides [9] shows how complete markets can be used to construct a representative agent of the sort needed for these models. The investment opportunity set in his model is assumed to not change over time.

3/ This approximate factor structure will in fact be imposed only on the securities in positive net supply.

4/ For overviews and critiques of the CAPM, see Ross [36] and Roll [33].

5/ We use a.s. to denote with probability one or almost surely.

6/ This relationship between mean-variance efficiency and the claim (ρ) that represents the price system is developed in Chamberlain and Rothschild [6].

7/ See Chung and Williams [7, p. 6].

8/ Alternatively, we could specify that $\theta_t(\omega)$ is constant over the interval $t_{n-1} < t \leq t_n$, in order to obtain a simple predictable process as in Harrison and Pliska [20].

9/ The "almost surely" qualification will be left implicit in all assertions involving conditional expectations.

[10]/For a treatment of the material in this section, see, for example, Chung and Williams [7] or Durrett [15].

[11]/Stochastic processes U and V are indistinguishable if $P\{U_t = V_t, 0 \leq t \leq T\} = 1$. The variance process $\langle X \rangle$ is only defined up to indistinguishability, and so we shall leave the "almost surely" qualification implicit in all assertions involving variance processes. We shall follow the same convention for assertions involving covariance processes, stochastic integrals, or, as in footnote 9, conditional expectations.

[12]/The predictable σ-field Π is the σ-field of subsets of $[0, T] \times \Omega$ generated by the sets of the form $\{0\} \times A_0$ and $(s,t] \times A$, where $A_0 \varepsilon F_0$ and $A \varepsilon F_s$ for $0 \leq s < t$ in R. A stochastic process, which should be regarded as a mapping from $[0, T] \times \Omega$ to R, is predictable if it is measurable with respect to Π.

[13]/Here and throughout the paper we take $F^{\underline{B}}$ to be a standard filtration that has been completed so that $F_0^{\underline{B}}$ contains all the P-null sets in F.

[14]/That we can choose α to be a predictable process follows from [7, Lemma 3.5 and Theorem 3.6].

[15]/Changing the numeraire in models of this sort is treated in [19, 20, 21, 13].

[16]/The simple strategies of Section 3.1 are predictable since all martingales adapted to F are continuous.

[17]/Working with a very general consumption space, Huang [21] obtains this result by means of a somewhat different argument.

18/ See [12, pp. 356-359] for the N-dimensional version of the martingale projection theorem.

19/ The development here corresponds to that in [5] for the static case.

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