ADDRESS

*Delivered * by the President, Professor H. H. Plaskett, on Astronomical Spectroscopy*

Introduction.—The stars are unrivalled sources of gently excited spectra, and through these spectra provide indispensable clues to our knowledge of atomic structure. Thus Huggins's discovery of the ultra-violet lines of hydrogen first made clear an obvious regularity in the wave-lengths of these lines, and led in turn to Balmer's law and Rydberg's identification of spectroscopic terms. Lockyer's recognition that a single atom could give normal and enhanced spectra, and A. Fowler's discovery that $\lambda 4686$ and the Pickering series, first found in O-type stars, were enhanced spectra of helium, each played a vital part in Bohr's application of the quantum theory to the nucleus atom and his demonstration that enhanced spectra, with series characterized by Rydberg constants $4R$, $9R$ etc. were due to ionized atoms which had lost one, two or more electrons. Again Saha's theory of thermal ionization and his description of the Harvard spectral classes as a temperature sequence, together with the work of R. H. Fowler and Milne on the significance of line maxima in this sequence, greatly stimulated the analysis into spectroscopic terms of the complex spectra and led to A. Fowler's and Russell's powerful contributions to this field. Lines of unknown origin, found in the spectra of the gaseous nebulae, the solar corona and various stars, led to Bowen's discovery that normally forbidden transitions between stationary states were occurring in the atoms in these sources ; the step-by-step identification of these lines by Bowen, Edlén, Merrill and others resulted in an intensive study of extreme ultra-violet spectra, notably by Edlén himself, and led to the discovery of further regularities among the stationary states in stripped atoms. The fact that in the solar spectrum alone there are between $\lambda 2914$ and $\lambda 13945$ some 8112 lines without identification † clearly indicates that the stars may still have some contributions to make to our knowledge of atomic structure. These investigations, in which astronomers have taken an honourable part, and which culminated in the development of quantum-mechanics, have had widespread consequences for physics and astronomy. With the solution, however, of its major problems, this phase of spectroscopy must be regarded as nearly ended, and the interest of physicists, at least, has tended to turn elsewhere.

To the physicist spectroscopy is only one weapon in a rich armoury. To the astronomer, on the other hand, spectroscopy provides almost the only means of learning about the physics of stellar atmospheres, and it is therefore not surprising that largely in his hands spectroscopy is passing into a second phase to be described as *astronomical spectroscopy*. By astronomical spectroscopy I shall mean those methods, involving an accurate knowledge of the atomic line-absorption coefficient and of the transfer of radiation in a gaseous medium, which permit a determination of the numbers of atoms in various stages of excitation and ionization in the stellar atmosphere. It is thus a quantitative technique for investigation in astrophysics just as X-ray analysis and electron diffraction are quantitative techniques for the investigation of the physics of crystals.

* The Address was a descriptive account of the topics here dealt with at greater length and in a slightly more systematic form.

† Private communication from Dr Charlotte Moore-Sitterly.

Astronomical spectroscopy may be said to have begun with the work of Schwarzschild on radiative transfer in continuous and line spectra, and in its early days significant contributions to it were made by Schuster, by Milne and by J. Q. Stewart. To A. Unsöld belongs, however, the honour of presenting in 1927 * the first papers on astronomical spectroscopy proper, while his *Physik der Sternatmosphären* with its complete list of references and its elegant presentation of theory and observation is the standard text-book of the subject. In these papers Unsöld was able to show from his own observations of the profiles of the *Na* D-lines across the disk of the Sun that the lines were formed closely in accordance with the equation of transfer set up by Schuster and Schwarzschild, and from the classical theory of dispersion he succeeded in finding the number of *Na* atoms in the ground state per square cm. column of the solar atmosphere. Following these papers development was rapid, and I need only remind you of Eddington's† clarification of the mode of formation of solar and stellar absorption lines, our Medallist's ‡ establishment of the curve of growth in the Sun, and Struve and Elvey's § discovery of "turbulence" in stellar atmospheres to show how soon astronomical spectroscopy established itself as a new mode of investigation.

Twenty years have elapsed since the publication of Unsöld's papers, and it is not inappropriate to attempt now to assess some of the results of this new mode of attack. I propose therefore this evening to take full advantage of the freedom given by Council to a President in his Annual Address and to discuss some of the work which has been done on astronomical spectroscopy. Since the whole subject is intimately bound up with the atomic line-absorption coefficient and the equation of transfer, I shall devote the first two parts of the address to these topics. In the third part I shall discuss some of the work which has been done on the curve of growth, and shall conclude by indicating some directions where, in my opinion, further investigation is needed.

1. *The Atomic Line-Absorption Coefficient*

1.1. *The Classical Oscillator.*—The classical oscillator is a fictitious mechanism of peculiar importance since alone among mechanisms obeying the simple laws of classical mechanics and electrodynamics it absorbs and emits a discrete frequency or line. It may therefore be anticipated that it will mimic in its behaviour some of the properties of the actual atoms obeying the laws of quantum-mechanics. The oscillator consists of an electron of charge ϵ and mass m, elastically attracted to a centre of force through which it oscillates in simple harmonic motion with a constant frequency ν_0. In the course of its motion the electron is accelerated, and in accordance with electrodynamics radiates energy of this frequency. The electron is therefore damped and its equation of motion is given by

$$\ddot{x}+(2\pi\nu_0)^2x+\gamma_0\dot{x}=0,$$

where

$$\gamma_0=\frac{8\pi^2\epsilon^2\nu_0^2}{3mc^3} \qquad (1)$$

is the classical damping constant. The amplitude of oscillation is

$$x(t)=x_0e^{-\gamma/2t}\cos(2\pi\nu_0t-\phi).$$

The oscillator will therefore emit a wave-train of constant frequency ν_0, and this wave-

* A. Unsöld, *Z. Phys.*, **44**, 793, 1927; **46**, 765, 1928.
† A. S. Eddington, *M.N.*, **89**, 620, 1929.
‡ M. Minnaert and G. F. W. Mulders, *Z. Astrophys.*, **1**, 192, 1930; **2**, 165, 1931.
§ O. Struve and C. T. Elvey, *Ap. J.*, **79**, 409, 1934.

train, observed at any point in its path, will show with the passage of time an exponential decrease of amplitude. It is only a homogeneous wave-train of infinite duration and constant amplitude which the spectroscope shows as a line of the frequency ν_0, while a non-homogeneous wave-train, such as that emitted by the classical oscillator, is analysed into the sum of an infinite number of homogeneous wave-trains of varying frequency and amplitude. Such an analysis is effected by Fourier's integral theorem, so that

$$x(t)=2x_0\int_0^\infty d\nu\int_0^\infty e^{-\frac{\gamma_0}{2}\tau}\cos 2\pi\nu_0\tau\times\cos 2\pi\nu(\tau-t)d\tau$$

$$=x_0\int_0^\infty\left[\frac{1}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}\right)^2}\right]^{\frac{1}{2}}\cos(2\pi\nu t-\psi)d\nu$$

neglecting terms with $(\nu+\nu_0)^2$ in the denominator. The intensity, $I_\nu d\nu$, proportional to the square of the amplitude, of the homogeneous wave-train with frequency between ν and $\nu+d\nu$ is therefore

$$I_\nu d\nu=\frac{K}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}\right)^2}d\nu,$$

where K is a constant of proportionality. If $I_0=\int_0^\infty I_\nu\, d\nu$ is the total intensity of the emitted line, we find

$$I_\nu d\nu=I_0\frac{\gamma_0}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}\right)^2}d\nu=I_0\frac{\delta_0}{\pi}\frac{1}{(\nu-\nu_0)^2+\delta_0^2}d\nu, \tag{2}$$

so that the damped wave emitted by the classical oscillator is analysed by the spectroscope into a broadened line whose half half-width (the distance from the centre of the line to the point where the intensity is one-half that at the centre) is given by

$$\delta_0=\frac{\gamma_0}{4\pi}=5\cdot08\times10^6\text{ sec.}^{-1}=0\cdot000059\text{ A}. \tag{3}$$

This half half-width, δ_0, is well below the limits of ordinary spectroscopic resolution, and it might therefore be thought that the widened line predicted by the classical oscillator was of no practical importance. However, beyond the limits $\pm\delta_0$ the intensity decreases only as $(\nu-\nu_0)^{-2}$, that is exceedingly slowly, so that the wings of the predicted emission line should be readily observable.

What is true of the emission of a classical oscillator is also true of its absorption. It may be shown* that on the passage of an electromagnetic wave through a medium of $\tilde{N}$ classical oscillators per unit volume there is exponential absorption and dispersion, the absorption-coefficient being given by

$$k_\nu d\nu=\frac{\pi\epsilon^2}{mc}\tilde{N}\frac{\gamma_0}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}\right)^2}d\nu=\frac{\pi\epsilon^2}{mc}\tilde{N}\frac{\delta_0}{\pi}\frac{1}{(\nu-\nu_0)^2+\delta_0^2}d\nu, \tag{4}$$

where γ_0 and δ_0 are defined as before. This absorption arises because the energy of the wave-train is used up in making the classical oscillators perform forced vibrations.

Broadening of the absorption or emission line given by a classical oscillator arises, however, not only from damping but also from collisions (Lorentz broadening). If the

* A. Unsöld, *Physik der Sternatmosphären*, p. 148, 1938.

effect of a collision is to quench the radiation emitted by a classical oscillator, then the amplitude of the emitted wave-train at any point in its path may be represented by

$$x(t)=x_0 e^{-\frac{\gamma_0}{2}t}\cos 2\pi\nu_0 t \quad \text{for} \quad 0\leqslant t<T$$
$$=0 \qquad \text{for } t<0,\ t\geqslant T,$$

where the wave-train commences at $t=0$, and is interrupted by a collision at $t=T$. Submitting this damped and interrupted wave train to a Fourier analysis we find

$$x(t)=2x_0\int_0^\infty d\nu\int_0^T e^{-\frac{\gamma_0}{2}\tau}\cos 2\pi\nu_0\tau\times\cos 2\pi\nu(\tau-t)d\tau$$

$$=x_0\int_0^\infty\left[\frac{1+e^{-\gamma_0 T}-2e^{-\frac{\gamma_0}{2}T}\cos 2\pi(\nu-\nu_0)T}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}\right)^2}\right]^{\frac{1}{2}}\cos(2\pi\nu t-\psi)d\nu$$

neglecting as before terms with $(\nu+\nu_0)^2$ in the denominator. The intensity of the homogeneous wave-train with a frequency between ν and $\nu+d\nu$ will be proportional to the square of the amplitude of that component, that is proportional to the expression in square brackets. If T_0 is the average time between collisions, then the fraction of the number of collisions between T and $T+dT$ will be

$$\frac{1}{T_0}e^{-T/T_0}dT.$$

Hence the average intensity of the homogeneous wave-train with frequency between ν and $\nu+d\nu$ will be given by

$$\bar{I}_\nu d\nu=\frac{Kd\nu}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}\right)^2}\int_0^\infty\frac{1}{T_0}e^{-T/T_0}\left[1+e^{-\gamma_0 T}-2e^{-\frac{\gamma_0}{2}T}\cos 2\pi(\nu-\nu_0)T\right]dT$$

$$=I_0\frac{2\left(\frac{\gamma_0}{2}+\frac{1}{T_0}\right)d\nu}{4\pi^2(\nu-\nu_0)^2+\left(\frac{\gamma_0}{2}+\frac{1}{T_0}\right)^2}=I_0\frac{\delta_0+\delta_c}{\pi}\frac{d\nu}{(\nu-\nu_0)^2+(\delta_0+\delta_c)^2} \qquad (5)$$

where I_0 is the total intensity of the emission line, $\delta_c=1/2\pi T_0$ and as before $\delta_0=\gamma_0/4\pi$. The collisional half half-width is therefore added to the damping half half-width, and a greatly broadened line may be produced by the classical oscillator if T_0 is sufficiently small, that is if the impact diameter of the oscillator is sufficiently large. Since T_0 decreases linearly with the pressure, the collisional half half-width must increase linearly with the pressure.

Are these characteristics of the emission and absorption lines of the classical oscillator actually observed? The line given by the classical oscillator is excited under the same conditions as is the resonance line of an actual quantum-atom, so that observation of a resonance line should provide a test. In a classical experiment Minkowski * observed the resonance D-lines produced by *Na* vapour in a metre-long absorption tube backed by a continuous source. With a Rowland concave grating used in the second order and having a practical resolving power of the order of 120,000, he found the profiles of the two absorption lines by accurate photographic spectrophotometry. A determination of the equivalent number of classical oscillators by magneto-rotation enabled him

* R. Minkowski, *Z. Phys.*, **36**, 839, 1926.

to calculate the half half-width of the greatly broadened lines produced by the long column of absorbing vapour. For pressures below 0·0087 mm. the half half-width of both lines was found to be 0·000058 A., in excellent agreement with the damping half half-width given in equation (3). Above this pressure the half half-width was greater than the damping value, clearly indicating the onset of Lorentz collisional-broadening.

1.2. *The Quantum-Atom — Natural Line Width.*—Minkowski's results, confirmed by those of many other workers, show that there are properties of the quantum-atom additional to those originally postulated by Bohr. Two steps are required to incorporate the behaviour of the classical oscillator into that of the quantum-atom. First consider a cylinder of unit cross-section and small thickness dz containing $\tilde{N}$ classical oscillators per cm.^{3}. If $I_\nu d\nu d\omega$ is the flux normally incident on the face of this cylinder within a solid angle $d\omega$, then the loss of flux per unit solid angle due to absorption of the classical oscillators will be

$$dI_\nu d\nu = -\tilde{N} a_\nu I_\nu d\nu dz,$$

where a_ν, the absorption coefficient for a single classical oscillator, is given by equation 1.1 (4) as

$$a_\nu d\nu = \frac{\pi\epsilon^2}{mc}\frac{\delta_0}{\pi}\frac{d\nu}{(\nu-\nu_0)^2+\delta_0^2}. \tag{6}$$

The total energy lost in the cylinder, integrating over all frequencies, will thus be

$$dI_\nu = \int_0^\infty (dI_\nu) d\nu = -\tilde{N}\frac{\pi\epsilon^2}{mc} I_\nu dz, \tag{7}$$

a somewhat remarkable result since the half half-width, δ_0, has disappeared. Next consider a geometrically identical cylinder, only now containing per cm.^{3} N_1 quantum-atoms in the ground state; by a transition from state 1 to state 2 these atoms can absorb a quantum $h\nu_0$ of radiation of the same mean frequency as that absorbed by the classical oscillators. Introducing the atomic Einstein-coefficient, B_{12}, where the number of transitions upwards per unit time is

$$\frac{dN_1}{dt} = -N_1 B_{12} u_\nu,$$

u_ν being the energy density of the radiation field, we find the total loss of flux per unit solid angle due to transitions from state 1 to 2, each absorbing the quantum $h\nu_0$, to be

$$dI_\nu = -N_1 B_{12} h\nu_0 \frac{I_\nu}{c} dz. \tag{8}$$

Equating the right-hand sides of (7) and (8), and assuming one quantum-atom equivalent to f classical oscillators, that is $\tilde{N} = N_1 f$, we get

$$N_1 f \frac{\pi\epsilon^2}{mc} = N_1 \frac{h\nu_0}{c} B_{12}\,;\; f = \frac{mh\nu_0}{\pi\epsilon^2} B_{12} = \frac{1}{3}\frac{q_2}{q_1}\frac{A_{21}}{\gamma_0}. \tag{9}$$

This value of f, the classical oscillator strength, required to make the total absorption of the quantum-atoms equal to that of the classical oscillators, is found from the well-known relation between B_{21} and A_{21}, defined by $\dfrac{dN_2}{dt} = -N_2 A_{21}$, namely

$$\frac{A_{21}}{B_{12}} = \frac{q_1}{q_2}\frac{8\pi h\nu^3}{c^3}, \tag{10}$$

where q_1, q_2 are the statistical weights of states 1, 2, and from the value of the classical damping constant given by equation 1.1 (1).

The second step is so to modify the quantum-atom as to give the line width. Since Bohr's frequency condition requires that the quantum $h\nu$ emitted or absorbed is equal to the difference of the energies of the two states concerned, it follows that a broadened line can only originate from transitions between broadened stationary states. It is, however, already clear from the uncertainty-principle, $\tau \Delta E \sim h/2\pi$, where τ is the life-time in the state, that the shorter the life-time, the greater must be the range of energy covered by the state. From the definition of the Einstein coefficient A_{21}, it follows that the average life-time in state 2 must be

$$\tau = \int_0^{N_2} t dN_2 \Big/ \int_0^{N_2} dN_2 = \int_0^\infty t A_{21} e^{-A_{21}t} dt \Big/ \int_0^\infty A_{21} e^{-A_{21}t} dt = 1/A_{21},$$

and from Wien's canal-ray experiments and from quantum mechanical calculations this is known to be of the order of 10^{-8} sec. Hence we may expect the excited states to be broadened, but the ground state, where the atom remains undisturbed for long periods of time, to be infinitely narrow. The resonance line of the quantum-atom, which bears so close a resemblance to the line emitted or absorbed by the classical oscillator, must therefore result from a transition between an infinitely narrow ground state and a broadened upper state. Let $W(E_2)dE_2$ be the probability that an atom in the broadened state 2 lies in a sub-level with an energy between E_2 and E_2+dE_2, and let a_{21}, b_{21}, and b_{12} be microscopic transition coefficients between the sub-levels of state 2 and the infinitely narrow state 1, where in accordance with Einstein's derivation of Planck's law

$$\frac{a_{21}}{b_{21}} = \frac{8\pi h\nu^3}{c^3} \; ; \; q_1 b_{12} = q_2 W(E_2) dE_2 \times b_{21}.$$

What form must the probability distribution $W(E_2)dE_2$ take in order that the quantum-atom will give the same absorption as the classical oscillator? Considering again our cylinder of thickness dz the flux absorbed will be per unit solid angle

$$N_1 f a_\nu d\nu \times I_\nu dz = N_1 \frac{h\nu}{c} b_{12} I_\nu dz = N_1 \frac{h\nu}{c} \frac{q_2}{q_1} W(E_2) dE_2 \frac{c^3}{8\pi h\nu^3} a_{21} \times I_\nu dZ,$$

where the left-hand side is the absorption for the $\tilde{N} = N_1 f$ classical oscillators and the right-hand side is the corresponding quantity for the actual quantum-atoms with the broadened upper state 2. Substituting the values of a_ν and f from equations (6) and (9), we find

$$W(E_2)dE_2 = \frac{\delta}{\pi} \frac{1}{(\nu-\nu_0)^2 + \delta^2} \frac{A_{21}}{a_{21}} d\nu.$$

Since by Bohr's frequency condition $h\nu = E_2 - E_1 : h\nu_0 = E_2^0 - E_1$ it follows that

$$W(E_2)dE_2 = \frac{\delta}{\pi} \frac{dE_2/h}{\left(\frac{E_2 - E_2^0}{h}\right)^2 + \delta^2} \frac{A_{21}}{a_{21}},$$

But

$$A_{21} = \int_0^\infty a_{21} W(E)_2 dE_2 = a_{21} \int_0^\infty W(E_2) dE_2 = a_{21}$$

if we assume by analogy with the sum rule for multiplets that a_{21} is the same for all sub-levels of state 2. Hence

$$W(E_2)dE_2 = \frac{\delta}{\pi} \frac{dE_2/h}{\left(\frac{E_2 - E_2^0}{h}\right)^2 + \delta^2}, \tag{11}$$

where

$$\int_0^\infty W(E_2)dE_2=1$$

as it should.

From the derivation of equation (11) it follows that $\delta=\gamma_0/4\pi$, the half half-width of the line emitted or absorbed by the classical oscillator. The mean time, τ_0, of dying-away of the radiation emitted by the classical oscillator is

$$\tau_0=\int_0^\infty te^{-\gamma_0 t}dt\bigg/\int_0^\infty e^{-\gamma_0 t}dt=1/\gamma_0,$$

so that if we equate the mean life-time in the excited state with the mean time of dying-away for the classical oscillator, it follows that the value of δ in our expression (11) is $\delta=A_{21}/4\pi$. This may agree with the classical value δ_0, as is the case for the *Na* D-lines in Minkowski's experiment, but need not necessarily do so. From this definition of the δ occurring in (11), it follows for a state $j>2$ that

$$\delta=\sum_{i=1}^{j-1} A_{ji}/4\pi,$$

where the summation is to be carried out over all lower states i for which transition from j are possible.

This elementary formulation of the complete quantum-mechanical theory of line broadening, due to Weisskopf and Wigner*, is sufficient to enable us to understand how in the actual quantum-atom the width of the line arises from the width of the upper state—a width which is inversely proportional to the life-time in that state.

When we come to consider lines which originate from transitions between two broadened levels, say i and j, the problem becomes slightly more complicated. By a further application of the process used in deriving an expression for $W(E_2)dE_2$ we find that the values of the microscopic transition coefficients for this problem are

$$\left.\begin{aligned} a_{ji}&=A_{ji}W(E_i)dE_i,\\ b_{ji}&=B_{ji}W(E_i)dE_i,\\ b_{ij}&=B_{ij}W(E_j)dE_j.\end{aligned}\right\}\qquad (12)$$

Now let us suppose the level j to be naturally excited, that is the distribution of atoms among the sub-levels of j is to be given by $W(E_j)$. Then the intensity of frequency ν given by a transition between a definite sub-level in j and a definite sub-level in i will be

$$h\nu\times N_k W(E_j)dE_j\times a_{ji}=N_jA_{ji}h\nu W(E_i)dE_i\times W(E_j)dE_j.$$

The total intensity of this component of the line will be found by integrating this expression over all values of E_j, subject to the condition that $E_j-E_i=h\nu$ is a constant. The resulting intensity-distribution in the emission line is found to be

$$I_\nu d\nu=\frac{h\nu}{4\pi}N_jA_{ji}\frac{\delta_i+\delta_j}{\pi}\frac{d\nu}{(\nu-\nu_0)^2+(\delta_i+\delta_j)^2},\qquad (13)$$

or the half half-width is the sum of the half half-widths of the two levels concerned. The same result is true for the absorption coefficient when the lower level, i, is naturally excited. When the initial level is not naturally excited the problem becomes even more complicated, but has been solved by Spitzer.† Thus in the simple case of monochromatic absorption from the broadened i level, but far from the centre, the resultant

* V. Weisskopf and E. Wigner, *Z. Phys.*, **63**, 54, 1930; V. Weisskopf, *Observatory*, **56**, 291, 1933.
† L. Spitzer, *M.N.*, **96**, 794, 1936.

transition from j to i gives rise to two emission components.* This is an example of so-called non-coherent scattering, which as Houtgast† has observationally shown, probably plays a large part in the formation of solar absorption lines.

1.3. *The Quantum-Atom—Doppler Width.*—So far we have treated the quantum-atom, and equally the classical oscillator, as being at rest. If the atom is in motion in the line of sight with the velocity u, the central frequency ν_0 of the absorption coefficient will be displaced by Doppler effect to $\nu_0-\frac{u}{c}\nu_0$. If $\phi(u)du$ is the fraction of atoms with a velocity between u and $u+du$ in the line of sight, then in thermodynamical equilibrium by Maxwell's law

$$\phi(u)du=\left(\frac{M}{2\pi kT}\right)^{\frac{1}{2}} e^{-Mu^2/2kT}du,$$

where M is the mass of the atom and k is Boltzmann's constant. The atomic absorption coefficient will then be obtained by averaging over all possible velocities, or

$$a_\nu=\frac{\pi\epsilon^2}{mc}f\int_{-\infty}^{\infty}\phi(u)\frac{\delta}{\pi}\frac{du}{\left(\nu-\nu_0+\frac{u}{c}\nu_0\right)^2+\delta^2}=\frac{\pi\epsilon^2}{mc}f\frac{1}{b\sqrt{\pi}}\left[e^{-\omega^2}-\frac{2\delta}{b\sqrt{\pi}}(1-2\omega F(\omega))\right],$$

where $$b=\sqrt{\frac{2kT}{M}}\frac{\nu_0}{c} : \omega=\frac{\nu-\nu_0}{b} : F(\omega)=e^{-\omega^2}\int_0^{\omega}e^{\omega^2}d\omega,$$

an expression first derived by Voigt. It will be noted that ω is simply $(\nu-\nu_0)$ expressed in units of b, while b itself is a measure of the width of the line due to Doppler effect alone. Under stellar conditions b has values between 0·01 A. and 0·1 A. A useful approximation to this expression for the absorption coefficient is given by

$$a_\nu=\frac{\pi\epsilon^2}{mc}f\frac{1}{b\sqrt{\pi}}\left[e^{-\omega^2}+\frac{\delta}{b\sqrt{\pi}}\frac{1}{\omega^2}\right]. \qquad (14)$$

The first term of this expression is to be used for small values of $\omega=\frac{\nu-\nu_0}{b}$, while for large values only the second term is to be used. The absorption coefficient consists therefore of an error-curve core, and outside of this core the natural line-widening takes charge giving wings decreasing as $(\nu-\nu_0)^{-2}$. It should be noted that Doppler widening also results in non-coherent scattering ‡, the absorbed quantum and the subsequently emitted quantum having different frequencies except under very restrictive circumstances.

1.4. *The Quantum-Atom—Impact-Width.*—Weisskopf § has shown from a quantum-mechanical calculation that the effect of impacts on the quantum-atom will be to produce exactly the same form of broadening as found by Lorentz for the classical oscillator. A similar conclusion had been reached earlier by Lenz from the correspondence-principle. It is clear, however, that the actual mechanism must be very different, since Bohr's frequency condition requires that the frequency absorbed or emitted is determined by the energies of the two levels concerned. It must therefore be supposed that the emitting or absorbing atom and the perturber form, during the time of the

* V. Weisskopf, *Observatory*, **56**, 291, 1933.
† J. Houtgast, Dissertation, Utrecht, 1942.
‡ L. G. Henyey, *Proc. Nat. Acad. Sc.*, **26**, 50, 1940.
§ V. Weisskopf, *Z. Phys.*, **75**, 287, 1932 ; *Phys. Z.*, **34**, 9, 1933.

collision, a single system, giving rise to a continuum of energy levels. For a time of collision short in comparison with the life-time in the excited state, this continuum will be different for the process of absorption and the subsequent process of emission, so that once again we shall get non-coherent scattering.

Impact-broadening, which is correctly given equally by the Lorentz theory and by quantum-mechanics, both of which lead to equation 1.1 (5), implies on the Lorentz theory an interruption of the wave-train by the collision, or a quenching of the radiation as by a collision of the second kind. It is well known from experiment, however, that impact-broadening can occur in the absence of quenching. To meet this difficulty Lenz * points out that a small percentage change in the 10^{-15} sec. period of the emitted radiation, produced by the perturbing field of the colliding atom during the average 10^{-13} sec. which the collision lasts, will be sufficient to produce a large change of phase between the wave-trains emitted before and after collision. Such non-coherent wave-trains act on Fourier analysis in precisely the same way as interrupted wave-trains, so that Lorentz or impact-broadening can occur in the absence of quenching. The impact diameter for such broadening is therefore that distance between the centres of the colliding atoms for which the perturbing field produces a perceptible change of phase. Weisskopf † arbitrarily takes a phase shift of unity as sufficient to produce non-coherent wave-trains, so that if $\Delta\nu(r)$ is the change of frequency when the perturber is at a distance r, then

$$\psi = \int_{-\infty}^{\infty} 2\pi\Delta\nu(r)dt \doteq 1.$$

If σ is the impact diameter and $\bar{v}$ the mean relative velocity of the colliding particles, then for a straight-line trajectory of the perturber

$$r = \sqrt{\bar{v}^2t^2 + \sigma^2}.$$

Two cases may now be distinguished. When the collisions are between identical oscillators or atoms we get a perturbation produced by dipole-interaction, so that

$$\Delta\nu(r) = \frac{C}{r^3},$$

where $C = \frac{\epsilon^2}{16\pi^2 m\nu_0} f$. Hence

$$\psi = \int_{-\infty}^{\infty} \frac{2\pi C dt}{(\bar{v}^2t^2+\sigma^2)^{\frac{3}{2}}} \doteq 1, \quad \text{or} \quad \sigma \doteq \sqrt{\frac{4\pi C}{\bar{v}}}. \tag{15}$$

When the collisions are between the radiating atom and a foreign atom, as generally holds in stellar atmospheres, the perturbing field is produced by Van der Waals's forces, so that

$$\Delta v(r) = \frac{C}{r^6}$$

and

$$\psi = \int_{-\infty}^{\infty} \frac{2\pi C dt}{(\bar{v}^2t^2 + \sigma^2)^3} \doteq 1, \quad \text{or} \quad \sigma \doteq \sqrt[5]{\frac{6\pi^2}{8}\frac{C}{\bar{v}}}. \tag{16}$$

In both these cases we know that the collisional half half-width is given by

$$\delta_c = \frac{1}{2\pi T_0},$$

* W. Lenz, *Z. Phys.*, **25**, 299, 1924.
† V. Weisskopf, *Z. Phys.*, **75**, 287, 1932.

where from the kinetic theory the mean time, T_0, between collisions is

$$\frac{1}{T_0}=2\sigma^2N\sqrt{2\pi kT\times\frac{M_1+M_2}{M_1M_2}},$$

M_1 and M_2 being the masses of the colliding atoms, and N the number of atoms per unit volume. Hence

$$\delta_c=\frac{\sigma^2N}{\pi}\sqrt{2\pi kT\times\frac{M_1+M_2}{M_1M_2}} \qquad (17)$$

from which we may either find δ_c from the value of σ given by (15) or (16), or alternatively can compute the impact diameter from the observed half half-width, δ_c. Minkowski's observations, earlier described, of the collisional half half-width in a *Na* absorption tube give excellent agreement between the observed σ and that predicted from (15). Subsequently Minkowski * observed the collisional half half-width for *Na* in an atmosphere of argon, and found under certain conditions a value of σ of $7\cdot9\times10^{-8}$ cm. A calculation of σ from (16) by Weisskopf † gave upper and lower limits to its value of $7\cdot6\times10^{-8}$ cm. and 10^{-7} cm., a not unsatisfactory agreement in view of the difficulty of evaluating the quantity C in (16).

1.5. *The Quantum-Atom—Statistical Broadening and a Summary.*—Statistical broadening of the atomic absorption coefficient ‡ in its limiting form arises when the radiating atom and the perturbers are at rest. The probability that the nearest perturber lies in a spherical shell centred on the radiating atom at a distance between r and $r+dr$ may be shown to be

$$W(r)dr=e^{-(r/r_0)^3}d(r/r_0)^3,$$

where r_0 is the average distance between perturbers. Hence, when N is the number of perturbers per unit volume

$$\tfrac{4}{3}\pi r_0^3N=1.$$

If the displacement in frequency produced by a perturber is

$$\Delta\nu(r)=C/r^n,$$

then the normal displacement, $\Delta\nu_0$, will be given by

$$\Delta\nu_0=C/r_0^n=C(\tfrac{4}{3}\pi N)^{n/3},$$

and in terms of $\Delta\nu_0$ the probability distribution will be

$$W(\nu)d\nu=e^{-(\Delta\nu_0/\Delta\nu)^{3/n}}\cdot\frac{3}{n}\left(\frac{\Delta\nu_0}{\Delta\nu}\right)^{\frac{3}{n}+1}\frac{d\nu}{\Delta\nu_0}.$$

If the radiating atom is subject to a linear Stark effect ($n=2$), as is for example hydrogen, then the nearest ion or electron with a charge of ϵ will produce a displacement

$$\Delta\nu=C/r^2=C\,.\,F/\epsilon,$$

where F is the resulting field-strength. The normal field-strength, F_0, produced by a perturber at r_0, will clearly be

$$F_0=\frac{\epsilon}{r_0^2}=\epsilon\left(\frac{4\pi}{3}N\right)^{\frac{2}{3}}=2\cdot60\epsilon N^{\frac{2}{3}}.$$

* R. Minkowski, *Z. Phys.*, **55**, 16, 1929.

† V. Weisskopf, *Phys. Z.*, **34**, 14, 1933.

‡ H. Margenau, *Phys. Rev.*, **40**, 387, 1932; H. Margenau and W. W. Watson, *Rev. Mod. Phys.*, **8**, 22, 1936.

From the theory of the linear Stark effect

$$C=\frac{3h}{8\pi^2 m}n_k,$$

where n_k is a quantum number. The original single line is thus split up into a number of components, and each component may be regarded as being smeared out in accordance with the probability-distribution given above. The resulting line profile will be found by adding these components, each weighted according to its theoretically known intensity. Reference may be made to the complete theory of Stark broadening given by Pannekoek and Verwey * where, following Holtsmark, the effect of ions or electrons, in addition to the nearest, is also included.

The difficult problem of deciding under what conditions the impact-broadening of Section 1.4 gives place to statistical broadening has been discussed by Burkhardt † and by Spitzer.‡ An analysis of their results, with special reference to stellar atmospheres, has been given by Unsöld § in the course of an admirable review of the problem of pressure-broadening. Apart from hydrogen where statistical broadening due to the linear Stark-effect ($n=2$) of ions is dominant, all other elements (with the possible exception of helium) show impact-broadening due primarily to van der Waals's forces ($n=6$).

Summarizing this lengthy discussion, the atomic line-absorption coefficient for a quantum-atom at rest, making transitions between the ground state and some higher state j, is given by

$$a_\nu d\nu=\frac{\pi\epsilon^2}{mc}f\frac{\delta+\delta_c}{\pi}\frac{d\nu}{(\nu-\nu_0)^2+(\delta+\delta_c)^2}, \tag{18}$$

where

$$\delta=\delta_j=\sum_{i=1}^{j-1}A_{ji}/4\pi, \text{ and } \delta_c=\frac{\sigma^2 N}{\pi}\sqrt{2\pi kT\times\frac{M_1+M_2}{M_1 M_2}},$$

σ being the impact diameter which under ordinary astronomical conditions may be found from equation (16). If the transition takes place between two broadened levels, i and j, where state i is naturally excited, then $\delta=\delta_i+\delta_j$. When the atoms have a Maxwellian velocity distribution, the atomic line-absorption coefficient is approximately given by

$$a_\nu d\nu=\frac{\pi\epsilon^2}{mc}f\frac{1}{b\sqrt{\pi}}\left[e^{-\omega^2}+\frac{\delta}{b\sqrt{\pi}}\frac{1}{\omega^2}\right]d\nu, \tag{19}$$

where

$$b=\sqrt{\frac{2kT}{M}}\frac{\nu_0}{c}, \quad \omega=\frac{\nu-\nu_0}{b}.$$

2. *The Equation of Transfer*

2.1. *The General Equation.*—To apply our atomic line-absorption coefficient to find the number of atoms per unit volume in a gaseous medium, we must find a relation between the intensity at any frequency ν in the emergent radiation and the absorption and emission coefficients. Consider the flux of radiation emerging within an elementary solid angle $d\omega$ from an element of area $d\sigma$ on the boundary of the medium; this is made up of the sum of the contributions, each weakened by the appropriate absorption, from

* A. Pannekoek and S. Verwey, *Proc. Amsterdam Acad.*, **38**, No. 5, 1935.
† G. Burkhardt, *Z. Phys.*, **115**, 592, 1940.
‡ L. Spitzer, *Phys. Rev.*, **58**, 348, 1940.
§ A. Unsöld, *V.J.S. Ast. Ges.*, **78**, 213, 1944.

the volume-elements of the medium which lie inside the cone of solid angle $d\omega$ on the medium-side of the vertex at $d\sigma$. Our first problem is therefore to find the gains and losses within one such volume-element in the medium, and this gives us the fundamental differential equation of transfer. Our second problem is to sum up the contributions from these volume-elements, that is to integrate the differential equation.

To find the equation of transfer, let us measure distance from the vertex of the cone into the medium by the coordinate r, the origin occurring at the vertex $d\sigma$. A volume-element of the gaseous medium at a distance r from the vertex will have a surface area $dS=r^2 d\omega$, and a thickness dr, dr being small compared with dS. Let $I_\nu(r, \psi)$ be the specific intensity of the radiation field at the point r and in a direction making an angle ψ with the negative direction of r. The flux normally incident on the surface dS and proceeding so as to strike $d\sigma$ on the boundary will be

$$I_\nu(r, 0)d\nu\, dS \frac{d\sigma}{r^2} = I_\nu(r, 0)d\nu\, d\sigma\, d\omega.$$

The loss due to absorption in the thickness dr of the volume-element will be

$$(k_0+k_\nu)I_\nu(r, 0)d\nu\, d\sigma\, d\omega\, dr,$$

where k_0 is the coefficient of continuous absorption of frequency ν in the medium at the position r, and $k_\nu = N\alpha_\nu$ is the coefficient of line absorption at the same frequency and position. The gain in the flux proceeding to strike $d\sigma$ from the emission in the volume-element will be

$$(j_0+j_\nu)d\nu\, dS\, dr \frac{d\sigma}{r^2} = (j_0+j_\nu)d\nu\, d\sigma\, d\omega\, dr,$$

where j_0 and j_ν are respectively the coefficients of continuous and line-emission (flux per unit volume per unit solid angle). The net increase in flux in the direction of r decreasing will therefore be

$$dI_\nu(r, 0)d\nu\, d\sigma\, d\omega = [(k_0+k_\nu)I_\nu(r, 0) - (j_0+j_\nu)]d\nu\, d\sigma\, d\omega\, dr$$

or

$$\frac{dI_\nu(r, 0)}{dr} = (k_0+k_\nu)I_\nu(r, 0) - (j_0+j_\nu), \qquad (1)$$

a differential equation which is valid for each volume-element of the medium lying inside the cone of solid angle $d\omega$ with a vertex at $d\sigma$ on the boundary.

The equation of transfer (1) is a non-homogeneous differential equation of the first order with a boundary condition given by the fact that there must be no radiation flowing into the surface element $d\sigma$ from outside the medium. Introducing an optical depth

$$t = \int_0^r (k_0+k_\nu)dr,$$

we can formally write down the solution of our differential equation in the form of an integral equation

$$I_\nu(0, 0) = I_\nu(t_1, 0)e^{-t_1} + \int_0^{t_1} \frac{j_0+j_\nu}{k_0+k} e^{-t} dt, \qquad (2)$$

where for sufficiently large values of t_1, $I_\nu(t_1, 0)e^{-t_1}$ equals zero. It follows from (2) that to find the emergent intensity we must first know the coefficients of emission and absorption as functions of optical depth, that is we must know the numbers of atoms in the required stationary states as functions of depth. This, however, is the very information we hope to obtain as a result of astronomical spectroscopy, and we seem to have reached an impasse. The usual way out of this difficulty is to assume the stellar atmosphere to be built on some simple model, that is to assume a specific mode of variation of the coefficients of emission and absorption as functions of optical depth. We may then carry out the integration indicated in (2), and from the observed intensity $I_\nu(0, 0)$ and the known values of the absorption and emission coefficients given in Section 1 we may find the numbers of atoms as functions of depth. If the results so obtained from a number of different absorption lines are consistent with each other and with the assumptions on which the model is based, we may feel confident that we have made a first step towards understanding the structure of the stellar atmosphere concerned.

2.2. *A Particular Form of Equation.*—Some of the difficulties involved in the integration of the equation of transfer disappear if the equation is simplified. The usual simplifying assumption is that of coherent scattering, that is the quantum of absorbed radiation is assumed to retain its identity while imprisoned by the atom, and ultimately to be re-emitted with its frequency unchanged. From our discussion in Section 1 we know that atoms do not behave this way, but in so difficult a problem it is justifiable to make this exploratory assumption. Let us commence by writing the coefficient of line emission in the form $j_\nu = j_\nu' + j_\nu''$. The first term, j_ν', is that part of the line-emission which results from atoms getting into the upper state of the line concerned by collisions of the first kind, by captures or by jumps from higher levels. If the atom which has thus got into the upper state now relapses to the lower state of the line concerned it will not necessarily, even on the assumption of coherent scattering, emit the same frequency as the one we are studying. In j_ν' we are, however, only concerned with the thus excited atoms which do emit this frequency. The second term, j_ν'' is that part of the line emission which results from the excitation of atoms to the upper state by absorption of the quantum $h\nu$ from the atoms in the lower state of the line concerned. Of these the fraction ϵ will suffer a collision of the second kind, or be otherwise prevented (by photo-ionization or by excitation to higher levels) from making a contribution to the line emission, while the fraction $1-\epsilon$ must necessarily emit, by the assumption of coherent scattering, a quantum of precisely the same frequency as that previously absorbed.

The term j_ν'' may be readily evaluated. Considering once again our elementary volume (Section 2.1) the flux incident on the face dS at an angle ψ with the normal and within the solid angle $d\Omega$ will be

$$I_\nu(r, \psi) d\nu \, dS \cos \theta d\Omega.$$

The energy absorbed will therefore be (since the thickness is $\sec\psi \times dr$)

$$k_\nu I_\nu(r, \psi) d\nu \, dS \, d\Omega \, dr.$$

Of the excited atoms so produced the fraction $1-\epsilon$ will spontaneously emit the quantum $h\nu$ and of these the fraction $\frac{\gamma(\psi)}{4\pi}\frac{d\sigma}{r^2}$ will emit their radiation in such a direction as to strike $d\sigma$. For isotropic scattering $\gamma(\psi)=1$, while for scattering according to Rayleigh's law $\gamma(\psi)=\frac{3}{4}(1+\cos^2\psi)$; in either event $\int \gamma(\psi)\frac{d\Omega}{4\pi}=1$. Hence the total useful flux emitted will be found by integrating over all directions, or

$$j_\nu'' \, d\nu \, d\sigma \, d\omega \, dr = (1-\epsilon) k_\nu \, d\nu \, d\sigma \, d\omega \, dr \int_{\text{sphere}} \gamma(\psi) I_\nu(r, \psi) \frac{d\Omega}{4\pi}.$$

It is usual to express the integral by the symbol J_ν, so that

$$j_\nu''=(1-\epsilon)k_\nu J_\nu.$$

If further the gaseous medium has a plane boundary and is stratified in layers plane-parallel to the boundary, we may introduce in place of r a new coordinate x measured inwards from $x=0$ at the boundary and normal to it. With this new coordinate the thickness dr of our volume-element will be given by $\sec\theta dx$, where θ is the angle between the outgoing direction and the negative direction of x. Our equation of transfer (1), with these new coordinates and on the assumption of coherent scattering, thus takes the form

$$\cos\theta\frac{dI_\nu(x,\theta)}{dx}=(k_0+k_\nu)I_\nu(x,\theta)-[j_0+j_\nu'+(1-\epsilon)k_\nu J_\nu]. \tag{3}$$

This form of the equation of transfer has a simple application to laboratory problems. Consider a tube of length x_1, such as that used by Minkowski, containing uniformly distributed atoms all in their ground state and backed by a source of continuous spectrum of specific intensity, or surface brightness, I_0 at the frequency ν. If the gas is cold, there is no continuous absorption or emission so that $k_0=j_0=0$; further, since there is no means whereby atoms may get into the excited state except by absorption of the quantum $h\nu$, we may put $j_\nu'=0$. The equation of transfer thus takes the form

$$\cos\theta\frac{dI_\nu(x,\theta)}{dx}=k_\nu I_\nu(x,\theta)-(1-\epsilon)k_\nu J_\nu. \tag{4}$$

Let us assume that the excited atoms are all de-excited by collisions of the second kind so that $\epsilon=1$. Then on integration of (4) we get for the emergent intensity in the line

$$I_\nu(0,\theta)=I_0e^{-k_\nu x_1\sec\theta}. \tag{5}$$

This is the usual exponential law of absorption, the absorbed energy being used to heat up the gas. This result is also nearly true when $\epsilon=0$ (pure scattering), provided an approximately parallel beam from the continuous source passes through the tube. This follows because the fraction of scattered radiation which gets into the small solid angle of the approximately parallel beam is itself very small. An exact treatment of this not simple problem for purely scattered radiation is given by Milne *, only with different boundary conditions, in his paper on planetary nebulae.

2.3. *Eddington's Equation of Transfer.*—In a stellar atmosphere, which may be reduced to a plane-parallel layer problem, we have a gas composed of atoms of different ionization potentials and free electrons, through which passes a flux of radiation from the interior of the star. Each volume-element of the gas gives rise to continuous absorption and emission due to photo-ionization and recombination of the atoms of low ionization potential and due to free-free electron switches, and also to line absorption and emission of the atoms in which we are interested when we study a particular absorption line. Eddington † was the first to apply the simplified equation of transfer (3) to this problem. Not only did this imply, as he was careful to point out, coherent scattering, but he introduced explicitly the additional assumptions that $j_\nu'=0$ (atoms excited only by the absorption of the quantum $h\nu$) and that $\gamma(\psi)=1$ (isotropic scattering). If the atmosphere is in local thermodynamic equilibrium we may put the coefficient of continuous emission, $j_0=(k_0+\epsilon k_\nu)B_\nu$, where B_ν is the specific intensity of black-body radiation

* E. A. Milne, *Z. Astrophys.*, **1**, 98, 1930.
† A. S. Eddington, *M.N.*, **89**, 620, 1929.

given by Planck's law for the temperature T appropriate to the depth x in the atmosphere. With these assumptions Eddington wrote equation (3) in the form

$$\cos\theta \frac{dI_\nu(x, \theta)}{dx} = (k_0+k_\nu)I_\nu(x, \theta) - (k_0+\epsilon k_\nu)B_\nu(x) - (1-\epsilon)k_\nu J_\nu(x). \qquad (6)$$

An equation of transfer of this form was applied by Milne * to a quite different problem in stellar atmospheres, and he was the first to integrate it ; consequently (6) is sometimes described as the Milne-Eddington equation.

Commencing with Milne the approximate integration of this equation of transfer has been effected in a variety of ways. One of the most useful, since it allows for arbitrary variations with x of the quantities k_ν/k_0 and $(1-\epsilon)k_\nu J_\nu$, is due to Strömgren.† Following Eddington he assumes that the specific intensity of the radiation-field may be represented by

$$I_\nu(x, \theta) = a + b\cos\theta + c\cos^3\theta + \ldots,$$

and introduces the symbols

$$J_\nu(x) = \int_{\text{sphere}} I_\nu(x, \theta)\frac{d\omega}{4\pi}; \quad F_\nu(x) = \int_{\text{sphere}} I_\nu(x, \theta)\cos\theta\frac{d\omega}{\pi}; \quad K_\nu(x) = \int_{\text{sphere}} I_\nu(x, \theta)\cos^2\theta\frac{d\omega}{4\pi}.$$

Of these $\pi F_\nu(x)$ is the net flux across unit area in the stellar atmosphere, and $F_\nu(0)$ is the net flux per unit solid angle at the boundary ; the latter is also equal to the average surface brightness of the hemisphere of the star exposed to the observer. If we now multiply equation (6) though by $d\omega$ and by *cos* θ $d\omega$ respectively and integrate over the sphere we get

$$\frac{d\frac{1}{4}F_\nu(x)}{dx} = (k_0+\epsilon k_\nu)(J_\nu - B_\nu); \quad \frac{dK_\nu(x)}{dx} = (k_0+k_\nu)\frac{F_\nu(x)}{4}. \qquad (7)$$

Introduce an optical depth defined by $\tau_\nu = \int_0^x (k_0+k_\nu)dx$, as well as the symbols $\eta = k_\nu/k_0$ and $\lambda = \frac{1+\epsilon\eta}{1+\eta}$. By virtue of our definition of J_ν and K_ν, and the assumption about the angular dependence of $I_\nu(x, \theta)$, it follows that $K_\nu = \frac{1}{3}J_\nu$. Hence equations (7) become

$$\frac{d^2J_\nu}{d\tau_\nu^2} = 3\lambda(J_\nu - B_\nu). \qquad (8)$$

This non-homogeneous second order differential equation, apart from the assumption about the angular dependence of $I_\nu(x, \theta)$, is exactly equivalent to the equation of transfer (6) ; in particular no assumption has yet had to be made about the dependence of λ and B_ν upon optical depth τ_ν. To find the specific intensity at any frequency in the emergent absorption line we must integrate (8) subject to two boundary conditions. First in the deep interior as $\tau_\nu \to \infty$, $J_\nu - B_\nu$ must not increase exponentially, and secondly at $\tau_\nu = 0$ there must be no flux of radiation from outside the star. Following Eddington it is assumed that $I_\nu(0, \theta) = a$, a constant, so that this second boundary condition can be expressed in the form $J_\nu(0) = \frac{1}{2}F_\nu(0)$.

Subject to these boundary conditions Strömgren now proceeds to integrate (8) by the method of variation of parameters. For our immediate purpose it will, however, suffice to consider the simplest model atmosphere where λ, that is η and ϵ are each independent of optical depth. From the theory of a stellar atmosphere in radiative

* E. A. Milne, *Phil. Trans.*, A, **223**, 237, 1922.
† B. Strömgren, *Ap. J.*, **86**, 1, 1937.

equilibrium it is known that B_ν is with sufficient accuracy a linear function of the optical depth $\tau_0=\int_0^x k_0 dx$ in the continuous spectrum, or $B_\nu=B_0(1+\beta\tau_0)$. Since by virtue of our assumptions about the constancy of the absorption coefficients with depth

$$\frac{\tau_0}{\tau_\nu}=\frac{k_0\int dx}{(k_0+k_\nu)\int dx}=\frac{1}{1+\eta},$$

it follows that $B_\nu=B_0\left(1+\frac{\beta}{1+\eta}\tau_\nu\right)$, where B_0 and β are known when the effective temperature of the star and the variation of k_0 with frequency are known. Our equation (8) may therefore be written

$$\frac{d^2(J_\nu-B_\nu)}{d\tau_\nu^2}=3\lambda(J_\nu-B_\nu),$$

and on integration subject to the two boundary conditions we find the average surface brightness, $F_\nu(0)$, at the frequency ν in the line to be

$$F_\nu(0)=\tfrac{4}{3}B_0\frac{\sqrt{3\lambda}+\beta/(1+\eta)}{1+2\sqrt{3\lambda}/3}.$$

The average surface brightness, $F_0(0)$, at the same frequency in the continuous spectrum, where $\eta=0$ and $\lambda=1$, will be

$$F_\nu(0)=\tfrac{4}{3}B_0\frac{\sqrt{3}+\beta}{1+2\sqrt{3}/3}.$$

Hence the ratio of intensity, r_ν, in the line to that in the adjacent continuous spectrum will be

$$r_\nu=\frac{F_\nu(0)}{F_0(0)}=\frac{\sqrt{3\lambda}+\beta/(1+\eta)}{\sqrt{3}+\beta}\times\frac{1+\frac{2}{3}\sqrt{3}}{1+\frac{2}{3}\sqrt{3\lambda}}. \tag{9}$$

When $\epsilon=0$, that is when there are no collisions of the second kind, and when $\eta\ll 1$, Unsöld * has shown that (9) may be written in the form

$$r_\nu=1-\frac{\eta}{2}\left[\frac{1}{1+2/\sqrt{3}}+\frac{1}{1+\sqrt{3}/\beta}\right], \tag{10}$$

an interesting result since the strength of the absorption line is seen to depend upon $\eta=k_\nu/k_0$—the larger k_0 the weaker the absorption line. Equation (10) may also be written in the form

$$r_\nu=1-\mu N\alpha_\nu, \tag{11}$$

giving the required dependence of the observed intensity ratio, r_ν, upon the atomic absorption coefficient (equation 1.5 (19)) and the number of atoms per unit volume. In this expression the quantity μ clearly depends upon the continuous absorption coefficient, k_0, and certain known constants. Equation (11) also applies to a line formed in pure absorption $\epsilon=1$, only with a different definition of the constant part of μ.

3. *The Curve of Growth*

3.1. *Equivalent Width.*—The ideal, still far short of attainment, in astronomical spectroscopy must be the construction of spectroscopes with practical resolving power of at least 10^6. From the accurate profiles determined with such instruments, and from the atomic line-absorption coefficient introduced into the integrated equation of transfer,

* A. Unsöld, *Physik der Sternatmosphären*, p. 241, 1938.

each absorption line would yield three physically significant parameters. First from the Doppler-core of the line would be determined the quantity b, giving the kinetic temperature of thermal agitation of the atom concerned. Secondly from the line wings would be found the half half-width, $\delta+\delta_c$, from which, knowing the radiation damping half half-width δ, δ_c and with it the optical impact diameter, σ, could be ascertained. Finally from the intensity ratio, r_ν, at any frequency in the line, given the oscillator strength and the previously determined parameters, the ratio of the number of atoms in the lower state to the continuous absorption coefficient could be found. Unfortunately existing spectroscopes fall far short of this ideal, and even the large solar instruments have practical resolving powers less than 2×10^5. The result is that at present the profiles of only a few strong lines (Rowland number >5) can be determined in the solar spectrum, and they only with moderate precision.* For stellar spectra the situation is correspondingly worse since the faintness of star-light makes the use of powerful spectroscopes impossible.

Pending the design and construction of the powerful spectroscope of the future, it is fortunate that in the meantime much can be learned from the equivalent width of an absorption line. From the conservation of energy it is evident that the spectroscope can influence in no way the total energy which has been absorbed in the line, so that the quantity

$$\int_0^\infty (F_0(0)-F_\nu(0))d\nu = F_0(0)\int_0^\infty (1-r_\nu)d\nu$$

is the same for all spectroscopes. The equivalent width

$$W_\nu = \int_0^\infty (1-r_\nu)d\nu \qquad (1)$$

has the dimensions of a frequency, and is a convenient expression for the total energy lost in the line expressed in units of $F_0(0)$, the average surface brightness in the adjacent continuous spectrum. Equivalent widths are generally expressed in angstroms, where

$$\frac{W}{\lambda}=\frac{W_\nu}{\nu}.$$

W, the equivalent width in angstroms, is therefore the width of a rectangular profile of zero central intensity which absorbs the same total energy as the actual profile.

3.2. *The Theoretical Curve of Growth.*—For the purpose of astronomical spectroscopy we now require the functional relation between the equivalent width of the line on the one hand and the number of atoms and the half half-width on the other. Such a relation is known as the curve of growth, and in giving its theory I shall follow the work of Menzel †, merely noting that an equivalent treatment has been given by Unsöld ‡ which requires, however, the determination of an additional parameter from the line profile, and at that one which is very difficult to find even with the best existing spectroscopes. From equation 2.3 (11) the intensity ratio in the absorption-line is given by

$$r_\nu = 1-\mu N a_\nu = \frac{1}{1+\mu N a_\nu}, \qquad (2)$$

which involves the assumption that the greatly extended line wings, where α_ν is small, effectively determine the equivalent width of the line. From the value of α_ν in equation

* R. O. Redman, *M.N.*, **98**, 325, 1938.
† D. H. Menzel, *Ap. J.*, **84**, 462, 1936 ; *Pop. Astr.*, **47**, 6, 66, 124, 1939.
‡ A. Unsöld, *Physik der Sternatmosphären*, p. 264, 1938.

(1.5) (19), putting

$$X_0=\mu\frac{\pi\epsilon^2}{mc}Nf\frac{1}{b\sqrt{\pi}}, \tag{3}$$

we find the equivalent width to be given by

$$W_\nu=\int_0^\infty(1-r_\nu)d\nu=\int_0^\infty\frac{d\nu}{1+\left[X_0\left(e^{-\omega^2}+\frac{\delta}{b\sqrt{\pi}}\frac{1}{\omega^2}\right)\right]^{-1}}.$$

Changing the variable of integration to $\omega=\frac{\nu-\nu_0}{b}$ and putting $b=\frac{\nu_0}{c}v$, where $v=\sqrt{\frac{2kT}{M}}$, we get (since the integrand is an even function of ω)

$$\frac{W_\nu}{\nu_0}=2\frac{v}{c}\int_0^\infty\frac{d\omega}{1+\left[X_0\left(e^{-\omega^2}+\frac{\delta}{b\sqrt{\pi}}\frac{1}{\omega^2}\right)\right]^{-1}}=\frac{W}{\lambda}. \tag{4}$$

As it stands the integral cannot be evaluated analytically, and we are forced to carry out numerical integrations for varying values of X_0. Taking advantage, however, of the conditions under which equation 1.5 (19) is valid we may treat analytically certain limiting cases.

(i) $X_0<1$, i.e. few atoms per unit volume. Then

$$\frac{W}{\lambda}=\frac{2v}{c}\int_0^\infty\frac{d\omega}{1+1/X_0e^{-\omega^2}}\doteq\sqrt{\pi}\frac{v}{c}X_0. \tag{5}$$

(ii) $X_0>1$, i.e. more atoms per unit volume, but not so many that the damping wings need be considered. Then by an asymptotic expansion

$$\frac{W}{\lambda}=\frac{2v}{c}\int_0^\infty\frac{d\omega}{1+1/X_0e^{-\omega^2}}\doteq\frac{2v}{c}(ln X_0)^{\frac{1}{2}}. \tag{6}$$

(iii) $X_0\gg1$, i.e. so many atoms that the Doppler-core may be neglected. Then

$$\frac{W}{\lambda}=\frac{2v}{c}\int_0^\infty\frac{d\omega}{1+\left(X_0\frac{\delta}{b\sqrt{\pi}}\right)^{-1}\omega^2}=\pi^{\frac{3}{4}}\sqrt{X_0\frac{v}{c}\frac{\delta}{\nu_0}}. \tag{7}$$

Taking logarithms of both sides in the limiting cases (i) and (iii) we see that

$$\log\left(\frac{W}{\lambda}\frac{c}{v}\right)=\tfrac{1}{2}\log\pi+\log X_0, \tag{8}$$

$$\log\left(\frac{W}{\lambda}\frac{c}{v}\right)=\tfrac{3}{4}\log\pi+\tfrac{1}{2}\log\frac{\delta}{\nu_0}\frac{c}{v}+\tfrac{1}{2}\log X_0, \tag{9}$$

or the theoretical curve of growth, where $\log\frac{W}{\lambda}\frac{c}{v}$ is plotted against $\log X_0$, commences with a linear portion with a slope of unity (the Doppler-part), and ends with a second linear portion with a slope of one-half (the damping portion) whose intercept on the $\log\frac{W}{\lambda}\frac{c}{v}$-axis is determined by the half half-width. The intermediate portion of the curve, which is almost flat, originates in the nearly square-shouldered Doppler-core to the atomic line-absorption coefficient, so that after the Doppler-core is saturated an increase in the number of atoms per unit volume produces little change in the equivalent width. The onset of the flat part will clearly be fixed by the width of this Doppler-core, that is by the quantity b, or equivalently v.

3.3. *Numbers of Atoms and Excitation Temperatures.*—While Schütz * and van der Held † were the first to give the theory of the complete curve of growth for exponential absorption in the laboratory, it was our Medallist ‡ who first constructed an empirical curve of growth for the Sun. Following a suggestion of Russell, Adams and Moore §, he plotted for individual multiplets the logarithm of his measured equivalent widths against the logarithm of the relative oscillator strengths, the latter being taken from the theoretical intensities of the lines deduced from the sum-rule and the correspondence-principle (subsequently confirmed by quantum-mechanics). This gave him small sections of the empirical curve of growth, and these were united into a single curve by a legitimate sliding of the individual sections along the $(\log f)$ axis. As a result he was the first to establish empirically the wave-length dependence of the equivalent width, the theoretical interpretation of which is given, following Menzel, in the previous sub-section, and the first to find the then surprising result that the half half-width of the line was some nine times greater than that predicted from natural line-widening.

The two outstanding recent discussions of the solar curve of growth have been by Menzel, Baker and Goldberg || and by K. O. Wright.¶ The first authors used the accurate equivalent widths determined by Allen **, and for relative f-values the computations by Goldberg †† of the theoretical intensities of the multiplets as a whole in a transition array. The numerous lines in a complete array covered a large range in X_0, and thus allowed a very accurate determination of the empirical curve of growth. Considering the lines of a single element, the horizontal displacement of the various multiplets in the $\left(\log \frac{W}{\lambda}, \log X_0\right)$ diagram gave in effect the ratio of the number of atoms in the ground states of the multiplets, and hence from Boltzmann's law the excitation-temperature of the solar atmosphere. The vertical displacement of this empirical curve of growth to bring it into coincidence with the theoretical curve, in which $\log\left(\frac{W}{\lambda}\frac{c}{v}\right)$ is plotted against $\log X_0$, gives a determination of v, and so of the kinetic temperature of thermal agitation of the atoms, while the branch of the theoretical damping curve on which the empirical curve lies gives an average value for the half half-width of the lines. In this way Menzel, Baker and Goldberg found excitation-temperatures of 4150 ± 50 deg. K. for *Fe*, and 4350 ± 50 deg. K. for *Ti*, a value of v of 0·6 km. sec.$^{-1}$ corresponding to a kinetic temperature of 5740 deg. K., and half half-widths approximately ten times the radiation-damping value.

These surprisingly low excitation temperatures have been confirmed by the already cited investigation of K. O. Wright. In the construction of his curve of growth Allen's equivalent widths were again used, but supplemented by Wright's own measures of equivalent widths from our Medallist's invaluable *Atlas of the Solar Spectrum.*‡‡ No systematic difference was found between the equivalent widths from these two sources, while the accidental differences amounted to 7 per cent for lines with $W < 0{\cdot}05$ A.,

* W. Schütz, *Z. Phys.*, **64**, 682, 1930.
† E. F. M. v. d. Held, *Z. Phys.*, **70**, 508, 1931.
‡ M. Minnaert and G. F. W. Mulders, *Z. Astrophys.*, **1**, 192, 1930 ; **2**, 165, 1931.
§ H. N. Russell, W. S. Adams and C. E. Moore, *Ap. J.*, **68**, 1, 1928.
|| D. H. Menzel, J. G. Baker and L. Goldberg, *Ap. J.*, **87**, 81, 1938.
¶ K. O. Wright, *Ap. J.*, **99**, 249, 1944.
** C. W. Allen, *Memoirs Commonwealth Solar Obs.*, **1**, No. 5, 1934.
†† L. Goldberg, *Ap. J.*, **82**, 1, 1935.
‡‡ M. Minnaert, G. F. W. Mulders and J. Houtgast, Amsterdam, 1940.

and to less than 6 per cent for stronger lines. The distinguishing feature of Wright's curve of growth, however, is his use of relative f-values determined by R. B. and A. S. King * from vacuum-furnace spectra of *Fe* and *Ti*. Plots of Goldberg's theoretical f-values against the Kings's empirical values show systematic differences presumably due to failure of the Russell-Saunders's coupling assumed by Goldberg, a failure already indicated by the presence of intersystem-lines. With an empirical curve of growth so constructed Wright finds excitation-temperatures for *Ti* and *Fe* of 4550 deg. K. and 4850 deg. K. respectively, while by comparison with Menzel's theoretical curve of growth a value of v of 0·9 km.sec.$^{-1}$ (corresponding to a kinetic temperature of 13,000 deg. K.) and a half half-width fifteen times the radiation-damping value. The differences between the excitation-temperatures found by Menzel and his colleagues on the one hand and by Wright on the other are presumably a result of the use of different f-values, but both investigations agree in placing these temperatures well below the effective temperature (an integrated brightness-temperature) of the Sun. This is also well confirmed by measures of the equivalent widths of solar molecular lines.† The disturbing feature of this discrepancy lies in the fact that atoms are lifted to their lower states by the absorption of continuous radiation, and to some extent by collisions of the first kind, so that excitation-temperatures should be nearly the same as brightness-temperatures and kinetic temperatures. The failure to obtain even approximate equality is one of the two major puzzles presented by the curve of growth.

3.4. *The Half Half-Width*.—Our Medallist's discovery from the curve of growth that the half half-width of the solar lines is some nine times greater than their radiation-damping width has been fully confirmed, as we have seen, from more recent investigations. In a subsequent paper ‡ he has determined equivalent widths of members of the *Mg* series, $2^1P—n^1D$, in the solar spectrum, and has found that these strong lines, which lie on the damping part of the curve of the growth, show an increase of equivalent width with series number n. From equations 3.2 (7) and (3) we see that

$$W \propto \lambda^{\frac{3}{2}}\sqrt{f\times\delta}$$

neglecting the variation of μ with frequency, and since f decreases as $1/n^3$, it follows that the half half-width, δ, must be increasing very rapidly with series number. The greater the series number the greater the impact-diameter of the atom, and there can be little question that both in this phenomenon and in the swollen half half-widths revealed by the curve of growth we have to do with impact-widening adding itself, in accordance with equation 1.5 (18), to the natural line width.

Strömgren § has recently given a highly satisfactory, quantitative explanation of the large impact-widening shown by solar lines. Following Wildt ‖, Strömgren attributes the continuous absorption coefficient, k_0, to the photo-ionization of negative hydrogen ions. Since from equation 2.3 (10) and more clearly from 2.3 (11) the intensity ratio, r_ν, in a line depends upon the ratio $\eta=k_\nu/k_0$, it follows that lines due to different elements will give the relative abundance of negative hydrogen, and hence from Saha's ionization formula the relative abundance of atomic hydrogen. Hydrogen thus turns out to be nearly 10,000 times as abundant as all the metals combined, even greater than the

* R. B. and A. S. King, *Ap. J.*, **87**, 24, 1938.
† M. G. Adam, *M.N.*, **98**, 544, 1938.
‡ M. Minnaert and J. Genard, *Z. Astrophys.*, **10**, 377, 1935.
§ B. Strömgren, *Festschrift für E. Strömgren*, p. 218, 1940.
‖ R. Wildt, *Ap. J.*, **89**, 295, 1939 ; **90**, 611, 1939.

surprisingly high abundance predicted by Russell * in 1933. Strömgren now calculates from equation 1.4 (16) the optical impact diameter for *Na* atoms absorbing the D lines in an atmosphere of hydrogen, and finds a value of $\sigma = 4 \cdot 6 \times 10^{-8}$ cm. From the abundance of hydrogen and this value of σ, he is thus enabled by equation 1.4 (17) to calculate the collisional half half-width, δ_c. Allowing for the variation with depth in the number of negative hydrogen ions and hydrogen atoms which would be expected in an atmosphere in radiative and local thermodynamic equilibrium, he integrates the equation of transfer in accordance with the method developed by himself and mentioned in Section 2.3. From this integration, from the approximate quantum-mechanical calculation of the natural line half half-width, and from his own calculation of the collisional half half-width he is thus able to predict the profile of the D lines, and their equivalent widths. Good agreement with observation is obtained, and we may therefore regard it as established that the large half half-width of the solar lines is due to collisions with the abundant hydrogen atoms. A further confirmation of this result has been obtained by ten Bruggencate and Houtgast † from their determination of the impact-widths of two *Fe*-multiplets observed across the disc of the Sun.

3.5. *Stellar Atmospheres.*—In spite of their generally inadequate resolving power and dispersion, stellar spectroscopes can in principle give accurate equivalent widths. It is therefore possible, in the same manner as for the Sun, to construct empirical curves of growth, and to find quantitative values of excitation temperature, half half-width and of the parameter v—thus opening up an almost inexhaustible field for the quantitative study of stellar atmospheres. A striking example of such an investigation is that of Unsöld ‡ on the B0 star, τ Scorpii. From the plates obtained by Struve and himself with the Coudé and Cassegrain spectroscopes of the McDonald Observatory, Unsöld determines the equivalent widths of some 200 lines due to *H*, *He* I and II, *C* II and III, *N* II and III, *O* II and III, *Ne* II, *Mg* II, *Al* III, and *Si* III and IV. He uses theoretical oscillator strengths from multiplets and transition arrays, provisionally converted to absolute values from one or two lines which arise from transitions between hydrogen-like terms, and thus determines an empirical curve of growth for each stage of ionization of each element. His theoretical curve of growth is based on his own theory, but his results would have been essentially unchanged if Menzel's theory, as described in Section 3.2, had been used. From a comparison of the empirical and theoretical curves of growth he derives relative numbers of atoms of the various elements in the rth stage of ionization and the sth level of excitation. Owing to the high level of excitation of the lines occurring in this stellar spectrum, the number of atoms in the $(r+1)$th stage of ionization, found from an application of Boltzmann's and Saha's relations, is relatively insensitive to the assumed temperature. Assuming a temperature of 25,200 deg. K., he is thus able to derive provisional values for the number of atoms in the $(r+1)$th and the $(r+2)$th stages of ionization, provided lines from the two preceding stages are visible in the spectrum, and from these numbers by a further application of Saha's equation a provisional value for the electron-pressure and density. From the theory of Pannekoek, confirmed by the observations of Mohler and re-discussed by Inglis and Teller §, the last observable line of the Balmer series is determined by the Stark-broadening of the lines, that is by the density of the ions which in a B-type star is also

* H. N. Russell, *Ap. J.*, **78**, 239, 1933.
† P. ten Bruggencate and J. Houtgast, *Z. Astrophys.*, **20**, 149, 1940.
‡ A. Unsöld, *Z. Astrophys.*, **21**, 1, 22, 1941.
§ D. R. Inglis and E. Teller, *Ap. J.*, **90**, 439, 1939.

approximately the density of the electrons. This directly observed electron-density is greater than that found from the numbers of atoms in different stages of ionization, and hence the assumed temperature must be too low. By trial and error Unsöld thus finds the correct temperature to be 28,150 deg. K. With this temperature revised values for the relative numbers of atoms in different stages of ionization are obtained, and so finally the relative abundance (in atoms)

$$H : He : \text{Rest of elements} = 85 : 15 : 0{\cdot}24.$$

In particular $\log (H : Mg) = 4{\cdot}24$, in good agreement with Strömgren's value of 4·4 in the solar atmosphere. In a fourth paper * Unsöld confirms his $H : He$ ratio from the theory of Stark-broadening and the observed equivalent widths of the stronger lines of these two elements. Finally in a third paper † he discusses the ratio $k_0/\bar{k}$, of the continuous absorption coefficient at a given frequency to the Rosseland mean coefficient, comparing the observed value derived from the central intensities of the strong lines on the assumption that they are formed by pure absorption ($\epsilon = 1$), with the value predicted from the theory of the continuous absorption coefficient and the relative abundance of the elements.

An early and remarkable discovery by Struve and Elvey ‡ still dominates the literature of stellar curves of growth. Empirical curves of growth were found by them for the five stars—α Canis Majoris (A 2), α Lyrae (A0), α Cygni (cA2), α Persei (F 5), and ε Aurigae (F5p). In addition they considered the curve of growth for 17 Leporis (A0) determined by Hynek.§ For the first three stars a comparison of the empirical with the theoretical curve of growth yielded not unexpected values for the half half-width and the parameter v, but the remaining three stars showed a greatly extended Doppler-part corresponding to values of v of 7 km. sec.$^{-1}$ for α Persei, 20 km. sec.$^{-1}$ for ε Aurigae and 67 km. sec.$^{-1}$ for 17 Leporis, that is to kinetic temperatures respectively of 3×10^5, 2×10^6 and 3×10^7 deg. K. Such temperatures must be fictitious, and following a suggestion of McCrea ‖, Struve and Elvey propose that the parameter v in these stars is given by $v^2 = \frac{2kT}{M} + v_t$, where v_t is the velocity component in the line of sight of masses of gas in turbulent motion with a Maxwellian distribution.

On the other hand, it has been suggested that the large values of v for these three stars arose from incorrect construction of the empirical curves of growth. Struve and Elvey's work was completed before the theory of the curve of growth was completely understood, and they plotted $\log W$, instead of $\log W/\lambda$, against $\log X_0$, a procedure which, if the lines covered a large range of wave-lengths, would give a fictitious extension to the Doppler-part of the curve of growth. In a private communication, however, Struve has pointed out that the great majority of their lines lay in the narrow range λλ 4300–4600, so that this criticism loses much of its force. In this connection the recent investigation of the curve of growth in α Persei by K. O. Wright ¶ is of the first importance. With a dispersion averaging 7 A. per mm. he has measured with great care the equivalent widths of over 1000 lines. Using empirical oscillator strengths taken from his earlier work on the Sun, he finds values of v of 3·7 km. sec.$^{-1}$ for neutral

* A. Unsöld, *Z. Astrophys.*, **23**, 75, 1944.
† A. Unsöld, *Z. Astrophys.*, **21**, 229, 1942.
‡ O. Struve and C. T. Elvey, *Ap. J.*, **79**, 409, 1934.
§ J. A. Hynek, *Ap. J.*, **78**, 54, 1933.
‖ W. H. McCrea, *M.N.*, **89**, 718, 1929.
¶ K. O. Wright, *J.R.A.S. Canada*, **40**, 183, 1946.

and of 6·4 km. sec.$^{-1}$ for ionized lines, thus confirming the results of Struve and Elvey. There thus can be no question as to the reality of their discovery.

Its interpretation by " turbulence " is more open to question. In granulation, as observed on the Sun, we are familiar with a bodily motion of masses of gas in vertical streams, the motion being maintained by the energy of ionization of the abundant hydrogen, but the phenomenon of granulation does not lead to an increase in the equivalent widths of the spectrum lines. Each granule over the disk of the Sun contributes its own slightly displaced absorption line, and the sum of these displaced lines results in a washed-out profile of unchanged equivalent width. Only in the event of the moving masses of gas overlying each other and each making its own contribution to the observed absorption line will there be an increase of equivalent width.* Under these conditions the masses of gas will collide with each other, but these collisions will not be, as Milne † has pointed out, the elastic collisions contemplated in the kinetic theory. The collisions will be collisions between the individual atoms of the gas-masses, not of the gas-masses as units, and the result will be a dissipation of kinetic energy into energy of thermal agitation of the atoms. " Turbulence " therefore affords no escape from kinetic temperatures of the order of millions of degrees.

3.6. *Emission-Line Stars.*—Menzel ‡ has proposed an alternative interpretation of Struve and Elvey's curves of growth. It has long been known that many stellar lines are on the point of appearing in emission—a striking example being the *He* II line, λ 4686 which should be the strongest *He* II line in the absorption-line O-type stars, but is normally either weak, absent or appears in emission. Other examples are provided by the Balmer lines in Be stars where the early lines of the series appear in emission and the later in absorption. What is true of these stars may well be true in a less marked degree of any star. If now, as is normally assumed, the emission-line is formed above the much stronger absorption line, then as a first approximation we can regard the observed equivalent width as the algebraic sum of the absorption and emission component (assumed transparent), each with its own appropriate curves of growth. Menzel has constructed composite curves of growth in this way, and has found that the Doppler-parts of the curves, just as in Struve and Elvey's curves, are much extended. This is a most suggestive result, and provided emission-lines were formed above the absorption-lines it could provisionally be accepted as an adequate interpretation, pending a more exact treatment by integration of an appropriate equation of transfer through the stellar atmosphere. Unfortunately the evidence for the high-level production of emission is not conclusive. Consider for example the Wolf-Rayet stars—generally regarded as presenting the strongest evidence for emission-lines occurring in an extended envelope about the star.

To C. S. Beals § we owe our modern knowledge of the spectra of these stars. He is also the author of the currently accepted theory of their line-emission. These spectra are characterized by emission-lines of atomic origin with widths in some stars of 60 to 100 A. and with intensities some 10 to 20 times that of the adjacent continuous spectrum. The lines are due to *He* II (from the decrement in the Pickering series the Balmer lines of *H* are weak or absent), to *C* II, III, IV, *N* III, IV, V, *O* III, IV, V, VI and

* A. Unsöld, *Physik der Sternatmosphären*, p. 272, 1938; H. Zanstra, private communication.
† S. Chandrasekhar, *M.N.*, **94**, footnote on p. 16, 1933.
‡ D. H. Menzel, *Pop. Astr.*, **47**, 6, 66, 124, 1939.
§ C. S. Beals, *Publ. Dom. Astrophys. Obs., Victoria*, **4**, 271, 1930 ; **6**, 95, 1934 ; *J.R.A.S. Canada*, **34**, 169, 1940.

Si II, III, IV. The identification of the lines in the higher stages of ionization of carbon, nitrogen and oxygen is due to Edlén.* The stars may be classified in two parallel temperature-sequences, in one of which carbon is present to the exclusion of nitrogen (type WC), and in the other of which nitrogen excludes carbon (type WN). From careful spectrophotometry Beals has found that the profiles of these lines in some stars show a flat top, and has shown that these profiles can be interpreted as the sum of emission-lines each displaced by Doppler effect due to the uniform radial outward motion of the atoms. From the symmetry of the lines and the absence of occultation-effect†, it follows on this hypothesis that the star proper, which provides the continuous spectrum, must be small in comparison with the atmosphere of radially ejected atoms which provide the emission-lines. The atoms are thus bathed in a dilute radiation-field ‡, and cyclic transitions will tend to produce emission-lines. In the limit, such as strictly occurs only in the gaseous nebulae, we get pure ionization and recombination, and may apply Zanstra's § theory to find the ultra-violet brightness-temperature of the star. Beals finds the resulting temperatures to lie between 100,000 deg. K. for WN 5 and WC 6 and 60,000 deg. K. for WN 8 and WC 8.

Though the application of Zanstra's theory to the atmospheres of Wolf-Rayet stars is open to question, the picture of the process of emission is at least self-consistent. Beals, however, finds that the width of the emission-lines increases with decreasing ionization-potential of the carrier, and to account for this is forced to assume (i) that in their radial motion outwards the atoms are accelerated, and (ii) that progressive absorption of the ultra-violet continuous radiation from the star produces a stratification of emission similar to that shown by the planetary nebulae. From Chandrasekhar's investigation (previously cited) the first of these hypotheses leads, on any physically probable law of emission as a function of distance from the star, to rounded or peaked line-profiles, instead of the flat-topped ones so characteristically observed. The second hypothesis implies that the emission-line envelope has a great optical depth for the continuous ultra-violet radiation from the star, and it becomes difficult to see how enough of this ultra-violet radiation can escape from the envelope for the Wolf-Rayet star to act as the nucleus and exciting source for a planetary nebula—as in fact happens in such typical planetary nebulae of high surface-brightness as NGC I 418, NGC 6543, NGC 6572 and NGC II 5217. These difficulties are not necessarily insuperable, but they do render Beals's picture of the emission in Wolf-Rayet less attractive.

A detailed study has recently been made of the star HD 193576 (=V 444 Cygni). This star was discovered by O. C. Wilson ‖ to be a spectroscopic binary, consisting of a Wolf-Rayet component (WN 5·5) and an absorption-line O-type component (O6) ¶ moving in circular orbits in a period of 4·21 days. The mean velocity of the Wolf-Rayet component, according to Wilson, is 56 km. sec.$^{-1}$, and of the O-type component is -34 km. sec.$^{-1}$, the latter being about the velocity of the system as judged from the radial velocities of early-type stars in the same vicinity of the sky. He therefore concluded that the lines of the Wolf-Rayet component had a red-shift of 90 km. sec.$^{-1}$, and he offered some evidence ** that the Wolf-Rayet stars as a class showed a similar

* B. Edlén, *Z. Astrophys.*, **7**, 378, 1933.
† S. Chandrasekhar, *M.N.*, **94**, 522, 1934.
‡ S. Rosseland, *Ap. J.*, **63**, 218, 1926.
§ H. Zanstra, *Ap. J.*, **65**, 50, 1927; *Publ. Dom. Astrophys. Obs.*, *Victoria*, **4**, 209, 1931.
‖ O. C. Wilson, *Ap. J.*, **91**, 379, 1940.
¶ C. S. Beals, *M.N.*, **104**, 205, 1944.
** O. C. Wilson, *Ap. J.*, **91**, 394, 1940.

red-shift. He discards after examination the hypothesis that this red-shift is a result of absorption-lines lying to the violet of the emission lines, and concludes that it is due to an Einstein gravitational displacement. If this be true, other not inapplicable data for HD 193576 suggest that the emission-lines are formed at the surface of a star with a radius rather smaller than that of Jupiter. Subsequently Gaposchkin * has found that HD 193576 is an eclipsing variable, and has published a provisional photometric orbit. An accurate light-curve by Kron and Gordon † has, however, revealed a surprising feature for an eclipsing variable with a circular orbit, namely that the primary minimum, when the O-type star passes behind the Wolf-Rayet star, is nearly twice as wide as the secondary minimum. This feature of the light-curve has been discussed by Kopal, Russell and Mrs Shapley ‡, and these writers conclude that the Wolf-Rayet star must be surrounded by an extensive, semi-transparent envelope which itself emits no light, but which is sufficiently opaque to absorb some of the light from the O-type star as it passes behind. Since Kron's potassium hydride photoelectric cell is sensitive in a half width 4100 to 4800 A., containing according to Beals (*loc. cit.*) much the strongest emission-line, *He* II, λ 4686, in the Wolf-Rayet spectrum, it does not seem probable that the emission-lines can originate in this non-luminous envelope. A re-interpretation of the photometric orbit in the light of Beals's careful photometric study of variations in the spectrum may reverse this verdict, but at present at least the star HD 193576 does not give strong support § for the hypothesis that the emission-lines of Wolf-Rayet stars are necessarily formed high-up in the atmosphere.

On the other hand emission-line stars, notably the Me stars, are known where it is reasonably certain that the emission-lines are formed below the absorption-lines. The hydrogen emission-lines in these stars ‖ have a peculiar decrement with $H\delta$ as the strongest line closely followed by $H\gamma$ and $H\zeta$. The other hydrogen emission-lines, which are much weaker, occur where there are strong absorption lines due to *Ca* II (H line), *Fe* I, *V* I and heavy absorption-bands of *TiO*, all presumably overlying the emission-lines.

Returning now to Menzel's interpretation of the extended Doppler-part of Struve and Elvey's curves of growth, the reproduction of this anomaly by composite curves of growth cannot be regarded as entirely convincing. When the evidence for the high-level formation of emission-lines in Wolf-Rayet stars is suspect, and when the Me stars have their absorption-lines formed above the emission-lines, it is no longer legitimate to assume, even as a first approximation, that the equivalent width of a composite line will be given by the algebraic addition of its absorption- and emission-components. Thus, though Menzel's interpretation is physically more satisfactory than that of "turbulence", it can only be established by integration of an equation of transfer containing the part, j'_ν of the line-emission coefficient which Eddington omitted (Section 2.3). Until this has been done and the extended Doppler-part of the curve of growth obtained, Struve and Elvey's "turbulence" must be added to the low excitation-temperature of the Sun as the second major puzzle presented by the curve of growth.

* S. Gaposchkin, *Ap. J.*, **93**, 202, 1941.

† G. E. Kron and K. C. Gordon, *Ap. J.*, **97**, 311, 1943.

‡ Z. Kopal, *Ap. J.*, **100**, 204, 1944; H. N. Russell, *Ap. J.*, **100**, 213, 1944; Z. Kopal and M. B. Shapley, *Ap. J.*, **104**, 160, 1946.

§ O. C. Wilson, *Ap. J.*, **95**, 402, 1942.

‖ P. W. Merrill, *Spectra of Long-Period Variable Stars*, pp. 45, 71, 1940.

4. *Future Work in Astronomical Spectroscopy*

4.1. *The Future.*—In the curve of growth we have considered one application of astronomical spectroscopy to the problems of astrophysics. Another application, even more promising, is to the profiles of solar absorption-lines obtained by high-resolution spectroscopy, but here I can only refer to the invaluable pioneer work in this field by Shane, Redman and Houtgast.* Enough has been said, however, at least to indicate the great power of this new technique in astrophysics, and it only remains to consider where improvements can be made in its future application. On three points at least immediate research is required. These are the observational determination of equivalent widths, the values of atomic constants and the modification of the equation of transfer. Let us consider each of these.

4.2. *Determination of Equivalent Widths.*—Two requirements must be satisfied if an accurate equivalent width is to be obtained. In the first place the profile of the observed line, the resultant of the true profile and the slit-pattern of the spectroscope, must be determined by accurate photographic spectrophotometry, especial care being taken to avoid systematic errors arising both from faulty calibration and from Eberhard-effect. The spectrophotometry (and indeed the wave-length measurement) must meet the same high standards which are satisfied in the determination of true profiles by high-resolution spectroscopy. The accuracy, as might have been anticipated, increases with increase of dispersion, and it is no doubt due to this, as well as to the accurate spectrophotometry, that the equivalent widths of solar lines found by Allen and by K. O. Wright (from measures in the *Utrecht Atlas*) are in such relatively good agreement.

In the second place particular care must be taken in fitting the wings of the observed profile into the continuous background. Provided the resolving power of the spectroscope is sufficiently great, our Medallist's † method should always be followed. From equations 3.2 (2) and 1.5 (19), expressed in wave-lengths, it follows that in the *wings* of a line-profile

$$r_\lambda = \frac{1}{1 + C/(\Delta\lambda)^2},$$

where C is a constant. In general the blackening of the plate due to the continuous background varies across the width of the line, either because of a variation in intensity or sensitivity of the plate. If this variation is linear, and in a small wave-length region it can generally be so regarded, we may write the intensity-ratio in the line in the form

$$r_\lambda = \frac{I_\lambda}{I_0 + m \times \Delta\lambda},$$

where m is a constant to be found from measures of the continuous background on either side of the line and well separated from it. From these two equations we thus find

$$I_\lambda - m \times \Delta\lambda = I_0 - C \frac{I_\lambda}{(\Delta\lambda)^2}. \qquad (1)$$

Consequently if we plot $I_\lambda - m \times \Delta\lambda$ against $I_\lambda/(\Delta\lambda)^2$, we shall get a straight line whose slope is C and whose intercept on the axis of ordinates is the required value of I_0, the intensity of the true continuous spectrum at the centre of the line. The error in equivalent width which results from a small error in the intensity of the continuous

* C. D. Shane, *L. O. Bull.*, **16**, 76, 1932 ; **19**, 119, 1941; R. O. Redman, *M.N.*, **95**, 742, 1935; **98**, 325, 1938 (a review); J. Houtgast, Dissertation, Utrecht, 1942.

† M. Minnaert, *Z. Astrophys.*, **10**, 40, 1935.

background, such as may result from the usual method of drawing in the background by eye, may be approximately evaluated by this same method, and Minnaert finds that when the background is put in at $0{\cdot}96 I_0$, the measured equivalent width is some 23 per cent too small—a disastrously large error. On the other hand this method is not applicable when the line is so faint that the observed profile is wholly determined by the slit-pattern of the spectroscope. In this event, however, the faint lines are of a standard shape, and a constant times the equivalent width can be determined by the measurement of the central intensity; the constant can be evaluated by finding the equivalent widths of a number of faint lines by the ordinary method.

As long as high dispersion and reasonable resolving power can be used, as in solar spectra, these two requirements can be met and should suffice to give accurate equivalent widths. In the spectra of stars, particularly the interesting faint ones, high dispersion cannot be employed, and precautions must be doubled to be sure the two requirements are satisfied. Particularly insidious are errors due to Eberhard-effect and to instrumental blending of lines, and it is not surprising that in the early days of astronomical spectroscopy the equivalent widths of the same line given by different observers were occasionally in the ratio of two or three to one. The previously mentioned paper of K. O. Wright on α Persei is a good example of the care taken in the best modern work; particularly noteworthy is his use of a reasonable dispersion, averaging 7 A. per mm., and his neat method of overcoming plate-grain fluctuation for the faint lines which fix the vital Doppler-part of the curve of growth. That there is a systematic difference of 25 per cent between his equivalent widths and those determined slightly earlier by Miss Steel * for the same star is only evidence of how difficult the problem is. This very difficulty makes it desirable that some standard star should be chosen, and the equivalent widths of its lines determined with the highest possible accuracy by a number of observers: in view of Wright's work and Dunham's † line identifications, α Persei would be very suitable. Intercomparison of the equivalent widths determined by different observers should reveal the major sources of systematic error, and thus lead to substantial improvements in the technique employed for stellar spectra. Further the finally adopted definitive values of the equivalent widths would then be available as standards for future investigations, so that an observer wishing to try out some new method, or to employ a small-dispersion spectroscope for faint stars, would have a control on his technique.

4.3. *Laboratory Determination of Atomic Constants.*—The quantum-mechanical calculation of atomic constants is both difficult and, for elements other than hydrogen, approximate. Theoretical physicists can, therefore, scarcely be expected to initiate such investigations until accurate laboratory values are available as a stimulus and a guide. Without accurate atomic constants for the astrophysically important lines, astronomical spectroscopy is impossible, and the responsibility for their determination in the laboratory lies squarely with the astronomer. The constants with which we are concerned are the oscillator-strength, f, and the half half-widths, δ, due to radiation-damping, and δ_c, due to impact-broadening.

A valuable start on the determination of relative and absolute f-values has already been made at Mt. Wilson by R. B. King.‡ Using the vacuum-tube furnace of A. S. King,

* H. R. Steel, *Ap. J.*, **102**, 43, 1945.
† T. Dunham, jr., *Contributions Princeton Univ. Obs.*, No. 9, 1929.
‡ R. B. and A. S. King, *Ap. J.*, **82**, 377, 1935; **87**, 24, 1938; R. B. King and D. C. Stockbarger, *Ap. J.*, **91**, 488, 1940; R. B. King, *Ap. J.*, **94**, 27, 1941; **95**, 78, 1942.

backed by a source of continuous spectrum, and a concave-grating spectroscope with a dispersion of 1·86 A. per mm., R. B. King and his collaborators have measured the equivalent widths of faint lines of different elements. These lines lie on the Doppler-part of the curve of growth, so that for exponential absorption (equation 2.2 (5)) in the laboratory, equation 3.2 (5) is applicable if we substitute the length, H, of the absorption tube for our quantity μ. We thus get

$$\frac{W}{\lambda}=\frac{\pi\epsilon^2}{mc}N_i H f\frac{\lambda}{c},$$

where N_i is the number of atoms, in the lower of the two states concerned, per unit volume. If now we assume thermodynamical equilibrium in the furnace we have from Boltzmann's law

$$N_i=N_1\frac{q_i}{q_1}e^{-E_i/kT},$$

where E_i is the excitation potential of the level i. Hence for this particular line *

$$f=\left[\frac{mc^2}{\pi\epsilon^2}\frac{q_1}{q_i}e^{E_i/kT}\right]\frac{W}{\lambda^2}\frac{1}{N_1 H}, \tag{2}$$

where the quantity in square brackets contains constants known from the term-structure of the spectrum, and where the quantities W/λ^2 and $N_1 H$ have to be determined respectively from spectrophotometry and from the vapour-pressure of the element at the temperature of the furnace.

These exceedingly valuable investigations which, it is to be hoped, R. B. King will be able to extend to elements in addition to *Fe* and *Ti*, require independent confirmation. In particular it would be useful to employ Roschdestwensky's † exceedingly accurate "hook-method" to the determination of refractive index, as well as magneto-rotation methods. All three methods suffer unfortunately from the limitation that they determine $N_i f$, and absolute values of f can only be found from a vapour-pressure determination—a great source of inaccuracy since a vapour-pressure determination does not distinguish between the population of atoms, in which we are interested, and the population of temporary molecules, which may be formed in evaporation.

A method of determining the oscillator-strength, which is independent of N, is based on equation 1.2 (9). Since from its definition the Einstein transition coefficient, A_{ji}, is the reciprocal of the average time, τ, which the atom spends in state j before emitting the line of frequency ν_{ij}, we have

$$f=\tfrac{1}{3}\frac{q_j}{q_i}A_{ji}=\tfrac{1}{3}\frac{q_j}{q_i}\frac{1}{\tau}. \tag{3}$$

Wien's determination ‡ of average life-times, and therefore of oscillator-strengths, from the decay of radiation along a column of positive rays is well known, but is not likely to yield accurate results because of subsequent cascade-jumps down to the upper level concerned from captures of electrons on still higher levels. An atomic beam illuminated by a narrow pencil of exciting radiation of the required frequency, as in Koenig and Ellett's § experiment, should be exceedingly accurate, but unfortunately the method is limited to long life-times, that is small values of f, by the low speed of the atoms in

* D. H. Menzel and L. Goldberg, *Ap. J.*, **85**, 40, 1937.
† D. Roschdestwensky, *Ann. Phys.*, **39**, 307, 1912 ; *Trans. Opt. Inst. Leningrad*, **2**, No. 13, 1921.
‡ W. Wien, *Ann. Phys.*, **73**, 483, 1924.
§ H. D. Koenig and A. Ellett, *Phys. Rev.*, **39**, 576, 1932.

the atomic beam. The method might be extended to shorter life-times by the use of high optical magnification using a Burch reflecting microscope * with its long working distance. A more speculative possibility is the illumination of a stationary mass of gas by an exciting beam modulated by a Kerr-cell, and a determination of the decay-time of the thus excited radiation by two spectroscopes at different distances from the gas with their exposures regulated by two synchronous Kerr-cells. The trapping of the excited radiation † would have to be allowed for, while for lines other than resonance lines Wood's ‡ method of controlled optical excitation would have to be used. An admirable discussion of a wide variety of methods of determining life-times will be found in Mitchell and Zemansky §, and with this critical record of past successes and failures it should be possible to find a satisfactory method of direct determination of oscillator-strengths.

The direct determination of the natural half half-width, δ, in the absence of collisions is, thanks to the classical investigation of Minkowski, already cited, comparatively simple. To avoid broadening due to the collisions of atoms with themselves, for which the optical impact-diameter is very large, it is necessary to use exceedingly low vapour-pressures and therefore a very long absorbing tube. The considerable technical difficulties thus imposed could be minimized by the method of controlled multiple reflections || in a short tube. Similarly the use of spectroscopes of very high resolving power could be avoided by measuring equivalent width and finding the radiation-damping part of the curve of growth by step-by-step increases in the equivalent length of the tube.

The determination of the collisional half half-width, δ_c, for an atom in an atmosphere of foreign atoms is also well within the range of the astronomical spectroscopist by a repetition of Minkowski's work. The difficulty here is that the interesting atmosphere is one of hydrogen atoms, and since the dissociation potential of the hydrogen molecule is 4·45 electron volts such an atmosphere is very difficult to obtain in the laboratory. Wood ¶ has been able to pump hydrogen atoms out of the centre of his long discharge-tube, but their life-time is short and the method does not appear promising. Failing this or some similar method it will be necessary to find collisional half half-widths for the atom in an atmosphere of the various inert gases, and then, following Strömgren in his already quoted paper on collisional broadening in the Sun, semi-empirically to extrapolate to a hydrogen atmosphere.

4.4. *The Equation of Transfer.*—Eddington's equation of transfer, which has been the basis of all work in astronomical spectroscopy, suffers, as he clearly stated, from two limitations. It assumes first that the radiation is coherently scattered, and secondly that arrivals in the excited state by collisions of the first kind, by captures and by cascade-jumps down from higher states may be neglected. Since except under restrictive conditions, not fulfilled in stellar atmospheres, the re-emitted radiation is of different frequency from the absorbed, and since in stellar atmospheres giving emission or incipient emission-lines there must be arrivals in the excited state in addition to those resulting from the absorption of quanta by the atom in its lower state, it is clear that future progress in astronomical spectroscopy will depend upon the removal of these limitations.

* C. R. Burch, *Proc. Phys. Soc.*, **59**, 41, 1947.
† E. A. Milne, *J. Lond. Math. Soc.*, **1**, 40, 1926.
‡ R. W. Wood, *Physical Optics*, 3rd ed., p. 599, 1934.
§ A. G. G. Mitchell and M. W. Zemansky, *Resonance Radiation and Excited Atoms*, 1934.
|| H. R. Kratz and J. E. Mack, *J.O.S.A.*, **32**, 457, 1942.
¶ R. W. Wood, *Proc. Roy. Soc.*, A, **102**, 1, 1922.

A Fabian approach is probably the best in so difficult a problem, but it is of some interest at least to try a frontal attack. Consider for example the general and exact equation of transfer (equation 2.1 (1)) for an atmosphere stratified in plane parallel layers

$$\cos\theta\,\frac{dI_\nu(x,\theta)}{dx}=(k_0+k_\nu)I_\nu(x,\theta)-\{j_0(x)+j_\nu(x)\}, \qquad (4)$$

and let us evaluate the absorption and emission coefficients k_ν and j_ν. Designate the ground state of the atom by 1, the next higher state making transitions with the ground state by 2, while 3 is a still higher state from which transitions to 2 are possible. Consider transitions between states 2 and 3. Neglecting for simplicity the stimulated emissions, the microscopic transition coefficients are from equation 1.2 (12) given by

$$a_{32}=A_{32}\,\frac{\delta_2}{\pi}\,\frac{dE_2/h}{\delta_2^2+\left(\dfrac{E_2-E_2^0}{h}\right)^2},$$

$$b_{23}=B_{23}\,\frac{\delta_3}{\pi}\,\frac{dE_3/h}{\delta_3^2+\left(\dfrac{E_3-E_3^0}{h}\right)^2}.$$

Introducing Spitzer's * notation, the distance from the centre of an energy level in frequency units is

$$s_2=\frac{E_2-E_2^0}{h}\,;\quad s_3=\frac{E_3-E_3^0}{h}\,;\quad s_3-s_2=\frac{E_3-E_2}{h}-\frac{E_3^0-E_2^0}{h}=\nu_{23}-\nu_{23}^0=\Delta\nu,$$

so that the microscopic transition coefficient becomes

$$a_{32}=A_{32}\,\frac{\delta_2}{\pi}\,\frac{ds_2}{\delta_2^2+s_2^2}\,;\quad b_{23}=B_{23}\,\frac{\delta_3}{\pi}\,\frac{ds_3}{\delta_3^2+s_3^2}. \qquad (5)$$

Now let the number of atoms in state i between s_i and s_i+ds_i be

$$N_iW^*(s_i)ds_i,\quad\text{where}\quad\int_{-\infty}^{\infty}W^*(s_i)ds_i=1,$$

and where the asterisk is used to denote that $W^*(s_i)ds_i$ is the actual distribution of atoms in the level i, as opposed to the distribution resulting from natural excitation or thermodynamical equilibrium. Hence for atoms at rest the absorption-coefficient will be given by

$$k_\nu=k(\nu_{23}^0+\Delta\nu)=\frac{h\nu}{c}N_2B_{23}\frac{\delta_3}{\pi}\int_{-\infty}^{\infty}\frac{W^*(s_2)ds_2}{\delta_3^2+s_3^2}=\frac{h\nu}{c}N_2B_{23}\frac{\delta_3}{\pi}\int_{-\infty}^{\infty}\frac{W^*(s_2)ds_2}{\delta_3^2+(s_2+\Delta\nu)^2}, \qquad (6)$$

where the integration is to be carried out with respect to s_2 and subject to the condition that $\Delta\nu$ is constant. Similarly the volume-coefficient of emission will be given by

$$j_\nu=j(\nu_{23}^0+\Delta\nu)=\frac{h\nu}{4\pi}N_3A_{32}\frac{\delta_2}{\pi}\int_{-\infty}^{\infty}\frac{W^*(s_3)ds_3}{\delta_2^2+(s_3-\Delta\nu)^2}, \qquad (7)$$

where the integration is to be carried out with respect to s_3, subject to the condition that $\Delta\nu$ is constant.

If the atoms have a Maxwellian distribution of velocity, the contribution to the absorption coefficient of those atoms with a velocity between u and $u+du$ will be

$$k(\nu_{23}^0+\Delta\nu)du=\frac{h\nu}{c}N_2B_{23}\left(\frac{M}{2\pi kT}\right)^{\frac{1}{2}}\frac{\delta_3}{\pi}e^{-Mu^2/2kT}du\int_{-\infty}^{\infty}\frac{W^*(s_2)ds_2}{\delta_3^2+\left(s_2+\Delta\nu+\dfrac{u}{c}\nu_{23}^0\right)^2},$$

* L. Spitzer, *M.N.*, **96**, 794, 1936.

and the total absorption coefficient will be

$$k(\nu_{23}^0+\Delta\nu)=\frac{h\nu}{c}N_2B_{23}\left(\frac{M}{2\pi kT}\right)^{\frac{1}{2}}\frac{\delta_3}{\pi}\int_{-\infty}^{\infty}e^{-Mu^2/2kT}\,du\int_{-\infty}^{\infty}\frac{W^*(s_2)ds_2}{\delta_3^2+\left(s_2+\Delta\nu+\frac{u}{c}\nu_{23}^0\right)^2}.$$

Now substitute

$$y=\left(\frac{M}{2kT}\right)^{\frac{1}{2}}u,\ \text{so that}\ \frac{u}{c}\nu_{23}^0=by\ \text{where}\ b=\left(\frac{2kT}{M}\right)^{\frac{1}{2}}\frac{\nu_{23}^0}{c}\ ;\ \text{also put}\ \omega=\frac{\Delta\nu}{b},\ \text{and}\ a_3=\delta_3/b,$$

and we finally get

$$k(\nu_{23}^0+\Delta\nu)=\frac{h\nu}{c}N_2B_{23}\frac{1}{b\sqrt{\pi}}\frac{a_3}{\pi}\int_{-\infty}^{\infty}e^{-y^2}dy\int_{-\infty}^{\infty}\frac{W^*(s_2)ds_2}{a_3^2+\left(\frac{s_2}{b}+\omega+y\right)^2}. \qquad (8)$$

Similarly for atoms in motion the emission-coefficient becomes

$$j(\nu_{23}^0+\Delta\nu)=\frac{h\nu}{4\pi}N_3A_{32}\frac{1}{b\sqrt{\pi}}\frac{a_2}{\pi}\int_{-\infty}^{\infty}e^{-y^2}dy\int_{-\infty}^{\infty}\frac{W^*(s_3)ds_3}{a_2^2+\left(\frac{s_3}{b}-\omega-y\right)^2}. \qquad (9)$$

Equations (8) and (9), as may be readily seen, reduce to equations (6) and (7) for values of $\Delta\nu \gg b$, that is if we work well out in the line wings. In what follows we shall use the absorption and emission coefficients given by (6) and (7), thus implying this restriction on the value of $\Delta\nu$.

4.41. *Application to Resonance Line in Solar Atmosphere.*—Now let us apply the exact equation of transfer (4) and the absorption- and emission-coefficients derived above to a resonance line, resulting from a transition between states 1 and 2. Level 1 with its long life and its small impact-radius will be *assumed* to be infinitely narrow. Equations (6) and (7) thus become

$$k(\nu_{12}^0+\Delta\nu)=\frac{h\nu}{c}N_1B_{12}\frac{\delta_2}{\pi}\frac{1}{\delta_2^2+s_2^2}, \qquad (10)$$

$$j(\nu_{12}^0+\Delta\nu)=\frac{h\nu}{4\pi}N_2A_{21}W^*(s_2). \qquad (11)$$

Our *second assumption* will be that

$$W^*(s_2)=\frac{\delta_2^*}{\pi}\frac{1}{\delta_2^{*2}+s_2^2}. \qquad (12)$$

This is equivalent to assuming that all methods whereby the atoms get into state 2 finally result in a normal distribution among sub-levels characterized by the half half-width $\delta_2^*=\delta_2+\delta_c$, the sum of the natural and the collisional half half-widths. On this assumption equation (11) becomes

$$j(\nu_{12}^0+\Delta\nu)=\frac{h\nu}{4\pi}N_2A_{21}\frac{\delta_2^*}{\pi}\frac{1}{\delta_2^{*2}+(\Delta\nu)^2}, \qquad (13)$$

where

$$s_2=\frac{E_2-E_2^0}{h}=\nu_{12}-\nu_{12}^0=\Delta\nu.$$

Hence for two frequencies $\Delta\nu_1$ and $\Delta\nu_2$, each much larger than δ_2^* we have

$$\frac{j(\nu_{12}^0+\Delta\nu_1)}{j(\nu_{12}^0+\Delta\nu_2)}=\frac{\delta_2^{*2}+(\Delta\nu_2)^2}{\delta_2^{*2}+(\Delta\nu_1)^2}\doteq\left(\frac{\Delta\nu_2}{\Delta\nu_1}\right)^2. \qquad (14)$$

Similarly

$$\frac{k(\nu_{12}^0+\Delta\nu_1)}{k(\nu_{12}^0+\Delta\nu_2)}=\frac{\delta_2^2+(\Delta\nu_2)^2}{\delta_2^2+(\Delta\nu_1)^2}\doteq\left(\frac{\Delta\nu_2}{\Delta\nu_1}\right)^2. \tag{15}$$

Now returning to our differential equation (4), introduce the optical depth

$$\tau_0=\int_0^x k_0 dx$$

and the symbol

$$\eta=\frac{k(\nu_{12}^0+\Delta\nu)}{k_0}.$$

The equation of transfer now takes the form

$$\cos\theta\frac{dI_\nu(\tau_0,\theta)}{d\tau_0}=(1+\eta)I_\nu(\tau_0,\theta)-\frac{j_0+j(\nu_{12}^0+\Delta\nu)}{k_0}, \tag{16}$$

and this may be formally expressed as an integral equation

$$I_\nu(0,\theta)=\int_0^\infty\frac{j_0+j(\nu_{12}^0+\Delta\nu)}{k_0}e^{-\sec\theta\int_0^t(1+\eta)dt}\sec\theta\,dt. \tag{17}$$

This gives the surface-brightness in the line at the frequency $\nu_{12}^0+\Delta\nu$ as a function of position θ on the disk of the Sun. The surface-brightness in the adjacent continuous spectrum, where $j_\nu=k_\nu=0$, is given by

$$I_0(0,\theta)=\int_0^\infty\frac{j_0}{k_0}e^{-t\sec\theta}\sec\theta\,dt. \tag{18}$$

For some value $\Delta\nu_1$ assume now a hypothetical variation of the corresponding η_1 with optical depth τ_0, and as shown elsewhere * find by a solution of linear equations, one for each value of θ, the values of $(j_0+j_\nu)/k_0$ and j_0/k_0 which satisfy equations (17) and (18). Note that j_0/k_0 obtained as a function of optical depth, τ_0, is definitive, but that j_ν/k_0 depends upon the hypothetical variation of η_1 with τ_0.

Now repeat this process for some other frequency $\Delta\nu_2$ in the line-wing. From equation (15) the corresponding values of η_2 are given in terms of the originally assumed η_1 by

$$\eta_2=\left(\frac{\Delta\nu_1}{\Delta\nu_2}\right)^2\eta_1,$$

and if our hypothetical variation of η_1 were correct then by (14) the two values of j_ν should at each optical depth be in the ratio

$$\frac{j(\nu_{12}^0+\Delta\nu_2)}{j(\nu_{12}^0+\Delta\nu_1)}=\left(\frac{\Delta\nu_1}{\Delta\nu_2}\right)^2. \tag{14}$$

If this equation is not satisfied then we must commence the whole process again with a new hypothetical variation of η_1 with τ_0, and repeat, if necessary, again and again until (14) is true at each optical depth. Once this condition is satisfied, then by (10) and (13) we shall have found

$$\frac{j(\nu_{12}^0+\Delta\nu)}{k_0}=\frac{1}{k_0}\frac{h\nu}{4\pi}N_2A_{21}\frac{\delta_2^*}{\pi}\frac{1}{\delta_2^{*2}+(\Delta\nu)^2}, \tag{19}$$

$$\eta=\frac{1}{k_0}\frac{h\nu}{c}N_1B_{12}\frac{\delta_2}{\pi}\frac{1}{\delta_2^2+(\Delta\nu)^2} \tag{20}$$

as functions of optical depth in the atmosphere. Provided the atomic constants A_{21},

* H. H. Plaskett, *M.N.*, **101**, 3, 1941.

B_{12} and δ_2 are known, this means that we have found $N_2\delta_2^*/k_0$ and N_1/k_0 as functions of optical depth.

To make any further progress we may proceed by successive approximations. If we assume the atmosphere to be in local thermodynamic equilibrium, then

$$\frac{j_0}{k_0}=\frac{2h\nu^3}{c^2}e^{-h\nu/kT} \text{(giving the temperature, } T\text{, as a function of } \tau_0\text{)},$$

which is the form Planck's law takes when stimulated emissions are neglected. Then from (19) and (20) $j_\nu/k_0\eta$, which is known as a function of optical depth, is given by

$$\frac{j_\nu}{k_0}\frac{1}{\eta}=\frac{j_\nu}{k_\nu}=\frac{c}{4\pi}\frac{8\pi h\nu^3}{c^3}\frac{q_1}{q_2}\frac{N_2}{N_1}\frac{\delta_2^*}{\delta_2}\frac{\delta_2^2+(\Delta\nu)^2}{\delta_2^{*2}+(\Delta\nu)^2}, \tag{21}$$

using the relation 1.2 (10) between Einstein coefficients. But in local thermodynamic equilibrium

$$\frac{N_2}{N_1}=\frac{q_2}{q_1}e^{-h\nu/kT}, \tag{22}$$

so that

$$\frac{j_\nu}{k_0}\times\frac{1}{\eta}=\frac{2h\nu^3}{c^2}e^{-h\nu/kT}\frac{\delta_2^*}{\delta_2}\frac{\delta_2^2+(\Delta\nu)^2}{\delta_2^{*2}+(\Delta\nu)^2}=\frac{j_0}{k_0}\frac{\delta_2^*}{\delta_2}\frac{\delta_2^2+(\Delta\nu)^2}{\delta_2^{*2}+(\Delta\nu)^2},$$

from which we can find δ_2^* and thus, from equations (19) and (20), N_2/N_1 as functions of optical depth. If the latter agrees with (22), our assumption of local thermodynamical equilibrium is justified. Otherwise we may put the observed value of N_2/N_1 from the assumption of local thermodynamical equilibrium back in (21), and get a second approximation to it and to δ_2^*.

The method as described is sufficient to find N_i/N_1 and δ_1^* for any level i which gives an absorption-line by transitions from the ground state. For levels other than these, say a level j which makes transitions with a level i for which the population-ratio and the quantity δ_i^* are known from the foregoing procedure, explicit expressions for $k(\nu_{ij}^0+\Delta\nu)$ and $j(\nu_{ij}^0+\Delta\nu)$ can be derived from equations (6), (7) and (12) with δ_j^* and N_j as the only unknowns. These unknowns are to be derived as before from the absorption line ν_{ij}.

No originality is claimed for the foregoing discussion which is closely modelled on Spitzer's * fundamental paper on non-coherent scattering and follows a suggestion of ten Bruggencate.† If the two assumptions on which it is based are valid and if no error has been made, the discussion does, however, suggest that it may be possible, to remove the limitations of Eddington's form of the equation of transfer, at least for the solar atmosphere. The more difficult problem of a stellar atmosphere has not been examined, but it is to be hoped that this problem and the whole question of the equation of transfer will be discussed by theoretical astrophysicists.

Conclusion.—We have now hurriedly and imperfectly examined the theory, a few of the results and some of the future needs of astronomical spectroscopy. In the 20 years since Unsöld's papers astrophysics has made enormous strides. If this progress is to be maintained, however, it is essential that the applications of astronomical spectroscopy should be multiplied. Even to-day I get the impression, perhaps wrongly, that too many observational papers in astrophysics are based on eye-estimates of line-intensity

* L. Spitzer, *M.N.*, **96**, 794, 1936.
† P. ten Bruggencate, *Z. Astrophys.*, **4**, 159, 1932.

and on line-shapes from unreduced microphotometer tracings. At the best these are only exploratory methods, which can make no permanent contribution to the science of astrophysics, and which recall all too clearly Kelvin's remark—

> " I often say that when you can measure what you are speaking about and express it in numbers you know something about it ; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind: it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of *science*, whatever the matter may be ". (*Popular Lectures and Addresses*, 2nd ed., **1**, 80, 1891.)

If this be our text the applications of astronomical spectroscopy will multiply, and the future of astrophysics will be assured.