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Asymmetric bound states of differential equations in nonlinear optics

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Asymmetric Bound States of Differential Equations in Nonlinear Optics (*).

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1. - Introduction.

Bound states of a nonlinear Schrödinger equation modelling propagation in a medium with dielectric function n^2 can be found as solutions of a differential equation of the type

$$-u''(x) + \beta^2 u(x) = n^2(x, u^2(x))\, u(x), \qquad x \in \mathbb{R}, \tag{1}$$

that decay to zero at infinity, namely satisfying

$$\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} u'(x) = 0 . \tag{2}$$

Actually, solutions u of (1)-(2) correspond to the eigenstate

$$E(x, z) = e^{i\beta z} u(x)$$

propagating in the direction z and with waveguide index $\beta > 0$, see [7] (actually in such a paper the equations are Maxwell's). In particular, we

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are interested in the case considered in [1] when there is an internal layer with a linear response while the external medium is nonlinear and self-focusing. More precisely, the dielectric function n^2 is taken of the form

$$n^2(x, s) = \begin{cases} q^2 + c^2 & \text{if } |x| < d\,, \\ q^2 + s & \text{if } |x| > d\,, \end{cases} \tag{3}$$

where q, $c \in \mathbb{R}$ and $d > 0$ denotes the thickness of the internal layer. In spite of the fact that the problem inherits a symmetry, it has been shown in [1] that at certain value $\beta = \beta_0$ a family of asymmetric solutions of (1)-(2) bifurcates from the the branch of the symmetric ones. The stability analysis has been carried out in [4, 5]: the symmetric states become unstable for $\beta > \beta_0$, while the asymmetric states are the stable ones for β greater than a certain $\beta_1 > \beta_0$, see figure 1 below. Both the preceding results rely on the fact that the nonlinearity n^2 in (3) is piece-wise linear and independent of x and this specific feature permits to solve (1) explicitely.

The purpose of this Note is to investigate the same phenomenon described above for a class of equations (1) that, unlike the cited papers, cannot be integrated directly. We consider the case that the internal layer is thin and n^2 is still symmetric but has a rather general form and show the existence of asymmetric bound states of (1) provided d is sufficiently small, see Theorem 1. To achieve this result we use a method, variational in nature, discussed in some recent papers, see [2, 3], and related to the Poincaré-Melnikov theory of homoclinics. This abstract set up allows us also to discuss, for a slightly less general class of n^2 (but still including the model case (3)), the orbital stability of these bound states, see Theorem 8.

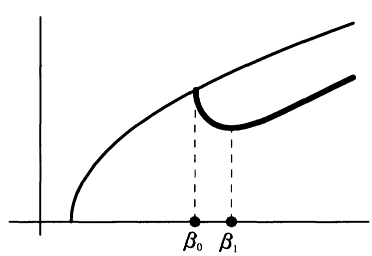

Fig. 1. – The curve in bold represents the aymmetric solutions.

2. – The main result.

Motivated by the preceding discussion, let us consider a thin layer of thickness $d = \varepsilon$ and a dielectric function of the type

$$n^2(x, s) = n_L^2(x) + n_{NL}^2(x, s),$$

with

$$\begin{cases} n_L^2(x) = q^2 + c^2 h(x/\varepsilon) \\ n_{NL}^2(x, s) = s - \alpha(x/\varepsilon, s). \end{cases} \tag{4}$$

We shall assume that $h\colon \mathbb{R} \to \mathbb{R}$ and $\alpha\colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfy:

(a) h is an even function, with $h(x) \geq 0$, $h \not\equiv 0$ and $h(x) \in \in L^1(\mathbb{R})$;

(b) α is even, with respect to $x \in \mathbb{R}$, with $\alpha(x, \cdot) \in C^1(\mathbb{R}^+)$, $\forall x \in \mathbb{R}$, and $\alpha(x, 0) \equiv 0$.

(c) There exists $\sigma > 0$ and $k \in L^1(\mathbb{R})$ such that $|\alpha_s'(x, s)| \leq k(x) s^\sigma$ $\forall s \geq 0$. Moreover, letting

$$a(s) = \int_{-\infty}^{+\infty} \alpha(x, s)\, dx,$$

one has that $a(s)$ is increasing and $a(s) \to +\infty$ as $s \to +\infty$.

We remark here that it is possible to change in the hypothesis (c) the power s^σ by any continuous function in s, and all the subsequent calculations remain valid.

To be consistent with the physical problem, h, α should also be such that n_L^2 is non-increasing and $n_{NL}^2(x, s)$ is non-decreasing in $x > 0$ and $s > 0$. However, we do not need such assumptions here. Letting $\chi(x)$ denote the characteristic function of $[-1, 1]$, the dielectric function n^2 fits into the Akhmediev setting provided

$$h(x) = \chi(x), \qquad \alpha(x, s) = \chi(x) \cdot s$$

and corresponds to a layered medium with dielectric function given by (3), with $d = \varepsilon$.

Substituting (4) into (1) and setting $\lambda = \beta^2 - q^2$, we find the equation

$$-u'' + \lambda u = u^3 + c^2 h(x/\varepsilon) u - \alpha(x/\varepsilon, u^2)\, u. \tag{5}$$

Solutions of (5) that decay at zero at infinity, namely satisfying (2), will be henceforth called *bound states.*

Equation (5) will be seen as a perturbation of

$$-u'' + \lambda u = u^3. \tag{6}$$

For all $\lambda > 0$, (6) has the positive symmetric solution

$$\phi_\lambda(x) = \sqrt{2\lambda}/\cosh(\sqrt{\lambda}\, x),$$

together with all its translates

$$\phi_\lambda(x+\theta), \qquad \theta \in \mathbb{R}.$$

To state our main result some further notation is in order. From (a), we can define

$$H = \int_{-\infty}^{+\infty} h(x)\, dx \in (0, +\infty).$$

From assumption (c) it follows that the equation

$$a(2\lambda) \equiv \int_{-\infty}^{+\infty} a(x, 2\lambda)\, dx = c^2 H \tag{7}$$

has a unique solution $\lambda_0 = \lambda_0(c) > 0$.

THEOREM 1. *Suppose that (a – c) hold and take* $\delta, \Lambda > 0$ *such that* $0 < \delta < \lambda_0 - \delta < \lambda_0 + \delta < \Lambda$. *Then there exists* $\varepsilon_0 = \varepsilon_0(\delta, \Lambda) > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *one has*:

1) *for all* $\lambda \in [\delta, \Lambda]$, *equation* (5) *has a symmetric bound state* $\overline{u}_\varepsilon$, *which satisfies*

$$\lim_{\varepsilon \to 0} \overline{u}_\varepsilon = \phi_\lambda \qquad \text{in } H^1(\mathbb{R})$$

2) *for all* $\lambda \in [\lambda_0 + \delta, \Lambda]$, *equation* (5) *has, in addition, a pair of asymmetric bound states* $v_\varepsilon^\pm$ *such that*

$$\lim_{\varepsilon \to 0} v_\varepsilon^\pm(x) = \phi_\lambda(x \pm \theta_\lambda) \qquad \text{in } H^1(\mathbb{R})$$

for some $\theta_\lambda > 0$.

The existence of the symmetric solution is well known, even in a much greater generality, see [7]. The existence of the asymmetric solutions will be proved in the sequel by means of some variational arguments introduced in [2, 3].

3. – Poincaré-Melnikov method.

We will prove Theorem 1 by using the results discussed in [2, 3] which are concerned with the existence of critical points of perturbed functionals of the form

$$f_\varepsilon(u) = \frac{1}{2}\|u\|^2 - F(u) + G(\varepsilon, u). \tag{8}$$

We assume that the reader is familiar with the cited papers. To put our problem into the preceding abstract frame, let us consider the Hilbert space $E = H^1(\mathbb{R})$ equipped with scalar product

$$(u\,|\,v) = \int_{\mathrm{Re}} [u'v' + \lambda uv]\,dx$$

and norm $\|u\|^2 = (u\,|\,u)$ and define

$$F(u) = \frac{1}{4}\int_{\mathbb{R}} u^4.$$

Obviously, $F \in C^\infty(E, \mathbb{R})$. Critical points of $f_0(u) = 1/2\,\|u\|^2 - F(u)$ are the bound states of the unperturbed problem (6). As remarked before, the functional f_0 has, for any fixed $\lambda > 0$, a one parameter family of critical points $Z = \{z_\theta = \phi_\lambda(\cdot + \theta)\,|\,\theta \in \mathbb{R}\}$. Such a Z is a smooth one dimensional manifold and the following non-degeneracy condition (see [6, p. 226]) is satisfied:

$$\operatorname{Ker} f_0''(z_\theta) = \operatorname{span}\{z_\theta'\}, \qquad \forall z_\theta \in Z. \tag{9}$$

Furthermore, since ϕ_λ decays exponentially to zero at infinity, then it is easy to see that for all $z \in Z$ the linear map $F''(z)$ is compact. Here, as usual, $F''(z)$ is defined by setting

$$(F''(z)\,v\,|\,w) = D^2F(z)[v, w].$$

In order to introduce the perturbation term G let us set

$$W(y, u) = \int_0^u a(y, s)\,ds - c^2 h(y)\,u. \tag{10}$$

Notice that $W(y, u^2(y))$ is in L^1 by hypotheses (a) and (c) and the inclusion $E \subset L^\infty(\mathbb{R})$. Furthermore, the change of variable $x = \varepsilon y$ yields:

$$\int_{\mathbb{R}} W\left(\frac{x}{\varepsilon}, u^2(x)\right) dx = \varepsilon \int_{\mathbb{R}} W(y, u^2(\varepsilon y))\, dy .$$

We set

$$\tilde{G}(\varepsilon, u) = \frac{1}{2} \int_{\mathbb{R}} W(y, u^2(\varepsilon y))\, dy$$

and

$$G(\varepsilon, u) = \begin{cases} \varepsilon \tilde{G}(\varepsilon, u) & \text{if } \varepsilon \neq 0 , \\ 0 & \text{if } \varepsilon = 0 . \end{cases}$$

With this notation, it turns out that bound states of (5) are the critical points of the Euler functional f_ε defined in (8).

Let $G'(\varepsilon, u)$ and $G''(\varepsilon, u)$ be defined by setting

$$(G'(\varepsilon, u) \,|\, v) = D_u G(\varepsilon, u)[v], \qquad \forall v \in E ,$$

$$(G''(\varepsilon, u)\, v \,|\, w) = D_{uu} G(\varepsilon, u)[v, w], \qquad \forall v, w \in E .$$

LEMMA 2. *$G \in C(\mathbb{R} \times E, \mathbb{R})$ and $G(0, u) = 0$ for all $u \in E$. Furthermore the following conditions hold:*

(G_1) *G is of class C^2 with respect to $u \in E$, $G'(0, u) = 0$ and $G''(0, u) = 0$ for all $u \in E$;*

(G_2) *the maps $(\varepsilon, u) \mapsto G'(\varepsilon, u)$ and $(\varepsilon, u) \mapsto G''(\varepsilon, u)$ are continuous as maps from $\mathbb{R} \times E$ to E, repectively to $L(E, E)$;*

(G_3) *for all $z \in Z$ the map $\varepsilon \mapsto \tilde{G}(\varepsilon, z)$ (and hence $\varepsilon \mapsto G(\varepsilon, z)$) is C^1.*

PROOF. Let $\varepsilon_n \to \varepsilon$ in $\mathbb{R}$ and $u_n \to u$ in E. From the embedding of E into $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we deduce that for every $y \in \mathbb{R}$,

$$|u_n(\varepsilon_n y) - u(\varepsilon y)| \leq |u_n(\varepsilon_n y) - u(\varepsilon_n y)| + |u(\varepsilon_n y) - u(\varepsilon y)| \to 0$$

whence

$$W(y, u_n^2(\varepsilon_n y)) \to W(y, u^2(\varepsilon y))$$

for all $y \in \mathbb{R}$. Since

$$|W(y, u_n^2(\varepsilon_n y)) - W(y, u^2(\varepsilon y))| \leqslant$$

$$\leqslant \frac{k(y)}{(\sigma+1)(\sigma+2)}[|u_n(\varepsilon_n y)|^{2\sigma+4} + |u(\varepsilon y)|^{2\sigma+4}] +$$

$$+ c^2 h(y)[|u_n(\varepsilon_n y)|^2 + |u(\varepsilon y)|^2] \leqslant C_1[k(y) + h(y)] \in L^1(\mathbb{R}),$$

one immediately deduces that $G(\varepsilon_n, u_n) \to G(\varepsilon, u)$.

By straight calculation we find

$$D_u G(\varepsilon, u)[v] = \varepsilon \int_{\mathbb{R}} W_u(y, u^2(\varepsilon y)) u(\varepsilon y)\, v(\varepsilon y)\, dy\ ,$$

$$D_{uu} G(\varepsilon, u)[v, w] = 2\varepsilon \int_{\mathbb{R}} W_{uu}(y, u^2(\varepsilon y)) u^2(\varepsilon y)\, v(\varepsilon y)\, w(\varepsilon y)\, dy +$$

$$+ \varepsilon \int_{\mathbb{R}} W_u(y, u^2(\varepsilon y)) v(\varepsilon y)\, w(\varepsilon y)\, dy\ ,$$

for every $v, w \in E$, and (G_1) follows directly.

The proof of (G_2) relies on the arguments of Lemma 4.1 of [3]. Let us prove the continuity of $(\varepsilon, u) \mapsto G'(\varepsilon, u)$. We have to show that

$$\|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| = \sup_{\|v\| \leqslant 1} |D_u G(\varepsilon_n, u_n)[v] - D_u G(\varepsilon, u)[v]| \to 0\ .$$

Setting

$$S_n(y) = \varepsilon_n W_u(y, u_n^2(\varepsilon_n y)) u_n(\varepsilon_n y) \text{ and } S(y) = \varepsilon W_u(y, u^2(\varepsilon y)) u(\varepsilon y),$$

there results

$$|S_n(y)\, v(\varepsilon_n y) - S(y)\, v(\varepsilon y)| \leqslant$$

$$\leqslant |S_n(y)\, v(\varepsilon_n y) - S_n(y)\, v(\varepsilon y)| + |S_n(y)\, v(\varepsilon y) - S(y)\, v(\varepsilon y)| \leqslant$$

$$\leqslant |S_n(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)| + \|v\|_\infty |S_n(y) - S(y)|\ .$$

Hence we find, for all $\|v\| \leq 1$,

$$|D_u G(\varepsilon_n, u_n)[v] - D_u G(\varepsilon, u)[v]| = \left| \int_{\mathbb{R}} (S_n(y)\, v(\varepsilon_n y) - S(y)\, v(\varepsilon y))dy \right| \leq$$

$$\leq \int_{\mathbb{R}} |S_n(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)|\, dy + \|v\|_\infty \int_{\mathbb{R}} |S_n(y) - S(y)|\, dy\,.$$

From this and since

$$|S_n(y)| \leq C_2[k(y) + h(y)] \equiv C_2\, \gamma(y) \in L^1,$$

we deduce:

$$\|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| = \sup_{\|v\| \leq 1} |D_u G(\varepsilon_n, u_n)[v] - D_u G(\varepsilon, u)[v]| \leq$$

$$\leq C_2 \sup_{\|v\| \leq 1} \int_{\mathbb{R}} |\gamma(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)|\, dy + C_3 \int_{\mathbb{R}} |S_n(y) - S(y)|\, dy\,.$$

Clearly, the latter integral tends to zero. As for the former, it can be uniformly estimated using the fact that $E \subset C^{0,\nu}$ for any $\nu \in (0, 1/2)$. Indeed, for any $M > 0$ and any $\|v\| \leq 1$ we find

$$\int_{\mathbb{R}} |\gamma(y)| \cdot |v(\varepsilon_n y) - v(\varepsilon y)|\, dy \leq$$

$$\leq C_4 \|v\|_{C^{0,\nu}} |\varepsilon_n - \varepsilon|^\nu \int_{|y| \leq M} |y^\nu \gamma(y)|\, dy + C_5 \|v\|_\infty \int_{|y| \geq M} \gamma(y)\, dy \leq$$

$$\leq C_6 |\varepsilon_n - \varepsilon|^\nu \int_{|y| \leq M} |y^\nu \gamma(y)|\, dy + C_7 \int_{|y| \geq M} \gamma(y)\, dy\,.$$

Taking limits as $n \to \infty$ we infer

$$\lim_{(\varepsilon_n, u_n) \to (\varepsilon, u)} \|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| \leq C_7 \int_{|y| \geq M} \gamma(y)\, dy\,.$$

Since M is arbitrary and $\gamma \in L^1$, it follows that

$$\|G'(\varepsilon_n, u_n) - G'(\varepsilon, u)\| \to 0\,,$$

as required. The continuity of G'' follows in a similar way.

Finally, to prove (G_3) it suffices to evaluate formally

$$D_\varepsilon \widetilde{G}(\varepsilon, u) = \int_{\mathbb{R}} W_u(y, u^2(\varepsilon y))\, u(\varepsilon y)\, u'(\varepsilon y)\, y\, dy\,,$$

and to observe that for $u = z_\theta$ we have from (a) and (c)

$$|W_u(y, z_\theta^2(\varepsilon y))\, z_\theta(\varepsilon y)\, z_\theta'(\varepsilon y)\, y| \leqslant \frac{k(y)}{\sigma+1} z_\theta^{2\sigma+3}\, |z_\theta' y| + c^2 h(y)\, z_\theta\, |z_\theta' y| \leqslant$$

$$\leqslant C_8[k(y) + h(y)] \in L^1(\mathbb{R})\,.$$

Thus, the theorem of derivation under the integral sign implies the assertion. ■

By Lemma 2, f_ε can be faced by the abstract setting discussed in [2, 3]. For the reader convenience, let us sketch the procedure. First, we seek w orthogonal to z_θ' satisfying

$$f_\varepsilon'(z_\theta + w) \in \operatorname{span}\{z_\theta'\}\,.$$

Considering the function

$$\Phi\colon \mathbb{R}\times\mathbb{R}\times E\times\mathbb{R} \to E\times\mathbb{R}\,,$$

$$\Phi(\varepsilon, \theta, w, \zeta) = (f_\varepsilon'(z_\theta + w) - \zeta z_\theta', (w|z_\theta'))\,,$$

we are lead to solve $\Phi(\varepsilon, \theta, w, \zeta) = 0$. An application of the Implicit Function Theorem yields

LEMMA 3. *For $\varepsilon > 0$ sufficiently small there exists a unique $w = w(\varepsilon, \theta)$, orthogonal to z_θ' and satisfying* (11). *Moreover there results*

$$w(\varepsilon, \theta) = \varepsilon w_0(\theta) + o(\varepsilon)\,, \tag{12}$$

and the symmetry property $w(\varepsilon, \theta)(x) = w(\varepsilon, -\theta)(-x)$, $\forall \theta, x \in \mathbb{R}$ (in particular, $w(\varepsilon, 0)$ is an even function of $x \in \mathbb{R}$).

PROOF. For a complete proof we refer to section 2 of [2] or to section 2 of [3]. Here we only point out that (G_3) implies the differentiability of w at $(0, \theta)$ and this gives rise to (20) with $w_0(\theta) = (\partial w/\partial\varepsilon)(0, \theta)$. Moreover, taking into account that h and a are even function with respect to $x \in \mathbb{R}$, one infers that the function $x \mapsto w(\varepsilon, -\theta)(-x)$ satisfies also the requirements for $w(\varepsilon, \theta)$, and the symmetry property follows. ■

Setting $Z_\varepsilon = \{z_\theta + w(\varepsilon, \theta)\}$, it turns out that Z_ε is (locally) diffeomorphic to Z and by (11) is a *natural constraint* for f_ε. This means that in a neighbourhood of Z the critical points of f_ε coincide with the the critical points of f_ε constrained on Z_ε.

Finally, let us evaluate f_ε on Z_ε. Using (12) and recalling that $f_0(z_\theta) = b$ as well as $f_0'(z_\theta) = 0$, for all $\theta \in \mathbb{R}$, there results:

$$f_\varepsilon(z_\theta + w) = f_0(z_\theta + w) + G(\varepsilon, z_\theta + w) =$$

$$= f_0(z_\theta) + \varepsilon f_0'(z_\theta)\, w_0 + o(\varepsilon) + \varepsilon[\widetilde{G}(\varepsilon, z_\theta) + O(\varepsilon)] = b + \varepsilon \widetilde{G}(\varepsilon, z_\theta) + o(\varepsilon).$$

As a consequence of (G_3) we infer $\widetilde{G}(\varepsilon, z_\theta) = \Gamma(\theta) + O(\varepsilon)$, where

$$\Gamma(\theta) = \widetilde{G}(0, z_\theta) = \frac{1}{2}\int_{\mathbb{R}} W(y, z_\theta^2(0))\, dy$$

and this yields

$$f_\varepsilon(z_\theta + w) = b + \varepsilon\Gamma(\theta) + o(\varepsilon).$$

In conclusion, we can state the following result:

THEOREM 4. *Suppose that there exist* $r > 0$ *and* $\theta^* \in \mathbb{R}$ *such that*

(13) *either* $\Gamma(\theta^*) < \min_{|\theta - \theta^*| = r} \Gamma(\theta)$, *or* $\Gamma(\theta^*) > \max_{|\theta - \theta^*| = r} \Gamma(\theta)$.

Then, for $\varepsilon > 0$ *sufficiently small, there exists* θ_ε, *with* $|\theta_\varepsilon - \theta^*| \le r$, *such that* f_ε *has a critical point* u_ε *of the form* $u_\varepsilon(x) = z_{\theta_\varepsilon} + O(\varepsilon)$.

REMARKS 5. (i) Theorem 3.3 is prompted for the application to the specific problem discussed here. For more general abstract results, we refer to [2, 3].

(ii) If Γ has a proper local minimum (or maximum) at θ^*, then $\theta_\varepsilon \to \theta^*$ as $\varepsilon \to 0$.

(iii) The function Γ is nothing but the primitive of the Melnikov function associated to (5). ■

4. – Proof of Theorem 1.

In order to apply Theorem 4 to our equation, we first recall that for the Melnikov primitive there results:

$$\Gamma(\theta) = \Gamma_\lambda(\theta) = \frac{1}{2}\int_{\mathbb{R}} W(y, z_\theta^2(0))\,dy = \frac{1}{2}\int_{\mathbb{R}} W(y, \phi_\lambda^2(\theta))\,dy$$

where we have used again the notation ϕ_λ to indicate the solutions of (6). Observe that

$$\Gamma''_\lambda(0) = \phi_\lambda(0)\,\phi''_\lambda(0)\left[\int_{-\infty}^{+\infty} a(y, \phi_\lambda^2(0))dy - c^2 H\right] = -2\lambda^2[a(2\lambda) - c^2 H].$$

Therefore $\Gamma''_\lambda(0) < 0$ whenever $\lambda > \lambda_0$. Observe also that

$$\Gamma_\lambda(\theta) = \frac{1}{2}\int_{\mathbb{R}}\left(\int_0^{\phi_\lambda^2(\theta)} a(y, s)\,ds - c^2 h(y)\,\phi_\lambda^2(\theta)\right)dy =$$

$$= \frac{1}{2}\left[\int_{\mathbb{R}}\int_0^{\phi_\lambda^2(\theta)} a(y, s)\,ds\,dy - c^2\phi_\lambda^2(\theta)\int_{\mathbb{R}} h(y)\,dy\right] =$$

$$= \frac{1}{2}\phi_\lambda^2(\theta)\left[\int_{\mathbb{R}}\int_0^1 a(y, \phi_\lambda^2(\theta)\,t)\,dt\,dy - c^2 H\right].$$

Then, one easily infers that

$$\lim_{\theta\to\pm\infty} \Gamma_\lambda(\theta) = 0\,,$$

with $\Gamma_\lambda(\theta) < 0$ for large values of $|\theta|$. It follows that the Melnikov primitive Γ_λ has, for these values of λ, 2 global minima $\theta_\lambda > 0$ and $-\theta_\lambda$. If $\lambda \in [\lambda_0 + \delta, \Lambda]$ there exists $r > 0$ independent of λ, such that Γ_λ satisfies (13) with $\theta^* = \pm\,\theta_\lambda$. Then such θ_λ gives rise, through Theorem 4, to a critical point $\theta_\lambda(\varepsilon)$ of f_ε on Z_ε and hence to a solution v_ε with

$$v_\varepsilon(x) \simeq \phi_\lambda(x + \theta_\lambda(\varepsilon))\,.$$

Since we can also take r such that $\theta_\lambda - r > 0$, this solution is asymmetric. Similar argument for $-\theta_\lambda$. For future reference, let us indicate how we can find in this frame the symmetric solution. Since Γ_λ is even, the value $\theta = 0$ is a critical point of Γ_λ for any $\lambda > 0$ and taking into account that $w(\varepsilon, 0)$ is even respect to x, this critical point gives rise to a symmetric solution $\overline{u}_\varepsilon$ of (5). It turns out that $\overline{u}_\varepsilon$ corresponds to a minimum of Γ_λ for $\lambda < \lambda_0 - \delta$, and a maximum of Γ_λ for $\lambda > \lambda_0 + \delta$. ■

REMARKS 6. (i) When $\alpha(x, s) = \alpha(x)\, s$ (that includes the Akhmediev model case) the Melnikov primitive becomes

$$\Gamma_\lambda(\theta) = \frac{1}{4} A \phi_\lambda^4(\theta) - \frac{1}{2} c^2 H \phi_\lambda^2(\theta),$$

where $A = \int_{\mathbb{R}} \alpha(x)\, dx$. Then $\lambda_0 = c^2 H/2A$ and for $\lambda > \lambda_0$ Γ_λ has precisely 3 nondegenerate critical points given by $\theta = 0$ and $\pm\theta_\lambda$. The latters are global proper minima and thus $\theta_\lambda(\varepsilon) \to \theta_\lambda$ and $v_\varepsilon \to \phi_\lambda(\cdot + \theta_\lambda)$. Let us notice that in the model case one has $\beta_0^2 = \lambda_0 + q^2 + O(\varepsilon)$. The graph of Γ_λ for different values of λ and the dependence of θ_λ on λ are indicated in figures 2 and 3 below.

(ii) We also point out that the maximum value of the function $\lambda \mapsto \theta_\lambda$ can be arbitrarily large, provided that λ_0 is sufficiently small. So, one can get «very asymmetric» bound states, by taking the data of the problem in such a way that $\lambda_0 = c^2 H/2A$ be small.

(iii) The existence of asymmetric bound states depends on the combined effect of αu^3 and $c^2 hu$. Indeed, if either $c = 0$ or $\alpha \equiv 0$, the Melnikov primitive Γ_λ has for all $\lambda > 0$ a unique critical point at $\theta = 0$. Therefore the preceding arguments show that (5) has, near Z, only symmetric solutions. These bound states turn out to be unstable (if $c = 0$), or stable (if $\alpha \equiv 0$), for all $\lambda > 0$, see Remark below. ■

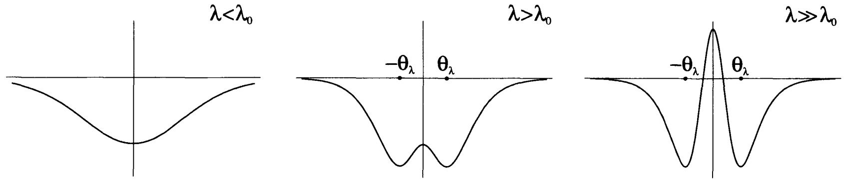

Fig. 2. – Graphs of $\Gamma_\lambda(\theta)$ for different values of λ.

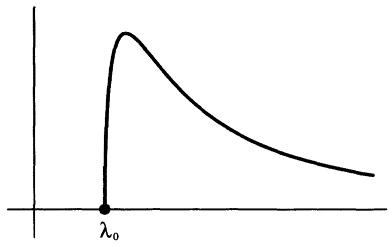

Fig. 3. – Dependence of θ_λ on $\lambda > \lambda_0$.

5. – Remarks on stability.

Here we shortly discuss the orbital stability of solitary waves $e^{i\lambda z} u_\varepsilon(x)$ corresponding to solutions found in Theorem 1. By «orbital stability» we mean that a solution $\psi(z, x)$ of the Schrödinger equation exists for all $z \geq 0$ and remains H^1-close to the solitary wave $e^{i\lambda z} u_\varepsilon(x)$ provided $\psi(0, x)$ is sufficiently near $u_\varepsilon(x)$ in H^1. See, for example, [4]. Since the results will depend on the value of λ, we will emphasize the dependence on λ by writing $u_{\varepsilon, \lambda}$ instead of u_ε.

We shall take $a(x, s) = a(x)\, s$. Our discussion relies on some results of [4] which, in the present setting, can be formulated as follows.

Let $u_{\varepsilon, \lambda}$ be a solution of (5) and consider the eigenvalues l of the linearized equation

$$-v'' + \lambda v - \left(3 u_{\varepsilon, \lambda}^2 + c^2 h\left(\frac{x}{\varepsilon} \right) - 3a\left(\frac{x}{\varepsilon} \right) u_{\varepsilon, \lambda}^2 \right) v = lv\,. \tag{14}$$

Let $N = N(u, \varepsilon, \lambda)$ denote the number of negative eigenvalues of (14) and let

$$\mu(\lambda) := \frac{\partial}{\partial \lambda} \int_{\mathbb{R}} |u_{\varepsilon, \lambda}(x)|^2 dx\,.$$

Then one has:

(A) $N = 1$ and $\mu(\lambda) > 0$ implies stability;

(B) $N = 1$ and $\mu(\lambda) < 0$ implies instability;

(C) $N = 2$ and $\mu(\lambda) > 0$ implies instability.

In all the cases, the rest of the spectrum of (14) is assumed to be positive and bounded away from zero. See Theorem 2 and Section 6. D of [4]-I for statements $(A), (B)$ and the Instability Theorem in [4]-II for the statement (C).

In the model case, namely when $\alpha(x) = h(x) = \chi(x/d)$, the characteristic function of the interval $[-d, d]$, the solitary wave corresponding to the symmetric mode becomes unstable for $\lambda > \lambda_0$. Moreover, there exists $\lambda_1 > \lambda_0$ such that the solitary wave corresponding to the asymmetric bound state is stable for $\lambda > \lambda_1$ and unstable for $\lambda \in (\lambda_0, \lambda_1)$. See [4, 5]. Actually, one shows by a direct calculation that $\mu(\lambda) > 0$ for all $\lambda > 0$ but when $u_{\varepsilon, \lambda}$ is asymmetric and $\lambda_0 < \lambda < \lambda_1$, see figure 1, where we have used the parameter β such that $\lambda = \beta^2 - q^2$. As for the spectral analysis, it is carried out by a phase plane analysis. This is no more possible in the more general case when $\alpha(x, s) = \alpha(x)\, s$ and it will be investigated by taking advantage of the variational approach discussed before.

We will use in the sequel the notation $\overline{u}_{\varepsilon, \lambda}$ for the symmetric solution, $v_{\varepsilon, \lambda}$ for the asymmetric one, and $z_{\lambda, 0}$ for $\phi_\lambda(\cdot + \theta)$. According to Remark 6-(i) we know that

$$\overline{u}_{\varepsilon, \lambda} = z_{\lambda, 0} + O(\varepsilon), \qquad v_{\varepsilon, \lambda} = z_{\lambda, \theta_\lambda} + O(\varepsilon).$$

LEMMA 7. *Take δ, Λ like in Theorem 1. Then there exists $\varepsilon_0' = \varepsilon_0'(\delta, \Lambda) > 0$ ($\varepsilon_0' \leq \varepsilon_0$) such that for all $\varepsilon \in (0, \varepsilon_0']$ one has*

1) *if $u_{\varepsilon, \lambda} = \overline{u}_{\varepsilon, \lambda}$,*

   (*a*) $\lambda \in [\delta, \lambda_0 - \delta] \Rightarrow N = 1$;

   (*b*) $\lambda \in [\lambda_0 + \delta, \Lambda] \Rightarrow N = 2$;

2) *if $u_{\varepsilon, \lambda} = v_{\varepsilon, \lambda}$ and $\lambda \in [\lambda_0 + \delta, \Lambda]$ then $N = 1$.*

In all the cases, the rest of the spectrum is positive and bounded away from zero.

PROOF. In the proof of this Lemma we let θ^* denote either 0 or $\pm\theta_\lambda$. The number of negative eigenvalues of (14), $N(u, \varepsilon, \lambda)$ equals the dimension of the subspace where $D^2 f_{\varepsilon, \lambda}(u_{\varepsilon, \lambda})$ is negative defined. Let first take $\varepsilon = 0$ and the corresponding family of solutions $z_{\lambda, \theta}$. By a straight calculation there results

$$D^2 f_{0,\lambda}(z_{\lambda,\theta})[z_{\lambda,\theta}, z_{\lambda,\theta}] < 0\,,$$
$$D^2 f_{0,\lambda}(z_{\lambda,\theta})[z'_{\lambda,\theta}, z'_{\lambda,\theta}] = 0\,,$$
$$D^2 f_{0,\lambda}(z_{\lambda,\theta})[v, v] > 0\,, \qquad \forall v \perp \operatorname{span}\{z_{\lambda,\theta}, z'_{\lambda,\theta}\}, \quad v \neq 0\,,$$

for every λ, θ. By the way, these relationships are related to the fact that $z_{\lambda,\theta}$ can be found as Mountain-Pass critical point of $f_{0,\lambda}$ and is degenerate because it appears together its translates. Let $J = [\delta, \lambda_0 - \delta] \cup \cup [\lambda_0 + \delta, \Lambda]$. Since the preceding inequalities are uniform for $\lambda \in J$ then, after a small perturbation, one has for all $\lambda \in J$:

$$D^2 f_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})[z_{\lambda,\theta^*}, z_{\lambda,\theta^*}] < 0\,,$$

as well as

$$D^2 f_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})[v, v] > 0\,, \qquad \forall v \perp \operatorname{span}\{z_{\lambda,\theta^*}, z'_{\lambda,\theta^*}\}\,, \qquad v \neq 0\,.$$

Next, using the properties of G and the fact that $\theta_\lambda(\varepsilon) \to \theta^*$ as $\varepsilon \to 0$, one can show, see Lemma 3.2 of [3]:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} D^2 f_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})[z'_{\lambda,\theta^*}, z'_{\lambda,\theta^*}] = \Gamma''_\lambda(\theta^*)\,. \tag{15}$$

According to Remark 6-(i), the critical points θ^* are nondegenerate for $\lambda \in J$ and hence (15) yields

$$\Gamma''_\lambda(\theta^*) > 0 \Rightarrow D^2 f_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})[z'_{\lambda,\theta^*}, z'_{\lambda,\theta^*}] > 0\,,$$

$$\Gamma''_\lambda(\theta^*) < 0 \Rightarrow D^2 f_{\varepsilon,\lambda}(u_{\varepsilon,\lambda})[z'_{\lambda,\theta^*}, z'_{\lambda,\theta^*}] < 0\,,$$

provided ε is sufficiently small. Recalling that $\overline{u}_{\varepsilon,\lambda}$ corresponds to a nondegenerate minimum (maximum) of Γ_λ provided that $\lambda \in [\delta, \lambda_0 - \delta]$ ($\lambda \in [\lambda_0 + \delta, \Lambda]$), while $v_{\varepsilon,\lambda}$ always corresponds to nondegenerate minima of Γ_λ for $\lambda \in [\lambda_0 + \delta, \Lambda]$, the Lemma follows. ■

THEOREM 8. *Let $a(x, s) = \alpha(x)\, s$ and h satisfy hypotheses $(a - c)$. Take δ, Λ like in Theorem* 1 *and suppose, like in the model case, that*

$$\frac{\partial}{\partial \lambda} \int_{\mathbb{R}} |\overline{u}_{\varepsilon,\lambda}(x)|^2 dx > 0\,, \qquad \forall \lambda > 0\,,$$

while

$$\frac{\partial}{\partial \lambda} \int_{\mathbb{R}} |v_{\varepsilon,\lambda}(x)|^2 dx < 0 , \qquad \forall \lambda \in [\lambda_0 + \delta, \lambda_1),$$

$$\frac{\partial}{\partial \lambda} \int_{\mathbb{R}} |v_{\varepsilon,\lambda}(x)|^2 dx > 0 , \qquad \forall \lambda \in (\lambda_1, \Lambda],$$

for some $\lambda_1 = \lambda_1(\varepsilon) \in (\lambda_0 + \delta, \Lambda)$. *Then:*

1) *the solitary waves corresponding to symmetric bound states* $\bar{u}_{\varepsilon,\lambda}$ *are stable for* $\lambda \in [\delta, \lambda_0 - \delta]$, *and unstable for* $\lambda \in [\lambda_0 + \delta, \Lambda]$

2) *the solitary waves corresponding to asymmetric bound states* $v_{\varepsilon,\lambda}$ *are unstable for* $\lambda \in [\lambda_0 + \delta, \lambda_1)$ *and stable for* $\lambda \in (\lambda_1, \Lambda]$.

PROOF. If $u_{\varepsilon,\lambda} = \bar{u}_{\varepsilon,\lambda}$ we have that $\mu(\lambda) > 0 \ \forall \lambda > 0$. Moreover, by Lemma 7-1) we infer

$$N = \begin{cases} 1 & \text{if } \lambda \in [\sigma, \lambda_0 - \delta], \\ 2 & \text{if } \lambda \in [\lambda_0 + \delta, \Lambda]. \end{cases}$$

Thus (A), resp. (C), implies stability, resp. instability. If $u_{\varepsilon,\lambda} = v_{\varepsilon,\lambda}$, Lemma 7-2) yields $N = 1$. Moreover, one has

$$\begin{cases} \mu(\lambda) < 0 & \text{if } \lambda \in [\lambda_0 + \delta, \lambda_1), \\ \mu(\lambda) > 0 & \text{if } \lambda \in (\lambda_1, \Lambda_1). \end{cases}$$

In the former case (B) implies instability, while in the latter stability follows from (A). ■

REMARK 9. Completing Remark 6-(iii), we point out that if either $c = 0$ or $\alpha \equiv 0$, the unique critical point $\theta = 0$ of Γ_λ is a maximum, respectively a minimum, and hence the corresponding (symmetric) solution is unstable, respectively stable. ■

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