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ASSOCIATED HYPERBOLIC AND FIBONACCI IDENTITIES

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It is well known that by using the substitutions

$$\cos X = \cosh x,\ \sin X = -i \sinh x,$$

where $i = \sqrt{-1}$, trigonometric identities give rise to hyperbolic ones and conversely. This results from Euler's formulas

$$\cos X = \cosh iX \quad \text{and} \quad \sin X = -i \sinh iX.$$

For instance, we have the relations

$$\cos^2 X + \sin^2 X = 1,\ \cosh^2 x - \sinh^2 x = 1$$

and

$$\sin 2X = 2 \sin X \cos X,\ \sinh 2x = 2 \sinh x \cosh x.$$

Also, we shall see that a simple substitution *automatically* associates some Fibonacci identities to a class of hyperbolic ones.

This note is more original in its form than in its conclusions. Similar methods have been used by Lucas [1], Amson [2], and Hoggatt & Bicknell [3].

I. THE HYPERBOLIC-FIBONACCI ASSOCIATION

The following notation will be essential:*

$$[A, B]_n = \begin{cases} A \text{ if } n \text{ is odd,} \\ B \text{ if } n \text{ is even.} \end{cases}$$

We start from Binet's formulas:

$$F_n = \frac{a^n - b^n}{a - b},\ L_n = a^n + b^n,$$

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*More generally $[u_1, u_2, \ldots, u_p]_n$ is equal to the u_i in the brackets such that $i = n$, modulo p [4].

where $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$ are the roots of the equation

$$X^2 - X - 1 = 0.$$

With $\alpha = \log a$, we have $a = e^{\alpha}$ and $b = -e^{-\alpha}$, and therefore

$$\frac{F_n}{2} = \frac{e^{\alpha n} - (-1)\ e^{-\alpha n}}{2\sqrt{5}},\ \frac{L_n}{2} = \frac{e^{\alpha n} + (-1)\ e^{-\alpha n}}{2}.$$

We now let $\alpha n = x$, then

$$\frac{\sqrt{5}F_n}{2} = [\cosh x,\ \sinh x]_n,\ \frac{L_n}{2} = [\sinh x,\ \cosh x]_n.$$

Substituting kn for n, k being an integer, we have

$$F_{kn} = \frac{2}{\sqrt{5}}[\cosh kx,\ \sinh kx]_{kn},\ L_{kn} = 2[\sinh kx,\ \cosh kx]_{kn}. \tag{1}$$

Substituting $n + m$ for n and putting $\alpha m = y$, we find that

$$\begin{aligned} F_{n+m} &= \frac{2}{\sqrt{5}}[\cosh(x+y),\ \sinh(x+y)]_{n+m}, \\ L_{n+m} &= 2[\sinh(x+y),\ \cosh(x+y)]_{n+m}. \end{aligned} \tag{2}$$

Equivalently, we have

$$\cosh kx = \frac{1}{2}[\sqrt{5}F_{kn},\ L_{kn}]_{kn},\ \sinh kx = \frac{1}{2}[L_{kn},\ \sqrt{5}F_{kn}]_{kn}, \tag{3}$$

and

$$\begin{aligned} \cosh(x+y) &= \frac{1}{2}[\sqrt{5}F_{n+m},\ L_{n+m}]_{n+m}, \\ \sinh(x+y) &= \frac{1}{2}[L_{n+m},\ \sqrt{5}F_{n+m}]_{n+m}. \end{aligned} \tag{4}$$

Formulas (2) and (4) also hold if we replace all the plus signs by minus signs.

Theorem 1

By substituting (1) and (2) on one side or (3) and (4) on the other, a Fibonacci identity gives one or several hyperbolic identities and conversely, provided that the indices or arguments have the form $kn \pm k'm$ or $kx \pm k'y$. The indices may be null or negative.

Remark

If we start with a Fibonacci identity, we must theoretically control the associated hyperbolic identity by other means, for then we pass from the particular to the general case. However, such an identity being true for $x = \alpha n$ and $y = \alpha m$ is probably true for all x and y, because

$$\alpha = \log \frac{1 + \sqrt{5}}{2}$$

is a transcendental number.

Since the hyperbolic identities are classic, we can easily establish some well-known Fibonacci identities.

II. DEVELOPMENT OF FIBONACCI IDENTITIES

Example 1

$$\sinh 2x = 2 \sinh x \cosh x$$

For all n, the substitution of (3) gives $F_{2n} = L_n F_n$.

Example 2

$$\cosh^2 x - \sinh^2 x = 1$$

The substitution of (3) gives:

$$\frac{5F_n^2}{4} - \frac{L_n^2}{4} = 1, \text{ if } n \text{ is odd}, \quad \text{and} \quad \frac{L_n^2}{4} - \frac{5F_n^2}{4} = 1, \text{ if } n \text{ is even}.$$

Thus for all n,

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{5}$$

Example 3

$$\sinh 5x = \sinh x \cosh\left(2x - \sqrt{5} \cosh x + \frac{3}{2}\right)\left(\cosh 2x + \sqrt{5} \cosh x + \frac{3}{2}\right).$$

By substitution

$$\frac{L_{5n}}{2} = 4 \frac{L_{2n}}{2}\left(\frac{L_{2n}}{2} - \sqrt{5} \frac{\sqrt{5}F_n}{2} + \frac{3}{2}\right)\left(\frac{L_{2n}}{2} + \sqrt{5} \frac{\sqrt{5}F_n}{2} + \frac{3}{2}\right), \text{ if } n \text{ is odd,}$$

and

$$\frac{\sqrt{5}F_{5n}}{2} = 4\,\frac{\sqrt{5}F_n}{2}\left(\frac{L_{2n}}{2} - \sqrt{5}\,\frac{L_n}{2} + \frac{3}{2}\right)\left(\frac{L_{2n}}{2} + \sqrt{5}\,\frac{L_n}{2} + \frac{3}{2}\right), \text{ if } n \text{ is even.}$$

Using $2n-1$ in place of n if n is odd and $2n$ in place of n if n is even, we find the following two distinct identities, which are valid for every n:

$$L_{10n-5} = L_{2n-1}(L_{4n-2} - 5F_{2n-1} + 3)(L_{4n-2} + 5F_{2n-1} + 3)$$

and

$$F_{10n} = F_{2n}(L_{4n} - \sqrt{5}L_{2n} + 3)(L_{4n} + \sqrt{5}L_{2n} + 3).$$

Example 4

$$(\cosh x + \sinh x)^k = \cosh kx + \sinh kx \quad (k \text{ an integer} \geqslant 0)$$

Examining three cases (n even, n odd and k even, n odd and k odd), we find for all n and k that

$$\left(\frac{L_n + \sqrt{5}F_n}{2}\right)^k = \frac{L_{kn} + \sqrt{5}F_{kn}}{2}. \tag{6}$$

Application: Suppose we wish to express L_{kn} and F_{kn} as functions of L_n and F_n. We could do this by separating the expanded form of the left side of (6) into those terms with or without the factor $\sqrt{5}$.

Instead, we use the well-known fact that F_{kn} is divisible by F_n to show that the integer F_{kn}/F_n is a function of L_n of the form

$$P(L_n) + (-1)^n Q(L_n),$$

where $P(X)$ and $Q(X)$ are polynomials whose parities are opposite to that of k. By (6) we see that F_{kn} has the form

$$\sum c_i F_n^i L_n^{k-i},$$

where i takes the odd values equal to k or less. Therefore, F_{kn}/F_n has the form

$$\sum c_i (F_n^2)^{k'} L_n^{k-i},$$

but, according to (5),

$$F_n^2 = \frac{1}{5}(L_n^2 - 4(-1)^n).$$

Thus for $k = 2, 3, 4, 5, 6$, the values of F_{kn}/F_n are, respectively:

$$L_n,\ L_n^2 - (-1)^n,\ L_n[L_n^2 - 2(-1)^n],\ L_n^4 - 3(-1)^n L_n^2 + 1,\ L_n[L_n^4 - 4(-1)^n L_n^2 + 3].$$

Example 5

$$\frac{1}{2} + \cosh 2x + \cosh 4x + \cosh 6x + \cdots + \cosh 2kx = \frac{\sinh(2k+1)x}{\sinh x} \tag{7}$$

Therefore,

$$1 + L_{2n} + L_{4n} + L_{6n} + \cdots + L_{2kn} = \left[\frac{L_{(2k+1)n}}{L_n},\ \frac{F_{(2k+1)n}}{F_n}\right]_n$$

If we replace n by $2n$, we get

$$1 + L_{4n} + L_{8n} + \cdots + L_{4kn} = \frac{F_{(4k+2)n}}{F_{2n}}.$$

If we substitute $X + (\pi/2)$ for X in the trigonometric identity associated with (7), we find a formula whose associated hyperbolic one is

$$\frac{1}{2} - \cosh 2x + \cosh 4x - \cosh 6x + \cdots + (-1)^k \cosh 2kx = \frac{(-1)^k \cosh(2x+1)x}{2 \cosh x}.$$

Hence,

$$1 - L_{2n} + L_{4n} - L_{6n} + \cdots + (-1)^k L_{2kn} = (-1)^k \left[\frac{F_{(2k+1)n}}{F_n},\ \frac{L_{(2k+1)n}}{L_n}\right]_n.$$

Application: We can use these two Fibonacci identities to prove that *for any odd k, L_{kn} is divisible by L_n.*

Example 6

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Using (3) and (4), we see that for all n and m,

$$\begin{cases} 2F_{n+m} = F_n L_m + L_n F_m, \\ 2L_{n+m} = L_n L_m + 5F_n F_m. \end{cases} \tag{8}$$

Note that for $m = \pm 1$, (8) becomes

$$L_n + F_n = 2F_{n+1},\ L_n - F_n = 2F_{n-1}.$$

Example 7

$$\cosh(x + y) + \cosh(x - y) = 2 \cosh x \cosh y$$

$$\sinh(x + y) + \sinh(x - y) = 2 \sinh x \cosh y$$

Examining the four cases (according to the parities of n and m), we find that

$$L_{n+m} + L_{n-m} = [5F_nF_m, L_nL_m]_m, \tag{9}$$

$$F_{n+m} + F_{n-m} = [L_nF_m, F_nL_m]_m.$$

In particular, for $m = 1$, (9) becomes $L_{n-1} + L_{n+1} = 5F_n$. It can also be shown that

$$L_{n+m} - L_{n-m} = [5F_nF_m, L_nL_m]_{m-1}$$

and

$$F_{n+m} - F_{n-m} = [L_nF_m, F_nL_m]_{m-1}.$$

Application: We shall establish the following proposition using the preceding identities.

Theorem 2

A number of the form $F_n \pm F_m$ or $L_n \pm L_m$ is never prime if the indices have the same parity and a difference greater than 4.

Proof: The proof goes as follows: Let $a = n + m$ and $b = n - m$, then

$$F_a + F_b = \left[L_{\frac{a+b}{2}}F_{\frac{a-b}{2}}, F_{\frac{a+b}{2}}L_{\frac{a-b}{2}}\right]_{\frac{a-b}{2}}.$$

Since $a - b > 4$, we have $\frac{a-b}{2} > 2$, so that there is no term

$$L_1 = F_1 = F_2 = 1$$

in the brackets. Hence, $F_a + F_b$ is composite.

A similar demonstration exists for $F_a - F_b$ or $L_a \pm L_b$.

Example 8

$$\sinh(x + y)\sinh(x - y) = \sinh^2 x - \sinh^2 y$$

$$\cosh(x + y)\cosh(x - y) = \cosh^2 x + \sinh^2 y$$

By substitution, we have:

a) if n and m are even,

$$F_{n+m}F_{n-m} = F_n^2 - F_m^2$$

$$L_{n+m}L_{n-m} = L_n^2 + 5F_m^2$$

$$5F_{n+m}F_{n-m} = L_n^2 - L_m^2;$$

b) if n and m are odd,

$$L_{n+m}L_{n-m} = 5F_n^2 + L_m^2;$$

c) if n is even and m is odd,

$$5F_{n+m}F_{n-m} = L_n^2 + L_m^2$$

$$L_{n+m}L_{n-m} = 5F_n^2 - L_m^2;$$

d) if n is odd and m is even,

$$F_{n+m}F_{n-m} = F_n^2 + F_m^2$$

$$L_{n+m}L_{n-m} = L_n^2 - 5F_m^2.$$

With the help of (5) the four expressions thus obtained for $F_{n+m}F_{n-m}$ and $L_{n+m}L_{n-m}$ can be condensed into two identities:

$$F_{n+m}F_{n-m} - F_n^2 = (-1)^{n+m+1}F_m^2,$$

and

$$L_{n+m}L_{n-m} - L_n^2 = (-1)^{n+m}L_m^2 - 4(-1)^n.$$

The first is the Catalan formula.

Letting $n = 1$ and $m = 2$, we see that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad \text{(Simson's formula)}$$

$$F_{n+2}F_{n-2} - F_n^2 = (-1)^{n+1},$$

$$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1},$$

and

$$L_{n+2}L_{n-2} - L_n^2 = 5(-1)^n.$$

Example 9

Our last example is of a Fibonacci-trigonometric transposition. In [5], it is shown that if a, b, and c are even integers, then

$$L_a L_b L_c = L_{a+b+c} + L_{b+c-a} + L_{c+a-b} + L_{a+b-c},$$

and

$$5F_a F_b F_c = F_{a+b+c} + F_{a-b-c} + F_{b-c-a} + F_{c-a-b}.$$

Using the hyperbolic transposition, (substitutions (1) and (2), we obtain

$$4 \cosh x \cosh y \cosh z = \cosh(x+y+z) + \cosh(y+z-x) + \cosh(z+x-y) + \cosh(x+y-z),$$

and

$$4 \sinh x \sinh y \sinh z = \sinh(x+y+z) + \sinh(x-y-z) + \sinh(y-z-x) + \sinh(z-x-y).$$

Now, applying the trigonometric transposition, we have

$$4 \cos X \cos Y \cos Z = \cos(X+Y+Z) + \cos(Y+Z-X) + \cos(Z+X-Y) + \cos(X+Y-Z),$$

and

$$4 \sin X \sin Y \sin Z = -\sin(X+Y+Z) + \sin(Y+Z-X) + \sin(Z+X-Y) + \sin(X+Y-Z).$$

III. GENERALIZATION

Let s be a positive integer with a and b the roots of the equation

$$X^2 - sX - 1 = 0, \text{ where } a = \frac{s + \sqrt{s^2+4}}{2}.$$

Consider the two generalized Fibonacci sequences given by

$$f_n = \frac{a^n - b^n}{a - b}, \quad \ell_n = a^n + b^n. \tag{10}$$

Let

$$\Delta = s^2 + 4, \ \alpha = \log a, \ \alpha n = x, \ \alpha m = y,$$

then,

$$\frac{f_n}{2} = \frac{a^n - b^n}{2\sqrt{\Delta}} = \frac{e^{\alpha n} - (-1)\ e^{-\alpha n}}{2\sqrt{\Delta}} = \frac{1}{\sqrt{\Delta}}[\cosh x, \sinh x]_n,$$

and

$$\frac{\ell_n}{2} = \frac{a^n + b^n}{2} = \frac{e^{\alpha n} + (-1)\ e^{-\alpha n}}{2} = [\sinh x, \cosh x]_n.$$

Hence,

$$\begin{cases} \cosh kx = \frac{1}{2}[\sqrt{\Delta}f_{kn},\ \ell_{kn}]_{kn},\ \sinh kx = \frac{1}{2}[\ell_{kn},\ \sqrt{\Delta}f_{kn}]_{kn}, \\ \cosh(x+y) = \frac{1}{2}[\sqrt{\Delta}f_{n+m},\ \ell_{n+m}]_{n+m},\ \sinh(x+y) = \frac{1}{2}[\ell_{n+m},\ \sqrt{\Delta}f_{n+m}]_{n+m}. \end{cases} \tag{11}$$

Theorem 3

To a hyperbolic identity with arguments of the form $k \pm k'm$ (n and m integers), the substitution formulas of (11) associate one or several generalized Fibonacci identities (the same as for F_n and L_n, with the restriction that the factor 5 or $\sqrt{5}$ is replaced by Δ or $\sqrt{\Delta}$).

For instance,

$$f_{2n} = \ell_n f_n,\ \ell_n^2 - \Delta f_n^2 = 4(-1)^n,\ f_{n+m} + f_{n-m} = [\ell_n f_m,\ f_n \ell_m]_m.$$

Note that the neighborly relations,

$$L_n + F_n = 2F_{n+1},\ L_n - F_n = 2F_{n-1},\ L_{n-1} + L_{n+1} = 5F_n,$$

$$L_{n-1} - F_{n+1} = L_{n+1} - F_{n-1} = 3F_n,\ L_n^2 - F_n^2 = 4F_{n-1}F_{n+1},$$

$$L_n^2 + F_n^2 = 2(F_{n-1}^2 + F_{n+1}^2),$$

do not hold for f_n and ℓ_n. However, for every s:

$$f_{n+1} + f_{n-1} = \ell_n.$$

The formulas of Simson and Catalan also hold for f_n and

$$\ell_{n+1}\ell_{n-1} - \ell_n^2 = \Delta(-1)^{n+1}.$$

Application: If we put $a = n + m$ and $b = n - m$, the formula

$$f_{n+m} + f_{n-m} = [\ell_n f_m,\ f_n \ell_m]_m$$

becomes

$$f_a + f_b = \left[\ell_{\frac{a+b}{2}} f_{\frac{a-b}{2}},\ f_{\frac{a+b}{2}} \ell_{\frac{a-b}{2}}\right]_{\frac{a-b}{2}}$$

if $a - b$ is even. Therefore:

Theorem 4

A number $f_a + f_b$ is not prime if $a - b$ is even and other than 2, and $f_a + f_{a+2}$ is a prime only if ℓ_{a+1} is a prime.

Proof: Note that

$$f_a + f_{a+2} = \ell_{a+1} \quad \text{and} \quad f_a + f_{a+4} = 3f_{a+2},$$

so if $a-b > 4$, there is no factor $f_1 = 1$ or $f_2 = \ell_1 = s = 1$ in the brackets, and if $a-b = 4$, then

$$f_a + f_b = f_{\frac{a+b}{2}}\ell_2.$$

Remarks

1) Under the same conditions, $\ell_a \pm \ell_b$ is not prime. Furthermore, if $a - b$ is even other than 2 or 4, $f_a - f_b$ is not prime.
2) An integer $F_a \pm 1$ is not prime for $n > 6$. (See [6].)

The latter remark is true, since $F_a \pm 1$ can be considered as $F_a \pm F_1$ if a is odd and $F_a \pm F_2$ if a is even. For $a > 6$, the difference $a - 1$ or $a - 2$ exceeds 4.

Recurrence: The generalized Fibonacci sequences can also be defined by

$$f_{n+2} = sf_{n+1} + f_n,\ f_0 = 0,\ f_1 = 1,$$

and

$$\ell_{n+2} = s\ell_{n+1} + \ell_n,\ \ell_0 = 2,\ \ell_1 = s.$$

This results directly from Binet's formulas (10). Note that for $s = 2$, the f_n are the "Pell numbers."

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