Electronic Journal of Probability

Electron. J. Probab. **18** (2013), no. 38, 1–24.
ISSN: 1083-6489 DOI: 10.1214/EJP.v18-2277

Assortativity and clustering of sparse random intersection graphs

Mindaugas Bloznelis* Jerzy Jaworski† Valentas Kurauskas*

Abstract

We consider sparse random intersection graphs with the property that the clustering coefficient does not vanish as the number of nodes tends to infinity. We find explicit asymptotic expressions for the correlation coefficient of degrees of adjacent nodes (called the assortativity coefficient), the expected number of common neighbours of adjacent nodes, and the expected degree of a neighbour of a node of a given degree k. These expressions are written in terms of the asymptotic degree distribution and, alternatively, in terms of the parameters defining the underlying random graph model.

Keywords: assortativity; clustering; power law; random graph; random intersection graph.
AMS MSC 2010: 05C80; 05C82; 91D30.
Submitted to EJP on August 27, 2012, final version accepted on March 8, 2013.

1 Introduction

Assortativity and clustering coefficients are commonly used characteristics describing statistical dependency of adjacency relations in real networks ([18], [2], [20]). The assortativity coefficient of a simple graph is the Pearson correlation coefficient between degrees of the endpoints of a randomly chosen edge. The clustering coefficient is the conditional probability that three randomly chosen vertices make up a triangle, given that the first two are neighbours of the third one.

It is known that many real networks have non-negligible assortativity and clustering coefficients, and a social network typically has a positive assortativity coefficient ([18], [21]). Furthermore, Newman et al. [21] remark that the clustering property (the property that the clustering coefficient attains a non-negligible value) of some social networks could be explained by the presence of a bipartite graph structure. For example, in the actor network two actors are adjacent whenever they have acted in the same film. Similarly, in the collaboration network authors are declared adjacent whenever they have coauthored a paper. These networks exploit the underlying bipartite graph structure: actors are linked to films, and authors to papers. Such networks are sometimes called affiliation networks.

*Faculty of Mathematics and Informatics, Vilnius University, 03225 Vilnius, Lithuania.
†Faculty of Mathematics and Computer Science, Adam Mickiewicz University, 60-614 Poznań, Poland

In this paper we study assortativity coefficient and its relation to the clustering coefficient in a theoretical model of an affiliation network, the so called random intersection graph. In a random intersection graph nodes are prescribed attributes and two nodes are declared adjacent whenever they share a certain number of attributes ([11], [15], see also [1], [13]). An attractive property of random intersection graphs is that they include power law degree distributions and have tunable clustering coefficient see [5], [6], [8], [12]. In the present paper we show that the assortativity coefficient of a random intersection graph is non-negative. It is positive in the case where the vertex degree distribution has a finite third moment and the clustering coefficient is positive. In this case we show explicit asymptotic expressions for the assortativity coefficient in terms of moments of the degree distribution as well as in terms of the parameters defining the random graph. Furthermore, we evaluate the average degree of a neighbour of a vertex of degree k, $k = 1, 2, \dots$, (called neighbour connectivity, see [16], [23]), and express it in terms of a related clustering characteristic, see (1.3) below.

Let us rigorously define the network characteristics studied in this paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite graph on the vertex set $\mathcal{V}$ and with the edge set $\mathcal{E}$. The number of neighbours of a vertex v is denoted $d(v)$. The number of common neighbours of vertices v_i and v_j is denoted $d(v_i, v_j)$. We are interested in the correlation between degrees $d(v_i)$ and $d(v_j)$ and the average value of $d(v_i, v_j)$ for adjacent pairs $v_i \sim v_j$ (here and below '$\sim$' denotes the adjacency relation of $\mathcal{G}$). We are also interested in the average values of $d(v_i)$ and $d(v_i, v_j)$ under the additional condition that the vertex v_j has degree $d(v_j) = k$.

In order to rigorously define the averaging operation we introduce the random pair of vertices (v_1^*, v_2^*) drawn uniformly at random from the set of ordered pairs of distinct vertices. By $\mathbf{E} f(v_1^*, v_2^*) = \frac{1}{N(N-1)} \sum_{i \neq j} f(v_i, v_j)$ we denote the average value of measurements $f(v_i, v_j)$ evaluated at each ordered pair (v_i, v_j), $i \neq j$. Here $N = |\mathcal{V}|$ denotes the total number of vertices. By $\mathbf{E}^* f(v_1^*, v_2^*) = p_{e*}^{-1} \mathbf{E} \left(f(v_1^*, v_2^*) \mathbb{I}_{\{v_1^* \sim v_2^*\}} \right)$ we denote the average value over ordered pairs of adjacent vertices. Here $p_{e*} = \mathbf{P}(v_1^* \sim v_2^*)$ denotes the edge probability and $\mathbb{I}_{\{v_i \sim v_j\}} = 1$, for $v_i \sim v_j$, and 0 otherwise. Furthermore, $\mathbf{E}^{*k} f(v_1^*, v_2^*) = p_{k*}^{-1} \mathbf{E} \left(f(v_1^*, v_2^*) \mathbb{I}_{\{v_1^* \sim v_2^*\}} \mathbb{I}_{\{d(v_2^*) = k\}} \right)$, denotes the average value over ordered pairs of adjacent vertices, where the second vertex is of degree k. Here $p_{k*} = \mathbf{P}(v_1^* \sim v_2^*,\, d(v_2^*) = k)$.

The average values of $d(v_i) d(v_j)$ and $d(v_i, v_j)$ on adjacent pairs $v_i \sim v_j$ are now defined as follows

$$g(\mathcal{G}) = \mathbf{E}^* d(v_1^*) d(v_2^*), \qquad h(\mathcal{G}) = \mathbf{E}^* d(v_1^*, v_2^*), \qquad h_k(\mathcal{G}) = \mathbf{E}^{*k} d(v_1^*, v_2^*).$$

We also define the average values

$$b(\mathcal{G}) = \mathbf{E}^* d(v_1^*), \qquad b'(\mathcal{G}) = \mathbf{E}^* d^2(v_1^*), \qquad b_k(\mathcal{G}) = \mathbf{E}^{*k} d(v_1^*)$$

and the correlation coefficient

$$r(\mathcal{G}) = \frac{g(\mathcal{G}) - b^2(\mathcal{G})}{b'(\mathcal{G}) - b^2(\mathcal{G})},$$

called the assortativity coefficient of $\mathcal{G}$, see [18], [19].

In the present paper we assume that our graph is an instance of a random graph. We consider two random intersection graph models: active intersection graph and passive intersection graph introduced in [10] (we refer to Sections 2 and 3 below for a detailed description). Let G denote an instance of a random intersection graph on N vertices. Here and below the number of vertices is non random. An argument bearing on the law of large numbers suggests that, for large N, we may approximate the characteristics $b(G)$, $b'(G)$, $b_k(G)$, $g(G)$, $h(G)$ and $h_k(G)$ defined for a given instance G, by the corresponding conditional expectations

$$b = \mathbf{E}^* d(v_1^*), \qquad b' = \mathbf{E}^* d^2(v_1^*), \qquad b_k = \mathbf{E}^{*k} d(v_1^*), \tag{1.1}$$
$$g = \mathbf{E}^* d(v_1^*) d(v_2^*), \qquad h = \mathbf{E}^* d(v_1^*, v_2^*), \qquad h_k = \mathbf{E}^{*k} d(v_1^*, v_2^*),$$

where now the expected values are taken with respect to the random instance G and the random pair (v_1^*, v_2^*). We assume that (v_1^*, v_2^*) is independent of G. Similarly, we may approximate $r(G)$ by $r = \frac{g-b^2}{b'-b^2}$.

The main results of this paper are explicit asymptotic expressions as $N \to +\infty$ for the correlation coefficient r, the neighbour connectivity b_k, and expected number of common neighbours h_k defined in (1.1). As a corollary we obtain that the random intersection graphs have tunable assortativity coefficient $r \geq 0$. Another interesting property is expressed by the identity

$$b_k - h_k = b - h + o(1) \qquad \text{as} \qquad N \to +\infty \tag{1.2}$$

saying that the average value of the difference $d(v_i) - d(v_i, v_j)$ of adjacent vertices $v_i \sim v_j$ is not sensitive to the conditioning on the neighbour degree $d(v_j) = k$. That is, a neigbour v_j of v_i may affect the average degree $d(v_i)$ only by increasing/decreasing the average number of common neighbours $d(v_i, v_j)$. It is relevant to mention that $h_k = (k-1)\alpha^{[k]}$, where $\alpha^{[k]} = \mathbf{P}(v_1^* \sim v_2^* | v_1^* \sim v_3^*, v_2^* \sim v_3^*, d(v_3^*) = k)$ measures the probability of an edge between two neighbours of a vertex of degree k. In particular, we have

$$b_k = (k-1)\alpha^{[k]} + b - h + o(1) \qquad \text{as} \qquad N \to +\infty. \tag{1.3}$$

The remaining part of the paper is organized as follows. In Section 2 we introduce the active random graph and present results for this model. The passive model is considered in Section 3. Section 4 contains proofs.

2 Active intersection graph

Let $s > 0$. Vertices $v_1, \dots, v_n$ of an active intersection graph are represented by subsets $D_1, \dots, D_n$ of a given ground set $W = \{w_1, \dots, w_m\}$. Elements of W are called attributes or keys. Vertices v_i and v_j are declared adjacent if they share at least s common attributes, i.e., we have $|D_i \cap D_j| \geq s$.

In the *active random intersection graph* $G_s(n, m, P)$ every vertex $v_i \in V = \{v_1, \dots, v_n\}$ selects its attribute set D_i independently at random ([11]) and all attributes have equal chances to belong to D_i, for each $i = 1, \dots, n$. We assume, in addition, that independent random sets $D_1, \dots, D_n$ have the same probability distribution such that

$$\mathbf{P}(D_i = A) = \binom{m}{|A|}^{-1} P(|A|), \tag{2.1}$$

for each $A \subset W$, where P is the common probability distribution of the sizes $X_i = |D_i|$, $1 \leq i \leq n$ of selected sets. We remark that X_i, $1 \leq i \leq n$ are independent random variables.

We are interested in the asymptotics of the assortativity coefficient r and moments (1.1) in the case where $G_s(n, m, P)$ is sparse and n, m are large. We address this question by considering a sequence of random graphs $\{G_s(n, m, P)\}_n$, where the integer s is fixed and where $m = m_n$ and $P = P_n$ depend on n. We remark that subsets of W of size s plays a special role, we call them joints: two vertices are adjacent if their attribute sets share at least one joint. Our conditions on P are formulated in terms of the number of joints $\binom{X_i}{s}$ available to the typical vertex v_i. We denote $a_k = \mathbf{E}\binom{X_1}{s}^k$. It is convenient to assume that as $n \to \infty$ the rescaled number of joints $Z_1 = \binom{m}{s}^{-1/2} n^{1/2} \binom{X_1}{s}$ converges in distribution. We also introduce the k-th moment condition *(i)* Z_1 *converges in distribution to some random variable* Z*;*

(ii-k) $0 < \mathbf{E}Z^k < \infty$ *and* $\lim_{n\to\infty} \mathbf{E}Z_1^k = \mathbf{E}Z^k$.

We remark that the distribution of Z, denoted P_Z, determines the asymptotic degree distribution of the sequence $\{G_s(n,m,P)\}_n$ (see [5], [6], [8], [25]). We have, under conditions (i), (ii-1) that

$$\lim_{m\to\infty} \mathbf{P}\left(d(v_1)=k\right) = p_k, \qquad p_k = (k!)^{-1}\mathbf{E}\left((z_1 Z)^k e^{-z_1 Z}\right), \qquad k = 0,1,\dots. \tag{2.2}$$

Here we denote $z_k = \mathbf{E}Z^k$. Let d_* be a random variable with the probability distribution $\mathbf{P}(d_* = k) = p_k$, $k = 0,1,\dots$. We call d_* the asymptotic degree. It follows from (2.2) that the asymptotic degree distribution is a Poisson mixture, i.e., the Poisson distribution with a random (intensity) parameter $z_1 Z$. For example, in the case where P_Z is degenerate, i.e., $\mathbf{P}(Z = z_1) = 1$, we obtain the Poisson asymptotic degree distribution. Furthermore, the asymptotic degree has a power law when P_Z does. We denote

$$\delta_i = \mathbf{E}d_*^i, \qquad \overline{\delta}_i = \mathbf{E}(d_*)_i, \qquad \text{where} \qquad (x)_i = x(x-1)\cdots(x-i+1). \tag{2.3}$$

Another important characteristic of the sequence $\{G_s(n,m,P)\}_n$ is the asymptotic ratio $\beta = \lim_{m\to\infty}\binom{m}{s}/n$. Together with P_Z it determines the first order asymptotics of the clustering coefficient $\alpha = \mathbf{P}(v_1 \sim v_2 | v_1 \sim v_3, v_2 \sim v_3)$, see [6], [8]. Under conditions (i), (ii-2), and

$$\binom{m}{s} n^{-1} \to \beta \in (0, +\infty) \tag{2.4}$$

we have

$$\alpha \;=\; \frac{a_1}{a_2} + o(1) \;=\; \frac{1}{\beta^{1/2}}\frac{\delta_1^{3/2}}{\delta_2 - \delta_1} + o(1). \tag{2.5}$$

Furthermore, we have $\alpha = o(1)$ in the case where $\binom{m}{s}n^{-1} \to +\infty$. We remark that $\alpha = o(1)$ also in the case where the second moment condition (ii-2) fails and we have $\mathbf{E}Z^2 = +\infty$, see [6].

To summarize, the clustering coefficient α does not vanish as $n, m \to \infty$ whenever the asymptotic degree distribution (equivalently P_Z) has finite second moment and $0 < \beta < \infty$.

Our Theorem 2.1, see also Remark 1, establishes similar properties of the assortativity coefficient r: it remains bounded away from zero whenever the asymptotic degree distribution (equivalently P_Z) has finite third moment and $0 < \beta < \infty$.

Theorem 2.1. *Let* $s > 0$ *be an integer. Let* $m, n \to \infty$*. Assume that* (i) *and* (2.4) *are satisfied. In the case where* (ii-3) *holds we have*

$$r \;=\; \frac{a_1}{\beta^{-1}(a_1 a_3 - a_2^2) + a_2} + o(1) \tag{2.6}$$

$$=\; \frac{1}{\sqrt{\beta}}\frac{\overline{\delta}_1^{5/2}}{\overline{\delta}_3\overline{\delta}_1 - \overline{\delta}_2^2 + \overline{\delta}_2\overline{\delta}_1} + o(1). \tag{2.7}$$

In the case where (ii-2) *holds and* $\mathbf{E}Z^3 = \infty$ *we have* $r = o(1)$.

We note that the inequality $a_1 a_3 \geq a_2^2$, which follows from Hölder's inequality, implies that the ratio in the right hand side of (2.6) is positive.

Remark 1. In the case where (i), (ii-2) hold and $\binom{m}{s}n^{-1} \to +\infty$ we have $r = o(1)$.

Our next result Theorem 2.2 shows a first order asymptotics of the neighbour connectivity b_k and the expected number of common neighbours h_k.

Theorem 2.2. *Let $s \geq 1$ and $k \geq 0$ be integers. Let $m, n \to \infty$. Assume that* (i), (ii-2) *and* (2.4) *hold. We have*

$$b = 1 + \beta^{-1} a_2 + o(1), \qquad h = \beta^{-1} a_1 + o(1) \tag{2.8}$$

and

$$h_{k+1} = \frac{a_1}{\beta} \frac{k}{k+1} \frac{p_k}{p_{k+1}} + o(1), \tag{2.9}$$

$$b_{k+1} = 1 + \beta^{-1}(a_2 - a_1) + h_{k+1} + o(1). \tag{2.10}$$

Here $a_1 = (\beta \delta_1)^{1/2} + o(1)$ and $a_2 = \beta \overline{\delta}_2 / \delta_1 + o(1)$.

We remark that the distribution of the random graph $G_s(n, m, P)$ is invariant under permutation of its vertices (we refer to this property as the symmetry property in what follows). Therefore, we have $b = \mathbf{E}(d(v_1)|v_1 \sim v_2)$ and $b_{k+1} = \mathbf{E}(d(v_1)|v_1 \sim v_2, d(v_2) = k+1)$. In particular, the increment $b_{k+1} - b$ shows how the degree of v_2 affects the average degree of its neighbour v_1. By (2.8), (2.10), we have $b_{k+1} - b = \frac{a_1}{\beta}\left(\frac{k}{k+1}\frac{p_k}{p_{k+1}} - 1\right) + o(1)$. In Examples 1 and 2 below we evaluate this quantity for a power law asymptotic degree distribution and the Poisson asymptotic degree distribution.

Example 1. Assume that the asymptotic degree distribution has a power law, i.e., for some $c > 0$ and $\gamma > 3$ we have $p_k = (c + o(1))k^{-\gamma}$ as $k \to +\infty$. Then

$$\frac{k}{k+1}\frac{p_k}{p_{k+1}} - 1 = \frac{\gamma - 1}{k} + o(k^{-1}).$$

Hence, for large k, we obtain as $n, m \to +\infty$ that $b_{k+1} - b \approx k^{-1}(\gamma - 1)(\delta_1/\beta)^{1/2}$.

Example 2. Assume that the asymptotic degree distribution is Poisson with mean $\lambda > 0$, i.e., $p_k = e^{-\lambda}\lambda^k/k!$. Then

$$\frac{k}{k+1}\frac{p_k}{p_{k+1}} - 1 = \frac{k}{\lambda} - 1$$

and, for large k, we obtain as $n, m \to +\infty$ that

$$b_{k+1} - b \approx (\lambda\beta)^{-1/2} k. \tag{2.11}$$

Our interpretation of (2.11) is as follows. We assume, for simplicity, that $s = 1$. We say that an attribute $w \in W$ realises the link $v_i \sim v_j$, whenever $w \in D_i \cap D_j$. We note that in a sparse intersection graph $G_1(n, m, P)$ each link is realised by a single attribute with a high probability. We also remark that in the case of the Poisson asymptotic degree distribution, the sizes of the random sets, defining intersection graph, are strongly concentrated about their mean value a_1. Now, by the symmetry property, every element of the attribute set D_2 of vertex v_2 realises about $k/|D_2| \approx k/a_1$ links to some neighbours of v_2 other than v_1. In particular, the attribute responsible for the link $v_1 \sim v_2$ attracts to v_1 some k/a_1 neighbours of v_2. Hence, $b_{k+1} - b \approx a_1^{-1} k \approx (\beta\lambda)^{-1/2} k$.

Finally, we remark that (2.8), (2.9), and (2.10) imply (1.2).

3 Passive intersection graph

A collection $D_1, \dots, D_n$ of subsets of a finite set $W = \{w_1, \dots, w_m\}$ defines the passive adjacency relation between elements of W: w_i and w_j are declared adjacent if $w_i, w_j \in D_k$ for some D_k. In this way we obtain a graph on the vertex set W, which we call the passive intersection graph, see [11]. We assume that $D_1, D_2, \dots, D_n$ are independent random subsets of W having the same probability distribution (2.1). In particular, their sizes $X_i = |D_i|$, $1 \le i \le n$ are independent random variables with the common distribution P. The *passive random intersection graph* defined by the collection $D_1, \dots, D_n$ is denoted $G_1^*(n, m, P)$.

We shall consider a sequence of passive graphs $\{G_1^*(n, m, P)\}_n$, where $P = P_n$ and $m = m_n$ depend on $n = 1, 2, \dots$. We remark that, in the case where $\beta_n = mn^{-1}$ is bounded and it is bounded away from zero as $n, m \to +\infty$, the vertex degree distribution can be approximated by a compound Poisson distribution ([6], [14]). More precisely, assuming that $\beta_n \to \beta \in (0, +\infty)$;

(iii) X_1 *converges in distribution to a random variable* Z;

(iv) $\mathbf{E}Z^{4/3} < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}X_1^{4/3} = \mathbf{E}Z^{4/3}$

it is shown in [6] that $d(w_1)$ converges in distribution to the compound Poisson random variable $d_{**} := \sum_{j=1}^{\Lambda} \tilde{Z}_j$. Here $\tilde{Z}_1$, $\tilde{Z}_2$,... are independent random variables with the distribution

$$\mathbf{P}(\tilde{Z}_1 = j) = (j+1)\mathbf{P}(Z = j+1)/\mathbf{E}Z, \qquad j = 0, 1, \dots,$$

in the case where $\mathbf{E}Z > 0$. In the case where $\mathbf{E}Z = 0$ we put $\mathbf{P}(\tilde{Z}_1 = 0) = 1$. The random variable Λ is independent of the sequence $\tilde{Z}_1$, $\tilde{Z}_2$,... and has Poisson distribution with mean $\mathbf{E}\Lambda = \beta^{-1}\mathbf{E}Z$.

We note that the asymptotic degree d_{**} has a power law whenever Z has a power law. Furthermore, we have $\mathbf{E}d_{**}^i < \infty \Leftrightarrow \mathbf{E}Z^{i+1} < \infty$, $i = 1, 2, \dots$.

In Theorems 3.1, 3.2 below we express the moments b, h, b_k, h_k and the assortativity coefficient $r = \frac{g-b^2}{b'-b^2}$ of the random graph $G_1^*(n, m, P)$ in terms of the moments

$$y_i = \mathbf{E}(X_1)_i \qquad \text{and} \qquad \delta_{*i} = \mathbf{E}d_{**}^i \qquad i = 1, 2, \dots.$$

Theorem 3.1. *Let* $n, m \to \infty$. *Assume that* (iii) *holds and*

(v) $\mathbf{P}(Z \ge 2) > 0$, $\mathbf{E}Z^4 < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}X_1^4 = \mathbf{E}Z^4$.

In the case where $\beta_n \to \beta \in (0, +\infty)$ *we have*

$$r = \frac{y_2y_4 + y_2y_3 - y_3^2}{y_2y_4 + y_2y_3 - y_3^2 + \beta_n^{-1}y_2^2(y_2 + y_3)} + o(1) \qquad (3.1)$$

$$= 1 - \frac{\delta_{*2}\delta_{*1}^2 - \delta_{*1}^4}{\delta_{*1}\delta_{*3} - \delta_{*2}^2} + o(1). \qquad (3.2)$$

In the case where $\beta_n \to +\infty$ *we have* $r = 1 - o(1)$. *In the case where* $\beta_n \to 0$ *and* $n\beta_n^3 \to +\infty$ *we have* $r = o(1)$.

Remark 2. We note that $y_* := y_2y_4 + y_2y_3 - y_3^2$ is always non-negative. Hence, for large n, m we have $r \ge 0$. To show that $y_* \ge 0$ we combine the identity $2y_* = \mathbf{E}y(X_1, X_2)$, where

$$y(i,j) = y'(i,j) + y'(j,i), \qquad y'(i,j) = (i)_2(j)_4 + (i)_2(j)_3 - (i)_3(j)_3,$$

with the simple inequality

$$y(i,j) = (i)_2(j)_2\left((i-2)^2 + (j-2)^2 - 2(i-2)(j-2)\right) \ge 0.$$

Remark 3. Assuming that $y_2 > 0$ and $y_2 = o(m\beta_n)$ as $m, n \to +\infty$, Godehardt et al. [12] showed the following expression for the clustering coefficient of $G_1^*(n, m, P)$

$$\alpha = \frac{\beta_n^{-2}m^{-1}y_2^3 + y_3}{\beta_n^{-1}y_2^2 + y_3} + o(1). \qquad (3.3)$$

Now, assuming that conditions (iii) and (v) hold we compare α and r using (3.1) and (3.3). For $\beta_n \to \beta \in (0, +\infty)$ we have $r < 1$ and $\alpha = \bigl(1 + y_2^2/(\beta y_3)\bigr)^{-1} + o(1) < 1$. In the case where $\beta_n \to +\infty$ we have $r = 1 - o(1)$ and $\alpha = 1 - o(1)$. In the case where $\beta_n \to 0$ and $n\beta_n^3 \to +\infty$ we have $r = o(1)$ and $\alpha = o(1)$.

Our last result Theorem 3.2 shows a first order asymptotics of the neighbour connectivity b_k and the expected number of common neighbours h_k in the passive random intersection graph.

Theorem 3.2. *Let $m, n \to \infty$. Assume that $\beta_n \to \beta \in (0, +\infty)$ and* (iii), (v) *hold. Then*

$$b = 1 + \beta_n^{-1} y_2 + y_2^{-1} y_3 + O(n^{-1}) = \delta_{*2}\delta_{*1}^{-1} + o(1), \tag{3.4}$$

$$h = y_2^{-1} y_3 + O(n^{-1}) = \delta_{*2}\delta_{*1}^{-1} - 1 - \delta_{*1} + o(1). \tag{3.5}$$

Assuming, in addition, that $\mathbf{P}(d_{**} = k) > 0$, where $k > 0$ is an integer, we have

$$h_k = k^{-1}\mathbf{E}(d_{2*}|d_{**} = k) + o(1), \tag{3.6}$$

$$b_k = 1 + \beta^{-1} y_2 + h_k + o(1) = 1 + \delta_{*1} + h_k + o(1). \tag{3.7}$$

Here $d_{2*} = \sum_{1 \le i \le \Lambda} (\tilde{Z}_i)_2$.

We remark that (3.4), (3.5), (3.6), (3.7) imply (1.2).

4 Proofs

Proofs for active and passive graphs are given in Section 4.1 and Section 4.2 respectively. We note that the probability distributions of $G_s(n, m, P)$ and $G_1^*(n, m, P)$ are invariant under permutations of the vertex sets. Therefore, for either of these models we have

$$b = \mathbf{E}_{12} d(\omega_1), \qquad h = \mathbf{E}_{12} d(\omega_1, \omega_2), \tag{4.1}$$
$$b_k = \mathbf{E}_{12}(d(\omega_2)|d(\omega_1) = k), \qquad h_k = \mathbf{E}_{12}(d(\omega_1, \omega_2)|d(\omega_1) = k).$$

Here $\omega_1 \neq \omega_2$ are arbitrary fixed vertices and $\mathbf{E}_{12}$ denotes the conditional expectation given the event $\omega_1 \sim \omega_2$. In the proof $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ (respectively, $\tilde{\mathbf{P}}_*$ and $\tilde{\mathbf{E}}_*$) denote the conditional probability and expectation given $X_1, \dots, X_n$ (respectively, $D_1, D_2, X_1, \dots, X_n$). Limits are taken as n and $m = m_n$ tend to infinity. We use the shorthand notation $f_k(\lambda) = e^{-\lambda}\lambda^k/k!$ for the Poisson probability.

4.1 Active graph

Before the proof we introduce some more notation. Then we state and prove auxiliary lemmas. Afterwards we prove Theorem 2.1, Remark 1 and Theorem 2.2.

The conditional expectation given D_1, D_2 is denoted $\mathbf{E}_*$. The conditional expectation given the event $v_1 \sim v_2$ is denoted $\mathbf{E}_{12}$. We denote

$$Y_i = \binom{X_i}{s}, \qquad d_i = d(v_i), \qquad d_i' = d_i - 1, \qquad d_{ij} = d(v_i, v_j),$$
$$\mathbb{I}_i = \mathbb{I}_{\{X_i < m^{1/4}\}}, \qquad \bar{\mathbb{I}}_i = 1 - \mathbb{I}_i, \qquad \eta_{ij} = 1 - \bar{\mathbb{I}}_i - \bar{\mathbb{I}}_j - (m^{1/2} - 1)^{-1} \tag{4.2}$$

and introduce events

$$\mathcal{E}_{ij}' = \{|D_i \cap D_j| = s\}, \qquad \mathcal{E}_{ij}'' = \{|D_i \cap D_j| \ge s + 1\}, \qquad \mathcal{E}_{ij} = \{|D_i \cap D_j| \ge s\}.$$

Observe that $\mathcal{E}_{ij}$ is the event that v_i and v_j are adjacent in $G_s(n, m, P)$. We denote

$$p_e = \mathbf{P}(\mathcal{E}_{ij}), \quad a_i = \mathbf{E} Y_1^i, \quad x_i = \mathbf{E} X_1^i, \quad z_i = \mathbf{E} Z^i, \quad \tilde{m} = \binom{m}{s}, \quad \beta_n = \frac{\tilde{m}}{n}. \tag{4.3}$$

We remark that the distributions of $X_i = X_{ni}$, $Y_i = Y_{ni}$ and $Z_i = Z_{ni} = (n/\tilde{m})^{1/2} Y_{ni}$ depend on n.

The following inequality is referred to as LeCam's lemma, see e.g., [26].

Lemma 4.1. *Let $S = \mathbb{I}_1 + \mathbb{I}_2 + \cdots + \mathbb{I}_n$ be the sum of independent random indicators with probabilities $\mathbf{P}(\mathbb{I}_i = 1) = p_i$. Let Λ be Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions P_S and P_Λ of S and Λ*

$$\sup_{A \subset \{0,1,2\dots\}} |\mathbf{P}(S \in A) - \mathbf{P}(\Lambda \in A)| \le 2 \sum_i p_i^2. \tag{4.4}$$

Lemma 4.2. ([6]) *Given integers $1 \le s \le k_1 \le k_2 \le m$, let D_1, D_2 be independent random subsets of the set $W = \{1, \dots, m\}$ such that D_1 (respectively D_2) is uniformly distributed in the class of subsets of W of size k_1 (respectively k_2). The probabilities $p' := \mathbf{P}(|D_1 \cap D_2| = s)$ and $p'' := \mathbf{P}(|D_1 \cap D_2| \ge s)$ satisfy*

$$\left(1 - \frac{(k_1 - s)(k_2 - s)}{m + 1 - k_1}\right) p^*_{k_1,k_2,s} \ \le \ p' \ \le \ p'' \le \ p^*_{k_1,k_2,s}, \tag{4.5}$$

Here we denote $p^*_{k_1,k_2,s} = \binom{k_1}{s}\binom{k_2}{s}\binom{m}{s}^{-1}$.

Lemma 4.3. *Let $s > 0$ be an integer. Let $m, n \to \infty$. Assume that conditions* (i) *and* (ii-3) *hold. Denote $\tilde{X}_{n1} = m^{-1/2} n^{1/(2s)} X_{n1} \mathbb{I}_{\{X_{n1} \ge s\}}$. We have*

$$\lim_{A \to +\infty} \sup_n \mathbf{E} Z_{n1}^3 \mathbb{I}_{\{Z_{n1} > A\}} = 0, \tag{4.6}$$

$$\sup_n \mathbf{E} \tilde{X}_{n1}^{3s} < \infty, \qquad \lim_{A \to +\infty} \sup_n \mathbf{E} \tilde{X}_{n1}^{3s} \mathbb{I}_{\{\tilde{X}_{n1} > A\}} = 0. \tag{4.7}$$

For any $0 \le u \le 3$ and any sequence $A_n \to +\infty$ as $n \to \infty$ we have

$$\mathbf{E} Z_{n1}^u \mathbb{I}_{\{Z_{n1} > A_n\}} = o(1), \qquad \mathbf{E} \tilde{X}_{n1}^{us} \mathbb{I}_{\{\tilde{X}_{n1} > A_n\}} = o(1). \tag{4.8}$$

Proof of Lemma 4.3. The uniform integrability property (4.6) of the sequence $\{Z_{n1}^3\}_n$ is a simple consequence of (i) and (ii-3), see, e.g., Remark 1 in [5]. The first and second identity of (4.7) follows from (ii-3) and (4.6) respectively. Finally, (4.8) follows from (4.6) and (4.7). □

Lemma 4.4. *In $G_s(n, m, P)$ the probabilities of events $\mathcal{E}_{ij} = \{v_i \sim v_j\}$, $\mathcal{E}'_{12}$, $\mathcal{E}''_{12}$, see* (4.2)*, and $\mathcal{B}_t = \{|D_t \cap (D_1 \cup D_2)| \ge s + 1\}$ satisfy the inequalities*

$$Y_1 Y_2 \tilde{m}^{-1} \eta_{12} \le \tilde{\mathbf{P}}(\mathcal{E}'_{12}) \le \tilde{\mathbf{P}}(\mathcal{E}_{12}) \le Y_1 Y_2 \tilde{m}^{-1}, \tag{4.9}$$

$$Y_i Y_j \tilde{m}^{-1} \eta_{ij} \le \tilde{\mathbf{P}}_*(\mathcal{E}_{ij}) = \tilde{\mathbf{P}}(\mathcal{E}_{ij}) \le Y_i Y_j \tilde{m}^{-1}, \qquad \text{for} \qquad \{i, j\} \ne \{1, 2\}, \tag{4.10}$$

$$\tilde{\mathbf{P}}(\mathcal{E}''_{12}) \le Y_1 Y_2 X_1 X_2 (\tilde{m} m)^{-1}, \tag{4.11}$$

$$\tilde{\mathbf{P}}_*(\mathcal{B}_t) \le 2^s \left((s+1)! \tilde{m} m\right)^{-1} Y_t X_t (X_1^{s+1} + X_2^{s+1}). \tag{4.12}$$

We recall that Y_i and η_{ij} are defined in (4.2).

Proof of Lemma 4.4. The right hand side of (4.9), (4.10) and inequality (4.11) are immediate consequences of (4.5). In order to show the left hand side inequality of (4.9) and (4.10) we apply the left hand side inequality of (4.5). We only prove (4.9). We have, see (4.2),

$$\tilde{\mathbf{P}}(\mathcal{E}'_{12}) = \tilde{\mathbf{E}} \mathbb{I}_{\mathcal{E}'_{12}} \ge \tilde{\mathbf{E}} \mathbb{I}_{\mathcal{E}'_{12}} \mathbb{I}_1 \mathbb{I}_2 \ge \tilde{m}^{-1} Y_1 Y_2 \mathbb{I}_1 \mathbb{I}_2 \big(1 - X_1 X_2 (m - X_1)^{-1}\big) \ge \tilde{m}^{-1} Y_1 Y_2 \eta_{12}. \tag{4.13}$$

In order to show (4.12) we apply the right-hand side inequality of (4.5) and write

$$\tilde{\mathbf{P}}_*(\mathcal{B}_t) \le \binom{|D_1 \cup D_2|}{s+1}\binom{|D_t|}{s+1}\binom{m}{s+1}^{-1} \le \binom{X_1+X_2}{s+1}\binom{X_t}{s+1}\binom{m}{s+1}^{-1}. \tag{4.14}$$

Invoking the inequalities $\binom{X_t}{s+1}\binom{m}{s+1}^{-1} = \frac{Y_t(X_t-s)}{\tilde{m}(m-s)} \le \frac{Y_t X_t}{\tilde{m}m}$ and

$$(X_1+X_2)_{s+1} \le (X_1+X_2)^{s+1} \le 2^s(X_1^{s+1}+X_2^{s+1})$$

we obtain (4.12). □

Lemma 4.5. *Assume that conditions of Theorem 2.2 are satisfied. Let $k \ge 0$ be an integer. For $d_1^* = \sum_{4\le t\le n} \mathbb{I}_{\mathcal{E}_{1t}}$ and $\Delta = \tilde{\mathbf{P}}_*(d_1^* = k) - f_k(\beta^{-1}a_1Y_1)$ we have*

$$\mathbf{E}_*|\Delta| \le R_1^* + R_2^* + R_3^* + R_4^*, \tag{4.15}$$

where $R_1^* = n\tilde{m}^{-1}\mathbf{E}_*Y_1Y_4|1-\eta_{14}|$ and

$$R_2^* = n^{1/2}\tilde{m}^{-1}a_2^{1/2}Y_1, \qquad R_3^* = a_1Y_1|(n-3)\tilde{m}^{-1} - \beta^{-1}|, \qquad R_4^* = 2n\tilde{m}^{-2}a_2Y_1^2.$$

We recall that $f_k(\lambda) = e^{-\lambda}\lambda^k/k!$.

Proof of Lemma 4.5. We denote $\tilde{S} = \tilde{\mathbf{E}}_*d_1^* = \sum_{4\le t\le n}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t})$ and $\tilde{S}_1 = \tilde{m}^{-1}\sum_{4\le t\le n} Y_t$ and write

$$\Delta = \Delta_1 + \Delta_2, \qquad \Delta_1 = \tilde{\mathbf{P}}_*(d_1^* = k) - f_k(\tilde{S}), \qquad \Delta_2 = f_k(\tilde{S}) - f_k(\beta^{-1}a_1Y_1).$$

We have, by Lemma 4.1, $|\Delta_1| \le 2\sum_{4\le t\le n}\tilde{\mathbf{P}}_*^2(\mathcal{E}_{1t})$. Invoking (4.10) we obtain $\mathbf{E}_*|\Delta_1| \le R_4^*$. Next, we apply the mean value theorem $|f_k(\lambda') - f_k(\lambda'')| \le |\lambda' - \lambda''|$ and write

$$|\Delta_2| \le |\tilde{S} - \beta^{-1}a_1Y_1| \le r_1^* + r_2^* + R_3^*, \tag{4.16}$$

where $r_1^* = |\tilde{S} - Y_1\tilde{S}_1|$ and $r_2^* = Y_1|\tilde{S}_1 - (n-3)\tilde{m}^{-1}a_1|$. Note that by (4.10),

$$r_1^* \le \sum_{4\le t\le n} |\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}) - \tilde{m}^{-1}Y_1Y_t| \le \sum_{4\le t\le n} \tilde{m}^{-1}Y_1Y_t|1-\eta_{1t}|$$

and, by symmetry, $\mathbf{E}_*r_1^* \le R_1^*$. Finally, we have

$$\mathbf{E}_*r_2^* = Y_1\mathbf{E}_*|\tilde{S}_1 - \mathbf{E}_*\tilde{S}_1| \le Y_1\left(\mathbf{E}_*(\tilde{S}_1 - \mathbf{E}_*\tilde{S}_1)^2\right)^{1/2} \le R_2^*.$$

□

Lemma 4.6. *Let $m, n \to \infty$. Assume* (i), (ii-3) *and* (2.4) *hold. Then*

$$\mathbf{E}_{12}d_1'd_2' = n\tilde{m}^{-1}a_1 + n^2\tilde{m}^{-2}a_2^2 + o(1), \tag{4.17}$$
$$\mathbf{E}_{12}d_1' = n\tilde{m}^{-1}a_2 + o(1), \tag{4.18}$$
$$\mathbf{E}_{12}(d_1')^2 = \mathbf{E}_{12}d_1' + n^2\tilde{m}^{-2}a_1a_3 + o(1), \tag{4.19}$$
$$\mathbf{E}_{12}d_{12} = n\tilde{m}^{-1}a_1 + o(1). \tag{4.20}$$

Proof of Lemma 4.6. Proof of (4.17). In order to prove (4.17) we write

$$\mathbf{E}_{12}\, d_1'd_2' = p_e^{-1}\mathbf{E}\varkappa, \qquad \varkappa := \mathbb{I}_{\mathcal{E}_{12}}d_1'd_2', \qquad p_e := \mathbf{P}(\mathcal{E}_{12}) \tag{4.21}$$

and invoke the identities

$$\mathbf{E}\varkappa = n\tilde{m}^{-2}a_1^3 + n^2\tilde{m}^{-3}a_1^2a_2^2 + o(\tilde{m}^{-1}), \tag{4.22}$$
$$p_e = \tilde{m}^{-1}a_1^2(1+o(1)). \tag{4.23}$$

Note that (4.23) follows from (4.10) and (4.8). Let us prove (4.22). To this aim we write

$$\mathbf{E}\varkappa = \mathbf{E}\left(\mathbb{I}_{\mathcal{E}_{12}}\tilde{\mathbf{E}}_*(d_1'd_2')\right) = \mathbf{E}(\tilde{\varkappa}_1 + \tilde{\varkappa}_2),$$

where $\tilde{\varkappa}_1 = \mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{E}}_* d_1'd_2'$ and $\tilde{\varkappa}_2 = \mathbb{I}_{\mathcal{E}_{12}''}\tilde{\mathbf{E}}_* d_1'd_2'$, and show that

$$\mathbf{E}\tilde{\varkappa}_1 = n\tilde{m}^{-2}a_1^3 + n^2\tilde{m}^{-3}a_1^2a_2^2 + o(\tilde{m}^{-1}), \qquad \mathbf{E}\tilde{\varkappa}_2 = o(\tilde{m}^{-1}). \tag{4.24}$$

Let us prove (4.24). Assuming that $\mathcal{E}_{12}$ holds we can write $d_i' = \sum_{t=3}^n \mathbb{I}_{\mathcal{E}_{it}}$, $i = 1, 2$, and

$$\tilde{\mathbf{E}}_* d_1'd_2' = S_1 + S_2, \qquad S_1 = \sum_{3\le t\le n}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}\cap\mathcal{E}_{2t}), \qquad S_2 = 2\sum_{3\le t<u\le n}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}\cap\mathcal{E}_{2u}). \tag{4.25}$$

To show the first identity of (4.24) we write $\mathbf{E}\tilde{\varkappa}_1 = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}S_1 + \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}S_2 =: I_1 + I_2$ and evaluate

$$I_1 = n\tilde{m}^{-2}a_1^3 + o(n\tilde{m}^{-2}), \qquad I_2 = n^2\tilde{m}^{-3}a_1^2a_2^2 + o(n^2\tilde{m}^{-3}). \tag{4.26}$$

We first evaluate I_1. Given $t \ge 3$, consider events

$$\mathcal{A}_t = \{|(D_1\cap D_2)\cap D_t| = s\} \qquad \text{and} \qquad \mathcal{B}_t = \{|D_t\cap(D_1\cup D_2)| \ge s+1\}. \tag{4.27}$$

Assuming that $\mathcal{E}_{12}'$ holds we have that $\mathcal{A}_t$ implies $\mathcal{E}_{1t}\cap\mathcal{E}_{2t}$ and $\mathcal{E}_{1t}\cap\mathcal{E}_{2t}$ implies $\mathcal{A}_t\cup\mathcal{B}_t$. Hence, $\tilde{\mathbf{P}}_*(\mathcal{A}_t) \le \tilde{\mathbf{P}}_*(\mathcal{E}_{1t}\cap\mathcal{E}_{2t}) \le \tilde{\mathbf{P}}_*(\mathcal{A}_t\cup\mathcal{B}_t)$. Now, we invoke the identity $\tilde{\mathbf{P}}_*(\mathcal{A}_t) = \tilde{m}^{-1}Y_t$ and write

$$\mathbb{I}_{\mathcal{E}_{12}'}\tilde{m}^{-1}Y_t = \mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{A}_t) \le \mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}\cap\mathcal{E}_{2t}) \le \mathbb{I}_{\mathcal{E}_{12}'}\left(\tilde{\mathbf{P}}_*(\mathcal{A}_t) + \tilde{\mathbf{P}}_*(\mathcal{B}_t)\right). \tag{4.28}$$

From (4.28) and (4.12) we obtain, by the symmetry property,

$$\frac{n-2}{\tilde{m}}\mathbf{P}(\mathcal{E}_{12}')\mathbf{E}Y_3 \le I_1 \le \frac{n-2}{\tilde{m}}\mathbf{P}(\mathcal{E}_{12}')\mathbf{E}Y_3 + \frac{n-2}{\tilde{m}m}\mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12}')R_1, \tag{4.29}$$

where $R_1 = Y_3X_3(X_1^{s+1} + X_2^{s+1})$. Next, we evaluate $\tilde{\mathbf{P}}(\mathcal{E}_{12}')$ and $\mathbf{P}(\mathcal{E}_{12}') = \mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12}')$ using (4.9):

$$\tilde{m}\mathbf{P}(\mathcal{E}_{12}')\mathbf{E}Y_3 = a_1^3 + o(1), \qquad \tilde{m}\mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12}')R_1 = O(1).$$

Combining these relations with (4.29) we obtain the first relation of (4.26).

Let us we evaluate I_2. We write

$$\tilde{\mathbf{E}}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}\cap\mathcal{E}_{2u}) = \tilde{\mathbf{E}}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t})\,\tilde{\mathbf{P}}_*(\mathcal{E}_{2u}) = \tilde{\mathbf{P}}(\mathcal{E}_{12}')\tilde{\mathbf{P}}(\mathcal{E}_{1t})\,\tilde{\mathbf{P}}(\mathcal{E}_{2u}) \tag{4.30}$$

and apply (4.9) to each probability in the right-hand side. We obtain

$$\tilde{m}^{-3}(Y_1^2Y_2^2Y_tY_u - R_{tu}) \le \tilde{\mathbf{P}}(\mathcal{E}_{12}')\tilde{\mathbf{P}}(\mathcal{E}_{1t})\,\tilde{\mathbf{P}}(\mathcal{E}_{2u}) \le \tilde{m}^{-3}Y_1^2Y_2^2Y_tY_u, \tag{4.31}$$

where $R_{tu} = Y_1^2Y_2^2Y_tY_u(1-\eta_{12}\eta_{1t}\eta_{2u})$ satisfies $\mathbf{E}R_{tu} = o(1)$, see (4.8). Now, by the symmetry property, we obtain from (4.31) the second relation of (4.26)

$$I_2 = (n-2)_2\mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12}')\tilde{\mathbf{P}}(\mathcal{E}_{1t})\,\tilde{\mathbf{P}}(\mathcal{E}_{2u}) = n^2\tilde{m}^{-3}a_1^2a_2^2 + o(n^2\tilde{m}^{-3}).$$

To prove the second bound of (4.24) we write, see (4.25), $\tilde{\varkappa}_2 = \mathbb{I}_{\mathcal{E}_{12}''}(S_1 + S_2)$ and show that

$$I_3 := \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}''}S_1 \le x_{2s+1}x_{s+1}x_sn/(\tilde{m}^2m), \qquad I_4 := \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}''}S_2 \le x_{2s+1}^2x_s^2n^2/(\tilde{m}^3m). \tag{4.32}$$

Here $x_{2s+1}, x_{s+1}, x_s = O(1)$, by (4.7). Let us prove (4.32). We have, see (4.9),

$$S_1 \le \sum_{3\le t\le n}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}) \le \sum_{3\le t\le n}Y_1Y_t\tilde{m}^{-1}. \tag{4.33}$$

Furthermore, by the symmetry property and (4.11), we obtain

$$I_3 = \mathbf{E}(\tilde{\mathbf{E}}\mathbb{I}_{\mathcal{E}''_{12}} S_1) = \mathbf{E}(\tilde{\mathbf{P}}(\mathcal{E}''_{12}) S_1) \le (n-2)(\tilde{m}^2 m)^{-1}\mathbf{E}Y_1^2 Y_2 Y_3 X_1 X_2.$$

Since the expected value in the right hand side does not exceed $x_{2s+1}x_{s+1}x_s$, we obtain the first bound of (4.32). In order to prove the second bound we write, cf. (4.30),

$$\tilde{\mathbf{E}}\mathbb{I}_{\mathcal{E}''_{12}}\tilde{\mathbf{P}}_*(\mathcal{E}_{1t}\cap\mathcal{E}_{2u}) = \tilde{\mathbf{P}}(\mathcal{E}''_{12})\tilde{\mathbf{P}}(\mathcal{E}_{1t})\,\tilde{\mathbf{P}}(\mathcal{E}_{2u}) \le \tilde{m}^{-3}m^{-1}Y_1^2 Y_2^2 Y_t Y_u X_1 X_2.$$

In the last step we used (4.9) and (4.11). Now, by the symmetry property, we obtain

$$I_4 = \mathbf{E}(\tilde{\mathbf{E}}\mathbb{I}_{\mathcal{E}''_{12}} S_2) \le (n-2)_2\tilde{m}^{-3}m^{-1}\mathbf{E}Y_1^2 Y_2^2 Y_3 Y_4 X_1 X_2 \le n^2\tilde{m}^{-3}m^{-1}x_{2s+1}^2 x_s^2.$$

Proof of (4.18). We write, by the symmetry property,

$$\mathbf{E}_{12}d'_1 = p_e^{-1}\mathbf{E}\sum_{3\le t\le n}\mathbb{I}_{\mathcal{E}_{1t}}\mathbb{I}_{\mathcal{E}_{12}} = (n-2)p_e^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{12}} \tag{4.34}$$

and evaluate using (4.9), (4.10)

$$\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}} = \mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12})\tilde{\mathbf{P}}(\mathcal{E}_{13}) = \tilde{m}^{-2}\mathbf{E}Y_1^2 Y_2 Y_3 + o(\tilde{m}^{-2}) = \tilde{m}^{-2}a_1^2 a_2 + o(\tilde{m}^{-2}).$$

Invoking this relation and (4.23) in (4.34) we obtain (4.18).

Proof of (4.19). Assuming that the event $\mathcal{E}_{12}$ holds we write

$$(d'_1)^2 = \Big(\sum_{3\le t\le n}\mathbb{I}_{\mathcal{E}_{1t}}\Big)^2 = d'_1 + 2\sum_{3\le t<u\le n}\mathbb{I}_{\mathcal{E}_{1t}}\mathbb{I}_{\mathcal{E}_{1u}}$$

and evaluate the expected value

$$\mathbf{E}_{12}(d'_1)^2 = \mathbf{E}_{12}d'_1 + p_e^{-1}(n-2)_2\varkappa^*. \tag{4.35}$$

Here $\varkappa^* = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{14}}$. We have

$$\varkappa^* = \mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12})\tilde{\mathbf{P}}(\mathcal{E}_{13})\tilde{\mathbf{P}}(\mathcal{E}_{14}) = \tilde{m}^{-3}\mathbf{E}Y_1^3 Y_2 Y_3 Y_4 + o(\tilde{m}^{-3}). \tag{4.36}$$

In the last step we used (4.9), (4.10). Now (4.23), (4.35) and (4.36) imply (4.19).

Proof of (4.20). We note that $d_{12} = \sum_{3\le t\le n}\mathbb{I}_{\mathcal{E}_{1t}}\mathbb{I}_{\mathcal{E}_{2t}}$ and $\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}d_{12} = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}S_1$, see (4.25). Next, we write

$$\mathbf{E}_{12}d_{12} = p_e^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}S_1 = p_e^{-1}(I_1 + I_3).$$

and evaluate the quantity in the right hand side using (4.23) and (4.26), (4.32).

□

Proof of Theorem 2.1. It is convenient to write r in the form

$$r = \eta/\xi, \qquad \text{where} \qquad \eta = \mathbf{E}_{12}d'_1 d'_2 - (\mathbf{E}_{12}d'_1)^2, \qquad \xi = \mathbf{E}_{12}(d'_1)^2 - (\mathbf{E}_{12}d'_1)^2. \tag{4.37}$$

In the case where (ii-3) holds we obtain (2.6) from (2.4), (4.17), (4.18), (4.19) and (4.37). Then we derive (2.7) from (2.6) using the identities

$$a_i = \beta^{i/2}z_i + o(1), \qquad \overline{\delta}_i = z_i z_1^i, \quad i = 1,2,3. \tag{4.38}$$

We recall that a_i and z_i are defined in (4.3).

Now we consider the case where (ii-2) holds and $\mathbf{E}Z^3 = \infty$. It suffices to show that

$$\eta = O(1) \qquad \text{and} \qquad \liminf \xi = +\infty. \tag{4.39}$$

Before the proof of (4.39) we remark that (4.23) holds under condition (ii-2). In order to prove the first bound of (4.39) we show that $\mathbf{E}_{12}d'_1d'_2 = O(1)$ and $\mathbf{E}_{12}d'_1 = O(1)$. To show the first bound we write $\mathbf{E}_{12}d'_1d'_2 = p_e^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}d'_1d'_2$ and evaluate

$$\begin{aligned} \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}d'_1d'_2 &= \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\sum_{3\le t\le n}\mathbb{I}_{\mathcal{E}_{1t}}\mathbb{I}_{\mathcal{E}_{2t}} + \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\sum_{3\le t,u\le n,\ t\neq u}\mathbb{I}_{\mathcal{E}_{1t}}\mathbb{I}_{\mathcal{E}_{2u}} \\ &= (n-2)\varkappa_1^* + (n-2)_2\varkappa_2^*, \end{aligned} \tag{4.40}$$

where

$$\varkappa_1^* = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}} \le \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}} \le \tilde{m}^{-2}a_2a_1^2 = O(n^{-2}), \tag{4.41}$$

$$\varkappa_2^* = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{24}} \le \tilde{m}^{-3}a_2^2a_1^2 = O(n^{-3}). \tag{4.42}$$

In the last step we used (4.9) and (4.10). We note that (4.23), (4.40) and (4.41), (4.42) imply $\mathbf{E}_{12}d'_1d'_2 = O(1)$. Similarly, the bound $\mathbf{E}_{12}d'_1 = O(1)$ follows from (4.23) and the simple bound, cf. (4.34),

$$\mathbf{E}_{12}d'_1 = p_e^{-1}(n-2)\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}} \le p_e^{-1}n\tilde{m}^{-2}a_2a_1^2. \tag{4.43}$$

In order to prove the second relation of (4.39) we show that $\liminf \mathbf{E}_{12}(d'_1)^2 = +\infty$. In view of (4.23) and (4.35) it suffices to show that $\liminf n^3\varkappa^* = +\infty$. It follows from the left-hand side inequality of (4.5) that

$$n^3\varkappa^* \ge n^3\mathbf{E}\mathbb{I}_1\mathbb{I}_2\mathbb{I}_3\mathbb{I}_4\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{14}} \ge \mathbf{E}\mathbb{I}_1\mathbb{I}_2\mathbb{I}_3\mathbb{I}_4Z_1^3Z_2Z_3Z_4(1-O(m^{-1/2}))^3, \tag{4.44}$$

where, by the independence of $Z_1,\dots,Z_4$, we have $\mathbf{E}\mathbb{I}_1\mathbb{I}_2\mathbb{I}_3\mathbb{I}_4Z_1^3Z_2Z_3Z_4 = \left(\mathbf{E}\mathbb{I}_1Z_1^3\right)\left(\mathbf{E}\mathbb{I}_2Z_2\right)^3$. Finally, (i) combined with (ii-2) imply $\mathbf{E}\mathbb{I}_2Z_2 = z_1 + o(1)$, and (i) combined with $\mathbf{E}Z^3 = \infty$ imply $\liminf \mathbf{E}\mathbb{I}_1Z_1^3 = +\infty$. □

Proof of Remark 1. Before the proof we introduce some notation and collect auxiliary inequalities. We denote

$$h = h_n = m^{1/2}n^{-1/(4s)}, \qquad \tilde{h} = \tilde{h}_n = \binom{h}{s}\beta_n^{-1/2}$$

and observe that, under the assumption of Remark 1, $\beta_n, h_n, \tilde{h}_n \to +\infty$ and $h_n = o(m^{1/2})$. We further denote

$$\mathbb{I}_{ih} = \mathbb{I}_{\{X_i<h\}}, \qquad \overline{\mathbb{I}}_{ih} = 1-\mathbb{I}_{ih}, \qquad \eta_{ijh} = 1-\overline{\mathbb{I}}_{ih}-\overline{\mathbb{I}}_{jh}-\varepsilon_h,$$

where $\varepsilon_h = h^2(m-h)^{-1}$, and remark that $\mathbb{I}_{ih} = \mathbb{I}_{\{Z_i<\tilde{h}\}}$ and $\varepsilon_h = o(1)$. We observe that conditions (i), (ii-k) imply, for any given $u \in (0,k]$, that

$$\mathbf{E}Z_1^u = z_u + o(1), \qquad \mathbf{E}Z_1^u\mathbb{I}_{1h} = z_u + o(1), \qquad \mathbf{E}Z_1^u\overline{\mathbb{I}}_{1h} = o(1). \tag{4.45}$$

Now from (4.5) we derive the inequalities

$$\mathbf{E}Z_1Z_2\eta_{12h} \le \mathbf{E}Z_1Z_2\mathbb{I}_{1h}\mathbb{I}_{2h}(1-\varepsilon_h) \le n\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{1h}\mathbb{I}_{2h} \le n\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}} \le \mathbf{E}Z_1Z_2. \tag{4.46}$$

Then invoking in (4.46) relations $\mathbf{E}Z_1 = z_1 + o(1)$ and $\mathbf{E}Z_1Z_2\eta_{12h} = z_1^2 + o(1)$, which follow from (4.45) for $u = 1$, we obtain the relation

$$np_e = n\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}} = z_1^2 + o(1). \tag{4.47}$$

Similarly, under conditions (i), (ii-2), we obtain the relations

$$n^2\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}} = z_1^2z_2 + o(1), \tag{4.48}$$

$$n^3\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{24}} = z_1^2z_2^2 + o(1), \tag{4.49}$$

and, under conditions (i), (ii-3), we obtain

$$n^3\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{14}} = z_1^3 z_3 + o(1). \tag{4.50}$$

Let us prove the bound $r = o(1)$ in the case where (i), (ii-2) hold and $\mathbf{E}Z^3 = +\infty$. In order to prove $r = o(1)$ we show (4.39). Proceeding as in (4.40), (4.41), (4.42), (4.43) and using (4.47) we show the bounds $\mathbf{E}_{12}d_1'd_2' = O(1)$ and $\mathbf{E}_{12}d_1' = O(1)$, which imply the first bound of (4.39). Next we show the second relation of (4.39). In view of (4.35) and (4.47) it suffices to prove that $\limsup n^3\varkappa^* = +\infty$. In the proof we proceed similarly as in (4.44) above, but now we use the product $\mathbb{I}_{1h}\mathbb{I}_{2h}\mathbb{I}_{3h}\mathbb{I}_{4h}$ instead of $\mathbb{I}_1\mathbb{I}_2\mathbb{I}_3\mathbb{I}_4$. We obtain

$$n^3\varkappa^* \geq \left(\mathbf{E}\mathbb{I}_{1h}Z_1^3\right)\left(\mathbf{E}\mathbb{I}_{2h}Z_2\right)^3(1-\varepsilon_h)^3.$$

Here $\mathbf{E}\mathbb{I}_{2h}Z_2 = z_1 + o(1)$, see (4.45). Furthermore, under conditions (i) and $\mathbf{E}Z^3 = +\infty$ we have $\mathbf{E}\mathbb{I}_{1h}Z_1^3 \to +\infty$. Hence, $n^3\varkappa^* \to +\infty$.

Now we prove the bound $r = o(1)$ in the case where (i), (ii-3) hold. We shall show that

$$\eta = o(1) \qquad \text{and} \qquad \liminf \xi > 0. \tag{4.51}$$

Let us prove the second inequality of (4.51). Combining the first identity of (4.43) with (4.47) and (4.48) we obtain

$$\mathbf{E}_{12}d_1' = z_2 + o(1). \tag{4.52}$$

Next, combining (4.35) with (4.47) and (4.50) we obtain

$$\mathbf{E}_{12}(d_1')^2 = \mathbf{E}_{12}d_1' + z_1z_3 + o(1). \tag{4.53}$$

It follows from (4.52), (4.53) and the inequality $z_1z_3 \geq z_2^2$, which follows from Hoelder's inequality, that $\xi = z_2 + z_1z_3 - z_2^2 + o(1) \geq z_2 + o(1)$. We have proved the second inequality of (4.51).

Let us prove the first bound of (4.51). In view of (4.40) and (4.52) it suffices to show that

$$p_e^{-1}n^2\varkappa_2^* = z_2^2 + o(1), \qquad p_e^{-1}n\varkappa_1^* = o(1). \tag{4.54}$$

We note that the first relation of (4.54) follows from (4.47), (4.49). To prove the second bound of (4.54) we need to show that $\varkappa_1^* = o(n^{-2})$. We split

$$\varkappa_1^* = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}} + \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}}$$

and estimate, using (4.10) and (4.11),

$$\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}} \leq \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{13}} \leq \tilde{m}^{-2}m^{-1}\mathbf{E}Y_1^2Y_2X_1X_2Y_3 = O(n^{-2-s^{-1}}).$$

In the last step we combined the inequality $Y_i^u \leq X_i^u\mathbb{I}_{\{X_i \geq s\}}$ and (4.7). Furthermore, using the right-hand side inequality of (4.28) we write

$$\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}} \leq \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{m}^{-1}Y_3 + \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{B}_3)$$

and estimate, by (4.9) and (4.12),

$$\begin{aligned}
&\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{m}^{-1}Y_3 \leq \tilde{m}^{-2}\mathbf{E}Y_1Y_2Y_3 = O(n^{-2}\beta_n^{-1/2}),\\
&\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{B}_3) \leq \tilde{m}^{-2}m^{-1}\mathbf{E}Y_1Y_2Y_3X_3(X_1^{s+1} + X_2^{s+1}) = O(n^{-2-s^{-1}}).
\end{aligned} \tag{4.55}$$

□

Proof of Theorem 2.2. Relations (2.8) follow from (4.1) and (4.18), (4.20).

Before the proof of (2.9) and (2.10) we introduce some notation. Given two sequences of real numbers $\{A_n\}$ and $\{B_n\}$ we write $A_n \simeq B_n$ (respectively $A_n \simeq 0$) to denote the fact that $A_n - B_n = o(n^{-2})$ (respectively $A_n = o(n^{-2})$). We denote $p_* = \mathbf{P}(v_1 \sim v_2,\, d_1' = k)$ and introduce random variables, see (4.2), $\mathbb{I}^* = \mathbb{I}_1\mathbb{I}_2$, $\overline{\mathbb{I}}^* = 1 - \mathbb{I}^*$, and

$$\tau_1 = \mathbb{I}_{\mathcal{E}_{12}}\tau, \qquad \tau_2 = \mathbb{I}_{\mathcal{E}_{12}''}\tau, \qquad \tau_3 = \mathbb{I}_{\mathcal{E}_{12}'}\overline{\mathbb{I}}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}}\mathbb{I}_{\{d_1^*=k\}}, \qquad \tau_4 = \mathbb{I}_{\mathcal{E}_{12}'}\tau^*, \qquad \tau_5 = \mathbb{I}_{\mathcal{E}_{12}''}\tau^*.$$

Here $\tau = \mathbb{I}_{\mathcal{E}_{23}}\mathbb{I}_{\{d_1'=k\}}$ and $\tau^* = \mathbb{I}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{E}_{23}}\mathbb{I}_{\{d_1^*=k-1\}}$, and $d_1^* = \sum_{4\le t\le n}\mathbb{I}_{\mathcal{E}_{1t}}$. We remark that the identity $\mathbb{I}_{\mathcal{E}_{12}} = \mathbb{I}_{\mathcal{E}_{12}'} + \mathbb{I}_{\mathcal{E}_{12}''}$ in combination with $1 = \mathbb{I}_{\mathcal{E}_{13}} + \overline{\mathbb{I}}_{\mathcal{E}_{13}}$ implies

$$\tau_1 = \tau_2 + \tau_3 + \tau_4. \tag{4.56}$$

Proof of (2.9), (2.10). In view of (4.1) we can write

$$\begin{aligned} h_{k+1} &= \mathbf{E}_{12}(d_{12}|d_1' = k) = p_*^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1'=k\}}d_{12}, \\ b_{k+1} - 1 &= \mathbf{E}_{12}(d_2'|d_1' = k) = p_*^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1'=k\}}d_2'. \end{aligned} \tag{4.57}$$

Furthermore, by the symmetry property, we have

$$\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1'=k\}}d_{12} = (n-2)\mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\tau^*, \qquad \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1'=k\}}d_2' = (n-2)\mathbf{E}\tau_1. \tag{4.58}$$

We note that (4.57), (4.58) combined with the identities $\mathbb{I}_{\mathcal{E}_{12}}\tau^* = \tau_4 + \tau_5$ and (4.56) imply

$$h_{k+1} = (n-2)p_*^{-1}\mathbf{E}(\tau_4 + \tau_5), \qquad b_{k+1} - 1 = (n-2)p_*^{-1}\mathbf{E}(\tau_2 + \tau_3 + \tau_4), \tag{4.59}$$

and observe that (2.9), (2.10) follow from (4.59) and the relations

$$\begin{aligned} p_* &= n^{-1}(k+1)p_{k+1} + o(n^{-1}), && (4.60)\\ \mathbf{E}\tau_3 &= n^{-2}\beta^{-1}(k+1)(a_2 - a_1)p_{k+1} + o(n^{-2}), && (4.61)\\ \mathbf{E}\tau_4 &= n^{-2}\beta^{-1}ka_1p_k + o(n^{-2}), && (4.62)\\ \mathbf{E}\tau_i &= o(n^{-2}), \qquad i = 2, 5. && (4.63) \end{aligned}$$

It remains to prove (4.60), (4.61), (4.62), (4.63).

In order to show (4.63) we combine the inequalities

$$\tau_i \le \mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{23}} = \mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{23}}(\mathbb{I}^* + \overline{\mathbb{I}}^*) \le \mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{23}}\mathbb{I}^* + \mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{23}}\overline{\mathbb{I}}^*$$

with the inequalities, which follow from (4.10) and (4.11),

$$\begin{aligned} \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}''}\mathbb{I}_{\mathcal{E}_{23}}\mathbb{I}^* &\le \mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12}'')\tilde{\mathbf{P}}_*(\mathcal{E}_{23})\mathbb{I}^* \le (\tilde{m}^2m)^{-1}\mathbf{E}Y_1Y_2^2Y_3X_1X_2\mathbb{I}^* = O(n^{-2}m^{-1/2}), && (4.64)\\ \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{E}_{23}}\overline{\mathbb{I}}^* &\le \mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12})\tilde{\mathbf{P}}_*(\mathcal{E}_{23})\overline{\mathbb{I}}^* \le \tilde{m}^{-2}\mathbf{E}Y_1Y_2^2Y_3\overline{\mathbb{I}}^* = o(n^{-2}). && (4.65) \end{aligned}$$

In the last step of (4.64) we use the inequality $X_1X_2\mathbb{I}^* \le m^{1/2}$. In the last step of (4.65) we use the bound $\mathbf{E}Y_1Y_2^2Y_3\overline{\mathbb{I}}^* = o(1)$, which holds under conditions (i), (ii-2). Indeed $Y_1Y_2Y_3$ is uniformly integrable as $n \to +\infty$ and $Y_1Y_2^2Y_3\overline{\mathbb{I}}^* = o(1)$ almost surely.

Proof of (4.62). We have

$$\mathbf{E}\tau_4 = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}'}\tilde{\mathbf{P}}_*(\mathcal{E}_{23}\cap\mathcal{E}_{13})\tilde{\mathbf{P}}_*(d_1^* = k-1). \tag{4.66}$$

We first replace in (4.66) the probability $\tilde{\mathbf{P}}_*(\mathcal{E}_{23}\cap\mathcal{E}_{13})$ by $\tilde{\mathbf{P}}_*(\mathcal{A}_3) = Y_3/\tilde{m}$ using (4.27), (4.28). Then we replace $\tilde{\mathbf{P}}_*(d_1^* = k-1)$ by $f_{k-1}(\beta^{-1}a_1Y_1)$ using Lemma 4.5. Finally, we replace $\mathbb{I}_{\mathcal{E}'_{12}}$ by $\tilde{m}^{-1}Y_1Y_2$ using (4.9). We obtain

$$\begin{aligned}
\mathbf{E}\tau_4 &\simeq \tilde{m}^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}Y_3\tilde{\mathbf{P}}_*(d_1^* = k-1) && (4.67)\\
&\simeq \tilde{m}^{-1}\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}Y_3 f_{k-1}(\beta^{-1}a_1Y_1) && (4.68)\\
&\simeq \tilde{m}^{-2}\mathbf{E}Y_1Y_2Y_3 f_{k-1}(\beta^{-1}a_1Y_1) && (4.69)\\
&= n^{-2}\beta_n^{-2}a_1^2\mathbf{E}Y_1 f_{k-1}(\beta^{-1}a_1Y_1). && (4.70)
\end{aligned}$$

Here (4.67) follows from the bound $\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}_3) = o(n^{-2})$. We remark that this bound follows from (4.55), but under stronger moment condition (ii-3). To show this bound under moment condition (ii-2) of the present theorem we write

$$\mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}_3) = \mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}_3)(\mathbb{I}^* + \overline{\mathbb{I}}^*) \le \mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}_3)\mathbb{I}^* + \mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}'_3)\overline{\mathbb{I}}^*,$$

where $\mathcal{B}'_3 = \{D_3 \cap (D_1 \cup D_2)| \ge s\}$, and estimate, see (4.9), (4.12), (4.14),

$$\begin{aligned}
\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}_3)\mathbb{I}^* &\le \tilde{m}^{-2}m^{-1}\mathbf{E}Y_1Y_2Y_3X_3(X_1^{s+1} + X_2^{s+1})\mathbb{I}^*\\
&\le \tilde{m}^{-2}m^{-3/4}\mathbf{E}Y_1Y_2Y_3X_3(X_1^{s} + X_2^{s})\\
&= O(n^{-2}m^{-3/4}),\\
\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{P}_*(\mathcal{B}'_3)\overline{\mathbb{I}}^* &\le \tilde{m}^{-2}\mathbf{E}Y_1Y_2Y_3(X_1^{s} + X_2^{s})\overline{\mathbb{I}}^*\\
&\le o(n^{-2}).
\end{aligned}$$

Furthermore, (4.68) follows from the bounds $\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}Y_3R_j^* = o(n^{-1})$, $1 \le j \le 4$, see (4.15). We show these bound using (4.9). For $1 \le j \le 3$ the proof is obvious. For $j = 4$ we need to show that $\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}Y_1^2Y_3 = o(1)$. For this purpose we write (using the inequality $\mathbb{I}_1Y_1 \le \mathbb{I}_1 m^{s/4}$)

$$\mathbb{I}_{\mathcal{E}'_{12}}Y_1^2Y_3 = \mathbb{I}_{\mathcal{E}'_{12}}Y_1^2Y_3(\mathbb{I}_1 + \overline{\mathbb{I}}_1) \le m^{s/4}\mathbb{I}_{\mathcal{E}'_{12}}Y_1Y_3\mathbb{I}_1 + Y_1^2Y_3\overline{\mathbb{I}}_1$$

and note that the expected values of both summands in the right hand side tend to zero as $n \to +\infty$. Finally, (4.69) follows from (4.9) and implies directly (4.70).

Now we derive (4.62) from (4.70). We observe that

$$k^{-1}\beta^{-1}a_1\mathbf{E}Y_1 f_{k-1}(\beta^{-1}a_1Y_1) = \mathbf{E}f_k(\beta^{-1}a_1Y_1) \to \mathbf{E}f_k(z_1Z)$$

(here we use the fact that the weak convergence of distributions (i) implies the convergence of expectations of smooth functions). Furthermore, by (2.2), $\mathbf{E}f_k(z_1Z) = p_k$. Hence, (4.70) implies

$$\mathbf{E}\tau_4 \simeq n^{-2}\beta^{-1}ka_1\mathbf{E}f_k(z_1Z) = n^{-2}\beta^{-1}ka_1p_k.$$

Proof of (4.61). Introduce the event $\mathcal{C} = \{D_3 \cap (D_1 \setminus D_2) = \emptyset\}$, probability $\tilde{p} = \tilde{\mathbf{P}}_*(\mathcal{E}'_{23} \cap \mathcal{C} \cap \overline{\mathcal{E}}_{13})$, and random variable $H = \tilde{m}^{-1}(Y_2 - 1)Y_3$. We obtain (4.61) in several steps. We show that

$$\begin{aligned}
\mathbf{E}\tau_3 &\simeq \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{p}\mathbb{I}_{\{d_1^*=k\}} && (4.71)\\
&\simeq \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}H\mathbb{I}_{\{d_1^*=k\}} && (4.72)\\
&\simeq \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}Hf_k(\beta^{-1}a_1Y_1) && (4.73)\\
&\simeq \tilde{m}^{-1}\mathbf{E}Y_1Y_2Hf_k(\beta^{-1}a_1Y_1) && (4.74)\\
&\simeq \tilde{m}^{-2}(a_2 - a_1)(k+1)\beta p_{k+1}. && (4.75)
\end{aligned}$$

We note that (4.71) is obtained by replacing $\mathbb{I}_{\mathcal{E}_{23}}$ by the product $\mathbb{I}_{\mathcal{E}'_{23}}\mathbb{I}_{\mathcal{C}}$ in the formula defining τ_3. In order to bound the error of this replacement we apply the inequality

$$\mathbb{I}_{\mathcal{E}'_{23}}\mathbb{I}_{\mathcal{C}} \le \mathbb{I}_{\mathcal{E}_{23}} \le \mathbb{I}_{\mathcal{E}'_{23}}\mathbb{I}_{\mathcal{C}} + \mathbb{I}_{\mathcal{B}_3}. \tag{4.76}$$

and invoke the bound $\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\overline{\mathbb{I}}_{\mathcal{E}_{13}}\mathbb{I}_{\mathcal{B}_3}\mathbb{I}_{\{d'_1=k\}} \le \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{\mathbf{P}}_*(\mathcal{B}_3) = o(n^{-2})$, see the proof of (4.67) above. We remark that the left hand side inequality of (4.76) is obvious. The right hand side inequality holds because the event $\mathcal{E}_{23}$ implies $(\mathcal{E}'_{23}\cap\mathcal{C})\cup\mathcal{B}_3$.

In (4.72) we replace $\tilde{p}$ by H. To prove (4.72) we show that

$$\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{p}\mathbb{I}_{\{d^*_1=k\}} \simeq \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{p}\mathbb{I}_{\{d^*_1=k\}}\mathbb{I}_1 \simeq \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}H\mathbb{I}_{\{d^*_1=k\}}\mathbb{I}_1 \simeq \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}H\mathbb{I}_{\{d^*_1=k\}}. \tag{4.77}$$

We remark that the first and third relations follow from the simple bounds, see (4.9), (4.10),

$$\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\tilde{p}\mathbb{I}_{\{d^*_1=k\}}\overline{\mathbb{I}}_1 \le \mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}\mathbb{I}_{\mathcal{E}'_{23}}\overline{\mathbb{I}}_1 \le \tilde{m}^{-2}\mathbf{E}Y_1Y_2^2Y_3\overline{\mathbb{I}}_1 = o(n^{-2}),$$
$$\mathbf{E}\mathbb{I}_{\mathcal{E}'_{12}}|H|\mathbb{I}_{\{d^*_1=k\}}\overline{\mathbb{I}}_1 \le \tilde{m}^{-1}\mathbf{E}Y_1Y_2|H|\overline{\mathbb{I}}_1 = o(n^{-2}).$$

In order to show the second relation of (4.77) we split

$$\tilde{p} = \tilde{\mathbf{P}}_*(\overline{\mathcal{E}}_{13}|\mathcal{E}'_{23}\cap\mathcal{C})\,\tilde{\mathbf{P}}_*(\mathcal{E}'_{23}|\mathcal{C})\,\tilde{\mathbf{P}}_*(\mathcal{C}) =: \tilde{p}_1\tilde{p}_2\tilde{p}_3 \tag{4.78}$$

and observe that $\tilde{p}_1$ is the probability that the random subset $D_3\cap D_2$ (of size s) of D_2 does not match the subset $D_1\cap D_2$ (we note that $|D_1\cap D_2| = s$, since the event $\mathcal{E}'_{12}$ holds). Hence, $\tilde{p}_1 = 1 - Y_2^{-1}$. Furthermore, from (4.5) we obtain

$$\tilde{p}_3 = 1 - \tilde{\mathbf{P}}_*(D_3\cap(D_1\setminus D_2)\neq\emptyset) \ge 1 - \tilde{\mathbf{P}}_*(D_3\cap D_1\neq\emptyset) \ge 1 - m^{-1}X_1X_3. \tag{4.79}$$

Finally, $\tilde{p}_2$ is the probability that the random subset D_3 of $W\setminus(D_1\setminus D_2)$ intersects with D_2 in exactly s elements. Taking into account that the event $\mathcal{E}'_{12}$ holds we obtain (see (4.9), (4.13))

$$\tilde{m}_1^{-1}Y_2Y_3\mathbb{I}_2\mathbb{I}_3(1 - m^{1/2}/(m' - X_2)) \le \tilde{p}_2 \le \tilde{m}_1^{-1}Y_2Y_3. \tag{4.80}$$

Here we denote $\tilde{m}_1 := \binom{m'}{s}$ and $m' = |W\setminus(D_1\setminus D_2)| = m - (X_1 - s)$. We remark that on the event $\{X_1 < m^{1/4}\}$ we have $m' = m - O(m^{1/4})$. Hence, for large m, (4.80) implies

$$\tilde{m}^{-1}Y_2Y_3\eta_{23}\mathbb{I}_1 \le \tilde{p}_2\mathbb{I}_1 \le \tilde{m}^{-1}Y_2Y_3\mathbb{I}_1(1 + m^{-3/4}(s + o(1))). \tag{4.81}$$

Now, collecting (4.79), (4.81), and the identity $\tilde{p}_1 = 1 - Y_2^{-1}$ in (4.78) we obtain the inequalities

$$\mathbb{I}_{\mathcal{E}'_{12}}\mathbb{I}_1\eta_{23}H(1 - m^{-1}X_1X_3) \le \mathbb{I}_{\mathcal{E}'_{12}}\mathbb{I}_1\tilde{p} \le \mathbb{I}_{\mathcal{E}'_{12}}\mathbb{I}_1H(1 + O(m^{-3/4})) \tag{4.82}$$

that imply the second relation of (4.77).

In the proof of (4.73), (4.74), (4.75) we apply the same argument as in (4.68), (4.69), (4.70) above.

Proof of (4.60). We write

$$p_* = \mathbf{E}\mathbb{I}_{\mathcal{E}_{12}}\tilde{\mathbf{P}}_*(d'_1 = k) = \mathbf{E}\tilde{\mathbf{P}}(\mathcal{E}_{12})\tilde{\mathbf{P}}_*(d'_1 = k)$$

and in the integrand of the right hand side we replace $\tilde{\mathbf{P}}_*(d'_1 = k)$ by $f_k(\beta^{-1}a_1Y_1)$ and $\tilde{\mathbf{P}}(\mathcal{E}_{12})$ by $\tilde{m}^{-1}Y_1Y_2$ using (4.15) and (4.9), respectively. □

4.2 Passive graph

Before the proof we introduce some more notation. Then we present auxiliary lemmas. Afterwards we prove Theorems 3.1, 3.2.

By $\mathbf{E}_{ij}$ we denote the conditional expectation given the event $\mathcal{E}_{ij} = \{w_i \sim w_j\}$. Furthermore, we denote

$$p_e = \mathbf{P}(\mathcal{E}_{ij}), \quad D_{ij} = D_i\cap D_j, \quad X_{ij} = |D_{ij}|, \quad x_i = \mathbf{E}X_1^i, \quad y_i = \mathbf{E}(X_1)_i, \quad u_i = \mathbf{E}(Z)_i.$$

For $w \in W$, we denote $\mathbb{I}_i(w) = \mathbb{I}_{\{w \in D_i\}}$ and $\overline{\mathbb{I}}_i(w) = 1 - \mathbb{I}_i(w)$, and introduce random variables

$$L(w) = \sum_{1 \le i \le n} l_i(w), \qquad l_i(w) = \mathbb{I}_i(w)(X_i - 1),$$
$$Q(w) = \sum_{1 \le i < j \le n} q_{ij}(w), \qquad q_{ij}(w) = \mathbb{I}_i(w)\mathbb{I}_j(w)(X_{ij} - 1),$$
$$S_1 = \sum_{1 \le i \le n} s_i, \qquad S_2 = \sum_{1 \le i < j \le n} s_i s_j, \qquad s_i = \mathbb{I}_i(w_1)\mathbb{I}_i(w_2).$$

We say that two vertices $w_i, w_j \in W$ are linked by D_k if $w_i, w_j \in D_k$. In particular, a set D_k defines $\binom{X_k}{2}$ links between its elements. We note that $L_t = L(w_t)$ counts the number of links incident to w_t. Similarly, $Q_t = Q(w_t)$ counts the number of different parallel links incident to w_t (a parallel link between w' and w'' is realized by a pair of sets D_i, D_j such that $w', w'' \in D_i \cap D_j$). Furthermore, S_1 counts the number of links connecting w_1 and w_2 and S_2 counts the number of different pairs of links connecting w_1 and w_2. We denote the degree $d_t = d(w_t)$ and introduce event $\mathcal{L}_t = \{L_t = d_t\}$.

Lemma 4.7. *The factorial moments $\overline{\delta}_{*i} = \mathbf{E}(d_{**})_i$ and $u_i = \mathbf{E}(Z)_i$ satisfy the identities*

$$\overline{\delta}_{*1} = \beta^{-1}u_2, \qquad \overline{\delta}_{*2} = \beta^{-2}u_2^2 + \beta^{-1}u_3, \qquad \overline{\delta}_{*3} = \beta^{-3}u_2^3 + 3\beta^{-2}u_2u_3 + \beta^{-1}u_4. \tag{4.83}$$

Proof of Lemma 4.7. We only show the third identity of (4.83). The proof of the first and second identities is similar, but simpler. We color $z = z_1 + \cdots + z_r$ distinct balls using r different colors so that z_i balls receive i-th color. The number of triples of balls

$$\binom{z}{3} = \sum_{i \in [r]} \binom{z_i}{3} + \sum_{i \in [r]} \binom{z_i}{2} \sum_{j \in [r] \setminus \{i\}} z_j + \sum_{\{i,j,k\} \subset [r]} z_i z_j z_k. \tag{4.84}$$

Here the first sum counts triples of the same color, the second sum counts triples having two different colors, etc. We apply (4.84) to the random variable $\binom{d_{**}}{3}$, where $d_{**} = \tilde{Z}_1 + \cdots + \tilde{Z}_\Lambda$. We obtain, by the symmetry property,

$$\mathbf{E}\binom{d_{**}}{3} = \mathbf{E}\Lambda\mathbf{E}\binom{\tilde{Z}_1}{3} + \mathbf{E}(\Lambda)_2\mathbf{E}\binom{\tilde{Z}_1}{2}\mathbf{E}\tilde{Z}_1 + \mathbf{E}\binom{\Lambda}{3}(\mathbf{E}\tilde{Z}_1)^3.$$

Now invoking the simple identities $\mathbf{E}(\Lambda)_i = (\mathbf{E}\Lambda)^i = (u_1\beta^{-1})^i$ and $\mathbf{E}(\tilde{Z}_1)_i = u_{i+1}u_1^{-1}$ we obtain the third identity of (4.83). ☐

Lemma 4.8. *We have*

$$\mathbf{E}S_1 = n^{-1}\beta_n^{-2}y_2 + R_1', \tag{4.85}$$
$$\mathbf{E}L_1S_1 = n^{-1}\beta_n^{-2}(y_2 + y_3) + n^{-1}\beta_n^{-3}y_2^2 + R_2', \tag{4.86}$$
$$\mathbf{E}L_1L_1S_1 = n^{-1}\beta_n^{-2}(y_2 + 3y_3 + y_4) + 3n^{-1}\beta_n^{-3}y_2(y_2 + y_3) + n^{-1}\beta_n^{-4}y_2^3 + R_3', \tag{4.87}$$
$$\mathbf{E}L_1L_2S_1 = n^{-1}\beta_n^{-2}(y_4 + 3y_3 + y_2) + 2n^{-1}\beta_n^{-3}y_2(y_3 + y_2) + n^{-1}\beta_n^{-4}y_2^3 + R_4', \tag{4.88}$$

where, for some absolute constant $c > 0$, we have $|R_1'| \le cn^{-2}\beta_n^{-3}x_2$ and

$$|R_2'| \le cn^{-2}(\beta_n^{-3} + \beta_n^{-4})x_4,$$
$$|R_j'| \le cn^{-2}\beta_n^{-3}(1 + \beta_n^{-1} + x_2 + \beta_n^{-2}x_2)x_4, \qquad j = 3, 4.$$

Proof of Lemma 4.8. We only show (4.88). The proof of remaining identities is similar or simpler. We write, for $t = 1, 2$, $L_t = L(w_t) = l_1(w_t) + L_t'$ and denote $\overline{\tau}_j = \tilde{\mathbf{E}}s_j =$

$(m)_2^{-1}(X_j)_2$. We have, by the symmetry property,

$$\begin{aligned} \mathbf{E}L_1L_2S_1 &= n\mathbf{E}s_1L_1L_2, \\ \mathbf{E}s_1L_1L_2 &= \mathbf{E}s_1l_1(w_1)l_1(w_2) + 2\mathbf{E}s_1l_1(w_1)L_2' + \mathbf{E}s_1L_1'L_2', \\ \mathbf{E}s_1L_1'L_2' &= (n-1)\mathbf{E}s_1l_2(w_1)l_2(w_2) + (n-1)_2\mathbf{E}s_1l_2(w_1)l_3(w_2), \\ \mathbf{E}s_1l_1(w_1)L_2' &= (n-1)\mathbf{E}s_1l_1(w_1)l_2(w_2). \end{aligned} \tag{4.89}$$

A straightforward calculation shows that

$$\begin{aligned} \tilde{\mathbf{E}}s_1l_1(w_1)l_1(w_2) &= (X_1-1)^2\overline{\tau}_1 = (m)_2^{-1}\left((X_1)_4 + 3(X_1)_3 + (X_1)_2\right), \\ \tilde{\mathbf{E}}s_1l_1(w_1)l_2(w_2) &= m^{-1}(X_1-1)(X_2-1)X_2\overline{\tau}_1 = m^{-1}(m)_2^{-1}\left((X_1)_3 + (X_1)_2\right)(X_2)_2, \\ \tilde{\mathbf{E}}s_1l_2(w_1)l_2(w_2) &= (X_2-1)^2\overline{\tau}_1\overline{\tau}_2 = (m)_2^{-2}(X_1)_2\left((X_2)_4 + 3(X_2)_3 + (X_2)_2\right), \\ \tilde{\mathbf{E}}s_1l_2(w_1)l_3(w_2) &= m^{-2}(X_2)_2(X_3)_2\overline{\tau}_1 = m^{-2}(m)_2^{-1}(X_1)_2(X_2)_2(X_3)_2. \end{aligned}$$

Invoking these expressions in the identity $\mathbf{E}s_1l_i(w_t)l_j(w_u) = \mathbf{E}\tilde{\mathbf{E}}s_1l_i(w_t)l_j(w_u)$ we obtain expressions for the moments $\mathbf{E}s_1l_i(w_t)l_j(w_u)$. Substituting them in (4.89) we obtain (4.88). □

Lemma 4.9. *We have*

$$\mathbf{E}S_2 \le 0.5n^{-2}\beta_n^{-4}x_2^2, \tag{4.90}$$
$$\mathbf{E}L_1S_2 \le n^{-2}\beta_n^{-4}x_2x_3 + 0.5n^{-2}\beta_n^{-5}x_2^3, \tag{4.91}$$
$$\mathbf{E}Q_1S_1 \le n^{-2}\beta_n^{-4}x_2x_3 + 0.5n^{-2}\beta_n^{-5}x_2^3, \tag{4.92}$$
$$\mathbf{E}L_1Q_2S_1 = \mathbf{E}L_2Q_1S_1 \le n^{-2}\beta_n^{-4}(2x_2x_4 + 1.5\beta_n^{-1}x_2^2x_3 + 0.5\beta_n^{-2}x_2^4) + n^{-3}\beta_n^{-6}x_2^2x_4, \tag{4.93}$$
$$\mathbf{E}L_1Q_1S_1 \le n^{-2}\beta_n^{-4}(x_3^2 + x_2x_4) + 2.5n^{-2}\beta_n^{-5}x_2^2x_3 + 0.5n^{-2}\beta_n^{-6}x_2^4, \tag{4.94}$$
$$\mathbf{E}L_1L_1S_2 \le n^{-2}\beta_n^{-4}(x_3^2 + x_2x_4) + 2.5n^{-2}\beta_n^{-5}x_2^2x_3 + 0.5n^{-2}\beta_n^{-6}x_2^4, \tag{4.95}$$
$$\mathbf{E}L_1L_2S_2 \le n^{-2}\beta_n^{-4}(x_2x_4 + x_3^2 + 2\beta_n^{-1}x_2^2x_3 + 0.5\beta_n^{-2}x_2^4) + n^{-3}0.5\beta_n^{-6}x_2^2x_4, \tag{4.96}$$
$$\mathbf{E}Q_1\mathbb{I}_1(w_1)(X_1-1)_2 \le 4n^{-2}\beta_n^{-3}y_2(y_3 + y_4 + \beta^{-1}y_2y_3). \tag{4.97}$$

Proof of Lemma 4.9. We only prove (4.93). The proof of remaining inequalities is similar or simpler. In the proof we use the shorthand notation $l_i = l_i(w_1)$ and $q_{ij} = q_{ij}(w_2)$.

To prove (4.93) we write, by the symmetry property,

$$\begin{aligned} \mathbf{E}Q_2L_1S_1 &= \tbinom{n}{2}\mathbf{E}q_{12}L_1S_1 \\ \mathbf{E}q_{12}L_1S_1 &= 2\mathbf{E}q_{12}l_1S_1 + (n-2)\mathbf{E}q_{12}l_3S_1, \\ \mathbf{E}q_{12}l_1S_1 &= \mathbf{E}q_{12}l_1s_1 + \mathbf{E}q_{12}l_1s_2 + (n-2)\mathbf{E}q_{12}l_1s_3, \\ \mathbf{E}q_{12}l_3S_1 &= \mathbf{E}q_{12}l_3s_1 + \mathbf{E}q_{12}l_3s_2 + \mathbf{E}q_{12}l_3s_3 + (n-3)\mathbf{E}q_{12}l_3s_4 \end{aligned}$$

and invoke the inequalities

$$\begin{aligned} &\mathbf{E}q_{12}l_1s_j \le m^{-4}x_2x_4, \qquad \mathbf{E}q_{12}l_3s_j \le m^{-5}x_2^2x_3, \qquad j = 1,2, \\ &\mathbf{E}q_{12}l_1s_3 \le m^{-6}x_2^2x_4, \qquad \mathbf{E}q_{12}l_3s_3 \le m^{-5}x_2^2x_3, \qquad \mathbf{E}q_{12}l_3s_4 \le m^{-6}x_2^4. \end{aligned}$$

These inequalities follow from the identity $\mathbf{E}q_{12}l_is_j = \mathbf{E}\tilde{\mathbf{E}}q_{12}l_is_j$ and the upper bounds for the conditional expectations $\tilde{\mathbf{E}}q_{12}l_is_j$ constructed below.

For $i = 1$ and $j = 1,2$, we have

$$\tilde{\mathbf{E}}q_{12}l_1s_j \le \tilde{\mathbf{E}}q_{12}l_1 = (X_1-1)\tilde{\mathbf{E}}q_{12}\mathbb{I}_1(w_1) \le m^{-4}X_1^4X_2^2. \tag{4.98}$$

In the first inequality we use $s_j \le 1$. In the second inequality we use the inequality

$$\tilde{\mathbf{E}}q_{12}\mathbb{I}_1(w_1) = \eta\xi \le m^{-4}X_1^3X_2^2. \tag{4.99}$$

Here $\eta = \tilde{\mathbf{E}}\left(X_{12}-1\middle|\mathbb{I}_1(w_1)\mathbb{I}_1(w_2)\mathbb{I}_2(w_2)=1\right)$ and $\xi = \tilde{\mathbf{P}}\left(\mathbb{I}_1(w_1)\mathbb{I}_1(w_2)\mathbb{I}_2(w_2)=1\right)$. We note that given X_1, X_2, D_1, the random variable η evaluates the expected number of elements of $D_1 \setminus \{w_2\}$ that belong to the random subset $D_2 \setminus \{w_2\}$ (of size $X_2 - 1$). Hence, we have $\eta = (m-1)^{-1}(X_1-1)(X_2-1)$. Furthermore, the probability

$$\xi = \tilde{\mathbf{P}}(w_1, w_2 \in D_1) \times \tilde{\mathbf{P}}(w_2 \in D_2) = \frac{\binom{X_1}{2}}{\binom{m}{2}} \times \frac{X_2}{m}.$$

Combining obtained expressions for η and ξ we easily obtain (4.99).

For $i=1$ and $j=3$, we write, by the independence of D_1, D_2 and D_3,

$$\tilde{\mathbf{E}}q_{12}l_1s_3 = (\tilde{\mathbf{E}}q_{12}l_1)(\tilde{\mathbf{E}}s_3) \le m^{-6}X_1^4X_2^2X_3^2.$$

In the last step we used $\tilde{\mathbf{E}}s_3 = (X_3)_2(m)_2^{-1}$ and $\tilde{\mathbf{E}}q_{12}l_1 \le m^{-4}X_1^4X_2^2$, see (4.98).

For $i=3$ and $j=1,2$, we write $\tilde{\mathbf{E}}q_{12}l_3s_j = (\tilde{\mathbf{E}}q_{12}\mathbb{I}_j(w_1))(\tilde{\mathbf{E}}l_3)$, by the independence of D_1, D_2 and D_3. Invoking the inequalities

$$\tilde{\mathbf{E}}l_3 = (X_3-1)\tilde{\mathbf{P}}(w_1 \in D_3) \le m^{-1}X_3^2, \qquad \tilde{\mathbf{E}}q_{12}\mathbb{I}_1(w_1) = \eta\xi \le m^{-4}X_1^3X_2^2,$$

see (4.99), we obtain $\tilde{\mathbf{E}}q_{12}l_3s_1 \le m^{-5}X_1^3X_2^2X_3^2$. Similarly, $\tilde{\mathbf{E}}q_{12}l_3s_2 \le m^{-5}X_1^2X_2^3X_3^2$.

For $i,j=3$, we split $\tilde{\mathbf{E}}(q_{12}l_3s_3) = (\tilde{\mathbf{E}}q_{12})(\tilde{\mathbf{E}}l_3s_3)$ and write $\tilde{\mathbf{E}}q_{12} = \eta_1\xi_1$. Here

$$\eta_1 = \tilde{\mathbf{E}}(X_{12}-1|\mathbb{I}_1(w_2)\mathbb{I}_2(w_2)=1), \qquad \xi_1 = \tilde{\mathbf{P}}(\mathbb{I}_1(w_2)\mathbb{I}_2(w_2)=1).$$

Invoking the identities $\eta_1 = (m-1)^{-1}(X_1-1)(X_2-1)$ and $\xi_1 = m^{-2}X_1X_2$ we obtain

$$\tilde{\mathbf{E}}q_{12} = \eta_1\xi_1 \le m^{-3}X_1^2X_2^2. \tag{4.100}$$

Combining (4.100) with the identities $\tilde{\mathbf{E}}l_3s_3 = (X_3-1)\tilde{\mathbf{E}}s_3 = (X_3-1)(X_3)_2(m)_2^{-1}$ we obtain the inequality $\tilde{\mathbf{E}}q_{12}l_3s_3 \le m^{-5}X_1^2X_2^2X_3^3$.

For $i=3$ and $j=4$ we write by the independence of D_1, D_2, D_3, D_4, and (4.100)

$$\tilde{\mathbf{E}}q_{12}l_3s_4 = \left(\tilde{\mathbf{E}}q_{12}\right)\left(\tilde{\mathbf{E}}l_3\right)\left(\tilde{\mathbf{E}}s_4\right) \le \left(m^{-3}X_1^2X_2^2\right)\left(m^{-1}X_3^2\right)\left(m^{-2}X_4^2\right) = m^{-6}X_1^2X_2^2X_3^2X_4^2.$$

□

Proof of Theorem 3.1. In order to show (3.1) we write

$$r = \frac{\mathbf{E}_{12}d_1d_2 - (\mathbf{E}_{12}d_1)^2}{\mathbf{E}_{12}d_1^2 - (\mathbf{E}_{12}d_1)^2} = \frac{p_e\mathbf{E}d_1d_2\mathbb{I}_{\mathcal{E}_{12}} - (\mathbf{E}d_1\mathbb{I}_{\mathcal{E}_{12}})^2}{p_e\mathbf{E}d_1^2\mathbb{I}_{\mathcal{E}_{12}} - (\mathbf{E}d_1\mathbb{I}_{\mathcal{E}_{12}})^2} \tag{4.101}$$

and invoke the expressions

$$\begin{aligned} p_e &= \mathbf{E}S_1 + O(n^{-2}\beta_n^{-4}), \\ \mathbf{E}d_1\mathbb{I}_{\mathcal{E}_{12}} &= \mathbf{E}L_1S_1 + O(n^{-2}\beta_n^{-4}(1+\beta_n^{-1})), \\ \mathbf{E}d_1^2\mathbb{I}_{\mathcal{E}_{12}} &= \mathbf{E}L_1^2S_1 + O(n^{-2}\beta_n^{-4}(1+\beta_n^{-2})), \\ \mathbf{E}d_1d_2\mathbb{I}_{\mathcal{E}_{12}} &= \mathbf{E}L_1L_2S_1 + O(n^{-2}\beta_n^{-4}(1+\beta_n^{-2})). \end{aligned} \tag{4.102}$$

Now the identities of Lemma 4.8 complete the proof of (3.1).

Let us prove (4.102). We first write, by the inclusion-exclusion,

$$S_1 - S_2 \le \mathbb{I}_{\mathcal{E}_{12}} \le S_1, \tag{4.103}$$

$$L_t - Q_t \le d_t \le L_t. \tag{4.104}$$

Then we derive from (4.104) the inequalities

$$0 \le L_1L_2 - d_1d_2 \le L_1Q_2 + L_2Q_1 \qquad \text{and} \qquad 0 \le L_1^2 - d_1^2 \le 2L_1Q_1, \tag{4.105}$$

which, in combination with (4.103) and (4.104), imply the inequalities

$$0 \le L_1 S_1 - d_1 \mathbb{I}_{\mathcal{E}_{12}} \le L_1 S_2 + Q_1 S_1, \tag{4.106}$$
$$0 \le L_1^2 S_1 - d_1^2 \mathbb{I}_{\mathcal{E}_{12}} \le L_1^2 S_2 + 2L_1 Q_1 S_1,$$
$$0 \le L_1 L_2 S_1 - d_1 d_2 \mathbb{I}_{\mathcal{E}_{12}} \le L_1 L_2 S_2 + L_1 Q_2 S_1 + L_2 Q_1 S_1.$$

Finally, invoking the upper bounds for the expected values of the quantities in the right hand sides of (4.106) shown in Lemma 4.9, we obtain (4.102).

Now we derive (3.2) from (3.1). Firstly, using the fact that (iii), (v) imply the convergence of moments $\mathbf{E}(X_1)_i \to \mathbf{E}(Z)_i$, for $i = 2, 3, 4$, we replace the moments y_i by $u_i = \mathbf{E}(Z)_i$ in (3.1). Secondly, we replace u_i by their expressions via δ_{*i}. For this purpose we solve for u_2, u_3, u_4 from (4.83) and invoke the identities

$$\overline{\delta}_{*1} = \delta_{*1}, \qquad \overline{\delta}_{*2} = \delta_{*2} - \delta_{*1}, \qquad \overline{\delta}_{*3} = \delta_{*3} - 3\delta_{*2} + 2\delta_{*1}. \tag{4.107}$$

For $\beta_n \to +\infty$ relation (3.1) remains valid and it implies $r = 1 + o(1)$.

For $\beta_n \to 0$ the condition $n\beta_n^3 \to +\infty$ on the rate of decay of β_n ensures that the remainder terms of (4.102) and Lemma 4.8 are negligibly small. In particular, we derive (3.1) using the same argument as above. Letting $\beta_n \to 0$ in (3.1) we obtain the bound $r = o(1)$. □

Proof of Theorem 3.2. Before the proof we introduce some notation. We denote

$$H = \sum_{1 \le i \le n} \mathbb{I}_i(w_1)(X_i - 1)_2, \qquad p_{ke} = \mathbf{P}(w_2 \sim w_1, d_1 = k).$$

Given $w_i, w_j \in W$ we write $d_{ij} = d(w_i, w_j)$. A common neighbour w of w_i and w_j is called black if $\{w, w_i, w_j\} \subset D_r$ for some $1 \le r \le n$, otherwise it is called red. Let d'_{ij} and d''_{ij} denote the numbers of black and red common neighbours, so that $d'_{ij} + d''_{ij} = d_{ij}$. Let w_* be a vertex drawn uniformly at random from the set $W' = W \setminus \{w_1\}$. By d'_{1*} we denote the number of black common neighbours of w_1 and w_*. By $\mathcal{E}_{1*}$ we denote the event $\{w_1 \sim w_*\}$. We assume that w_* is independent of the collection of random sets $D_1 \ldots, D_n$ defining the adjacency relation of our graph.

In the proof we use the identity, which follows from (4.85), (4.102),

$$p_e = n^{-1}\beta_n^{-2} y_2 + O(n^{-2}). \tag{4.108}$$

We also use the identities, which follow from (4.83) and (4.107)

$$1 + \beta^{-1} u_2 + u_2^{-1} u_3 = \delta_{*2}\delta_{*1}^{-1}, \qquad \beta^{-1} u_2 = \delta_{*1}. \tag{4.109}$$

We remark that (4.109) in combination with relations $y_i \to u_i$ as $n, m \to +\infty$, imply the right hand side relations of (3.4), (3.5) and (3.7).

Now we prove the left hand side relations of (3.4), (3.5) and (3.7), and the relation (3.6).

In order to show (3.4) we write $b = p_e^{-1}\mathbf{E} d_1 \mathbb{I}_{\mathcal{E}_{12}}$ and invoke identities (4.102), (4.86) and (4.108).

Proof of (3.5). We write $h = p_e^{-1}\mathbf{E} d_{12} \mathbb{I}_{\mathcal{E}_{12}}$ and evaluate

$$\mathbf{E} d_{12} \mathbb{I}_{\mathcal{E}_{12}} = n^{-1}\beta_n^{-2} y_3 + O(n^{-2}). \tag{4.110}$$

Combining (4.108) with (4.110) we obtain (3.5). Let us show (4.110). Using the identity

$$d_{12} = d'_{12} + d''_{12} = d'_{12}\mathbb{I}_{\mathcal{L}_1} + d'_{12}\overline{\mathbb{I}}_{\mathcal{L}_1} + d''_{12} \tag{4.111}$$

we write

$$\mathbf{E}d_{12}\mathbb{I}_{\mathcal{E}_{12}} = \mathbf{E}d'_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\mathcal{L}_1} + R_1 + R_2, \tag{4.112}$$

where $R_1 = \mathbf{E}d''_{12}\mathbb{I}_{\mathcal{E}_{12}}$ and $R_2 = \mathbf{E}\bar{\mathbb{I}}_{\mathcal{L}_1}d'_{12}\mathbb{I}_{\mathcal{E}_{12}}$. Next, we observe that $\mathbf{E}\mathbb{I}_{\mathcal{L}_1}d'_{12}\mathbb{I}_{\mathcal{E}_{12}} = \mathbf{E}\mathbb{I}_{\mathcal{L}_1}d'_{1j}\mathbb{I}_{\mathcal{E}_{1j}}$, for $2 \le j \le n$, and write

$$\mathbf{E}\mathbb{I}_{\mathcal{L}_1}d'_{12}\mathbb{I}_{\mathcal{E}_{12}} = \mathbf{E}\mathbb{I}_{\mathcal{L}_1}d'_{1*}\mathbb{I}_{\mathcal{E}_{1*}} = \mathbf{E}\mathbb{I}_{\mathcal{L}_1}H(m-1)^{-1}. \tag{4.113}$$

We explain the second identity of (4.113). We observe that $H(m-1)^{-1}$ is the conditional expectation of $d'_{1*}\mathbb{I}_{\mathcal{E}_{1*}}$ given $D_1, \dots, D_n$. Indeed, any pair of sets D_i, D_j containing w_1 intersects in the single point w_1, since the event $\mathcal{L}_1$ holds. Consequently, each D_i containing w_1 produces $X_i - 2$ black common neighbours provided that w_* hits D_i. Since the probability that w_* hits D_i equals $(X_i - 1)/(m-1)$, the set D_i contributes (on average) $(m-1)^{-1}\mathbb{I}_i(w_1)(X_i - 1)_2$ black vertices to d'_{1*}.

Now, by the symmetry property, we write the right-hand side of (4.113) in the form

$$\frac{n}{m-1}\mathbf{E}\mathbb{I}_{\mathcal{L}_1}\mathbb{I}_1(w_1)(X_1-1)_2 = \frac{n}{m-1}\mathbf{E}\mathbb{I}_1(w_1)(X_1-1)_2 - R_3 = \frac{n}{(m)_2}y_3 - R_3, \tag{4.114}$$

where, $R_3 = \frac{n}{m-1}\mathbf{E}\bar{\mathbb{I}}_{\mathcal{L}_1}\mathbb{I}_1(w_1)(X_1-1)_2$. Finally, we observe that (4.110) follows from (4.112), (4.113), (4.114) and the bounds $R_i = O(n^{-2})$, $i = 1, 2, 3$, which are proved below.

In order to bound R_i, $i = 1, 2$, we use the inequalities

$$d'_{12} \le d_1 \le L_1, \qquad \mathbb{I}_{\mathcal{E}_{12}} \le S_1, \qquad \bar{\mathbb{I}}_{\mathcal{L}_1} = \mathbb{I}_{\{L_1 \ne d_1\}} = \mathbb{I}_{\{Q_1 \ge 1\}} \le Q_1 \tag{4.115}$$

and write $R_2 \le \mathbf{E}Q_1L_1S_1$ and $R_3 \le n(m-1)^{-1}\mathbf{E}Q_1\mathbb{I}_1(w_1)(X_1-1)_2$. Then we apply (4.94) and (4.97). In order to bound R_1 we observe, that the number of red common neighbours of w_1, w_2 produced by the pair of sets D_i, D_j is

$$a_{ij} = \left(\mathbb{I}_i(w_1)\mathbb{I}_j(w_2)\bar{\mathbb{I}}_j(w_1)\bar{\mathbb{I}}_i(w_2) + \mathbb{I}_j(w_1)\mathbb{I}_i(w_2)\bar{\mathbb{I}}_i(w_1)\bar{\mathbb{I}}_j(w_2)\right)X_{ij}.$$

Hence, on the event $w_1, w_2 \in D_1$ we have $d''_{12} \le \sum_{2 \le i < j \le n} a_{ij}$, since elements of $D_1 \setminus \{w_1, w_2\}$ are black common neighbours of w_1, w_2. >From this inequality and the inequality $\mathbb{I}_{\mathcal{E}_{12}} \le S_1$ we obtain

$$R_1 \le \mathbf{E}d''_{12}S_1 = n\mathbf{E}d''_{12}s_1 \le n\binom{n-1}{2}\mathbf{E}s_1a_{23}. \tag{4.116}$$

Furthermore, invoking in (4.116) identities

$$\mathbf{E}(s_1a_{23}) = \mathbf{E}\tilde{\mathbf{E}}(s_1a_{23}) = \mathbf{E}\left(\tilde{\mathbf{E}}s_1\right)\left(\tilde{\mathbf{E}}a_{23}\right), \qquad \tilde{\mathbf{E}}s_1 = (X_1)_2/(m)_2$$

and inequalities

$$\tilde{\mathbf{E}}a_{23} = 2\mathbf{E}\mathbb{I}_2(w_1)\mathbb{I}_3(w_2)\bar{\mathbb{I}}_3(w_1)\bar{\mathbb{I}}_2(w_2)X_{23} \le 2\frac{X_2}{m}\frac{X_3}{m}\frac{(X_2-1)(X_3-1)}{m-2}$$

we obtain $R_1 = O(n^{-2})$.

Proof of (3.6). In the proof we use the fact that the random vector (H, L_1) converges in distribution to (d_{2*}, d_{**}) as $n \to +\infty$. We recall that H is described after (4.113). The proof of this fact is similar to that of the convergence in distribution of $L_1 = \sum_{1 \le i \le n}\mathbb{I}_i(w_1)(X_i - 1)$ to the random variable d_{**}, see Theorems 5 and 7 of [6]. We note that the convergence in distribution of (H, L_1) implies the convergence in distribution of $H\mathbb{I}_{\{L_1 = k\}}$ to $d_{2*}\mathbb{I}_{\{d_{**} = k\}}$. Furthermore, since under condition (v) the first moment $\mathbf{E}H$ is uniformly bounded as $n \to +\infty$ and $\mathbf{E}d_{2*} < \infty$, we obtain the convergence of moments

$$\mathbf{E}H\mathbb{I}_{\{L_1 = k\}} \to \mathbf{E}d_{2*}\mathbb{I}_{\{d_{**} = k\}} \qquad \text{as} \qquad n \to \infty. \tag{4.117}$$

In order to prove (3.6) we write

$$h_k = \mathbf{E}(d_{12}|w_1 \sim w_2, d_1 = k) = p_{ke}^{-1}\mathbf{E}d_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1=k\}}$$

and show that

$$p_{ke} = km^{-1}\mathbf{P}(d_{**} = k) + o(n^{-1}), \tag{4.118}$$

$$\mathbf{E}d_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1=k\}} = m^{-1}\mathbf{E}H\mathbb{I}_{\{L_1=k\}} + o(n^{-1}). \tag{4.119}$$

We remark that (4.117) in combination with (4.118) and (4.119) implies (3.6).

Let us show (4.118). In view of the identities $p_{ke} = \mathbf{P}(w_i \sim w_1, d_1 = k)$, $2 \le i \le n$, we can write

$$p_{ke} = \mathbf{P}(w_* \sim w_1, d_1 = k) = \mathbf{P}(w_* \sim w_1|d_1 = k)\mathbf{P}(d_1 = k).$$

Now, from the simple identity $\mathbf{P}(w_* \sim w_1|d_1 = k) = k(m-1)^{-1}$ and the approximation $\mathbf{P}(d_1 = k) = \mathbf{P}(d_{**} = k) + o(1)$, see [6], we obtain (4.118).

Let us show (4.119). Using (4.111) we obtain, cf. (4.112),

$$\mathbf{E}d_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1=k\}} = \mathbf{E}d'_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1=k\}}\mathbb{I}_{\mathcal{L}_1} + O(n^{-2}). \tag{4.120}$$

Furthermore, proceeding as in (4.113), we obtain

$$\mathbf{E}d'_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_1=k\}}\mathbb{I}_{\mathcal{L}_1} = \mathbf{E}d'_{1*}\mathbb{I}_{\mathcal{E}_{1*}}\mathbb{I}_{\{d_1=k\}}\mathbb{I}_{\mathcal{L}_1} = (m-1)^{-1}\mathbf{E}H\mathbb{I}_{\{d_1=k\}}\mathbb{I}_{\mathcal{L}_1}. \tag{4.121}$$

Next, we invoke identity $\mathbf{E}H\mathbb{I}_{\{d_1=k\}}\mathbb{I}_{\mathcal{L}_1} = \mathbf{E}H\mathbb{I}_{\{L_1=k\}}\mathbb{I}_{\mathcal{L}_1}$ and approximate, cf. (4.114),

$$(m-1)^{-1}\mathbf{E}H\mathbb{I}_{\{L_1=k\}}\mathbb{I}_{\mathcal{L}_1} = (m-1)^{-1}\mathbf{E}H\mathbb{I}_{\{L_1=k\}} + O(n^{-2}). \tag{4.122}$$

Combining (4.120), (4.121) and (4.122) we obtain (4.119).

Proof of (3.7). Let $\bar{d}_{12}$ denote the number of neighbours of w_1, which are not adjacent to w_2, and let $\bar{h}_k = \mathbf{E}(\bar{d}_{12}|w_1 \sim w_2, d_2 = k)$. We obtain (3.7) from the identity

$$b_k = \mathbf{E}(d_1|w_1 \sim w_2, d_2 = k) = 1 + h_k + \bar{h}_k$$

and the relation $\bar{h}_k = \beta_n^{-1}y_2 + o(1)$. In order to prove this relation we write

$$\bar{h}_k = p_{ke}^{-1}\tau, \qquad \text{where} \qquad \tau = \mathbf{E}\bar{d}_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}},$$

and combine (4.118) with the identity

$$\tau = km^{-1}\beta_n^{-1}y_2\mathbf{P}(d_{**} = k) + o(n^{-1}). \tag{4.123}$$

It remains to prove (4.123). In the proof we use the shorthand notation

$$\eta_i = \mathbb{I}_i(w_1)\bar{\mathbb{I}}_i(w_2)(X_i - 1), \qquad \eta'_i = \eta_i\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}\mathbb{I}_{\mathcal{L}_1}, \qquad \eta''_i = \eta_i\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}.$$

Let us prove (4.123). Using the identity $1 = \mathbb{I}_{\mathcal{L}_1} + \bar{\mathbb{I}}_{\mathcal{L}_1}$ we write

$$\tau = \mathbf{E}\bar{d}_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}\mathbb{I}_{\mathcal{L}_1} + R_4, \qquad R_4 = \mathbf{E}\bar{d}_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}\bar{\mathbb{I}}_{\mathcal{L}_1}.$$

Next, assuming that the event $\mathcal{L}_1$ holds, we invoke the identity $\bar{d}_{12} = \sum_{1 \le i \le n}\eta_i$ and obtain

$$\mathbf{E}\bar{d}_{12}\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}\mathbb{I}_{\mathcal{L}_1} = \mathbf{E}\sum_{1 \le i \le n}\eta'_i = n\mathbf{E}\eta'_1.$$

In the last step we used the symmetry property. Furthermore, from the identity

$$\mathbf{E}\eta'_1 = \mathbf{E}\eta''_1 - R_5, \qquad R_5 = \mathbf{E}\eta_1\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}\bar{\mathbb{I}}_{\mathcal{L}_1},$$

we obtain $\tau = n\mathbf{E}\eta_1'' + R_4 - nR_5$. We note that inequalities $\overline{d}_{12} \le d_1 \le L_1$ and (4.115) imply

$$R_4 \le \mathbf{E}L_1S_1Q_1, \qquad R_5 \le \mathbf{E}\mathbb{I}_1(w_1)(X_1-1)S_1Q_1 = n^{-1}\mathbf{E}L_1S_1Q_1.$$

Now, from (4.94) we obtain $R_4 = O(n^{-2})$ and $R_5 = O(n^{-3})$. Hence, we have $\tau = n\mathbf{E}\eta_1'' + O(n^{-2})$. Finally, invoking the relation

$$\mathbf{E}\eta_1'' = km^{-2}y_2\mathbf{P}(d_{**}=k) + o(n^{-2}), \tag{4.124}$$

we obtain (4.123). To show (4.124) we write

$$\mathbf{E}\eta_1'' = \mathbf{E}\eta_1\kappa, \qquad \kappa = \mathbf{E}\left(\mathbb{I}_{\mathcal{E}_{12}}\mathbb{I}_{\{d_2=k\}}\middle|D_1\right), \tag{4.125}$$

and observe that on the event $w_2 \notin D_1$ the quantity κ evaluates the probability of the event $\{w_1 \sim w_2, d_2 = k\}$ in the passive random intersection graph defined by the sets $D_2, \dots, D_3$ (i.e., the random graph $G_1^*(n-1,m,P)$). We then apply (4.118) to the graph $G_1^*(n-1,m,P)$ and obtain $\kappa = km^{-1}\mathbf{P}(d_{**}=k)+o(n^{-1})$. Here the remainder term does not depend on D_1. Substitution of this identity in (4.125) gives

$$\mathbf{E}\eta_1'' = \left(km^{-1}\mathbf{P}(d_{**}=k) + o(n^{-1})\right)\mathbf{E}\eta_1.$$

The following identities complete the proof of (4.124)

$$\begin{aligned}\mathbf{E}\eta_1 &= \mathbf{E}\mathbb{I}_1(w_1)(X_1-1) - \mathbf{E}\mathbb{I}_1(w_1)\mathbb{I}_1(w_2)(X_1-1)\\ &= m^{-1}y_2 - (m)_2^{-1}(y_3+y_2).\end{aligned}$$

□

References

[1] Barbour, A. D. and Reinert, G.: The shortest distance in random multi-type intersection graphs, *Random Structures and Algorithms* **39** (2011), 179–209. MR-2850268

[2] Barrat, A. and Weigt M.: On the properties of small-world networks, *The European Physical Journal B* **13** (2000), 547–560.

[3] Behrisch, M.: Component evolution in random intersection graphs, *The Electronical Journal of Combinatorics* **14**(1) (2007). MR-2285821

[4] Blackburn, S. and Gerke, S.: Connectivity of the uniform random intersection graph, *Discrete Mathematics* **309** (2009), 5130-5140. MR-2548914

[5] Bloznelis, M.: Degree distribution of a typical vertex in a general random intersection graph, *Lithuanian Mathematical Journal* **48** (2008), 38-45. MR-2398169

[6] Bloznelis, M.: Degree and clustering coefficient in sparse random intersection graphs, to appear in *Annals of Applied Probability*.

[7] Britton, T., Deijfen, M., Lindholm, M. and Lagerås, N. A.: Epidemics on random graphs with tunable clustering. *Journal of Applied Probability* **45** (2008), 743-756. MR-2455182

[8] Deijfen, M. and Kets, W.:, Random intersection graphs with tunable degree distribution and clustering, *Probab. Engrg. Inform. Sci.* **23** (2009), 661–674. MR-2535025

[9] Eschenauer, L. and Gligor, V. D.: *A key-management scheme for distributed sensor networks*, in: Proceedings of the 9th ACM Conference on Computer and Communications Security (2002), 41–47.

[10] Godehardt, E. and Jaworski, J.: Two models of random intersection graphs and their applications, *Electronic Notes in Discrete Mathematics* **10** (2001), 129–132. MR-2154493

[11] Godehardt, E. and Jaworski, J.: *Two models of random intersection graphs for classification*, in: Studies in Classification, Data Analysis and Knowledge Organization, Springer, Berlin–Heidelberg–New York, 2003, pp. 67–81. MR-2074223

[12] Godehardt, E., Jaworski, J. and Rybarczyk, K.: *Clustering coefficients of random intersection graphs*, in: Studies in Classification, Data Analysis and Knowledge Organization, Springer, Berlin–Heidelberg–New York, 2012, 243–253.

[13] Guillaume, J. L. and Latapy, M.: Bipartite structure of all complex networks, *Inform. Process. Lett.* **90** (2004), 215–221. MR-2054656

[14] Jaworski, J. and Stark, D.: The vertex degree distribution of passive random intersection graph models, *Combinatorics, Probability and Computing* **17** (2008), 549–558. MR-2433940

[15] Karoński, M., Scheinerman, E. R. and Singer-Cohen, K. B.: On random intersection graphs: The subgraph problem, *Combinatorics, Probability and Computing* **8** (1999), 131–159. MR-1684626

[16] Lee, S. H., Kim, P. J. and Jeong, H.: Statistical properties of sampled networks, *Physical Review E* **73** (2006), 016102.

[17] Newman, M. E. J., Strogatz, S. H. and Watts, D. J.: Random graphs with arbitrary degree distributions and their applications, *Physical Review E* **64** (2001), 026118.

[18] Newman, M. E. J.: Assortative Mixing in Networks, *Physical Review Letters* **89** (2002), 208701.

[19] Newman, M. E. J.: Mixing patterns in networks, *Physical Review E* **67** (2003), 026126. MR-1975193

[20] Newman, M. E. J.: Properties of highly clustered networks, *Physical Review E* **68** (2003), 026121.

[21] Newman, M. E. J. Watts, D. J. and Strogatz, S.H. : Random graph models of social networks, *Proc. Natl. Acad. Sci. USA* **99** (Suppl. 1) (2002), 2566–2572.

[22] Nikoletseas, S., Raptopoulos, C. and Spirakis, P. G.: On the independence number and hamiltonicity of uniform random intersection graphs, *Theoretical Computer Science* **412** (2011), 6750-6760. MR-2885093

[23] Pastor-Satorras, R., Vázquez, A. and Vespignani, A.: Dynamical and correlation properties of the internet, *Phys. Rev. Lett.* **87** (2001), 258701.

[24] Rybarczyk, K.: Diameter, connectivity, and phase transition of the uniform random intersection graph, *Discrete Mathematics* **311** (2011), 1998–2019. MR-2818878

[25] Stark, D.: The vertex degree distribution of random intersection graphs, *Random Structures Algorithms* **24** (2004), 249–258. MR-2068868

[26] Steele, J. M.: Le Cam's inequality and Poisson approximations, *The American Mathematical Monthly* **101** (1994), 48–54. MR-1252705

[27] Strogatz, S. H. and Watts, D. J.: Collective dynamics of small-world networks, *Nature* **393** (1998), 440–442.

[28] Yagan, O. and Makowski, A. M.: Zero-one laws for connectivity in random key graphs, *IEEE Transactions on Information Theory* **58** (2012), 2983 - 2999. MR-2952528

Acknowledgments. We thank the anonymous referee for valuable comments that helped to improve the manuscript. The work of M. Bloznelis and V. Kurauskas was supported in part by the Research Council of Lithuania grant MIP-053/2011. J. Jaworski acknowledges the support by National Science Centre - DEC-2011/01/B/ST1/03943.