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ASSOCIATIVE ALGEBRAS SATISFYING A SEMIGROUP IDENTITY

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Abstract. Denote by $(R, \cdot)$ the multiplicative semigroup of an associative algebra R over an infinite field, and let $(R, \circ)$ represent R when viewed as a semigroup via the circle operation $x \circ y = x + y + xy$. In this paper we characterize the existence of an identity in these semigroups in terms of the Lie structure of R. Namely, we prove that the following conditions on R are equivalent: the semigroup $(R, \circ)$ satisfies an identity; the semigroup $(R, \cdot)$ satisfies a reduced identity; and, the associated Lie algebra of R satisfies the Engel condition. When R is finitely generated these conditions are each equivalent to R being upper Lie nilpotent.

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1. Introduction and statement of results. A well-known result due to Levitzki [**3**] states that every finitely generated bounded nil ring is nilpotent. Not long ago, Zel'manov proved the Lie-theoretic analogue: every finitely generated Lie ring satisfying the bounded Engel condition is nilpotent [**19**]. The corresponding problem in the category of groups is the famous Burnside problem. The construction by Adian and Novikov of infinite finitely generated groups of finite exponent provided a negative solution to this problem. See [**1**].

The Burnside problem has some natural generalizations. For example, the problem of whether or not every Engel group is locally nilpotent remains open [**17**]. Because every nilpotent group is known to satisfy a semigroup identity [**5**,**8**], a weaker version of this problem has also been posed: does every Engel group satisfy a semigroup identity [**6**, Problem 2.82]? Even the following question remains open: can an Engel group contain a free (noncommutative) subsemigroup? See [**10**].

Recently, the present authors settled the ring-theoretic analogues of these problems.

Recall that a ring R satisfies the *Engel identity* of degree n if and only if

$$e_n := [x, \underbrace{y, y, \dots, y}_{n}]$$

is identically zero in R; whereas, R is said to be *upper Lie nilpotent* if the descending central series of associative ideals $\{\gamma^i(R)\}$ in R defined by $\gamma^1(R) = R$, $\gamma^{i+1}(R) = \langle[\gamma^i(R), R]\rangle$ reaches zero in finitely many steps. In addition to the usual multiplicative semigroup, $(R, \cdot)$, R forms a semigroup, denoted by $(R, \circ)$, under the circle operation $x \circ y = x + y + xy$. We proved in [**14**] that every finitely generated associative ring R satisfying the Engel condition is upper Lie nilpotent. From this result we were able to infer that whenever R satisfies an Engel identity then both the associated circle and multiplicative semigroups of R must satisfy a so-called Morse identity.

Define sequences f_n and g_n by $f_1(x, y) = xy$, $g_1(x, y) = yx$, and

$$f_{n+1}(x, y, x_3, \ldots, x_{n+2}) = f_n(x, y, x_3, \ldots, x_{n+1})x_{n+2}g_n(x, y, x_3, \ldots, x_{n+1}),$$

$$g_{n+1}(x, y, x_3, \ldots, x_{n+2}) = g_n(x, y, x_3, \ldots, x_{n+1})x_{n+2}f_n(x, y, x_3, \ldots, x_{n+1}),$$

for all $n \geq 1$. The nth *Mal'cev identity* **[5]** is the semigroup identity

$$f_n(x, y, x_3, \ldots, x_{n+1}) = g_n(x, y, x_3, \ldots, x_{n+1}),$$

while the nth *Morse identity* $u_n(x, y) = v_n(x, y)$ **[7]** is the nth Mal'cev identity with $x_3 = \cdots = x_{n+1} = 1$.

Consequently, neither $(R, \cdot)$ nor $(R, \circ)$ can contain a free subsemigroup if R satisfies an Engel identity.

The problem of characterizing finitely generated groups satisfying an arbitrary semigroup identity has been studied by several authors (see, for example, **[4]**, **[18]** and **[16]**). Because this class of groups contains the Burnside groups, this problem is highly nontrivial, especially in the light of the recent construction by Ol'shanskii and Storozhev of a 2-generated group satisfying a semigroup identity that is not a periodic extension of a locally soluble group **[9]**.

Algebras over fields of characteristic zero which satisfy a circle semigroup law, and a more general semigroup condition called *collapsibility*, were studied previously by the first author in **[12]**. In sharp contrast to the combinatorial methods employed in this paper, the techniques used in **[12]** rely heavily on deep structure theorems from both group and ring theory. In this article we study associative algebras that satisfy an arbitrary semigroup identity. In fact, we obtain a partial converse to our result in **[14]**.

Throughout the remainder of this paper, K will denote an infinite commutative domain and R an associative K-algebra on which the action of K is torsion-free; (this occurs, for example, when K is an infinite field). All identical relations in algebraic objects will be assumed to be nontrivial unless otherwise stated. A semigroup S satisfies an identity if and only if there are distinct words u, v in the free semigroup on

$$X = \{x = x_1, y = x_2, x_3, x_4, \ldots\}$$

so that $u = v$ in S. The semigroup identity is *left reduced* if the first letters of u and v are different, *right reduced* if the last letters of u and v are different and simply *reduced* if it is both left and right reduced. In other words, $u = v$ is reduced if and only if uv^{-1} and $v^{-1}u$ are reduced words in the free group on X. If $(R, \cdot)$ (respectively $(R, \circ)$) satisfies an identity we often say that R satisfies a *semigroup identity* (respectively, a *circle semigroup identity*). Clearly each of these corresponds to a polynomial identity in R. A generalization of a multiplicative semigroup identity in R is a *binomial identity*, a polynomial identity of the form $\alpha_1 u_1 + \alpha_2 u_2 = 0$, where u_1, u_2 are monomials and $\alpha_1, \alpha_2 \in K$. The various types of reduced binomial identities are defined in the obvious way.

Tasić and the first author proved in [**13**] that R is Lie nilpotent of class at most n if and only if $(R, \circ)$ satisfies the nth Mal'cev identity. The main result in the present article further demonstrates the close relationship between the Lie structure of R and semigroup properties of R.

THEOREM 1.1. *Let R be a K-algebra. Then the following statements are equivalent.*

(i) *R satisfies a circle semigroup identity;*
(ii) *R satisfies a reduced semigroup identity;*
(iii) *R satisfies a reduced binomial identity;*
(iv) *R satisfies an identity of the form $\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0$, $\alpha_i \in K, \alpha_0 \neq 0, \alpha_n \neq 0$;*
(v) *R satisfies an Engel identity;*
(vi) *$(R, \circ)$ satisfies a Morse identity.*

Furthermore, for any two conditions A, B from (i)–(vi), our proof gives (sometimes theoretical) bounds for the degree of the identity in B in terms of the degree of the identity in A. In particular, these bounds do not depend on R, K or the characteristic of K. Notice, too, that since every finite semigroup (in particular $(R, \circ)$, where R is a finite ring) satisfies an identity, some hypothesis on the coefficient ring K is required. The following example demonstrates that the distinction between reduced and arbitrary multiplicative semigroup identities is also necessary.

EXAMPLE 1.2. Let R be the subalgebra of the matrix algebra $M_2(K)$ spanned by the matrix units e_{11} and e_{12}. Then $[R, R] \subseteq Ke_{12}$, and so R satisfies the semigroup identity $[x, y]z = xyz - yxz = 0$. R does not satisfy any Engel identity, since $[e_{11}, e_{12}] = e_{12}$. Thus, by Theorem 1.1, R does not satisfy any reduced semigroup identity, nor any circle semigroup identity.

THEOREM 1.3. *Let R be a K-algebra, where $K = p > 0$. Then the following statements are equivalent.*

(i) *R satisfies a semigroup identity;*
(ii) *R satisfies a binomial identity;*
(iii) *R satisfies an identity of the form $\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0$, $\alpha_i \in K$;*
(iv) *R satisfies an identity of the form $y^m e_m y^m = 0$.*

We remark that the characteristic zero analogue of Theorem 1.3. is stated in [**2**]; however, their result corresponding to our implication (iv) $\Rightarrow$ (i) is not proved and does not seem obvious to the present authors.

The fact that R is non-unital is essential to Example 1.2, as indicated by the following proposition.

PROPOSITION 1.4. *Let R be a unital K-algebra. If R satisfies a semigroup identity then R satisfies the corresponding reduced semigroup identity.*

THEOREM 1.5. *There exists a function f, depending only on natural numbers d and n, such that if a K-algebra R satisfies a circle semigroup identity of degree n and R is generated over K by d elements then R is upper Lie nilpotent of index at most $f(d, n)$.*

2. Semigroup identities. Our hypotheses on K were chosen to imply, by the usual Vandermonde determinant argument, that every homogeneous component of a polynomial identity for R is also a polynomial identity for R (see [**11**, 6.4.14]). We shall use this key fact freely, without explicit mention.

By a *partial linear identity* we shall mean an identity of the form

$$\sum_{i=0}^{n} \alpha_i y^i x y^{n-i} = 0,$$

with $\alpha_i \in K$. Such an identity will be called *left reduced* if $\alpha_0 \neq 0$, *right reduced* if $\alpha_n \neq 0$ and *reduced* if it is both left and right reduced.

Proposition 2.1. *Let R be a K-algebra.*

(i) *If a semigroup S satisfies an identity in $x, y, x_3, \ldots$ that is left reduced, right reduced or reduced, then S satisfies an identity, of the same type, in x and y only.*

(ii) *If R satisfies a binomial identity then R is bounded nil or R satisfies a semigroup identity.*

(iii) *If R satisfies a binomial identity that is left reduced, right reduced or reduced, then R satisfies a partial linear identity of the same type.*

(iv) *If R satisfies the identity $y^n = 0$, then R satisfies $e_{2n-1} = 0$.*

Proof. Suppose without loss of generality that our left reduced identity has the form

$$x x_{i_1} \cdots x_{i_m} = y x_{j_1} \cdots x_{j_n}.$$

Recall that we identify $x = x_1$ and $y = x_2$. Substituting $x_i = xy^i$, $(i \geq 3)$, we obtain a left reduced identity in x and y only. If the original identity were right reduced as well, then $x_{i_m} \neq x_{j_n}$. Thus, by an appropriate permutation of the variables, we obtain an equivalent identity of the form

$$x_{k_1} \cdots x_{k_m} x = x_{l_1} \cdots x_{l_n} y.$$

Substituting $x_i = xy^i$, $(i \geq 3)$, into this identity and then concatenating on the right with the 2-variable left reduced identity yields the 2-variable reduced identity:

$$x x_{i_1} \cdots x_{i_m} x_{k_1} \cdots x_{k_m} x = y x_{j_1} \cdots x_{j_n} x_{l_1} \cdots x_{l_n} y.$$

This and symmetry prove (i).

Next, given a binomial identity $\alpha_1 u_1 + \alpha_2 u_2 = 0$ holding in R, set all variables equal, to y say. If the identity is not homogeneous, then separating components shows that R is bounded nil. On the other hand, if it is homogeneous then $(\alpha_1 + \alpha_2)y^n = 0$, for some n, so that either R is bounded nil or $\alpha_1 = -\alpha_2$, in which case $u_1 - u_2 = 0$ holds in R. This proves (ii).

In order to prove (iii), suppose that R satisfies a given binomial identity. Observe from (ii) that either R is bounded nil, in which case R satisfies a partial linear identity by (1) in the proof of (iv) below, or R satisfies a semigroup identity,

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which by (i) can be taken to be of the form $u(x,y) - v(x,y) = 0$. Thus we may assume that R is not bounded nil, and hence that the semigroup identity is homogeneous. We assert that the homogeneous component of degree 1 in x of the identity $u(x+y,y) - v(x+y,y) = 0$ is a partial linear identity. To see why it is nontrivial, write $u = au'$, $v = av'$, where a has length m and $u' = v'$ is a left reduced equation. If (as we may assume without loss of generality) u' starts with x and v' with y, then in the expansion of $u(x+y,y)$ there is precisely one monomial starting with $y^m x$, whereas no monomial in the expansion of $v(x+y,y)$ begins with $y^m x$. This, and symmetry, yields (iii).

To prove the well-known fact (iv), let l, r denote respectively the K-linear operators of left and right multiplication by y. Then, since l and r commute, we obtain

$$e_m = (r-l)^m(x) = \sum_{i=0}^{m}(-1)^i\binom{m}{i}l^i r^{m-i}(x) = \sum_{i=0}^{m}(-1)^i\binom{m}{i}y^i x y^{m-i}. \qquad (1)$$

Thus if $m = 2n-1$ and R satisfies $y^n = 0$, then every term in the sum on the right is zero.

Proposition 1.4. is a consequence of the following result.

Proposition 2.2. *Let R be a K-algebra.*

(i) *If $(R, \circ)$ satisfies a semigroup identity, then $(R, \cdot)$ satisfies the same identity.*

(ii) *If $(R, \circ)$ satisfies a semigroup identity then $(R, \circ)$ satisfies the corresponding reduced identity.*

(iii) *If R is unital, then $(R, \circ) \cong (R, \cdot)$.*

Proof. Let S be the unital hull of R; that is, $S = R$ if R is unital and $S = K1 \oplus R$ if R is nonunital. The map $\iota\colon r \mapsto 1 + r$ is an injective semigroup map from $(R, \circ)$ into $(S, \cdot)$ that is onto if (and only if) $R = S$. This proves (iii). The image under ι of an identity in $(R, \circ)$ is an identity in $(1 + R, \cdot) \subseteq (S, \cdot)$. Only the bottom degree homogeneous component of this identity involves 1 and the other homogeneous components yield identities in $(R, \cdot)$. The highest degree component is precisely the original identity, yielding (i).

Assume that $u(x,y) = v(x,y)$ is an identity for $(R, \circ)$ of degree n. Write $u = au'b$, $v = av'b$, where $u' = v'$ is a reduced equation. We show that $u' = v'$ also holds in $(R, \circ)$. It suffices, by symmetry and by induction on the maximum length of a and b, to prove this in the case when $a = x$ and b is empty. The identity $xu'(x,y) = xv'(x,y)$ in $(R, \circ)$ is equivalent to the polynomial identity

$$(1+x)u'(1+x, 1+y) - (1+x)v'(1+x, 1+y) = 0$$

in R. Let m be an even integer with $m \geq n+1$. Then multiplying the last identity on the left by $1 - x + x^2 - \cdots + x^m$ yields the polynomial identity

$$(1+x^{m+1})u'(1+x, 1+y) - (1+x^{m+1})v'(1+x, 1+y) = 0.$$

Separating homogeneous components and using the fact that x^{m+1} has higher x-degree than u' and v', we obtain the polynomial identity

$$u'(1+x, 1+y) - v'(1+x, 1+y) = 0$$

in R, which is equivalent to $u' = v'$ holding in $(R, \circ)$. This proves (ii). □

The following lemma is crucial to our main theorems and is best possible in view of Example 1.2. A simpler argument, as in [**2**], is available in characteristic zero. That argument fails in positive characteristic, where the situation is more delicate.

LEMMA 2.3. *Suppose that R satisfies $y^m \alpha y^k = 0$, where $\alpha = \sum_{i=0}^{n} \alpha_i y^i x y^{n-i}$.*

(i) *If α is right reduced, then R satisfies $y^{m+n} e_n y^k = 0$.*
(ii) *If α is left reduced, then R satisfies $y^m e_n y^{n+k} = 0$.*
(iii) *If α is reduced, then R satisfies $y^m e_{3n-1} y^k = 0$.*

Proof. By symmetry, the proof of (ii) is entirely analogous to that of (i). If the conclusions of (i) and (ii) hold, then the conclusion of (iii) follows from equation (1):

$$y^m e_{3n-1} y^k = y^m \sum_{i=0}^{2n-1} (-1)^i \binom{2n-1}{i} y^i e_n y^{2n-1-i} y^k = 0.$$

Thus it suffices to prove the conclusion of (i).

First assume that $m = k = 0$. Make the substitution $y \mapsto y(y+1)$. Expanding $\sum_{i=0}^{n} \alpha_i y^i (y+1)^i x y^{n-i} (y+1)^{n-i} = 0$ by the binomial theorem and separating homogeneous components yields identities $v_0 = 0, \ldots, v_n = 0$ for R, where v_r is homogeneous of degree $n + r$ in y. We claim that

$$\sum_{r=0}^{n} (-1)^r v_r y^{n-r} = \alpha_n y^n e_n. \tag{2}$$

To establish equation (2), it suffices to show that the coefficients of $y^a x y^{2n-a}$ on each side are equal, whenever $0 \le a \le 2n$.

First note that by equation (1), the coefficient of $y^a x y^{2n-a}$ in $y^n e_n$ is $(-1)^{a-n}\binom{n}{a-n}$ if $a \ge n$ and 0 otherwise. With the usual convention on binomial coefficients, the expression $(-1)^{a-n}\binom{n}{a-n}$ is valid for all a. Using the same convention we may sum over all values of any index occurring.

Now we calculate the coefficient of $y^a x y^{2n-a}$ in $v_r y^{n-r}$ or, what is the same, the coefficient of $y^a x y^{n+r-a}$ in v_r. The binomial theorem expansion above shows that the coefficient of $y^s x y^t$ is precisely $\sum_{i+j=n} \alpha_i \binom{i}{s-i}\binom{j}{t-j}$. Putting $s = a$ and $t = r + n - a$, we obtain the desired coefficient as $\sum_i \alpha_i \binom{i}{a-i}\binom{n-i}{r-(a-i)}$.

It follows that

$$\begin{aligned}
&\sum_r(-1)^r\sum_i\alpha_i\binom{i}{a-i}\binom{n-i}{r-(a-i)}\\
&=\sum_i\alpha_i\binom{i}{a-i}\sum_r(-1)^r\binom{n-i}{r-(a-i)}\\
&=\sum_i\alpha_i\binom{i}{a-i}(-1)^{a-i}\sum_s(-1)^s\binom{n-i}{s}\\
&=(-1)^{a-n}\binom{n}{a-n}\alpha_n,
\end{aligned}$$

since the inner sum has the value zero, unless $n-i=0$, and 1 otherwise. This proves (i) in the case $m=k=0$.

In the general case, where m and k are not necessarily zero, the substitution $y\mapsto y(y+1)$ into the original identity yields an identity

$$\sum_{r+s+t\leq m+n+k} c_{rst}y^s(y^m v_r y^k)y^t=0, \tag{3}$$

for some coefficients $c_{rst}\in K$. For $0\leq a\leq n$, consider the homogeneous component of (3) of degree $m+n+k+a$ in y. The only v_r occurring have $r\leq a$ and the only term involving v_a is precisely $y^m v_a y^k$. By induction on a, $y^m v_r y^k=0$ is an identity in R for all $r<a$, and hence so is $y^m v_a y^k=0$. We may now proceed exactly as in the special case above and the conclusion follows. □

2.1. Unital algebras. In case R is unital, more information can be obtained. Note that $e_n(x,y)=x(\mathrm{ad}y)^n=x(\mathrm{ad}(y+1))^n=e_n(x,y+1)$. Thus by substituting $y\mapsto y+1$ into the result of (i) or (ii) in Lemma 2.3. and separating out the component of degree n in y we obtain $e_n=0$ in R.

In the rest of this subsection (which is not essential to the main results of the paper) we give a characterization (for unital K-algebras) of the Engel identities.

For each $m\geq 0$, let W_m be the K-submodule of $K\langle x,y\rangle$ with basis all monomials y^ixy^j such that $i+j=m$, and let $V_n=\sum_{m=0}^n W_m$ and $V=\sum_{n\geq 0}V_n$. Note that W_0 is spanned by the monomial x, and that, for $n\geq 1$, e_n is a reduced element of W_n.

Define the difference operator Δ on V by $\Delta\alpha(x,y)=\alpha(x,y+1)-\alpha(x,y)$. Note that $\Delta\colon V_n\to V_{n-1}$, and that the homogeneous component of degree $n-1$ in y of $\Delta\alpha$ is simply the Hausdorff derivative $\partial\alpha/\partial y$ with respect to y (that is, the unique K-derivation of $K\langle x,y\rangle$ sending y to 1 and x to 0).

PROPOSITION 2.4. *Let R be a unital K-algebra, and $\alpha\in W_n$.*

(i) $\Delta\alpha=0$ *if and only if α is a scalar multiple of e_n.*
(ii) *If* $\mathrm{char}\,K=0$*, then $\partial\alpha/\partial y=0$ if and only if α is a scalar multiple of e_n.*

Proof. Given $\alpha(x,y)=\sum_{i=0}^n\alpha_iy^ixy^{n-i}$, expand $\alpha(x,y+1)$ by the binomial theorem. The coefficient of y^sxy^t in $\alpha(x,y+1)$ is given by

$$[y^s x y^t] = \begin{cases} \sum_i \alpha_i \binom{i}{s}\binom{n-i}{t} & (s+t<n), \\ 0 & (s+t=n). \end{cases} \tag{4}$$

Now $\Delta\alpha = 0$ if and only if the coefficients of all monomials $y^i x y^j$, for $i+j \le n-1$, are zero. This gives a system of linear equations in the $n+1$ unknowns $\alpha_0, \dots, \alpha_n$. We claim that the coefficient matrix M has rank exactly n. Indeed, by equation (4), the submatrix of rows corresponding to the components of xy^m, $0 \le m \le n-1$, has the form

$$\begin{bmatrix} * & 1 & 0 & 0 & \cdots & 0 \\ * & * & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ * & * & \cdots & * & 1 & 0 \\ * & * & * & \cdots & * & 1 \end{bmatrix},$$

which shows that the rank is at least n. However the rank is not $n+1$ since, as observed above, the coefficient vector $\alpha_i = (-1)^i\binom{n}{i}$ of e_n is in the kernel of M. This proves (i).

To prove (ii), it suffices to show that in characteristic zero, the submatrix of M consisting of all rows corresponding to monomials $y^s x y^t$, with $s+t = n-1$, has rank n. By equation (4), this submatrix has the form

$$\begin{bmatrix} n & 1 & 0 & 0 & \cdots & 0 \\ 0 & n-1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & n-2 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 & n \end{bmatrix}.$$

Since char $K = 0$ the submatrix consisting of the first n columns is nonsingular and (ii) follows.

3. Proofs of theorems. We first prove Theorem 1.1. The implication (ii) $\Rightarrow$ (iii) is obvious. Also (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) follow from Proposition 2.2. and Proposition 2.1, respectively. By Lemma 2.3 with $m = k = 0$, (iv) and (v) are equivalent. Suppose then that R satisfies $e_n = 0$. Let $x, y \in R$. By [**14**], the subalgebra T of R generated by x and y is Lie nilpotent of class m depending on n only. An easy induction on m shows that T, and hence R, satisfies the Morse identity, in the circle sense, of degree m. Indeed,

$$u_m - v_m = [u_1, v_1, v_2, \dots, v_{m-1}].$$

This proves (v) $\Rightarrow$ (vi). The last implication (vi) $\Rightarrow$(i) is obvious.

We now prove Theorem 1.3. The implication (i) $\Rightarrow$ (ii) is obvious; (ii) $\Rightarrow$ (iii) follows from Proposition 2.1, and (iii) $\Rightarrow$ (iv) can be deduced from Lemma 2.3. If $K = p > 0$ then (iv) $\Rightarrow$ (i), since by increasing m if necessary we may assume that $m = p^t$, so that

$$y^{p^t}xy^{2p^t} - y^{2p^t}xy^{p^t} = y^{p^t}e_{p^t}y^{p^t} = 0.$$

Finally, Theorem 1.5 follows from the quantitative form of Theorem 1.1 and [**14**, Theorem].

4. Comments. In an earlier version of this paper, we asked the following questions about an arbitrary ring R. These questions arose naturally from the work above, and the converses had been shown to hold in [**14**].

- If a ring R satisfies a reduced semigroup identity, does R necessarily satisfy an Engel identity?
- If a ring R satisfies a reduced circle semigroup identity, does R necessarily satisfy an Engel identity?

We are indebted to Ol'ga Paison for showing us that the answer to both is no. We now present her example.

Let p be a prime, let F be a field of order p^2 and let R be the subring of $M_2(F)$ consisting of all elements of the form $ae_{11} + a^p e_{22} + be_{12}$, for $a, b \in F$. Then R does not satisfy any Engel identity. To see this, choose $a \in F$ with $a^p \neq a$. Let $x = ae_{11} + a^p e_{22}$ and $y = e_{12}$. Then, for sufficiently large even integers s, we have $[x^{p^s}, y] = (a - a^p)e_{12} \neq 0$. On the other hand, the only idempotents of R are 0 and 1, and so R satisfies a reduced (circle) semigroup identity in view of the following result.

Proposition 4.1. *Let R be a finite ring. Then*

(i) $(R, \cdot)$ *and* $(R, \circ)$ *satisfy an identity of the form* $x^t \equiv x^{2t}$.
(ii) *If all idempotents of R are central, then R satisfies a reduced semigroup identity.*

Proof. The conclusion of part (i) is true for every finite semigroup S. First, every element of S is periodic. Furthermore, every periodic element in a semigroup has some power which is an idempotent. To see this, note that for a fixed $x \in S$, $x^m = x^{m+a}$, for some $m, a > 0$. This implies that, for all $n \geq 1$ and all $s \geq m$, $x^s = x^{s+na}$. Choose t_x such that $t_x \geq m$ and a divides t_x. Then $(x^{t_x})^2 = x^{t_x}$. The desired global identity follows directly from this equation, since S satisfies $x^t \equiv x^{2t}$ with $t = \prod_{x \in S} t_x$.

Now by (i), there is some t for which x^t is an idempotent, for each $x \in R$. Thus if all idempotents of R are central, R satisfies the identity $x^t y \equiv yx^t$, yielding (ii).

In [**2**] it was shown (using arguments special to characteristic zero) that the K-algebra R satisfies a partial linear identity if and only if the algebra of 2×2 upper triangular matrices over K is not in the variety generated by R. Perhaps this is true in all characteristics.

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