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ASSOCIATE AND CONJUGATE MINIMAL IMMERSIONS IN $M \times R$

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Abstract. We establish the definition of associate and conjugate conformal minimal isometric immersions into the product spaces, where the first factor is a Riemannian surface and the other is the set of real numbers. When the Gaussian curvature of the first factor is nonpositive, we prove that an associate surface of a minimal vertical graph over a convex domain is still a vertical graph. This generalizes a well-known result due to R. Krust. Focusing the case when the first factor is the hyperbolic plane, it is known that in certain class of surfaces, two minimal isometric immersions are associate. We show that this is not true in general. In the product ambient space, when the first factor is either the hyperbolic plane or the two-sphere, we prove that the conformal metric and the Hopf quadratic differential determine a simply connected minimal conformal immersion, up to an isometry of the ambient space. For these two product spaces, we derive the existence of the minimal associate family.

1. Introduction. A beautiful phenomenon for minimal surfaces in Euclidean space is the existence of a 1-parameter family of minimal isometric surfaces connecting the catenoid and the helicoid, which are said to be *associate*. Also, it is a well-known fact that any two conformal isometric minimal surfaces in a space form are associate. What happens in other 3-dimensional manifolds ?

Our objective of this paper is to discuss the same phenomenon in the product space, $\boldsymbol{M} \times \boldsymbol{R}$, establishing a definition of associate minimal immersions. We specialize in the situations when $\boldsymbol{M} = \boldsymbol{H}^2$, the hyperbolic plane, and $\boldsymbol{M} = \boldsymbol{S}^2$, the sphere, where surprising facts occur. We will prove some existence and uniqueness results explained below. We begin with relevant definitions.

Let $\boldsymbol{M}$ be a two dimensional Riemannian manifold. Let (x, y, t) be local coordinates in $\boldsymbol{M} \times \boldsymbol{R}$, where $z = x + iy$ are conformal coordinates on $\boldsymbol{M}$ and $t \in \boldsymbol{R}$. Let $\sigma^2|dz|^2$ be the metric on $\boldsymbol{M}$, so that $ds^2 = \sigma^2|dz|^2 + dt^2$ is the metric on the product space $\boldsymbol{M} \times \boldsymbol{R}$. Let $\Omega \subset \boldsymbol{C}$ be a simply connected domain of the plane, $w = u + iv \in \Omega$. We recall that if $X : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$, $w \mapsto (h(w), f(w))$, $w \in \Omega$, is a conformal minimal immersion, then $h : \Omega \to (\boldsymbol{M}, \sigma^2|dz|^2)$ is a harmonic map. We recall also that for any harmonic map $h : \Omega \subset \boldsymbol{C} \to \boldsymbol{M}$ there exists a related Hopf holomorphic quadratic differential $Q(h)$. Two conformal immersions $X = (h, f)$ and $X^* = (h^*, f^*) : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ are said to be *associate* if they are isometric and if the Hopf quadratic differentials satisfy the relation $Q(h^*) = e^{2i\theta} Q(h)$ for a real number θ. If $Q(h^*) = -Q(h)$, then the two immersions are said to be *conjugate*. Observe that if $\boldsymbol{M} = \boldsymbol{R}^2$, then, locally, any conformal and minimal immersion has an associated family, see Remark 17.

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In this paper, we will show that there exist two conformal minimal immersions $X, Y : \boldsymbol{D} \to \boldsymbol{H}^2 \times \boldsymbol{R}$ which are isometric each other, with constant Gaussian curvature $K \equiv -1$, but are not associate (see Example 18), where $\boldsymbol{D}$ is the unit disk. We will prove also that the vertical cylinder over a planar geodesic in $\boldsymbol{H}^2 \times \boldsymbol{R}$ is the only minimal surface with constant Gaussian curvature $K \equiv 0$ (Corollary 5).

One of our principal results is a uniqueness theorem in $\boldsymbol{H}^2 \times \boldsymbol{R}$ or $\boldsymbol{S}^2 \times \boldsymbol{R}$, showing that the conformal metric and the Hopf quadratic differential determine a minimal conformal immersion, up to an isometry of the ambient space, see Theorem 6. We will derive the existence of the minimal associate family in $\boldsymbol{H}^2 \times \boldsymbol{R}$ and $\boldsymbol{S}^2 \times \boldsymbol{R}$, in Corollary 10, by establishing an existence result (Theorem 7). The associate minimal family in $\boldsymbol{H}^2 \times \boldsymbol{R}$ or $\boldsymbol{S}^2 \times \boldsymbol{R}$ is derived by another approach in [4].

The first author has constructed examples of minimal surfaces in $\boldsymbol{H}^2 \times \boldsymbol{R}$ and in $\boldsymbol{S}^2 \times \boldsymbol{R}$, which generalize the family of Riemann's minimal examples of $\boldsymbol{R}^3$. He classified and constructed all examples foliated by horizontal constant curvature curves. Some of them have Gaussian curvature $K \equiv -1$. This family is parametrized by two parameters (c, d) and the example corresponding to (c, d) is conjugate to the one parametrized by (d, c) ([8]). The second and third authors proved that any two minimal isometric screw motion immersions in $\boldsymbol{H}^2 \times \boldsymbol{R}$ and $\boldsymbol{S}^2 \times \boldsymbol{R}$ are associate, see [15]. The second author proved that any two minimal isometric parabolic screw motion immersions into $\boldsymbol{H}^2 \times \boldsymbol{R}$ are associate. On the other hand, he proved that there exist families of associate hyperbolic screw motion immersions, but there exist also isometric non-associate hyperbolic screw motion immersions, see [14]. There exist hyperbolic screw motion surfaces associate to parabolic screw motion surfaces (Example 16).

Several questions arise: We point out the problem of the existence of the associate minimal family in $\boldsymbol{M} \times \boldsymbol{R}$, for any 2-dimensional Riemannian manifold $\boldsymbol{M}$. Also, we may ask in which general assumptions isometric immersions must be associate.

The second principal result is a generalization of Krust's theorem (see [5, Volume I, page 118] and applications therein) which states that an associate surface of a minimal vertical graph in $\boldsymbol{R}^3$ on a convex domain is a vertical graph. This theorem is true in $\boldsymbol{M} \times \boldsymbol{R}$ when $K_{\boldsymbol{M}} \le 0$ (Theorem 14), where $K_{\boldsymbol{M}}$ is the Gaussian curvature of $\boldsymbol{M}$.

For related works on minimal surfaces in $\boldsymbol{M} \times \boldsymbol{R}$, see for instance Daniel [4], Nelli and Rosenberg [12], Meeks and Rosenberg [11] and Rosenberg [13].

For related works on harmonic maps between surfaces the reader can see, for instance, Han [7], Schoen and Yau [16], Tam and Wan [17] and Wan [18].

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2. Preliminarias. We consider $X : \Omega \subset \boldsymbol{R}^2 \to \boldsymbol{M} \times \boldsymbol{R}$ a minimal surface conformally immersed in a product space, where $\boldsymbol{M}$ is a complete Riemannian two-manifold with metric $\mu = \sigma^2(z)|dz|^2$ and Gaussian curvature $K_{\boldsymbol{M}}$. First we fix some notation. Let us denote $|v|_\sigma = \sigma|v|$, $\langle v_1, v_2\rangle_\sigma = \sigma^2\langle v_1, v_2\rangle$, where $|v|$ and $\langle v_1, v_2\rangle$ stand for the standard norm and inner product in $\boldsymbol{R}^2$, respectively. Let us find $w = u + iv$ as conformal parameters of Ω, i.e., $ds_X^2 = \lambda^2|dw|^2$. We denote by $X = (h, f)$ the immersion, where $h(w) \in \boldsymbol{M}$ and $f(w) \in \boldsymbol{R}$.

Assume that $\boldsymbol{M}$ is isometrically embedded in $\boldsymbol{R}^k$. By definition (see Lawson [10]) the mean curvature vector in $\boldsymbol{R}^k$ is given by

$$2\overrightarrow{H} = (\Delta X)^{T_X \boldsymbol{M}\times \boldsymbol{R}} = ((\Delta h)^{T_h \boldsymbol{M}}, \Delta f),$$

where $h = (h_1, \dots, h_k)$. Since X is minimal, $h : \Omega \to \boldsymbol{M}$ is a harmonic map from Ω to the complete Riemannian manifold $\boldsymbol{M}$, and f is a real harmonic function. If $(U, \sigma^2(z)|dz|^2)$ is a local parametrization of $\boldsymbol{M}$, the harmonic map equation in the complex coordinate $z = x + iy$ of $\boldsymbol{M}$ (see [16, page 8]) is written as

$$h_{w\bar{w}} + 2(\log\sigma \circ h)_z h_w h_{\bar{w}} = 0. \tag{1}$$

In the theory of *harmonic map* there are two important classical objects to investigate. One is the holomorphic *Hopf quadratic differential* associate to h:

$$Q(h) = (\sigma \circ h)^2 h_w \bar{h}_w (dw)^2 := \phi(w) dw^2. \tag{2}$$

The other is the *complex coefficient of dilatation* (see Ahlfors [2]) of a quasiconformal map:

$$a(w) = \frac{\overline{h_{\bar{w}}}}{h_w}.$$

Since we consider a conformal immersion, we have $(f_w)^2 = -\phi(w)$ from (see [15]):

$$|h_u|_\sigma^2 + (f_u)^2 = |h_v|_\sigma^2 + (f_v)^2,$$

$$\langle h_u, h_v\rangle_\sigma + f_u \cdot f_v = 0.$$

We define η as the holomorphic one-form $\eta = \pm 2i\sqrt{\phi(w)}dw$, when ϕ has only even zeros. The sign is chosen so that we have

$$f = \operatorname{Re}\int_w \eta. \tag{3}$$

Assume that X is a conformal immersion and let N be the Gauss map in $\boldsymbol{M} \times \boldsymbol{R}$. Then, setting $N = N_1\partial/\partial x + N_2\partial/\partial y + N_3\partial/\partial t$, where $x + iy$ are isothermic coordinates of $\boldsymbol{M}$ and t is a coordinate of $\boldsymbol{R}$, we have (see [15]):

$$N := (N_1, N_2, N_3) = \frac{((2/\sigma)\operatorname{Re} g, (2/\sigma)\operatorname{Im} g, |g|^2 - 1)}{|g|^2 + 1}, \tag{4}$$

where

$$g := \frac{f_w h_{\bar{w}} - f_{\bar{w}} h_w}{\sigma |h_{\bar{w}}|(|h_w| + |h_{\bar{w}}|)}. \tag{5}$$

We remark that

$$g^2 = -\frac{h_w}{\overline{h_{\bar{w}}}} = -\frac{1}{a}. \tag{6}$$

Using Equations (2), (3) and (6), we can express the differential dh as follows:

$$dh = h_{\bar{w}} d\bar{w} + h_w dw = \frac{1}{2\sigma}\overline{g^{-1}\eta} - \frac{1}{2\sigma} g\eta. \tag{7}$$

The induced metric $ds_X^2 = \lambda^2|dw|^2$ is given by [15]:

$$ds_X^2 = (|h_w|_\sigma + |h_{\bar w}|_\sigma)^2|dw|^2 \,. \tag{8}$$

Thus, combining these equations together, we derive the metric in terms of g and η:

$$ds_X^2 = \frac{1}{4}(|g|^{-1} + |g|)^2|\eta|^2 = (|\sqrt{a}| + |\sqrt{a}|^{-1})^2|\phi||dw|^2 \,. \tag{9}$$

In the case of minimal surfaces X conformally immersed in $\boldsymbol{R}^3 = \boldsymbol{R}^2 \times \boldsymbol{R}$, the data (g, η) are classical Weierstrass data:

$$X(w) = (h, f) = \left(\frac{1}{2}\overline{\int_w g^{-1}\eta} - \frac{1}{2}\int_w g\eta, \operatorname{Re}\int_w \eta\right). \tag{10}$$

In particular g is the Gauss map.

The main difference is in the fact that g is no more a meromorphic map when the ambient space is $\boldsymbol{M} \times \boldsymbol{R}$. In order to study g, it is more convenient occasionally to consider the complex function $\omega + i\psi$ defined by

$$g := -ie^{\omega + i\psi} \,, \tag{11}$$

where ω and ψ are $\boldsymbol{R}$-valued functions. It is a well-known fact (see [16, page 9]) that harmonic mappings satisfy the Bochner formula:

$$\Delta_0 \log \frac{|h_w|}{|h_{\bar w}|} = -2K_{\boldsymbol{M}} J(h) \,, \tag{12}$$

where $J(h) = \sigma^2(|h_w|^2 - |h_{\bar w}|^2)$ is the Jacobian of h with $|h_w|^2 = h_w\overline{h_w}$ and Δ_0 denotes the Laplacian in the Euclidean metric. Hence, taking into account of (2), (6), (11) and (12), we have

$$\Delta_0\omega = -2K_{\boldsymbol{M}} \sinh(2\omega)|\phi| \,. \tag{13}$$

With these conventions, notice that the metric and the third coordinate of the Gauss map N are given respectively by

$$ds_X^2 = 4\cosh^2\omega|\phi||dw|^2 \quad \text{and} \quad N_3 = \tanh\omega \,. \tag{14}$$

On account of the discussion above, we deduce the following

PROPOSITION 1. *Let $h : \Omega \to \boldsymbol{M}$ be a harmonic mapping from a simply connected domain $\Omega \subset \boldsymbol{C}$ such that the holomorphic quadratic differential $Q(h)$ does not vanish or has zeros with even order. Then there exists a complex map $g = -ie^{\omega+i\psi}$ and a holomorphic one-form $\eta = \pm 2i\sqrt{Q(h)}$ such that, setting $f = \operatorname{Re}\int \eta$, the map $X := (h, f) : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ is a conformal and minimal (possibly branched) immersion. The third component of the unit normal vector is given by $N_3 = \tanh\omega$. The metric of the immersion is given by* (8) *or* (9):

$$ds_X^2 = \cosh^2\omega|\eta|^2 \,,$$

where ω is a solution of the sinh-Gordon equation

$$\Delta_0\omega = -2K_{\boldsymbol{M}} \sinh(2\omega)|\phi| \,.$$

REMARK 2. We remark that the branch points of X are among the zeros of $Q(h)$. Therefore, the branch points are isolated. Note also that the poles of ω are among the zeros of $Q(h)$.

PROOF OF PROPOSITION 1. We deduce from the hypothesis that we can solve in f the equation $(f_w)^2 = -(\sigma \circ h)^2 h_w \bar{h}_w$, (since Ω is simply connected). Therefore the real function f is harmonic and the map $X := (h, f) : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ is a conformal and minimal (possibly branched) immersion.

Observe that $X^* := (h, -f)$ also defines a conformal and minimal (possibly branched) immersion into $\boldsymbol{M} \times \boldsymbol{R}$, which is isometric to X with $g^* = -g$ and $\eta^* = -\eta$. □

REMARK 3. Denote the quadratic differential of Abresh and Rosenberg [1] by Q_{A-R} and keep the notation in Proposition 1. Then, using the relation $(f_w)^2 = -(\sigma \circ h)^2 h_w \bar{h}_w$, a straightforward computation shows that we have $Q_{A-R} = -2Q(h)$.

LEMMA 4. *Let $X = (h, f) : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ be a conformal immersion. Let $N = (N_1, N_2, N_3)$ be the Gauss map of X. Let K (resp. K_{ext}) be the intrinsic (resp. extrinsic) curvature of X. Denote by $K_{\boldsymbol{M}}$ the Gaussian curvature of $\boldsymbol{M}$. Then the Gauss equation of X reads as*

$$\text{(Gauss Equation)} \qquad K(w) - K_{\text{ext}}(X(w)) = K_{\boldsymbol{M}}(h(w)) N_3^2(w)$$

for each $w \in \Omega$.

PROOF. As usual, $z = x + iy$ are local conformal coordinates of $\boldsymbol{M}$ and t is the coordinate on $\boldsymbol{R}$. We denote by $\bar{R}$ the curvature tensor of $\boldsymbol{M} \times \boldsymbol{R}$, that is,

$$\bar{R}(A, B)C = \bar{\nabla}_A \bar{\nabla}_B C - \bar{\nabla}_B \bar{\nabla}_A C - \bar{\nabla}_{[A,B]} C$$

for any vector fields A, B, C on $\boldsymbol{M} \times \boldsymbol{R}$, where $\bar{\nabla}$ is the Riemannian connection on $\boldsymbol{M} \times \boldsymbol{R}$.

As X is a conformal immersion, the induced metric on Ω has the form $ds_X^2 = \lambda^2(w)|dw^2|$ with $\lambda = (\sigma \circ h)(|h_w| + |\bar{h}_w|)$. The Gauss equation is given by

$$K(w) - K_{\text{ext}}(X(w)) = \frac{\langle \bar{R}(X_u, X_v)X_v \,;\, X_u \rangle}{\lambda^4}(w)\,,$$

where $\langle\ ;\ \rangle$ is the scalar product on $\boldsymbol{M} \times \boldsymbol{R}$, $X_u = \partial X/\partial u = (\operatorname{Re} h)_u \partial_x + (\operatorname{Im} h)_u \partial_y + f_u \partial_t$ and so on. A tedious but straightforward computation shows that

$$\begin{aligned}
\bar{R}(\partial_x, \partial_y)\partial_x &= \Delta_0 \log(\sigma)\partial_y\,,\\
\bar{R}(\partial_x, \partial_y)\partial_y &= -\Delta_0 \log(\sigma)\partial_x\,,\\
\bar{R}(\partial_x, \partial_x)\partial_* &= \bar{R}(\partial_y, \partial_y)\partial_* = 0\,,\\
\bar{R}(\partial_t, \partial_*)\partial_* = \bar{R}(\partial_*, \partial_t)\partial_* &= \bar{R}(\partial_*, \partial_*)\partial_t = 0\,,
\end{aligned}$$

where ∂_* stands for any vector field among ∂_x, ∂_y or ∂_t, and Δ_0 is the Euclidean Laplacian. We deduce that

$$\langle \bar{R}(X_u, X_v)X_v; X_u \rangle = -\sigma^2 \Delta_0 \log(\sigma)(|h_w|^2 - |h_{\bar{w}}|^2)^2.$$

Let us observe that $K_{\boldsymbol{M}} = -\Delta_0 \log(\sigma)/\sigma^2$, so that we deduce from Equations (4) and (6) that $N_3 = (|h_w| - |h_{\bar{w}}|)/(|h_w| + |h_{\bar{w}}|)$. Now, using the expression of λ, we obtain the result. $\square$

Notice that given a geodesic $\Gamma \subset \boldsymbol{H}^2 \times \{0\}$, the vertical cylinder $\mathcal{C}$ over Γ defined by $\mathcal{C} := \{(x, y, t)\,;\ x+iy \in \Gamma,\ t \in \boldsymbol{R}\} \subset \boldsymbol{H}^2 \times \boldsymbol{R}$ is a minimal surface with Gaussian curvature $K \equiv 0$. We now deduce the following.

COROLLARY 5. *Let $X = (h, f) : \Omega \to \boldsymbol{H}^2 \times \boldsymbol{R}$ be a conformal and minimal immersion. Let $w \in \Omega$ be such that $K(w) = 0$, where K stands for the Gaussian curvature of X (that is, the intrinsic curvature).*

Then the tangent plane of $X(\Omega)$ at $X(w)$ is vertical. Therefore, if $K \equiv 0$, then $X(\Omega)$ is part of a vertical cylinder over a planar geodesic plane of $\boldsymbol{H}^2 \times \boldsymbol{R}$, that is, there exists a geodesic Γ of $\boldsymbol{H}^2 \times \{t\}$ such that $X(\Omega) \subset \Gamma \times \boldsymbol{R}$.

PROOF. As $\boldsymbol{M} = \boldsymbol{H}^2$, we have $K_{\boldsymbol{M}} \equiv -1$. Using the Gauss equation in Lemma 4, we deduce that if $K(w) = 0$ at some point $w \in \Omega$, then

$$K_{\mathrm{ext}}(X(w)) = N_3^2(w)\,. \tag{$*$}$$

Recall that the extrinsic curvature K_{ext} is the ratio between the determinants of the second and the first fundamental forms of X. Therefore, as X is a minimal immersion, we have $K_{\mathrm{ext}}(X(w)) \le 0$ at any point w. Using $(*)$, we obtain that $N_3^2(w) = 0$, that is, the tangent plane is vertical at $X(w)$.

Furthermore, if $K \equiv 0$, we deduce that at each point the tangent plane is vertical. Using this fact, we get that at any point $X(w)$ the intersection of $X(\Omega)$ with the vertical plane at $X(w)$ spanned by $N(w)$ and ∂_t is part of a vertical straight line. We deduce that there exists a planar curve $\Gamma \subset \boldsymbol{H}^2 \times \{0\}$ such that $X(\Omega) \subset \Gamma \times \boldsymbol{R}$. Again, as X is minimal, we obtain that the curvature of Γ always vanishes, that is, Γ is a geodesic of $\boldsymbol{H}^2$. $\square$

2.1. Harmonic maps and CMC surfaces in Minkowski space. We now make some comments about an existence theorem of spacelike CMC surfaces in Minkowski 3-space and their relation with harmonic maps, inferred by Akutagawa and Nishikawa [3].

We denote by $\boldsymbol{R}^{2,1}$ the Minkowski 3-space, that is, $\boldsymbol{R}^3$ equipped with the Lorentzian metric $\bar{\nu} = dx_1^2 + dx_2^2 - dx_3^2$, where (x_1, x_2, x_3) are the coordinates in $\boldsymbol{R}^3$. We consider the hyperboloid $\mathcal{H}$ in $\boldsymbol{R}^{2,1}$ defined by

$$\mathcal{H} = \{(x_1, x_2, x_3) \in \boldsymbol{R}^{2,1};\ x_1^2 + x_2^2 - x_3^2 = -1\}\,.$$

Since $\mathcal{H}$ has two connected components, we call $\mathcal{H}_+$ the component for which $x_3 \ge 1$, and $\mathcal{H}_-$ the other component. It is well-known that the restriction of $\bar{\nu}$ to $\mathcal{H}_+$ is a regular metric ν_+ and that $(\mathcal{H}_+, \nu_+)$ is isometric to the hyperbolic plane $\boldsymbol{H}^2$. We define in the same way the metric ν_- on $\mathcal{H}_-$ and $(\mathcal{H}_-, \nu_-)$ is also isometric to the hyperbolic plane $\boldsymbol{H}^2$. Throughout this paper, we always choose as model for $\boldsymbol{H}^2$ the unit disk $\boldsymbol{D}$ equipped with the metric $\sigma^2|dz|^2 = (4/(1-|z|^2)^2)|dz|^2$. The isometries $\Pi_+ : \mathcal{H}_+ \to \boldsymbol{D}$ and $\Pi_- : \mathcal{H}_- \to \boldsymbol{D}$ are given by

$$\Pi_+(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 + x_3} \quad \text{for any } (x_1, x_2, x_3) \in \mathcal{H}_+ ,$$
$$\Pi_-(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3} \quad \text{for any } (x_1, x_2, x_3) \in \mathcal{H}_- .$$

Indeed, Π_+ (resp. Π_-) is the stereographic projection from the south pole $(0, 0, -1) \in \mathcal{H}_-$ (resp. from the north pole $(0, 0, 1) \in \mathcal{H}_+$). Observe that in [3], keeping their notation, ψ_2 is the conjugate of the stereographic projection from the south pole, that is, $\psi_2 = \overline{\Pi_+}$.

Let $\Omega \subset \boldsymbol{C}$ be a connected and simply connected open subset with $w = u + iv$ as the coordinates on Ω. An immersion $X : \Omega \to \boldsymbol{R}^{2,1}$ is said to be a *spacelike immersion* if for every point $p \in \Omega$ the restriction of $\bar{\nu}$ at the tangent space $T_pX(\Omega)$ is a positive definite metric. In this paper we only consider spacelike immersions. Since we are concerned with CMC spacelike surfaces in Minkowski 3-space, observe that, up to a dilatation, we can consider only spacelike mean curvature 1 surfaces.

Let $X : \Omega \to \boldsymbol{R}^{2,1}$ be a spacelike immersion. For each $p \in \Omega$ there is a unique vector $N(p) \in \mathcal{H}$ such that $(X_u, X_v, N)(p)$ is a positively oriented basis and $N(p)$ is orthogonal to $X_u(p)$ and $X_v(p)$. This defines a map $N : \Omega \to \mathcal{H}$ called the *Gauss map*.

Let $h : \Omega \to \boldsymbol{H}^2$ be a harmonic map, that is, h satisfies (1):

$$h_{w\bar{w}} + \frac{2\bar{h}}{1 - |h|^2} h_w h_{\bar{w}} = 0 .$$

We assume that neither h nor $\bar{h}$ is holomorphic. It is shown in [3, Theorem 6.1], that given such h there exists an (possibly branched) immersion $X_+ : \Omega \to \boldsymbol{R}^{2,1}$ such that the Gauss map is $N_+ = \Pi_+^{-1} \circ h$. Furthermore, the mean curvature is constant and equals to 1, and the induced metric on Ω is

$$\nu_+ = \frac{4|h_w|^2}{(1 - |h|^2)^2} |dw|^2 .$$

We remark the correspondence between our notation and the notation of [3]: $N_+ = G$ and $h = \overline{\Psi_2}$. The map X_+ is unique up to a translation. In the same way there exists a unique, up to a translation, (possibly branched) immersion $X_- : \Omega \to \boldsymbol{R}^{2,1}$ such that the Gauss map is $N_- = \Pi_-^{-1} \circ h$. Furthermore, the mean curvature is constant and equals to 1 and the induced metric on Ω is

$$\nu_- = \frac{4|\bar{h}_w|^2}{(1 - |h|^2)^2} |dw|^2 ,$$

with $N_- = G$ and $h = \Psi_1$. Let us note that these two (branched) immersions are not isometric and that the Gauss map of X_+ (resp. X_-) takes values in $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). In this paper we are only concerned with the immersion X_+.

3. Minimal immersions in $\boldsymbol{M} \times \boldsymbol{R}$. Next, we suppose that $\boldsymbol{M} = \boldsymbol{R}^2$, $\boldsymbol{H}^2$ or $\boldsymbol{S}^2$. In the case where $\boldsymbol{M} = \boldsymbol{R}^2$, we have $\sigma(z) \equiv 1$. If $\boldsymbol{M} = \boldsymbol{H}^2$, we consider the unit disk model $\boldsymbol{D}$, and then $\sigma(z) = 2/(1 - |z|^2)$ for every $z \in \boldsymbol{D}$. Finally, if $\boldsymbol{M} = \boldsymbol{S}^2$, we can choose coordinate charts $\boldsymbol{R}^2$, given by the stereographic projections with respect to the north pole and the south pole, and in both cases we have $\sigma(z) = 2/(1+|z|^2)$ for every $z \in \boldsymbol{R}^2$.

THEOREM 6. *Let $\Omega \subset \boldsymbol{C}$ be a simply connected open set and consider two conformal minimal immersions $X, X^* : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ which are isometric each other. Let us call h (resp. h^*) the horizontal component of X (resp. X^*). Assume that h and h^* share the same Hopf quadratic differential. Then X and X^* are congruent.*

PROOF. We set $X = (h, f)$, where $h : \Omega \to \boldsymbol{M}$ is the horizontal component and $f : \Omega \to \boldsymbol{R}$ is the vertical component. Similarly, let us set $X^* = (h^*, f^*)$. We use the map $g = -ie^{\omega + i\psi}$ (resp. $g^* = -ie^{\omega^* + i\psi^*}$) associated to h (resp. h^*) defined in (11) and the one-form η (resp. η^*) defined in (3). As X and X^* are mutually isometric, we infer from (9) that

$$\frac{1}{4}(|g| + |g|^{-1})^2|\eta| = \frac{1}{4}(|g^*| + |g^*|^{-1})^2|\eta^*|\,.$$

Also, as h and h^* share the same Hopf quadratic differential $Q = \phi dw^2$, we have

$$|\eta| = 2|\phi|^{1/2} = |\eta^*|\,.$$

We deduce that

$$(*) \qquad |g| = |g^*|\,,$$

or

$$(**) \qquad |g| = |g^*|^{-1}\,.$$

If the case $(**)$ happens, we consider the new immersion $X^{**} : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ defined by $X^{**} = (\overline{h^*}, f^*)$. Now, the case $(*)$ happens, since the immersion X^{**} has data $g^{**} = -(g^*)^{-1}$ by (5) and $\eta^{**} = \eta^*$ by (3). Note that X^{**} and X are immersions isometric each other with the same Hopf quadratic differential. Therefore, up to an isometry of $\boldsymbol{M} \times \boldsymbol{R}$, we can assume that the case $(*)$ happens and then $\omega = \omega^*$.

Let us assume now that $\boldsymbol{M} = \boldsymbol{H}^2$. The case $\boldsymbol{M} = \boldsymbol{S}^2$ is similar, and the case $\boldsymbol{M} = \boldsymbol{R}^2$ (which is easy to show by Weierstrass representation) will be considered later.

Let us consider the Minkowski 3-space $\boldsymbol{R}^{2,1}$. As $h : \Omega \to \boldsymbol{H}^2$ is a harmonic map and Ω is simply connected, it is known that there exists a CMC 1 (possibly branched) immersion $\tilde{X} : \Omega \to \boldsymbol{R}^{2,1}$ such that the Gauss map is $\Pi_+^{-1} \circ h$. Furthermore, the induced metric on Ω is given by

$$ds^2_{\tilde{X}} = ((\sigma \circ h)|h_w|)^2\,|dw|^2 = e^{2\omega}|\phi||dw|^2\,,$$

see Subsection 2.1. Notice that ϕ can vanish only at isolated points, so there exists a simply connected open subset V of Ω such that $\tilde{X}$ defines a regular immersion from V into $\boldsymbol{R}^{2,1}$ and $ds^2_{\tilde{X}}$ defines a regular metric.

Furthermore, we deduce from Theorem 3.4 of [3] that the second fundamental form of $\tilde{X}$ is given uniquely in term of Q and $ds^2_{\tilde{X}}$. To see this, observe first that, setting $\tilde{\phi}(\tilde{X}) :=$

$(1/2)(b_{uu} - b_{vv} - i2b_{uv})$, where b_{uu}, b_{vv} and b_{uv} are the coefficients of the second fundamental form, we get from the relation (3.12) of [3] that

$$\tilde{\phi}(\tilde{X}) = (\sigma \circ h)^2 h_w \bar{h}_w = \phi\,, \tag{15}$$

that is, $\tilde{\phi}(\tilde{X})dw^2 = Q$. Pay attention to the fact that our convention is not the same as that in [3]. More precisely, considering the notation h_{11}, h_{12}, h_{22} and Ψ_2 of [3], we have $\Psi_2 = \bar{h}$, $b_{uu} = ((\sigma \circ h)|h_w|)^2\, h_{11}$ and so on. Therefore, $\tilde{\phi}(\tilde{X}) = (1/2)\,((\sigma \circ h)|h_w|)^2\,(h_{11} - h_{22} - i2h_{12})$.

Now, using the fact that $b_{uu} + b_{vv} = 2\,((\sigma \circ h)|h_w|)^2$ (because the mean curvature is 1), we deduce that

$$b_{uu}(\tilde{X}) = ((\sigma \circ h)|h_w|)^2 + \operatorname{Re} Q/dw^2 = e^{2\omega}|\phi| + \operatorname{Re}\phi\,, \tag{16}$$

$$b_{vv}(\tilde{X}) = ((\sigma \circ h)|h_w|)^2 - \operatorname{Re} Q/dw^2 = e^{2\omega}|\phi| - \operatorname{Re}\phi\,, \tag{17}$$

$$b_{uv}(\tilde{X}) = -\operatorname{Im} Q/dw^2 = -\operatorname{Im}\phi. \tag{18}$$

In the same way, there exists a unique (up to a translation) CMC 1 (possibly branched) immersion $\widetilde{X^*} : \Omega \to \boldsymbol{R}^{2,1}$ such that the Gauss map is $\Pi_+^{-1} \circ h^*$. We can assume that $\widetilde{X^*}$ defines a regular immersion on V. Notice that we have $Q(h^*) = Q$ and the identity $(*)$ as well. We deduce from the previous discussion that $\tilde{X}$ and $\widetilde{X^*}$ share the same induced metric on V and the same second fundamental form. Therefore, we infer from the fundamental theorem of surface theory in Minkowski 3-space that $\tilde{X}$ and $\widetilde{X^*}$ are equal up to a positive isometry Γ in $\boldsymbol{R}^{2,1}$, that is, $\widetilde{X^*} = \Gamma \circ \tilde{X}$. The restriction of Γ on $\boldsymbol{H}^2$ defines an isometry γ of $\boldsymbol{H}^2$, and we get $h^* = \gamma \circ h$ on V. By an argument of analyticity we have $h^* = \gamma \circ h$ on the entire Ω.

Let us return to $\boldsymbol{H}^2 \times \boldsymbol{R}$. As $f_w^* = \pm f_w$ in view of (3), we get that $f^* = \pm f + c$, where c is a real constant. Hence we obtain $X^* := (h^*, f^*) = (\gamma \circ h, \pm f + c)$, that is, X^* and X differ from an isometry of $\boldsymbol{H}^2 \times \boldsymbol{R}$.

In the case where $\boldsymbol{M} = \boldsymbol{S}^2$, the proof is similar: We use the fact that any harmonic map from Ω into $\boldsymbol{S}^2$ is the Gauss map of a unique (up to a translation) CMC 1 (possibly branched) immersion into $\boldsymbol{R}^3$, see [9].

Finally, let us consider the case where $\boldsymbol{M} = \boldsymbol{R}^2$. Let (g, η) (resp. (g^*, η^*)) be the Weierstrass representation of X (resp. X^*). Therefore X is given by $X = \big((1/2)\overline{\int g^{-1}\eta} - (1/2)\int g\eta, \operatorname{Re}\int \eta\big)$. As $|g^*| = |g|$, we deduce that there exists a real number θ such that $g^* = e^{i\theta}g$. Furthermore, we have $\eta = \pm\eta^*$, since $(f_z)^2 = (f_z^*)^2 = -\phi$. In consequence, we have $(g^*, \eta^*) = (e^{i\theta}g, \pm\eta)$ and we deduce that X^* differs from X by an isometry of $\boldsymbol{R}^2 \times \boldsymbol{R}$. □

There is also an existence result of minimal immersions into $\boldsymbol{M} \times \boldsymbol{R}$, where $\boldsymbol{M} = \boldsymbol{H}^2, \boldsymbol{S}^2$ or $\boldsymbol{R}^2$.

THEOREM 7. *Let $\Omega \subset \boldsymbol{C}$ be a simply connected domain. Let $ds^2 = \lambda^2(w)|dw|^2$ be a conformal metric on Ω and $Q = \phi(w)dw^2$ a holomorphic quadratic differential on Ω with zeros (if any) of even order. Assume that $\boldsymbol{M} = \boldsymbol{H}^2, \boldsymbol{S}^2$ or $\boldsymbol{R}^2$.*

Then there exists a conformal and minimal immersion $X : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ *such that, setting* $X := (h, f)$, *the Hopf quadratic differential of* h *is* Q (*that is*, $Q(h) = \phi(w)dw^2$) *and such that the induced metric* ds_X^2 *is*

$$ds_X^2 = ds^2 = \lambda^2(w)|dw|^2$$

if and only if λ *satisfies* $\lambda^2 - 4|\phi| \geq 0$ *and*

$$\Delta_0 \omega = -2K_{\boldsymbol{M}} \sinh 2\omega |\phi| \tag{19}$$

in which $\phi \neq 0$, *where* $K_{\boldsymbol{M}}$ *is the* (*constant*) *Gaussian curvature of* $\boldsymbol{M}$, Δ_0 *is the Euclidean Laplacian and*

$$\omega := \log \frac{\lambda + \sqrt{\lambda^2 - 4|\phi|}}{2} - \frac{1}{2} \log |\phi| .$$

PROOF. Observe that each zero of ϕ corresponds to a pole of ω and that $e^{2\omega}|\phi||dw|^2$ is positive definite on the whole Ω. We first consider the case where $K_{\boldsymbol{M}} = -1$, that is, $\boldsymbol{M} = \boldsymbol{H}^2$. Let us assume that λ satisfies (19). Consider the symmetric 2-tensor $\mathrm{II} := b_{uu}du^2 + 2b_{uv}dudv + b_{vv}dv^2$ on Ω, where b_{uu}, b_{uv} and b_{vv} are given by

$$\begin{cases} b_{uu} + b_{vv} = 2e^{2\omega}|\phi| , \\ b_{uu} - b_{vv} = 2\mathrm{Re}(\phi) , \\ b_{uv} = -\mathrm{Im}(\phi) . \end{cases} \tag{20}$$

The Gauss equation for the pair ($e^{2\omega}|\phi||dw|^2$, II) in $\boldsymbol{R}^{2,1}$ is written as

$$\Delta_0 \omega = 2 \sinh(2\omega)|\phi| ,$$

which is satisfied by our assumption. The Codazzi-Mainardi equations are also satisfied, since ϕ is holomorphic. Therefore the fundamental theorem of surface theory in $\boldsymbol{R}^{2,1}$ assures that there exists an immersion $\tilde{X} : \Omega \to \boldsymbol{R}^{2,1}$ such that the induced metric on Ω is $ds_{\tilde{X}}^2 = e^{2\omega}|\phi||dw|^2$ and the second fundamental form is II. Now Equations (20) show that the immersion has constant mean curvature 1.

Up to an isometry of $\boldsymbol{R}^{2,1}$ we can assume that the Gauss map N of $\tilde{X}$ takes values in $\mathcal{H}_+$. Therefore, $h := \Pi_+ \circ N : \Omega \to \boldsymbol{H}^2$ is a harmonic mapping such that its Hopf quadratic differential is the same as $\tilde{X}$ so that $Q(h) = \tilde{\phi}(\tilde{X})dw^2$, as we have seen in the proof of Theorem 6, see the relation (15). Equations (20) show that

$$Q = \frac{1}{2}(b_{uu} - b_{vv} - i2b_{uv})dw^2 := \tilde{Q}(\tilde{X}) .$$

That is, Q is the classical Hopf quadratic differential of CMC surfaces in $\boldsymbol{R}^{2,1}$. Therefore, we obtain $Q(h) = Q$. Moreover, we have

$$ds_{\tilde{X}}^2 = ((\sigma \circ h)|h_w|)^2 \, |dw|^2 ,$$

deducing that $e^{2\omega}|\phi| = ((\sigma \circ h)|h_w|)^2$.

Now, we apply Proposition 1 which states that there exists a conformal and minimal immersion $X = (h, f) : \Omega \to \boldsymbol{H}^2 \times \boldsymbol{R}$, with the induced metric

$$ds_X^2 = (\sigma^2 \circ h)(|h_w| + |\bar{h}_w|)^2 |dw|^2 .$$

Finally, using the fact that $(\sigma \circ h)|\bar{h}_w| = |\phi|/(\sigma \circ h)|h_w|$, we easily compute that

$$(\sigma^2 \circ h)(|h_w| + |\bar{h}_w|)^2 = 4\cosh^2 \omega |\phi| |dw|^2 = \lambda^2 ,$$

that is, $ds_X^2 = \lambda^2 |dw|^2$ as desired.

Conversely, suppose that such an immersion exists. Then, by (14), we have

$$\lambda^2 = 4\cosh^2 \omega |\phi| .$$

A simple computation shows that we have

$$\omega = \omega_1 := \frac{1}{2} \log \frac{|\bar{h}_w|}{|h_w|} \quad \text{or} \quad \omega = -\omega_1 = \frac{1}{2} \log \frac{|h_w|}{|\bar{h}_w|} .$$

Note that Equation (19) is the Bochner formula (12). This completes the proof in the case where $\boldsymbol{M} = \boldsymbol{H}^2$.

If $\boldsymbol{M} = \boldsymbol{S}^2$ (and then $K_{\boldsymbol{M}} = 1$), the proof is analogous: We use the fact that for any constant mean curvature 1 immersion $\tilde{X} : \Omega \to \boldsymbol{R}^3$, its Gauss map $N : \Omega \to \boldsymbol{S}^2$ is harmonic, and conversely any harmonic map from Ω into $\boldsymbol{S}^2$ is the Gauss map of an (possibly branched) immersion into $\boldsymbol{R}^3$ with constant mean curvature 1 [9].

If $K_{\boldsymbol{M}} = 0$, that is, $\boldsymbol{M} = \boldsymbol{R}^2$, assume first that ω satisfies (19), that is, ω is a harmonic function. As Ω is simply connected, ω is the real part of a holomorphic function ψ on Ω: $\omega = \mathrm{Re}(\psi)$. We set

$$\eta := -2i\sqrt{\phi}dw \quad \text{and} \quad g := e^{\psi} .$$

Let $X = (h, f) : \Omega \to \boldsymbol{R}^2 \times \boldsymbol{R}$ be the conformal and minimal immersion given by the Weierstrass representation (g, η), see (10). It is straightforward to verify that we have $h_w dw = -(1/2)g\eta$ and $\bar{h}_w dw = (1/2)g^{-1}\eta$. Hence we obtain

$$Q(h) := h_w \bar{h}_w dw^2 = -\frac{1}{4}\eta^2 = Q ,$$

and the induced metric is given by

$$ds_X^2 := \frac{1}{4}(|g^{-1}| + |g|)^2 |\eta|^2 = \lambda^2(w)|dw|^2 .$$

Conversely, assume that such a minimal immersion $X = (h, f) : \Omega \to \boldsymbol{R}^2 \times \boldsymbol{R}$ exists. Let us denote by (g, η) its Weierstrass representation. Since $|\phi dw| = |\eta|^2$, we easily verify that $\lambda^2 - |\phi| \geq 0$. Moreover, we compute that we have $|g| = (\lambda - \sqrt{\lambda^2 - 4|\phi|})/(2\sqrt{|\phi|})$ or $|g| = (\lambda + \sqrt{\lambda^2 - 4|\phi|})/(2\sqrt{|\phi|})$. In both cases, ω is a harmonic function and then satisfies (19). $\square$

Observe that in the case where $K_{\boldsymbol{M}} = 0$ the result can be found in [6, Section 10.2]. We gave the proof for the sake of completeness. The cases where $K_{\boldsymbol{M}} = 1$ and $K_{\boldsymbol{M}} = -1$ were proved by Daniel [4], using different methods.

DEFINITION 8. Let M be a Riemann surface. Let $X, X^* : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ be two conformal minimal immersions, and set $X = (h, f)$ and $X = (h^*, f^*)$. For any $\theta \in \boldsymbol{R}$ we say that X and X^* are *θ-associate* (or simply *associate*) if they are isometric each other and if their Hopf quadratic differentials are related by $Q(h^*) = e^{2i\theta} Q(h)$. Namely, X and X^* are associate if and only if we have

$$(\sigma \circ h)(|h_w| + |\bar{h}_w|) = (\sigma \circ h^*)(|h^*_w| + |\overline{h^*}_w|) \quad \text{and}$$
$$(\sigma \circ h^*)^2 h^*_w \overline{h^*}_w = e^{2i\theta}(\sigma \circ h)^2 h_w \bar{h}_w \,,$$

where, in a local coordinate z, the metric on $\boldsymbol{M}$ is given by $\sigma^2(z)|dz|^2$.

In the case where $\boldsymbol{M} = \boldsymbol{R}^2$, $\boldsymbol{H}^2$ or $\boldsymbol{S}^2$, we deduce from Theorem 6 that given a conformal minimal immersion X, the θ-associate minimal immersion is uniquely determined up to an isometry of $\boldsymbol{M} \times \boldsymbol{R}$. Furthermore, if $\theta = \pi/2$, we say that X and X^* are *conjugate*.

REMARK 9. Two isometric immersions X and X^θ are associate up to an isometry if $\eta^\theta = e^{i\theta}\eta$ and by (9) $|g^\theta| + |g^\theta|^{-1} = |g| + |g|^{-1}$ (or equivalently, $\cosh \omega^\theta = \cosh \omega$). Then $\omega^\theta = \omega$ or $\omega^\theta = -\omega$. In particular, X and X^θ are associate if and only if $N_3(X) = N_3(X^*)$ or $N_3(X) = -N_3(X^*)$ (recall that $N_3(X) = \tanh \omega$) and $\eta^\theta = e^{i\theta}\eta$.

In fact, Daniel [4] proved that the associate family always exists in $\boldsymbol{H}^2 \times \boldsymbol{R}$ and $\boldsymbol{S}^2 \times \boldsymbol{R}$. In this situation, he gave an alternative definition of associate and conjugate isometric immersions, which turns out to be equivalent to our definition. We are going to give another proof of the existence of the associate family.

COROLLARY 10. *Let $X := (h, f) : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$ be any conformal and minimal immersion, where $M = \boldsymbol{H}^2$, $\boldsymbol{S}^2$ or $\boldsymbol{R}^2$. Then, for any $\theta \in \boldsymbol{R}$, there exists a θ-associate immersion $X_\theta := (h_\theta, f_\theta) : \Omega \to \boldsymbol{M} \times \boldsymbol{R}$. Furthermore, $X_0 = X$ and X_θ is unique up to isometries of $\boldsymbol{M} \times \boldsymbol{R}$.*

PROOF. Set $Q(h) = \phi(w)dw^2$, and let ds^2_X be the conformal metric induced on Ω by X. We deduce from Theorem 7 that the pair (ds^2_X, ϕ) satisfies Condition (19). Therefore, for any $\theta \in \boldsymbol{R}$, the pair $(ds^2_X, e^{2i\theta}\phi)$ also satisfies Condition (19). Finally, we infer with Theorem 7 that there exists a θ-associate immersion. □

4. Vertical minimal graph. In this section, we study geometric properties of minimal graphs and their associate family. Recall from Introduction that we introduce some "Weierstrass" data for minimal surfaces (g, η) with $g = -ie^{\omega + i\psi}$ and $\eta = -2i\sqrt{\phi}$. When X is a minimal surface of $\boldsymbol{R}^3$, $\omega + i\psi$ is meromorphic. In the other cases ω satisfies the sinh-Gordon equation (13). In the following Lemma, we determine how the function $\omega + i\psi$ deviates from being meromorphic (since ω can have infinite value at each zero of ϕ). We express the same expression for the associate family. In this case (Remark 9), up to an isometry, we have $g^\theta = -ie^{\omega + i\psi^\theta}$ and $\eta^\theta = e^{i\theta}\eta$ $(\omega^\theta = \omega)$.

LEMMA 11. *Let $h : \Omega \to (U, \sigma^2|dz|^2)$ be a harmonic map with holomorphic Hopf quadratic differential $Q = \phi(w)(dw)^2$ with zeros (if any) of even order and coefficient of* *dilatation* $a(z) = e^{-2(\omega+i\psi)} = g^{-2}$. *If we define* $\sqrt{\phi} = |\phi|^{1/2}e^{i\beta}$ *and identify* σ *with* $\sigma \circ h$, *then*

$$(\omega + i\psi)_{\bar{w}} = |\phi|^{1/2}e^{-i\beta}\left(\sinh\omega\left\langle \frac{\nabla \log\sigma}{\sigma}, e^{i\psi} \right\rangle + i\cosh\omega\left\langle \frac{\nabla \log\sigma}{\sigma}, ie^{i\psi} \right\rangle \right). \tag{21}$$

COROLLARY 12. *Let* $X = (h, f)$ *be a minimal surface and* $X^\theta = (h^\theta, \eta^\theta)$ *the associate family of* X *defined in Definition* 8. *If we define the map* $\omega^\theta + i\psi^\theta$ *and denote* $\sigma^\theta = \sigma \circ h^\theta$, *then we have* $\omega^\theta = \omega$ (*up to an isometry*) *and*

$$(\omega + i\psi^\theta)_{\bar{w}} = |\phi|^{1/2}e^{-i(\beta+\theta)}\left(\sinh\omega\left\langle \frac{\nabla \log\sigma^\theta}{\sigma^\theta}, e^{i\psi^\theta} \right\rangle + i\cosh\omega\left\langle \frac{\nabla \log\sigma^\theta}{\sigma^\theta}, ie^{i\psi^\theta} \right\rangle \right).$$

PROOF. In complex coordinate w, using (7) and (11) and assuming $\eta = -2i\sqrt{\phi}$, we derive

$$h_w = \frac{\sqrt{\phi}e^{\omega+i\psi}}{\sigma} \quad \text{and} \quad h_{\bar{w}} = \frac{\overline{\sqrt{\phi}}e^{-\omega+i\psi}}{\sigma},$$

while

$$h^\theta_w = \frac{e^{i\theta}\sqrt{\phi}e^{\omega+i\psi^\theta}}{\sigma^\theta} \quad \text{and} \quad h^\theta_{\bar{w}} = \frac{e^{-i\theta}\overline{\sqrt{\phi}}e^{-\omega+i\psi^\theta}}{\sigma^\theta}.$$

Inserting these expressions in the harmonic equation (1), we obtain

$$(\omega + i\psi)_{\bar{w}} = -\sigma\left(\frac{1}{\sigma \circ h}\right)_{\bar{w}} - 2(\log\sigma)_z h_{\bar{w}},$$

$$(\omega + i\psi^\theta)_{\bar{w}} = -\sigma^\theta\left(\frac{1}{\sigma^\theta}\right)_{\bar{w}} - 2(\log\sigma^\theta)_z h_{\bar{w}}.$$

Now note that

$$-\sigma\left(\frac{1}{\sigma}\right)_{\bar{w}} = (\log\sigma)_{\bar{w}} = (\log\sigma)_z h_{\bar{w}} + (\log\sigma)_{\bar{z}}\bar{h}_{\bar{w}},$$

where $2(\log\sigma)_z = (\log\sigma)_x - i(\log\sigma)_y$ and $\bar{h}_{\bar{z}} = \overline{h_z}$. Collecting these equations, we obtain

$$(\omega + i\psi)_{\bar{w}} = (\log\sigma)_{\bar{z}}\bar{h}_{\bar{w}} - (\log\sigma)_z h_{\bar{w}},$$

which yields

$$\begin{aligned}(\omega + i\psi)_{\bar{w}} = \frac{|\phi|^{1/2}e^{-i\beta}}{\sigma}&(\sinh\omega(\cos\psi(\log\sigma)_x + \sin\psi(\log\sigma)_y)\\ &+ i\cosh\omega(\cos\psi(\log\sigma)_y - \sin\psi(\log\sigma)_x)).\end{aligned}$$

Since X^θ is isometric to X, as in the proof of Theorem 6, we can assume, up to an isometry, that $|g| = |g^\theta|$, that is, $\omega^\theta = \omega$. Then the same equation applied to h^θ yields that

$$(\omega + i\psi^\theta)_{\bar{w}} = (\log\sigma^\theta)_{\bar{z}}\bar{h}^\theta_{\bar{w}} - (\log\sigma^\theta)_z h^\theta_{\bar{w}}.$$

Therefore, we obtain

$$(\omega + i\psi^\theta)_{\bar{w}} = \frac{|\phi|^{1/2} e^{-i(\beta+\theta)}}{\sigma^\theta}(\sinh\omega(\cos\psi^\theta(\log\sigma^\theta)_y + \sin\psi^\theta(\log\sigma^\theta)_y) \\ - i\cosh\omega(\cos\psi^\theta(\log\sigma^\theta)_y - \sin\psi^\theta(\log\sigma^\theta)_y)). \qquad \square$$

We consider the projection $F : \boldsymbol{M} \times \boldsymbol{R} \to \boldsymbol{M} \times \{0\}$, thus $F \circ X = h$. Now, let us consider a curve $\gamma : [0, l] \to \Omega \subset \boldsymbol{C}$ parametrized by arclength such that $\gamma'(t) = e^{i\alpha(t)}$ in $\Omega \subset \boldsymbol{C}$. We will compute in the following the curvatures in $\boldsymbol{M}$ of the planar curves $F \circ X(\gamma) = h(\gamma)$ and $F \circ X^\theta(\gamma) = h^\theta(\gamma)$. Analogous computation appears in [8] in the particular case where $\alpha = 0$ and $\alpha = \pi/2$.

PROPOSITION 13. *Let γ be a curve in Ω and consider the images $h(\gamma)$ and $h^\theta(\gamma)$ in $\boldsymbol{M}$. Then the curvatures of $h(\gamma)$ and $h^\theta(\gamma)$ are given respectively by*

$$k(h(\gamma)) = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v + G_t}{2|\phi|^{1/2}R}, \tag{22}$$

$$k(h^\theta(\gamma)) = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v + G^\theta_t}{2|\phi|^{1/2}R^\theta}, \tag{23}$$

where

$$Re^{iG} = \cos(\alpha+\beta)\cosh\omega + i\sin(\alpha+\beta)\sinh\omega,$$
$$R^\theta e^{iG^\theta} = \cos(\alpha+\beta+\theta)\cosh\omega + i\sin(\alpha+\beta+\theta)\sinh\omega.$$

PROOF. We apply Formula (7) with $g = -ie^{\omega+i\psi}$ and $\eta = -2i\sqrt{\phi}dz$ for X, and $g^\theta = -ie^{\omega+i\psi^\theta}$ and $\eta^\theta = e^{i\theta}\eta = -2ie^{i\theta}\sqrt{\phi}dz$ for X^θ. Recall that $\sqrt{\phi} = |\phi|^{1/2}e^{i\beta}$. Then we get

$$\frac{dh(\gamma)}{dt} = \frac{2|\phi|^{1/2}}{\sigma}\cosh(\omega + i\alpha + i\beta)e^{i\psi}$$
$$= \frac{2|\phi|^{1/2}}{\sigma}(\cos(\alpha+\beta)\cosh\omega + i\sin(\alpha+\beta)\sinh\omega)e^{i\psi},$$
$$\frac{dh(\gamma)}{dt} = \frac{2|\phi|^{1/2}}{\sigma}Re^{i(\psi+G)},$$

$$\frac{dh^\theta(\gamma)}{dt} = \frac{2|\phi|^{1/2}}{\sigma}\cosh(\omega + i\beta + i\alpha + i\theta)e^{i\psi^\theta}$$
$$= \frac{2|\phi|^{1/2}}{\sigma}(\cos(\alpha+\beta+\theta)\cosh\omega + i\sin(\alpha+\beta+\theta)\sinh\omega)e^{i\psi^\theta},$$
$$\frac{dh^\theta}{dt} = \frac{2|\phi|^{1/2}}{\sigma}Re^{i(\psi^\theta+G^\theta)}.$$

If k is the curvature of a curve γ in $(U, \sigma^2(z)|dz|^2)$ and k_e is the Euclidean curvature in $(U, |dz|^2)$, we get by a conformal change of the metric:

$$k = \frac{k_e}{\sigma} - \frac{\langle\nabla\sigma, n\rangle}{\sigma^2},$$

where n is the Euclidean normal to the curve γ such that (γ', n) is positively oriented. If s denotes the arclength of $h(\gamma)$ and s^θ the arclength of $h^\theta(\gamma)$, both for the Euclidean metric, we have

$$k_e(h(\gamma)) = \psi_s + G_s = \frac{\sigma}{2|\phi|^{1/2}R}(\cos\alpha\psi_u + \sin\alpha\psi_v) + G_s\,. \tag{24}$$

The Euclidean normal of $h(\gamma)$ (resp. $h^\theta(\gamma)$) is given by

$$n = (-\sin(\alpha+\beta)\sinh\omega + i\cos(\alpha+\beta)\cosh\omega)\frac{e^{i\psi}}{R}\,,$$

$$n^\theta = (-\sin(\alpha+\beta+\theta)\sinh\omega + i\cos(\alpha+\beta+\theta)\cosh\omega)\frac{e^{i\psi^\theta}}{R^\theta}\,,$$

and

$$\begin{aligned}\frac{\langle\nabla\sigma, n\rangle}{\sigma^2} = \frac{\langle\nabla\log\sigma, n\rangle}{\sigma} &= -\sin(\alpha+\beta)\frac{\sinh\omega}{R}\left\langle\frac{\nabla\log\sigma}{\sigma}, e^{i\psi}\right\rangle \\ &\quad + \cos(\alpha+\beta)\frac{\cosh\omega}{R}\left\langle\frac{\nabla\log\sigma}{\sigma}, ie^{i\psi}\right\rangle.\end{aligned}$$

Using Lemma 11 and the expression of n, we obtain

$$\frac{\langle\nabla\sigma, n\rangle}{\sigma^2} = \frac{\langle\nabla\log\sigma, n\rangle}{\sigma} = -\frac{1}{2|\phi|^{1/2}R}(\sin\alpha\omega_u - \sin\alpha\psi_v - \cos\alpha\omega_v - \cos\alpha\psi_u)\,.$$

Hence we deduce that

$$\frac{\psi_s}{\sigma} - \frac{\langle\nabla\sigma, n\rangle}{\sigma^2} = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v}{2|\phi|^{1/2}R}\,.$$

The same computation with X^θ yields that

$$\frac{\psi^\theta_{s^\theta}}{\sigma^\theta} - \frac{\langle\nabla\sigma^\theta, n\rangle}{(\sigma^\theta)^2} = \frac{\sin\alpha\omega_u - \cos\alpha\omega_v}{2|\phi|^{1/2}R^\theta}\,.$$

This completes the proof of the proposition, since $G_s = (\sigma/(2|\phi|^{1/2}R))G_t$. □

Now, we prove the generalization of Krust's theorem for minimal vertical graphs and associate family surfaces. Let $U \subset \boldsymbol{M}$ be an open set and $f(z)$ a smooth function on U. We say that F is a *vertical graph* in $\boldsymbol{M} \times \boldsymbol{R}$ if $F = \{(z,t) \in \boldsymbol{M} \times \boldsymbol{R}\,;\ t = f(z),\ z \in U\}$. The graph is an *entire vertical graph* if $U = \boldsymbol{M}$.

THEOREM 14. *Let $X(\Omega)$ be a vertical minimal graph on a convex domain $h(\Omega)$ in $\boldsymbol{M}$. Then the associate surface $X^\theta(\Omega)$ is a graph provided that $K_{\boldsymbol{M}} \le 0$.*

When $K_{\boldsymbol{M}} \equiv 0$, this is a result of Krust (see [5, page 188] and application therein).

PROOF. The proof is a direct application of the Gauss-Bonnet theorem with the fact that ω has no zero (X is a vertical graph). If we consider a smooth piece of an embedded curve Γ in $\boldsymbol{M}$ with end points p_1 and p_2, then if $p_1 = p_2$, Γ encloses a region A and

$$\int_A K_{\boldsymbol{M}} dV_\sigma + \int_\Gamma k(s)ds + \alpha = 2\pi\,, \tag{25}$$

where α is the exterior angle at $p_1 = p_2$, $-\pi \le \alpha \le \pi$. The Gauss-Bonnet formula (25) gives us, in the case where $K_M \le 0$, that

$$\pi \ge \alpha \ge 2\pi - \int_\Gamma k ds\,.$$

Now, if we assume that $X^\theta(\Omega)$ is not a graph, then there exist two points p_1 and p_2 of Ω with $h^\theta(p_1) = h^\theta(p_2)$. Since $h(\Omega)$ is convex, there is a geodesic in $h(\Omega)$ which can be lifted by a path γ in Ω. In summary, we assume that the curve $\gamma(t)$, $t \in [0, l]$, is parametrized by Euclidean arclength, $\gamma'(t) = e^{i\alpha}$, $h(\gamma)$ is a piece of a geodesic of $\boldsymbol{M}$, p_1 and p_2 are the end points of γ and $h^\theta(p_1) = h^\theta(p_2)$. We assume that $h^\theta(\gamma)$ is a closed embedded curve. If not, we can consider a subarc of γ with end points p'_1 and p'_2, with the image by h^θ smooth, embedded and $h^\theta(p'_1) = h^\theta(p'_2)$. In the case where this embedded subarc does not exist, it follows that all points are double, like a path where we go and back after an interior end point q. At q, $h^\theta(\gamma)$ is not immersed, so that the derivative at q is zero and then the tangent plane of X^θ is vertical. Then ω would have an interior zero (a contradiction with the vertical graph assumption).

We will apply the Gauss-Bonnet formula to prove that $\int_{h^\theta(\gamma)} k ds^* < \pi$ under the hypothesis that $h(\gamma)$ is a geodesic. It yields a contradiction with $\alpha \le \pi$ and then the horizontal curve $h^\theta(\gamma)$ is an embedded arc with $h^\theta(p_1) \ne h^\theta(p_2)$.

Since $h(\gamma)$ is a geodesic, using Formula (22) of Proposition 13, we have $\sin\alpha\omega_u - \cos\alpha\omega_v + G_t = 0$. Thus we obtain

$$k(h^\theta(\gamma)) = \frac{G_t^\theta - G_t}{2|\phi|^{1/2}R^\theta}\,.$$

Since $ds^* = 2|\phi|^{1/2}R^\theta dt$, we have

$$\int_{h^\theta(\gamma)} k ds^* = (G^\theta(l) - G(l)) - (G^\theta(0) - G(0))\,.$$

Now, we remark by a direct computation of the real and imaginary parts of

$$\frac{R^\theta}{R} e^{i(G^\theta - G)} = \frac{\cos(\alpha+\beta+\theta)\cosh\omega + i\,\sin(\alpha+\beta+\theta)\sinh\omega}{\cos(\alpha+\beta)\cosh\omega + i\,\sin(\alpha+\beta)\sinh\omega}$$

that

$$\tan(G^\theta(t) - G(t)) = \frac{\sinh(2\omega)\sin\theta}{2\cos\theta R^2 - \sin\theta\sin 2(\alpha+\beta)}\,.$$

Since X is a graph, ω has no interior zero, so that $\sinh(2\omega)\sin\theta$ cannot be zero for $\theta \in (0, \pi/2]$. This implies that modulo π we have $G^\theta(t) - G(t) \in (0, \pi)$ modulo π. □

The Example (16) shows that the conjugate surface of an entire vertical minimal graph in $\boldsymbol{H}^2 \times \boldsymbol{R}$ (which is a graph by Theorem 14) is not necessarily an entire graph. In this direction we obtain the following criterion in $\boldsymbol{H}^2 \times \boldsymbol{R}$.

THEOREM 15. *Let $X : \boldsymbol{D}^2 \to \boldsymbol{H}^2 \times \boldsymbol{R}$ be an entire vertical graph on $\boldsymbol{H}^2$. Assume that for any divergent path γ of finite Euclidean length in $\boldsymbol{D}$ we have*

$$\int_\gamma |\phi|^{1/2} dt < \infty .$$

Then the conjugate graph X^* is an entire graph.

PROOF. Recall that $f + i f^* = -2i \int^z \sqrt{\phi}$ is holomorphic. We consider a divergent path $\gamma(t)$ in $\boldsymbol{D}^2$ and its image $X(\gamma) = \Gamma$ in the graph (recall that $\gamma'(t) = e^{i\alpha}$)). Since X is a proper map, the length of Γ is infinite in X and

$$\ell(\Gamma) = \int_\gamma 2 \cosh \omega |\phi|^{1/2} dt = \infty . \tag{26}$$

Now, we show that the length of $h^* \circ \gamma$ is infinite, which proves the theorem. If X^* is not entire, one can find a diverging curve in $\boldsymbol{D}^2$ with $h^* \circ \gamma$ of finite length. To this end, we compute

$$\ell(h^* \circ \gamma) = \int_\gamma 2 |\phi|^{1/2} R^* dt ,$$

where $R^{*2} = \sin^2(\alpha + \beta) \cosh^2 \omega + \cos^2(\alpha + \beta) \sinh^2 \omega$ (recall $R^* = R^{\pi/2}$). Remark that

$$R^{*2} = \cosh^2 \omega - \cos^2(\alpha + \beta) .$$

Then, using (26) together with the hypothesis, we have

$$\ell(h^* \circ \gamma) \geq \ell(\Gamma) - \int_\gamma |\cos(\alpha + \beta)| |\phi|^{1/2} dt = \infty . \qquad \square$$

5. Examples.

EXAMPLE 16. Let us consider the Figure 1, where the left side shows the *Scherk type surface* in $\boldsymbol{H}^2 \times \boldsymbol{R}$ invariant by hyperbolic translations. It is a complete minimal graph over an unbounded domain in $\boldsymbol{H}^2$ defined by a complete geodesic γ in $\boldsymbol{H}^2 \times \{0\}$. In particular, it is not an entire graph. The graph takes values $\pm\infty$ on γ and value 0 on the asymptotic boundary. In the upper half-plane model of $\boldsymbol{H}^2 = \{(x, y) \in \boldsymbol{R}^2 \ ; \ y > 0\}$, there is a nice formula for the graph in $\boldsymbol{H}^2 \times \boldsymbol{R}$ as

$$t = \ln \left(\frac{\sqrt{x^2 + y^2} + y}{x} \right), \quad y > 0 , \quad x > 0 .$$

The right side of Figure 1 represents Scherk's conjugate minimal surface in $\boldsymbol{H}^2 \times \boldsymbol{R}$ (see Theorem 4.2 of [14]) which is given by the equation $t = x$.

It is invariant by parabolic screw-motions. It is an entire graph over $\boldsymbol{H}^2$. The second and third authors proved that in $\boldsymbol{H}^2 \times \boldsymbol{R}$, a catenoid is conjugate to a helicoid of pitch $\ell < 1$, see [15]. Surprisingly, a helicoid of pitch $\ell = 1$ is conjugate to a surface invariant by parabolic translations, see [4] or [14]. Furthermore, any helicoid with pitch $\ell > 1$ is conjugate to a minimal surface invariant by hyperbolic translations, see [14].

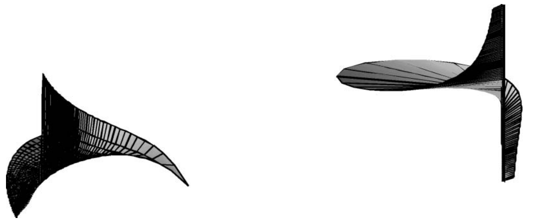

FIGURE 1. Scherk minimal surface invariant by hyperbolic translation and its conjugate in $\boldsymbol{H}^2 \times \boldsymbol{R}$.

REMARK 17. Assume that $\boldsymbol{M} = \boldsymbol{R}^2$, and consider $X, X^* : \Omega \to \boldsymbol{R}^2 \times \boldsymbol{R}$ two conformal minimal immersions. Let (g, η) (resp. (g^*, η^*)) be the Weierstrass representation of X (resp. X^*). We know that X and X^* are associate in the usual sense, that is, in the Euclidean space $\boldsymbol{R}^3$, if and only if $g^* = g$ and $\eta^* = e^{i\theta}\eta$ for a real number θ. Set $X = (h, f)$ and $X^* = (h^*, f^*)$. Since $Q(h) = -(\eta)^2/4$, we see that if X and X^* are associate in the usual sense, then there are associate in the meaning of Definition 8. Conversely, assume that X and X^* are associate in the sense of Definition 8. Then we have $\eta^* = \pm e^{i\theta}\eta$ and $|g^*| = |g|$ or $|g^*| = 1/|g|$. Therefore, there exists an isometry Γ of $\boldsymbol{R}^3$ such that X and $\Gamma \circ X^*$ are associate in the usual sense.

For example, the Weierstrass representations (g, η) and $(e^{i\theta}g, e^{i\theta}\eta)$ for $\theta \neq 2k\pi,\ k \in \mathbf{Z}$, are associate in the sense of Definition 8, but are not in the usual sense.

In consequence, in $\boldsymbol{R}^2 \times \boldsymbol{R}$ these two notions of associate minimal immersions are equivalent only up to an isometry of $\boldsymbol{R}^2 \times \boldsymbol{R}$.

It is known that any two isometric conformal minimal immersions in $\boldsymbol{R}^3$ are associate up to an isometry. Also, it is shown in [15] that any two isometric screw motion minimal complete immersions in $\boldsymbol{H}^2 \times \boldsymbol{R}$ are associate. The following example shows that this is no longer true for isometric immersions in $\boldsymbol{H}^2 \times \boldsymbol{R}$.

EXAMPLE 18. There is given in [15] an example of a complete minimal surface in $\boldsymbol{H}^2 \times \boldsymbol{R}$ with intrinsic curvature constant and equals to -1, $K \equiv -1$. Namely, from Formula (49) in Corollary 21 of [15], setting $H = 0,\ l = m = 1$ and d is any positive real number (keeping the same notation), we obtain $U(s) = \sqrt{1+d^2}\cosh(s)$, $s \in \boldsymbol{R}$. Consequently, from Theorem 19 in [15] we obtain (see (36), (37) and (38))

$$\begin{aligned} \rho(s) &= \operatorname{arcosh}(\sqrt{1+d^2}\cosh s)\,, \\ \lambda(\rho(s)) &= d\int \frac{U(s)}{U^2(s)-1}\,ds\,, \\ \varphi(s,\tau) &= \tau - d\int \frac{1}{U(s)(U^2(s)-1)}\,ds\,. \end{aligned}$$

Now, let us consider the map $T : \boldsymbol{R}^2 \to \boldsymbol{H}^2 \times \boldsymbol{R}$ defined for every $(s, \tau) \in \boldsymbol{R}^2$ by

$$T(s,\tau) = (\tanh(\rho(s)/2)\cos\varphi(s,\tau), \tanh(\rho(s)/2)\sin\varphi(s,\tau), \lambda(\rho(s)) + \varphi(s,\tau))\,,$$

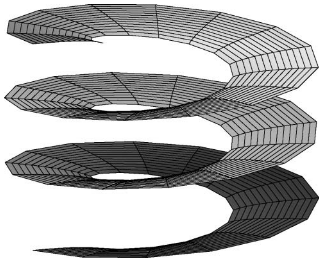

FIGURE 2. Embedded minimal screw motion surface in $\boldsymbol{H}^2 \times \boldsymbol{R}$ with Gaussian curvature $K \equiv -1$.

see Figure 2 (where $d = 1$).

It is shown that T is a regular minimal *embedding* with induced metric

$$ds_T^2 = ds^2 + U^2(s)d\tau^2 = ds^2 + (1 + d^2)\cosh^2(s)d\tau^2.$$

A straightforward computation shows that the intrinsic curvature is given by $K = -U''/U$. Therefore, we have $K \equiv -1$. By construction the surface $T(\boldsymbol{R}^2)$ is invariant by screw motions. The immersion T is not conformal, but, setting $r := \int (1/U(s))ds$, the new coordinates (r, τ) are conformal, that is, the immersion $\tilde{T}(r, \tau) := T(s, \tau)$ is conformal. Thus, the surface $T(\boldsymbol{R}^2)$ is isometric to the hyperbolic plane $(\boldsymbol{D}, \sigma^2(z)|dz|^2)$. Therefore, there exists a conformal minimal immersion $X : \boldsymbol{D} \to \boldsymbol{H}^2 \times \boldsymbol{R}$ such that the induced metric on $\boldsymbol{D}$ is the hyperbolic one and $X(\boldsymbol{D}) = T(\boldsymbol{R}^2)$. Clearly, the canonical immersion $Y : \boldsymbol{D} \to \boldsymbol{H}^2 \times \boldsymbol{R}$ defined by $Y(z) = (z, 0)$ is isometric to X. According to Remark 9, we deduce that X and Y are not associate, since the third component of the Gauss map of X is never equal to ± 1, as it is the case for Y.

It should be remarked that in [8] one can find other examples of complete minimal surfaces in $\boldsymbol{H}^2 \times \boldsymbol{R}$ with intrinsic curvature equal to -1.

REMARK 19. The second author has constructed in [14] new families of complete minimal immersions in $\boldsymbol{H}^2 \times \boldsymbol{R}$ invariant by parabolic or hyperbolic screw motions. It is shown (see Theorem 4.1) that any two minimal isometric parabolic screw motion immersions into $\boldsymbol{H}^2 \times \boldsymbol{R}$ are associate. However, this is no longer true for hyperbolic screw motion immersions. Indeed, there exist isometric minimal hyperbolic screw motion immersions into $\boldsymbol{H}^2 \times \boldsymbol{R}$ which are not associate, see Theorem 4.2.

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