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ASSOCIATION OF NORMAL RANDOM VARIABLES AND SLEPIAN'S INEQUALITY

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A simple extension of Slepian's inequality is given which implies the original inequality and the result that positively correlated normal random variables are associated. Extensions to elliptically contoured distributions are given.

I. Introduction. We give a simple proof of the result in Pitt (1981) that positively correlated normal random variables are associated. The proof is an adaptation of the original proof of Slepian's inequality in Slepian (1962), and extends to give similar results in the case of elliptically contoured distributions.

II. Normal random vectors. Let $X = (X_1, \cdots, X_n)$ be a mean zero n-dimensional normal random vector with $n \times n$ covariance matrix $\Sigma = (\sigma_{ij})$. By a smooth function we mean a C^2 function $h(x) = h(x_1, \cdots, x_n)$ which together with its first and second order derivatives satisfy a $O(|x|^N)$ growth condition at ∞, for some finite N.
We set

$$\mathscr{H}(\Sigma) = E_\Sigma h(X),$$

and we are interested in the manner that $\mathscr{H}(\Sigma)$ varies with Σ. Our main result is the following.

PROPOSITION 1. *Let h be a smooth function, and suppose that (i, j) is a pair of indices with $i \le j$ and such that*

$$\frac{\partial^2 h(x)}{\partial x_i \partial x_j} \ge 0 \quad \textit{for all } x. \tag{1}$$

Then $\mathscr{H}(\Sigma) = E_\Sigma h(X)$ is increasing in σ_{ij}.

PROOF. By standard approximation arguments it suffices to show

$$\frac{\partial \mathscr{H}(\Sigma)}{\partial \sigma_{ij}} \ge 0 \tag{2}$$

whenever Σ is nonsingular and $\partial^2 h / \partial x_i \partial x_j \ge 0$. Let $\phi(x) = \phi_\Sigma(x)$ be the mean zero normal density on R^n with covariance matrix Σ. Then

$$\frac{\partial \phi}{\partial \sigma_{ii}} = \frac{1}{2} \frac{\partial^2 \phi}{\partial x_i^2}, \quad \frac{\partial \phi}{\partial \sigma_{ij}} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \quad i \ne j. \tag{3}$$

See e.g. Plackett (1954). Using (3) and our assumptions on h which justify two integrations

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by parts we have

$$\frac{\partial \mathscr{H}(\Sigma)}{\partial \sigma_{ij}} = \int_{R^n} h(x) \frac{\partial \phi(x)}{\partial \sigma_{ij}}\, dx = \int_{R^n} \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \phi(x)\, dx \geq 0, \tag{4}$$

which completes the proof.

COROLLARY 1. (*Slepian*). $P_\Sigma\{\min_{1\le i\le n} X_i \geq C\}$ *and* $P_\Sigma\{\max_{1\le i\le n} X_i \leq C\}$ *are increasing in each* σ_{ij} *with* $i < j$.

PROOF. First apply Proposition 1 to functions h of the form $h(x_1, \cdots, x_n) = \prod_1^n f_i(x_i)$ where each f_i is a bounded nonnegative smooth increasing (decreasing) function. Then approximate the indicator function of the event $\{\min_{1\le i\le n} X_i \geq C\}$ ($\{\max_{1\le i\le n} X_i \leq C\}$) by such functions h.

COROLLARY 2. *Let* $h(x_1, \cdots, x_n) = f(x_1, \cdots, x_k) g(x_{k+1}, \cdots, x_n)$, *where* f *and* g *are both increasing or both decreasing. Then*

$$E_\Sigma f(X_1, \cdots, X_k) g(X_{k+1}, \cdots, X_n)$$

is increasing in σ_{ij} *for* $1 \le i \le k < j \le n$.

PROOF. If f and g are smooth then h is smooth and

$$\frac{\partial^2 h}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \geq 0 \quad \text{for } 1 \le i \le k < j \le n.$$

In this case the result follows from (2). The general case follows by approximations as in Pitt (1981).

COROLLARY 3. *Suppose* f *and* g *are as in Corollary* 2. *If* $\sigma_{ij} \geq 0$ *for* $1 \le i \le k < j \le n$, *then*

$$E_\Sigma f(X_1, \cdots, X_k) g(X_{k+1}, \cdots, X_n) \geq E_\Sigma f(X_1, \cdots, X_k) E_\Sigma g(X_{k+1}, \cdots, X_n).$$

The reverse inequality holds if $\sigma_{ij} \le 0$ *for* $1 \le i \le k < j \le n$.

PROOF. Define $\Gamma = (\gamma_{ij})$ by

$$\gamma_{ij} = \sigma_{ij} \quad \text{if } 1 \le i, j \le k \text{ or } k < i, j \le n$$
$$\gamma_{ij} = 0 \quad \text{otherwise.}$$

Corollary 3 then follows from Corollary 2 after observing that

$$E_\Gamma f(X_1, \cdots, X_k) g(X_{k+1}, \cdots, X_n) = E_\Sigma f(X_1, \cdots, X_k) E_\Sigma g(X_{k+1}, \cdots, X_n).$$

REMARK. The inequality in Corollary 3 when $\sigma_{ij} \le 0$ is due to Joag-dev and Proschan (1983).

COROLLARY 4. (Pitt). *Suppose that* $\sigma_{ij} \geq 0$ *for all* i *and* j. *Then* $X_1, \cdots, X_n$ *are associated, i.e.,*

$$E_\Sigma f(X_1, \cdots, X_n) g(X_1, \cdots, X_n) \geq E_\Sigma f(X_1, \cdots, X_n) E_\Sigma g(X_1, \cdots, X_n).$$

PROOF. Introduce dummy variables $X_{n+j} \equiv X_j$ for $j = 1, \cdots, n$, and apply Corollary 3.

III. Elliptically contoured distributions. Proposition 1 and Corollaries 1 and 2 remain valid for elliptically contoured distributions, as we now show. The original extension of Slepian's inequality to this case was given in Das Gupta, Eaton, Olkin, Perlman, Savage and Soble (1972).

Let $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ be the inner product in R^n, and let $\Sigma = (\sigma_{ij})$ be an invertible positive definite matrix. For the inverse matrix we will write $\Sigma^{-1} = (\sigma^{ij})$.

A probability density on R^n of the form

$$p_\Sigma(x) = |\Sigma|^{-1/2} p(\langle x, \Sigma^{-1}x\rangle)$$

is called eliptically contoured. Here $p(\lambda) \geq 0$ is defined on $[0, \infty)$ and is assumed to satisfy $\int_0^\infty \lambda^{n-1} p(\lambda^2)\, d\lambda < \infty$.

We write

$$\mathscr{H}(\Sigma) = \int_{R^n} h(x) p_\Sigma(x)\, dx,$$

and with this notation we will show that Proposition 1 remains valid.

It will suffice to establish (2) under the condition that $p(\lambda)$ is a bounded C^2 function with compact support. The proof is similar to that for normal variables in Proposition 1, but requires a substitute for the equation (4). This is supplied by Proposition 2.

For $\lambda \geq 0$ we set

$$F(\lambda) = \int_0^\lambda p(\xi)\, d\xi.$$

Let

$$F_\infty = \int_0^\infty p(\xi)\, d\xi,$$

and

$$G_\Sigma(x) = ½\, |\Sigma|^{-1/2} \{F_\infty - F(\langle x, \Sigma^{-1}x\rangle)\}.$$

PROPOSITION 2. *If $h(x)$ is smooth and $p(\lambda)$ is C^2 with compact support, then*

$$\frac{\partial}{\partial\sigma_{ij}} \int_{R^n} h(x) p_\Sigma(x)\, dx = \int_{R^n} \frac{\partial^2 h(x)}{\partial x_i \partial x_j} G_\Sigma(x)\, dx, \quad i \neq j. \tag{5}$$

PROOF. Recall the notation $\Sigma^{-1} = (\sigma^{ij})$. We will use the matrix identities

$$\begin{aligned}
&\frac{\partial}{\partial\sigma_{ii}} |\Sigma|^{-1/2} = -\frac{1}{2}\sigma^{ii} |\Sigma|^{-1/2},\\
&\frac{\partial}{\partial\sigma_{ij}} |\Sigma|^{-1/2} = -\sigma^{ij} |\Sigma|^{-1/2}, \quad i \neq j\\
&\frac{\partial}{\partial\sigma_{ii}} \langle x, \Sigma^{-1}x\rangle = (\textstyle\sum_{k=1}^{n} \sigma^{ik} x_k)^2,\\
&\frac{\partial}{\partial\sigma_{ij}} \langle x, \Sigma^{-1}x\rangle = -2(\textstyle\sum_{k=1}^{n} \sigma^{ik} x_k)(\sum_{l=1}^{n} \sigma^{jl} x_l), \quad i \neq j,
\end{aligned} \tag{6}$$

without further comment.

Calculating, we now have

$$\begin{aligned}
\frac{\partial p_\Sigma}{\partial\sigma_{ij}} &= -\sigma^{ij} p_\Sigma - 2|\Sigma|^{-1/2} p'(\langle x, \Sigma^{-1}x\rangle)(\textstyle\sum_{k=1}^{n} \sigma^{ik} x_k)(\sum_{l=1}^{n} \sigma^{jl} x_l)\\
&= -\sigma^{ij} p_\Sigma - (\textstyle\sum_{k=1}^{n} \sigma^{ik} x_k) \dfrac{\partial p_\Sigma}{\partial x_j}\\
&= -\dfrac{\partial}{\partial x_j} (\textstyle\sum_{k=1}^{n} \sigma^{ik} x_k) p_\Sigma.
\end{aligned} \tag{7}$$

Our assumptions on p justify integration by parts and we have

$$\frac{\partial}{\partial \sigma_{ij}} \int_{R^n} h(x) p_\Sigma(x)\, dx = \int_{R^n} \left(\textstyle\sum_{k=1}^n \sigma^{ik} x_k\right) \frac{\partial h(x)}{\partial x_j} p_\Sigma(x)\, dx$$

$$= \frac{1}{2|\Sigma|^{1/2}} \int_{R^n} \frac{\partial h(x)}{\partial x_j} \frac{\partial}{\partial x_i} F(\langle x, \Sigma^{-1} x\rangle)\, dx$$

$$= \int_{R^n} \frac{\partial^2 h(x)}{\partial x_i \partial x_j} G_\Sigma(x)\, dx,$$

which completes the proof of Proposition 2.

REMARKS. If $p_\Sigma = \phi_\Sigma$ is a normal density then $G_\Sigma = \phi_\Sigma$ so (5) is a generalization of (4). Since $G_\Sigma \geq 0$, (5) shows that Proposition 1 and Corollaries 1 and 2 remain valid for elliptically contoured distributions. Corollaries 3 and 4 do not directly extend to such distributions, however, since the normal family is the only elliptically contoured family in which zero correlations imply independence (Kelker, 1970). Aside from the normal family, an elliptically contoured distribution with all correlations nonnegative need not be associated. In fact, (c.f. Sampson, 1980), if all $\sigma_{ij} = 0$ for $i \neq j$, then except in the normal case, $P[X_i \leq a, X_j \leq b] - P[X_i \leq a]P[X_j \leq b]$ must assume both positive and negative values as (a, b) varies. Thus, by continuity, except in the normal case, an elliptically contoured density P_Σ with the $\sigma_{ij} > 0$ which is sufficiently close to being radical cannot be associated.

As was commented by the referee, an elementary derivation of (7) is possible which uses Fourier transforms but avoids matrix differentiation. In fact, the characteristic function $\hat{p}_\Sigma(\lambda)$ of p_Σ must have the form $\hat{p}_\Sigma(\lambda) = \psi(\langle \lambda, \Sigma\lambda\rangle)$. Thus for $i \neq j$ we have

$$\frac{\partial}{\partial \sigma_{ij}} \hat{p}_\Sigma(\lambda) = 2\lambda_i \lambda_j \psi'(\langle \lambda, \Sigma\lambda\rangle) = \lambda_j \left(\textstyle\sum_{k=1}^n \sigma^{ik} \frac{\partial}{\partial \lambda_k} \right) \hat{p}_\Sigma(\lambda).$$

Taking inverse Fourier transforms gives (7).

The calculations in the proof of Proposition 2 allow one extension which is perhaps worthwhile. Using (6) we can easily verify that

$$\frac{\partial}{\partial \sigma_{ii}} \int_{R^n} h(x) p_\Sigma(x)\, dx = \frac{1}{2} \int_{R^n} \frac{\partial^2 h(x)}{\partial x_i^2} G_\Sigma(x)\, dx. \tag{8}$$

Combining (5) and (8), it is elementary to prove the following.

PROPOSITION 3. *Let* Γ *and* Σ *be positive definite matrices and set* $A = \Sigma - \Gamma = (a_{ij})$. *Let* $\Sigma_t = \Gamma + tA$ *and let* $h(x)$ *be a smooth function satisfying*

$$\mathscr{A} h(x) = \textstyle\sum_{i,j=1}^n a_{ij} \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \geq 0. \tag{9}$$

Then

$$\int_{R^n} h(x) p_{\Sigma_t}(x)\, dx,$$

is an increasing function of t, $0 \leq t \leq 1$.

PROOF. Assuming, as before, that $p(\lambda)$ is a C^2 function with compact support, then (5) and (8) give

$$\frac{\partial}{\partial t} \int_{R^n} h(x) p_{\Sigma_t}(x)\, dx = \frac{1}{2} \int_{R^n} \mathscr{A} h(x) G_{\Sigma_t}(x)\, dx.$$

By (9) this is nonnegative.

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