Asymmetric First-Price Auctions with Uniform Distributions: Analytic Solutions to the General Case

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Abstract

We provide analytic solutions for any asymmetric first-price auction, both with and without a minimum bid m, for two buyers having valuations uniformly distributed on $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$. We show that our solutions are consistent with the previously known subcases (Griesmer et al., 1967 and Plum, 1992), which have $\underline{v}_1 = \underline{v}_2$. We also show that the solution is continuous in $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ and m. Several interesting examples are presented, including a class where the two bid functions are linear.

1 Introduction

While research on auctions, including asymmetric auctions, has grown significantly in recent years, there are still very few analytic solutions of first-price auctions outside the symmetric case. Surprisingly, even what is possibly the most natural environment to study has not been solved analytically, namely, the uniform case with two buyers and private values drawn from independent but asymmetric distributions. This is a fundamental case that would

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be useful to test conjectures, do comparative statics, or illustrate important features of auction design.

There does exist a solution to a special case, which dates back to Griesmer et al. (1967), with distributions $V_1 \sim U[0,1], V_2 \sim U[0,\beta]$ that has the following equilibrium inverse-bid functions:[1]

$$v_1(b) = \frac{2b\beta^2}{\beta^2 - b^2(1-\beta^2)}, \quad v_2(b) = \frac{2b\beta^2}{\beta^2 + b^2(1-\beta^2)}. \tag{1}$$

However, this result has the restriction that the distribution of buyers' values have the same lower end.[2] This is a substantial assumption: it restricts the asymmetry between the buyers to one dimension, namely, one distribution being a "stretch" of the other while ignoring different aspects of strength like "shifts". For example, if the distribution of one buyer's value is $U[0,1]$, then the second buyer could be considered stronger for having $U[0,2]$ or for having distribution $U[1/2,1]$. In the first case (stretching), he is stronger in the sense that he may have higher values (in $[1,2]$), while in the second case he is stronger in the sense that he cannot have low values (in $[0,1/2]$). These two notions of strength may yield different effects on the equilibrium, as was already observed by Maskin and Riley (2000). Likewise, the lack of a minimum bid is yet another dimension missing in the existing results. For instance, outside the symmetric case, linear bid functions can occur only in the presence of a minimum bid. Also as we shall show, the presence of a minimum bid may qualitatively affect the equilibrium. For instance, the two equilibrium bid functions may intersect both at the minimum bid and at an internal point (see Example 1). This means that the bidding of the buyers cannot be ordered such that one is more aggressive than the other. Rather, one is more aggressive in a part of the common region of values while the other buyer is more aggressive in the other region. This is despite the fact that starting from the minimum bid, one buyer's distribution of values

[1]For presentation purposes, we normalize the first bidder's distribution to be on [0,1].

[2]This result was later used by Lebrun (1998, 1999), Maskin and Riley (2000), and Cantillon (2002). Plum (1992) extends this analytic result to cover the power distribution $F_1(x) = x^\mu$ and $F_2(x) = \left(\frac{x}{\beta}\right)^\mu$. Note that these again have the same lower bound for the support of the two distributions.

stochastically dominates the other's distribution. This is the first example of such a phenomenon that we are aware of.

In this paper, we present analytic solutions for any asymmetric first-price auction, both with and without a minimum bid m, for two buyers having values uniformly distributed on $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$. We show that our solutions are consistent with previously known solutions of auctions with uniform distributions. As we explain later, our solution also covers the general case of uniform distributions with atoms at the lower end of the interval.

The mathematical expressions change in the different regions of the parameters $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ and m. While one change occurs when the minimum bid ceases to bind, surprisingly, we find another change occurs when $m = \max\{\underline{v}_1, \underline{v}_2\}$. Furthermore, as a function of the distributions, changes occur when a distribution shrinks to a single point (that buyer's value becomes commonly known) whereupon that buyer uses a mixed strategy in equilibrium. We prove that despite these changes, the solutions are still continuous in the parameters. As far as we are aware of, this issue of continuity was not addressed in this generality. A consequence of the continuity is that the profits are also continuous in the parameters.

Several interesting examples are presented, including a class where both bid functions are linear. In particular, given any minimum bid $m \geq 0$, there is a class of uniform distributions (where buyer 2's distribution stochastically dominates buyer 1's distribution) for which the bid functions $b_1(v) = \frac{v}{2} + \frac{m}{2}$ and $b_2(v) = \frac{v}{2} + \frac{m}{4}$ form the equilibrium. This provides a handy textbook example of linear equilibrium bidding in asymmetric auctions. Furthermore, we characterize the environments with uniform distributions that yield linear bid functions and provide a more general formula that becomes linear in those environments.

Besides the challenge of obtaining analytic solutions to a rather wide class of asymmetric first-price auctions, we hope that our results will improve our understanding of auctions and serve as a useful tool for future research on auctions. We also hope that this paper can help or at least inspire others to find analytic solutions of other classes of auction models.

In Section 2, we describe the model and provide initial results about the equilibrium and boundary conditions. We then derive the differential equations resulting from the first-order conditions of the equilibrium and appropriate boundary conditions. We show that this and the second-order conditions can be reduced to a single differential equation. In Section 3, we make use of these results to provide solutions that are distinct on the various regions of the parameters. In Section 4, we show that the solutions are continuous in the parameters. Some examples are then provided in Section 5 along with a short discussion in Section 6. Several of the proofs are given in the Appendix.

2 The Model and the Equilibrium Conditions

We consider a first-price, independent, private-value auction for an indivisible object with two buyers having two general uniform distributions: $U[\underline{v}_1, \overline{v}_1]$ for buyer 1 and $U[\underline{v}_2, \overline{v}_2]$ for buyer 2 (where $-\infty < \underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2 < \infty$, as a uniform distribution has a finite support). Without loss of generality, we assume that $\underline{v}_1 \leq \underline{v}_2$.

As usual, we are interested in the Bayes-Nash equilibrium of this game with incomplete information, that is, a pair of bidding strategies that are best replies to each other, given the beliefs of the buyers about the values of the object.

We allow for the possibility of a minimum bid m, which is assumed to be finite, to ensure that bids are bounded from below. The fact that the bids are bounded from below implies that no buyer wins by bidding less than $\underline{v}_1$ (the argument here is similar to the one made in Kaplan and Wettstein, 2000).[3] In particular, in equilibrium, there is no bid b lower than $\underline{v}_1$. Consequently, we shall assume from now on and without loss of generality that $m \geq \underline{v}_1$.

[3] The argument is along the following lines and by contradiction. Assume that there is a minimum bid m and that bidding below $\underline{v}_1$ has strictly positive probability of winning. From this, bidders must have strictly positive profits for all values including $\underline{v}_1$. Take b^* as the minimum possible equilibrium bid. The bidder bidding b^* must have no chance of winning since if not a slight increase in bid would yield a discrete jump in probability of winning. Since he has no chance of winning by bidding b^*, it follows that the bidder has zero expected profits, providing a contradiction.

Notice that when $m \geq \min\{\overline{v}_1, \overline{v}_2\}$, we have the trivial equilibrium of at most one buyer placing a bid at m. In addition, if $\underline{v}_2 \geq 2\overline{v}_1 - \underline{v}_1$, then any Nash equilibrium must have buyer 2 always bidding $\overline{v}_1$ (and hence always winning the object at price $\overline{v}_1$). Such an equilibrium reflects what Maskin and Riley (2000) refers to as the *Getty effect* where one bidder (the J. Paul Getty Museum) is so dominant that it always wins (in art auctions). The elimination of these trivial cases implies the following:

Lemma 1 *The set of parameters $\underline{v}_1, \underline{v}_2, \overline{v}_1, \overline{v}_2$, and m for which non-trivial equilibria may exist is defined by the following constraints:*

$$\begin{aligned} (i)\ \underline{v}_1 &< \overline{v}_1, \\ (ii)\ \underline{v}_1 &\leq \underline{v}_2 < \overline{v}_2, \\ (iii)\ \underline{v}_2 &< 2\overline{v}_1 - \underline{v}_1, \\ (iv)\ m &< \min\{\overline{v}_1, \overline{v}_2\}. \end{aligned}$$

In this set, we now look for strictly monotone, differentiable equilibrium bid functions $b_1(v)$ and $b_2(v)$. Denote the inverses of these bid functions as $v_1(b)$ with support $[\underline{b}_1, \overline{b}_1]$ and $v_2(b)$ with support $[\underline{b}_2, \overline{b}_2]$. Assume that (in equilibrium) a buyer with zero probability of winning bids his value (this includes any value below m).[4]

Lemma 2 *The interval of equilibrium bids (a subinterval of $[\underline{b}_i, \overline{b}_i]$) in which buyer i has a strictly positive probability of winning in equilibrium is the same for both buyers.*

Proof. Denote the interval of equilibrium bids in which buyer i has a strictly positive probability of winning as $(\underline{c}_i, \overline{c}_i]$. (Note that the lower end of

[4] Without this assumption a bidder with value v, who in equilibrium has zero probability of winning, can sometimes bid more than his value. Formally, this could still be part of a Bayes-Nash equilibrium and have a different allocation than other Bayes-Nash equilibria. Such equilibria can be eliminated, for example, by a trembling-hand argument, i.e., by assuming that each bidder i bids with positive density on $[\underline{v}_i, \overline{v}_i]$. While a bidder bidding below his value when he has zero probability of winning can also be supported in a Bayes-Nash equilibrium, the allocation is the same as the Bayes-Nash equilibrium where he bids his value. For simplicity, we may eliminate such equilibria.

the interval is open since the bid function of either buyer is strictly increasing.) First, observe that due to independence of the value distributions, in equilibrium when a buyer bids b_a he wins with a (weakly) higher probability than when he bids $b_b \leq b_a$. This implies $\bar{c}_i = \bar{b}_i$. Furthermore, increasing probability and continuity of the bid function imply that in equilibrium the set of bids used by a particular buyer with which he has a strictly positive probability of winning is indeed an interval. In addition, $\bar{b}_1 = \bar{b}_2$. Otherwise, if $\bar{b}_i > \bar{b}_j$, then there would be a small enough amount for buyer i (of value $\bar{v}_i$) to lower his bid from $\bar{b}_i$ without lowering his probability of winning. Since $\bar{c}_i = \bar{b}_i$, we have $\bar{c}_1 = \bar{c}_2$.

Finally, we show that $\underline{c}_1 = \underline{c}_2$. First note that for any bid above $\underline{c}_i$, buyer i has a positive probability of winning; hence, there is a positive probability that $b_j \leq \underline{c}_i$. Assume by contradiction that $\underline{c}_i < \underline{c}_j$. By definition, $b_j > \underline{c}_j$ with positive probability. It follows by continuity of buyer j's bid function that j bids b_j, where $\underline{c}_i < b_j < \underline{c}_j$, with strictly positive probability. Consider a bid b'_j such that $\underline{c}_i < b'_j < \underline{c}_j$. By continuity of buyer i's bid function between $\underline{c}_i$ and $\bar{c}_i$, there is a positive measure of b_i for which $\underline{c}_i \leq b_i < b'_j$. In other words, buyer j has a strictly positive probability of winning with bid b_j in contradiction to the definition of $\underline{c}_j$. Hence, $\underline{c}_1 = \underline{c}_2$. ■

In view of Lemma 2, denote by $(\underline{b}, \bar{b}]$ the region of equilibrium bids where if a buyer submits a bid, he has a strictly positive probability of winning in equilibrium. From our assumption that in equilibrium a buyer with zero probability of winning bids his value, it follows that $b_i(v_i) = v_i$ for $v_i < \underline{b}$ and by continuity $b_i(\underline{b}) = \underline{b}$ (if $\underline{b} \geq \underline{v}_i$).

Lemma 3 *In equilibrium,* $\underline{v}_1 = \underline{b}_1 \leq \max\{\underline{b}_2, m\} = \underline{b}$, *and*

$$\underline{b} = \max\{\frac{\underline{v}_1 + \underline{v}_2}{2}, m\}. \tag{2}$$

Proof. Since any bid b is such that $b \geq \underline{v}_1$ and no one bids above his value we have $\underline{b}_1 = \underline{v}_1$. Consequently, $\underline{b}_1 \leq \underline{b}_2$ and $\underline{b}_1 \leq \max\{\underline{b}_2, m\}$. We now show that $\underline{b} = \max\{\underline{b}_2, m\}$ by first showing that $\underline{b} \geq \max\{m, \underline{b}_2\}$ and then showing that $\underline{b} \leq \max\{m, \underline{b}_2\}$.

To show $\underline{b} \geq \max\{m, \underline{b}_2\}$, let us first show that $\underline{b} \geq \max\{m, \underline{b}_1, \underline{b}_2\}$. In fact, $\underline{b} < m$ is impossible since this would imply that a bid strictly less than the minimum bid m has a strictly positive probability of winning. Since by definition $(\underline{b}, \bar{b}]$ is a subinterval of both $[\underline{b}_1, \bar{b}_1]$ and $[\underline{b}_2, \bar{b}_2]$, it follows that $\underline{b}_i \leq \underline{b}$ for $i = 1, 2$, completing the proof of $\underline{b} \geq \max\{m, \underline{b}_1, \underline{b}_2\}$. Since $\underline{b}_1 \leq \underline{b}_2$, this also yields $\underline{b} \geq \max\{m, \underline{b}_2\}$.

Now let us show $\underline{b} \leq \max\{m, \underline{b}_2\}$. By contradiction assume that $\underline{b} > \max\{m, \underline{b}_2\}$. Since $\underline{b}_1 \leq \underline{b}_2$, by continuity of buyer 1's bid function, for some value, buyer 1 bids b_1' such that $\underline{b} > b_1' > \max\{m, \underline{b}_2\}$. However, by continuity of buyer 2's bid function, buyer 2 bids b_2 such that $b_1' > b_2 > \max\{m, \underline{b}_2\}$ with strictly positive probability. This implies that buyer 1 wins with strictly positive probability when bidding b_1', in contradiction to the definition of $\underline{b}$.

Now we solve for $\underline{b}$ in terms $\underline{v}_1$ and $\underline{v}_2$ when $m = \underline{v}_1$. In the interval $[\underline{b}, \bar{b}]$, buyer 2 with value v_2 solves the following maximization problem:

$$\max_b \left(\frac{v_1(b) - \underline{v}_1}{\overline{v}_1 - \underline{v}_1}\right)(v_2 - b).$$

Below $\underline{b}$ buyer 1 bids his value, and thus buyer 2 with value $v_2(\underline{b})$ must not benefit from bidding less than $\underline{b}$:

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (b - \underline{v}_1)(v_2(\underline{b}) - b), \ \forall\, b \leq \underline{b}.$$

This is true only if $\underline{b} \leq \frac{\underline{v}_1 + v_2(\underline{b})}{2}$. Similarly, buyer 2 with value $v_2(\underline{b})$ does not benefit from bidding more than $\underline{b}$:

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (v_1(b) - \underline{v}_1)(v_2(\underline{b}) - b), \ \forall\, b \geq \underline{b}. \tag{3}$$

However, since $v_1(b) \geq b$, we have

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (b - \underline{v}_1)(v_2(\underline{b}) - b), \ \forall\, b \geq \underline{b}. \tag{4}$$

This can happen only if $\underline{b} \geq \frac{\underline{v}_1 + v_2(\underline{b})}{2}$; therefore $\underline{b} = \frac{\underline{v}_1 + v_2(\underline{b})}{2}$. Since $m = \underline{v}_1$, we have $\underline{b}_2 \geq \underline{v}_1 = m$, which implies that $\underline{b} = \max\{m, \underline{b}_2\} = \underline{b}_2$, and hence $v_2(\underline{b}) = v_2(\underline{b}_2) = \underline{v}_2$. Thus,

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}. \tag{5}$$

With a minimum bid m, by definition $\underline{b} \geq m$. If $m \leq \frac{\underline{v}_1+\underline{v}_2}{2}$, (5) still holds. If $m \geq \frac{\underline{v}_1+\underline{v}_2}{2}$, then we have $\underline{b} = m$ (the first constraint (3) from above is not necessary and the second constraint (4) is satisfied). Therefore,

$$\underline{b} = \max\{\frac{\underline{v}_1 + \underline{v}_2}{2}, m\}.$$

■

2.1 The differential equations

In the interval $[\underline{b}, \bar{b}]$, the functions $v_1(b)$ and $v_2(b)$ must satisfy (by the first-order conditions of the maximization problems)

$$\begin{aligned} v_1'(b)(v_2(b) - b) &= v_1(b) - \underline{v}_1, \\ v_2'(b)(v_1(b) - b) &= v_2(b) - \underline{v}_2. \end{aligned} \tag{6}$$

Let us look now at the boundary conditions. As we noted above, $\underline{b}$ belongs to $[\underline{v}_1, \overline{v}_1]$. Furthermore, if $m \geq \underline{v}_2$, then $\underline{b} = m$. We must have, in equilibrium, the following:

B1 $v_1(\underline{b}) = \underline{b}$ (recall that a buyer bids his value when his probability of winning is zero).

B2 $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$ (this is the minimum value that gives buyer 2 a positive probability of winning).

B3 $v_1(\bar{b}) = \overline{v}_1$ and $v_2(\bar{b}) = \overline{v}_2$ (the highest bid of each buyer is reached for his highest value.)

Adding the equations in (6) together yields

$$v_1'(b)v_2(b) + v_2'(b)v_1(b) = [(v_1(b) + v_2(b) - (\underline{v}_1 + \underline{v}_2))b]'.$$

By integrating, we have

$$v_1(b) \cdot v_2(b) = b(v_1(b) + v_2(b)) - (\underline{v}_1 + \underline{v}_2) \cdot b + c. \tag{7}$$

Lemma 4 *The upper bound of the bid functions, $\bar{b}$, is given by*

$$\bar{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - c}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}, \tag{8}$$

where

$$c = \begin{cases} \frac{(\underline{v}_1+\underline{v}_2)^2}{4} & \text{if } \frac{\underline{v}_1+\underline{v}_2}{2} \geq m, \\ (\underline{v}_1 + \underline{v}_2)m - m^2 & \text{otherwise.} \end{cases} \tag{9}$$

Proof. Substituting the lower boundary condition B1 into (7) yields

$$v_2(\underline{b})\underline{b} = \underline{b}(v_2(\underline{b}) + \underline{b}) - (\underline{v}_1 + \underline{v}_2)\underline{b} + c.$$

This simplifies to

$$c = (\underline{v}_1 + \underline{v}_2)\underline{b} - \underline{b}^2.$$

From (2), we have (9). (Note that c, as a function of m, reaches its maximum at $m = \frac{\underline{v}_1+\underline{v}_2}{2}$.) Using B3 and (7) we have

$$\overline{v}_1 \cdot \overline{v}_2 = \bar{b}(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2) \cdot \bar{b} + c,$$

which yields (8). ■

2.1.1 Reduction to a single differential equation.

We can use (7) to find $v_2(b)$ in terms of $v_1(b)$ as follows:

$$v_2(b) = \frac{bv_1(b) - (\underline{v}_1 + \underline{v}_2)b + c}{v_1(b) - b}. \tag{10}$$

We can then rewrite the differential equation (6) as

$$v_1'(b) \cdot (\frac{bv_1(b) - b(\underline{v}_1 + \underline{v}_2) + c}{v_1(b) - b} - b) = v_1(b) - \underline{v}_1$$

or

$$v_1'(b) \cdot (-b(\underline{v}_1 + \underline{v}_2) + c + b^2) = (v_1(b) - \underline{v}_1)(v_1(b) - b). \tag{11}$$

Equations (9) and (11) and boundary condition $v_1(\bar{b}) = \overline{v}_1$ are used to find a solution for $v_1(b)$. With the solution of $v_1(b)$, equations (9) and (10) are then used to find $v_2(b)$. Although the differential equation is derived from the first-order conditions, any solution to it also satisfies the second-order conditions (see Appendix A.1), and hence is an equilibrium bid function. This is the method we follow to find the equilibrium bid functions in the next section.

3 Solutions

3.1 Auction without a minimum bid

The auction without a minimum bid has the same solution as an auction with a minimum bid m that satisfies $m \leq \frac{\underline{v}_1+\underline{v}_2}{2}$.

Proposition 1 *When* $m \leq (\underline{v}_1+\underline{v}_2)/2$, *the equilibrium inverse bid functions are given by*

$$v_1(b) = \underline{v}_1 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_2 + \underline{v}_1 - 2b)c_1 e^{\frac{\underline{v}_2-\underline{v}_1}{\underline{v}_2+\underline{v}_1-2b}} + 4(\underline{v}_2 - b)}, \tag{12}$$

$$v_2(b) = \underline{v}_2 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_1 + \underline{v}_2 - 2b)c_2 e^{\frac{\underline{v}_1-\underline{v}_2}{\underline{v}_1+\underline{v}_2-2b}} + 4(\underline{v}_1 - b)} \tag{13}$$

where

$$c_1 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_1-\underline{v}_1} + 4(\overline{b} - \underline{v}_2)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\overline{b}-\underline{b})}}, \tag{14}$$

$$c_2 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_2-\underline{v}_2} + 4(\overline{b} - \underline{v}_1)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_1-\underline{v}_2}{2(\overline{b}-\underline{b})}} \tag{15}$$

and

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}, \quad \overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\frac{\underline{v}_1+\underline{v}_2}{2})^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}. \tag{16}$$

Proof. In solving differential equation (11), we first have (by (9) and (8)) $c = \frac{(\underline{v}_1+\underline{v}_2)^2}{4}$ and

$$\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\frac{\underline{v}_1+\underline{v}_2}{2})^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}.$$

Rewrite equation (11) as

$$v_1'(b)\cdot(\underline{v}_1+\underline{v}_2-2b)^2 = 4(v_1(b)-\underline{v}_1)(v_1(b)-b).$$

Define now $\alpha \equiv \underline{v}_1+\underline{v}_2-2\underline{v}_1 = \underline{v}_2-\underline{v}_1$, $x \equiv b-\underline{v}_1$ and $D(x)$ such that

$$v_1(b) = \frac{\alpha^2}{D(x)} + \underline{v}_1. \tag{17}$$

We then have $v_1'(x) = -\frac{\alpha^2}{D(x)^2}D'(x)$, and equation (11) becomes

$$\begin{aligned}
D'(x)\cdot(\alpha-2x)^2 &= 4(D(x)x-\alpha^2),\\
D'(x)\cdot(\alpha-2x)^2 &= 4D(x)x-16x(\alpha-x)-4(\alpha-2x)^2,\\
(D'(x)+4)\cdot(\alpha-2x)^2 &= 4x(D(x)-4(\alpha-x)),
\end{aligned}$$

$$\begin{aligned}
\frac{D'(x)+4}{D(x)-4(\alpha-x)} &= \frac{4x}{(\alpha-2x)^2}\\
&= \frac{2\alpha}{(\alpha-2x)^2} - \frac{2}{\alpha-2x}.
\end{aligned}$$

By integrating both sides, we obtain

$$\ln(D(x)-4(\alpha-x)) = \frac{\alpha}{\alpha-2x} + \ln(\alpha-2x) + \ln c_1,$$

and taking the exponent of both sides yields

$$\begin{aligned}
D(x)-4(\alpha-x) &= (\alpha-2x)c_1e^{\frac{\alpha}{\alpha-2x}},\\
D(x) &= (\alpha-2x)c_1e^{\frac{\alpha}{\alpha-2x}} + 4(\alpha-x).
\end{aligned} \tag{18}$$

The upper boundary condition $v_1(\bar{b}) = \overline{v}_1$ determines c_1. When $b=\bar{b}$, we have $x = \overline{x} \equiv \bar{b}-\underline{v}_1$. From our definition we have $D(\overline{x}) = \frac{\alpha^2}{\overline{v}_1-\underline{v}_1}$. Hence the boundary condition becomes

$$c_1 = \frac{\frac{\alpha^2}{\overline{v}_1-\underline{v}_1} - 4(\alpha-(\bar{b}-\underline{v}_1))}{(\alpha-2(\bar{b}-\underline{v}_1))}e^{-\frac{\alpha}{\alpha-2(\bar{b}-\underline{v}_1)}},$$

which can be rewritten as (recall that in this case $\underline{b} = \frac{\underline{v}_1+\underline{v}_2}{2}$)

$$c_1 = \frac{\frac{(\underline{v}_2-\underline{v}_1)^2}{\overline{v}_1-\underline{v}_1} + 4(\overline{b} - \underline{v}_2)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_2-\underline{v}_1}{2(\overline{b}-\underline{b})}}.$$

Note that this depends only on the constants of the game $\underline{v}_i, \overline{v}_i$, since

$$\overline{b} - \underline{v}_2 = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)} - \underline{v}_2$$

and

$$\overline{b} - \underline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1+\underline{v}_2)^2}{4}}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)} - \frac{\underline{v}_1 + \underline{v}_2}{2}.$$

Equations (12) and (14) are obtained from equations (17) and (18) and the definitions of α, x. Finally, equations (13) and (15) are obtained from equations (12) and (14), respectively, by reversing the roles of $\underline{v}_1, \overline{v}_1$ with those of $\underline{v}_2, \overline{v}_2$. ■

3.2 A limit case where buyer 2's value is known (continuity as $\overline{v}_2 \longrightarrow \underline{v}_2$)

As a test of the above result let us relate it to the asymmetric situation treated by Kaplan and Zamir (KZ) (2000) (see also Martínez-Pardina, 2006), namely, the situation in which the value of one of the two buyers is common knowledge. For simplicity, we normalize this situation to $[\underline{v}_1, \overline{v}_1] = [0, 1]$ and $\underline{v}_2 = \overline{v}_2 = \beta$ where $0 < \beta < 2$ (when $\beta > 2$, the equilibrium is that buyer 2 bids 1 and wins with certainty).[5]

For this situation, KZ found that in the equilibrium of the first-price auction, buyer 1's inverse bid function is

$$v_1(b) = \frac{\beta^2}{4(\beta - b)}, \tag{19}$$

[5] When buyer 1's value is commonly known, the equilibrium is trivial in that buyer 2 wins the auction at buyer 1's value. (In this special situation, we also have to relax the assumption of buyer 1 bidding his value when he doesn't win to obtain an equilibrium.)

while buyer 2, whose value is known to be β, uses a mixed strategy given by the following cumulative probability distribution (with support from $\underline{b} = \frac{\beta}{2}$ to $\overline{b} = \beta - \frac{\beta^2}{4}$) :

$$F(b) = \frac{(2-\beta)\beta}{2(2b-\beta)} e^{-\frac{\beta}{2b-\beta} - \frac{2}{\beta-2}}. \tag{20}$$

Let us view this situation as a limiting case of our model where $[\underline{v}_1, \overline{v}_1] = [0,1]$,$[\underline{v}_2, \overline{v}_2] = [\beta, \beta+\varepsilon]$, and $\varepsilon \to 0$. This may be viewed as a continuity property of the solution as $\overline{v}_2 \longrightarrow \underline{v}_2$. To see that, we first write the probability distribution of the bids of buyer 2, which is

$$P(b_2(V_2, \varepsilon) \le b) = P(V_2 \le v_2(b, \varepsilon)) = \frac{v_2(b,\varepsilon) - \beta}{\varepsilon}.$$

(We use V_2 for the random value of buyer 2, denote $b_i(v, \varepsilon)$ as the bid function for bidder i when the distribution is $[\beta, \beta+\varepsilon]$, and denote $v_i(b, \varepsilon)$ as the respective inverse bid function.)

Proposition 2 *The equilibrium in KZ is a limit of our solution in the following sense:*

(i) The limit of buyer 1's bid function is that in KZ, namely,

$$\lim_{\varepsilon \to 0} v_1(b, \varepsilon) = \frac{\beta^2}{4(\beta - b)}.$$

(ii) The bid distribution of buyer 2 approaches the mixed bidding strategy in KZ, i.e.,

$$\lim_{\varepsilon \to 0} \frac{v_2(b,\varepsilon) - \beta}{\varepsilon} = F(b),$$

where F(b) is given by (20).

Proof. First, we observe that for $[\underline{v}_1, \overline{v}_1] = [0,1]$ and $\underline{v}_2 = \overline{v}_2 = \beta$ we obtain from our above equations for $\underline{b}$ and $\overline{b}$ ((5) and (16)) the correct range of bids: $\underline{b} = \frac{\beta}{2}$ and $\overline{b} = \beta - \frac{\beta^2}{4}$. Next, notice that $\overline{b} > \underline{b}$ whenever $\beta - \frac{\beta^2}{4} > \frac{\beta}{2}$ (i.e., $\beta < 2$). Assuming that this is indeed the case, we have a range of bids even when one buyer's value is known with near certainty. (This makes sense since it converges to a mixed-strategy equilibrium.) Now, using the analytic solution for buyer 1's inverse bid function, (12) and (14), with the distributions of $[\underline{v}_1, \overline{v}_1] = [0,1]$,$[\underline{v}_2, \overline{v}_2] = [\beta, \beta+\varepsilon]$, we have

$$\begin{aligned} v_1(b,\varepsilon) &= \frac{\beta^2}{(\beta-2b)c_1 e^{\frac{\beta}{\beta-2b}} + 4(\beta-b)}, \\ c_1(\varepsilon) &= \frac{\beta^2 - 4(\beta-\overline{b})}{(\beta-2\overline{b})} e^{-\frac{\beta}{\beta-2\overline{b}}}, \end{aligned}$$

where $\overline{b} = \overline{b}(\varepsilon) = \frac{\beta+\varepsilon-\frac{\beta^2}{4}}{1+\varepsilon}$.

We have

$$\lim_{\varepsilon\to 0} v_1(b,\varepsilon) = \frac{\beta^2}{(\beta-2b)\lim_{\varepsilon\to 0} c_1(\varepsilon) e^{\frac{\beta}{\beta-2b}} + 4(\beta-b)} = \frac{\beta^2}{4(\beta-b)},$$

since

$$\lim_{\varepsilon\to 0} c_1(\varepsilon) = \lim_{\varepsilon\to 0} \frac{\beta^2 - 4(\beta-\overline{b}(\varepsilon))}{(\beta-2\overline{b}(\varepsilon))} e^{-\frac{\beta}{\beta-2\overline{b}(\varepsilon)}} = 0.$$

Furthermore, using the analytic solution for buyer 2's inverse bid function, (13) and (15), we have

$$v_2(b,\varepsilon) = \frac{\beta^2/\varepsilon}{\left(\frac{4-\beta+\frac{\beta}{\varepsilon}}{\frac{1}{2}\beta-1}\right)(\beta-2b)\, e^{-\frac{\beta}{\beta-2b}} e^{-\frac{2}{2-\beta}} - 4b}. \tag{21}$$

And finally it can be verified (by straightforward calculation using (21) and (20)) that indeed

$$\lim_{\varepsilon\to 0} \frac{v_2(b,\varepsilon)-\beta}{\varepsilon} = F(b).$$

■

3.3 Auction with a minimum bid

When the minimum bid is binding, as in the case where $m > (\underline{v}_1+\underline{v}_2)/2$, equation (9) becomes $c = (\underline{v}_1+\underline{v}_2)m - m^2$ and (8) becomes $\overline{b} = \frac{\overline{v}_1\cdot\overline{v}_2 - (\underline{v}_1+\underline{v}_2)m + m^2}{(\overline{v}_1-\underline{v}_1)+(\overline{v}_2-\underline{v}_2)}$. Also, since $\underline{v}_1 \le \underline{v}_2$, we have $m > \underline{v}_1$. Now, we can rewrite the differential equation (11) as

$$v_1'(b) \cdot (b - m)(b + m - \underline{v}_1 - \underline{v}_2) = (v_1(b) - \underline{v}_1)(v_1(b) - b). \tag{22}$$

Notice that since $b \geq m$ and $2m > \underline{v}_1 + \underline{v}_2$, the coefficient of $v_1'(b)$ on the left-hand side of the above equation is positive. This leads to the following proposition:

Proposition 3 *The equilibrium inverse bid function for buyer 1 with minimum bid m such that $m > (\underline{v}_1 + \underline{v}_2)/2$ and $m \neq \underline{v}_2$ is given by*

$$v_1(b) = \underline{v}_1 + \frac{(m - \underline{v}_1)(m - \underline{v}_2)}{b - \underline{v}_2 + (b - m)^{\frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}} (b + m - \underline{v}_1 - \underline{v}_2)^{\frac{(m - \underline{v}_2)}{(m - \underline{v}_1) + (m - \underline{v}_2)}} c_1}, \tag{23}$$

where

$$c_1 = -\frac{(\overline{v}_1 - m)(\overline{v}_2 - \underline{v}_2)\left(\frac{(m - \underline{v}_2 + \overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - m)(\overline{v}_2 - m)}\right)^{\frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}}}{(\overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}. \tag{24}$$

Buyer 2's inverse bid function $v_2(b)$ is obtained from $v_1(b)$ by interchanging the roles of $\underline{v}_1, \overline{v}_1$ and $\underline{v}_2, \overline{v}_2$. The bounds of the bid functions are $\underline{b} = m$ and $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\underline{v}_1 + \underline{v}_2)m + m^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$.

Proof. The derivation of this solution of equation (22) is given in Appendix A.2. ■

3.3.1 Special case when $\underline{v}_1 = \underline{v}_2 = 0$ (with and without minimum bids).

Corollary 1 *The equilibrium inverse bid function for buyer 1 with minimum bid m and $\underline{v}_1 = \underline{v}_2 = 0$ is given by*

$$v_1(b) = \frac{m^2}{b + \sqrt{b^2 - m^2 c_1}},$$

$$c_1 = -\frac{(\overline{v}_1 - m)(\overline{v}_2)\left(\frac{(m+\overline{v}_1)(m+\overline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}\right)^{1/2}}{\overline{v}_1(m+\overline{v}_2)}.$$

Proof. Substituting $\underline{v}_1 = \underline{v}_2 = 0$ into the solution, equations (23) and (24) yields the result. ■

This is a special case that comes directly from substitution in our formula of Proposition 3. We are not aware of this solution elsewhere. Now we show that this solution agrees with other results. Taking $\lim_{m\to 0} v_1(b)$ and applying L'Hopital's rule yields

$$v_1(b) = \frac{2b\overline{v}_1^2\overline{v}_2^2}{\overline{v}_1^2\overline{v}_2^2 + b^2(\overline{v}_2^2 - \overline{v}_1^2)}.$$

Reversing the roles of $\overline{v}_1$ and $\overline{v}_2$ gives us

$$v_2(b) = \frac{2b\overline{v}_1^2\overline{v}_2^2}{\overline{v}_1^2\overline{v}_2^2 - b^2(\overline{v}_2^2 - \overline{v}_1^2)}.$$

Setting $\overline{v}_1 = 1$ and $\overline{v}_2 = \beta$ to find $v_1(b)$ and $v_2(b)$ yields equation (1), which is the result in Griesmer et al. (1967).

Furthermore, setting $\overline{v}_1 = \overline{v}_2 = 1$ yields the symmetric case with a minimum bid:

$$v_1(b) = \frac{m^2}{b + \sqrt{b^2 - m^2 c_1}},$$

$$c_1 = -\frac{(1-m)\frac{(m+1)}{(1-m)}}{(m+1)} = -1.$$

The limit as $m \to 0$ is $v_1(b) = 2b$, which agrees with the standard result.

3.4 The case when $m = \underline{v}_2$

Looking at the solution for the case of a minimum bid, the expressions $(m - \underline{v}_1)$ and $(m - \underline{v}_2)$ appear in the denominator (in the constant). Since we are in the case where $m > (\underline{v}_1 + \underline{v}_2)/2$ and $\underline{v}_2 \geq \underline{v}_1$, we have $m = \underline{v}_1$ only when $\underline{v}_1 = \underline{v}_2 = m$, which reduces to the case of no minimum bid. We are left to check the limit of our solution with a minimum bid as $m = \underline{v}_2$.

Proposition 4 *The equilibrium inverse bid function for buyer 1 with minimum bid m such that* $m = \underline{v}_2$ *and* $\underline{v}_2 > \underline{v}_1$ *is given by*

$$v_1(b) = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{b - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\left[c + \log\left(\frac{b - \underline{v}_1}{b - \underline{v}_2}\right)\right]} \tag{25}$$

where

$$\begin{aligned} c &= \frac{(\overline{v}_1 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{b} - \underline{v}_2)} - \log\left(\frac{\overline{b} - \underline{v}_1}{\overline{b} - \underline{v}_2}\right) \\ &= \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_2)} - \log\left(\frac{(\overline{v}_2 - \underline{v}_1)(\overline{v}_1 - \underline{v}_1)}{(\overline{v}_2 - \underline{v}_2)(\overline{v}_1 - \underline{v}_2)}\right). \end{aligned} \tag{26}$$

Again, the buyer 2's function $v_2(b)$ *is obtained from* $v_1(b)$ *by interchanging the roles of* $\underline{v}_1, \overline{v}_1$ *and* $\underline{v}_2, \overline{v}_2$. *The bounds of the bid functions are* $\underline{b} = m$ *and* $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$.

Proof. See the Appendix A.3. ■

3.5 A limit case as $\overline{v}_2 \longrightarrow \underline{v}_2$ with a minimum bid.

In this section, we check for continuity at the limit case when buyer 2's value is commonly known. This is an extension of the case treated in Section 3.2 in the presence of a minimum bid. We again use the normalization in Section 3.2, that is, $[\underline{v}_1, \overline{v}_1] = [0, 1]$ and $\underline{v}_2 = \beta$, $\overline{v}_2 = \beta + \varepsilon$. From substituting these into equations (23) and (24), it is clear that $c_1(\varepsilon) \to 0$, therefore

$$\lim_{\varepsilon \to 0} v_1(b, \varepsilon) = \frac{m(\beta - m)}{\beta - b}.$$

To find $v_2(b, \varepsilon)$, we again use (23) and (24) (but the roles of $\underline{v}_1, \overline{v}_1$ and $\underline{v}_2, \overline{v}_2$ reversed). Hence,

$$v_2(b, \varepsilon) = \beta + \frac{m(m - \beta)}{b + (b - m)^{\frac{m-\beta}{2m-\beta}}(b + m - \beta)^{\frac{m}{2m-\beta}} c_2(\varepsilon)},$$

where

$$c_2(\varepsilon) = -\frac{(\beta + \varepsilon - m)\left(\frac{(m+\varepsilon)(m-\beta+1)}{(1-m)(\beta-m)}\right)^{\frac{m-\beta}{2m-\beta}}}{\varepsilon(m - \beta + 1)}.$$

As in Section 3.2, buyer 2's strategy goes to a mixed strategy with cumulative distribution

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{v_2(b, \varepsilon) - \beta}{\varepsilon} &= \frac{m}{\frac{\left(\frac{m(m-\beta+1)}{(1-m)(\beta-m)}\right)^{\frac{m-\beta}{2m-\beta}}}{m-\beta+1}(b - m)^{\frac{m-\beta}{2m-\beta}}(b + m - \beta)^{\frac{m}{2m-\beta}}} \\
&= \frac{((1 - m)(\beta - m))^{\frac{m-\beta}{2m-\beta}}(m(m - \beta + 1))^{\frac{m}{2m-\beta}}}{(b - m)^{\frac{m-\beta}{2m-\beta}}(b + m - \beta)^{\frac{m}{2m-\beta}}}.
\end{aligned}$$

The following proposition shows us that the limit equals the equilibrium when $\varepsilon \to 0$.[6]

Proposition 5 *The limit of our solution with a minimum bid is an equilibrium when buyer 2's value is known with a minimum bid. Namely,*

$$\lim_{\varepsilon \to 0} v_1(b, \varepsilon) = \frac{m(\beta - m)}{\beta - b} \equiv v_1(b)$$

and

$$\lim_{\varepsilon \to 0} \frac{v_2(b, \varepsilon) - \beta}{\varepsilon} = \frac{((1 - m)(\beta - m))^{\frac{m-\beta}{2m-\beta}}(m(m - \beta + 1))^{\frac{m}{2m-\beta}}}{(b - m)^{\frac{m-\beta}{2m-\beta}}(b + m - \beta)^{\frac{m}{2m-\beta}}} \equiv G(b)$$

form the unique equilibrium when buyer 2's value is known where buyer 1 has inverse bid function $v_1(b)$ and buyer 2 bids with a mixed-strategy given by the cumulative distribution $G(b)$.

Proof. The limits were shown above. To show that $v_1(b)$ and $G(b)$ is in fact the unique equilibrium when buyer 2's value is known, we proceed to find this equilibrium directly. Since buyer 2 bids a mixed strategy, he must

[6] Also when $m \to \beta(< 1)$, the solution approaches the equilibrium that buyer 2 stays out of the auction and buyer 1 wins the auction (for all values above m). Also, when $m \to \beta/2$, this goes to the solution in Section 3.2 (when the minimum bid is not binding).

be indifferent to every point in his support including the minimum bid m. The following formula represents this.

$$v_1(b)(\beta - b) = m(\beta - m).$$

Hence, we have

$$v_1(b) = \frac{m(\beta - m)}{\beta - b}. \tag{27}$$

If the cumulative distribution of buyer 2's mixed strategy is $G(b)$, then the first-order conditions of buyer 1 yields

$$G'(b)(v_1(b) - b) = G(b).$$

By rewriting this equation and substituting for $v_1(b)$ using equation (27), we have

$$\frac{G'(b)}{G(b)} = \frac{1}{v_1(b) - b} = \frac{\beta - b}{m(\beta - m) - b\,(\beta - b)}.$$

We can solve this differential equation through the following steps:

$$\begin{aligned}
\int \frac{G'(b)}{G(b)}db &= \int \frac{\beta - b}{m(\beta - m) - b\,(\beta - b)}db = \int \frac{\beta - b}{(b + m - \beta)(b - m)}db, \\
\ln(G(b)) &= \frac{(\beta - m)\ln(b - m) - m\ln(b + m - \beta)}{2m - \beta} + c, \\
G(b) &= c_2(b - m)^{\frac{\beta - m}{2m - \beta}}(b + m - \beta)^{-\frac{m}{2m - \beta}}.
\end{aligned}$$

Note that for $\bar{b}$, we have

$$v_1(\bar{b})(\beta - \bar{b}) = m(\beta - m) = (\beta - \bar{b}),$$

which implies

$$\bar{b} = \beta - m(\beta - m).$$

This determines c_2 by using the equality $G(\bar{b}) = 1$:

$$1 = c_2(\bar{b} - m)^{\frac{\beta - m}{2m - \beta}}(\bar{b} + m - \beta)^{-\frac{m}{2m - \beta}}$$

Rewriting this gives us the expression for c_2:

$$c_2 = ((1-m)(\beta - m))^{-\frac{\beta - m}{2m-\beta}} (m(m+1-\beta))^{\frac{m}{2m-\beta}}.$$

■

We note that when $m \to \beta(< 1)$, the solution approaches the equilibrium that buyer 2 stays out of the auction and buyer 1 wins the auction (for all values above m). Also, it can be shown that when $m \to \beta/2$, this goes to the solution in Section 3.2 (when the minimum bid is not binding).

4 Continuity of the Equilibrium

In this section, we prove the following proposition.

Proposition 6 *The equilibrium bid functions are continuous in the parameters* $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ *and* m.

Proof. We shall prove the continuity of the inverse bid functions since this implies the continuity of the equilibrium bid functions. In Sections 3.2 and 3.5, we proved continuity when $\overline{v}_2 \to \underline{v}_2$ with and without a binding minimum bid m. When $\overline{v}_1 \to \underline{v}_1$ (and $m \leq \underline{v}_1$), the solution goes to the equilibrium where buyer 2 is bidding $\overline{v}_1$ and winning the auction.

Outside of these cases, we have found the equilibrium bid functions on four regions of the minimum bid m:

(1) For $m \leq (\underline{v}_1 + \underline{v}_2)/2$. This was the case of "no minimum bid", that is, the minimum bid is not binding in equilibrium. This equilibrium, given in equations (12) and (14), thus does not depend on m.

(2) For $m > (\underline{v}_1 + \underline{v}_2)/2$ and $m \neq \underline{v}_2$. The minimum bid is binding in equilibrium, and this equilibrium depends on m. It is given in equations (23) and (24).

(3) For $m > (\underline{v}_1 + \underline{v}_2)/2$ and $m = \underline{v}_2$. This equilibrium is listed in equations (25) and (26).

(4) For $\overline{v}_i \leq m \leq \overline{v}_j$. Buyer j bids m for all $m \leq v_j$ while buyer i bids his value v_i.

Within each of these regions the solution is continuous in $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$, and m. Hence, to check the continuity of the equilibrium, we need to check continuity between regions in the following cases:

(A) between regions 1 and 2. ($m = (\underline{v}_1 + \underline{v}_2)/2$).

(B) between regions 2 and 3. ($m = \underline{v}_2$ and $m > (\underline{v}_1 + \underline{v}_2)/2$).

(C) between regions 2 and 4. ($m = \min\{\overline{v}_1, \overline{v}_2\}$).

We do this as follows.

(A) Continuity at $m = (\underline{v}_1 + \underline{v}_2)/2$

Since for $m \leq (\underline{v}_1 + \underline{v}_2)/2$ the equilibrium does not depend on m, continuity is established by proving that the inverse bid function $v_1(b)$ given by (23) approaches that given by (12) as m approaches the critical value $(\underline{v}_1 + \underline{v}_2)/2$ from above. First, we verify that

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} (b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}} (b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}} = \frac{1}{2} e^{-\frac{\underline{v}_2-\underline{v}_1}{2b-\underline{v}_1-\underline{v}_2}} (2b-\underline{v}_1-\underline{v}_2),$$

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} \left(\frac{(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)} \right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}} = e^{2\frac{(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)(\underline{v}_2-\underline{v}_1)}{(2\overline{v}_1-\underline{v}_2-\underline{v}_1)(2\overline{v}_2-\underline{v}_2-\underline{v}_1)}}.$$

Using these in our solution for $v_1(b)$ and c_1 in equations (23) and (24), we have

$$\begin{aligned} \lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} v_1(b) = \underline{v}_1 + & \\ & \lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} \frac{(m-\underline{v}_1)(m-\underline{v}_2)}{b-\underline{v}_2+\frac{1}{2}e^{-\frac{\underline{v}_2-\underline{v}_1}{2b-\underline{v}_1-\underline{v}_2}}(2b-\underline{v}_1-\underline{v}_2)\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} c_1} \\ & = \underline{v}_1 + \frac{-(\underline{v}_2-\underline{v}_1)^2/4}{b-\underline{v}_2+\frac{1}{2}e^{-\frac{\underline{v}_2-\underline{v}_1}{2b-\underline{v}_1-\underline{v}_2}}(2b-\underline{v}_1-\underline{v}_2)\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} c_1}, \end{aligned} \tag{28}$$

$$\lim_{m \searrow (\underline{v}_1+\underline{v}_2)/2} c_1 = -\frac{(2\overline{v}_1-(\underline{v}_1+\underline{v}_2))(\overline{v}_2-\underline{v}_2)e^{2\frac{(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)(\underline{v}_2-\underline{v}_1)}{(2\overline{v}_1-\underline{v}_2-\underline{v}_1)(2\overline{v}_2-\underline{v}_2-\underline{v}_1)}}}{(\overline{v}_1-\underline{v}_1)(2\overline{v}_2-(\underline{v}_1+\underline{v}_2))}. \tag{29}$$

We now see that, indeed, this limit yields the equilibrium bid functions for the case of no minimum bid. Note that the range of bids is as follows:

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}, \quad \overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1 + \underline{v}_2)^2}{4}}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)}.$$

Notice that by (5) and (16) we have $\overline{b} - \underline{b} = \frac{1}{4} \frac{(2\overline{v}_1 - \underline{v}_2 - \underline{v}_1)(2\overline{v}_2 - \underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)}$ and that

$$\frac{(\underline{v}_2 - \underline{v}_1)^2}{\overline{v}_1 - \underline{v}_1} + 4(\overline{b} - \underline{v}_2) = \frac{(\overline{v}_2 - \underline{v}_2)(2\overline{v}_1 - (\underline{v}_1 + \underline{v}_2))}{(\overline{v}_1 - \underline{v}_1)(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}.$$

Using these two in equations (28) and (29) yields the equilibrium bid function without a minimum bid; namely, it establishes the equality between (28), (29) and (12), (14), respectively.

(B) Continuity at $m = \underline{v}_2$ (in the region $m > (\underline{v}_1 + \underline{v}_2)/2$)

To prove continuity at $m = \underline{v}_2$, we have to show that the limit of the functions in (23) and (24) as $m \to \underline{v}_2$ converges to the functions in (25) and (26) in Proposition 4. This we prove in Appendix A.4, which completes our proof of the continuity in m.

(C) Continuity at $m = \min\{\overline{v}_1, \overline{v}_2\}$ (between regions 2 and 4).

Here we examine the case where $\overline{v}_1 \geq \overline{v}_2$. In region 2, when $m \to \overline{v}_2$, by equation (24), $c_1 \to -\infty$. By examining equation (23). The only way that $v_1(b) > \underline{v}_1$ is for $b \to m$. Similarly, we can show the same for $v_2(b)$ when $\overline{v}_1 \leq \overline{v}_2$. ■

We note that continuity of the bid functions in the parameters also implies that the profits are continuous in the parameters (if $\underline{v}_1 < \overline{v}_1$ and $\underline{v}_2 < \overline{v}_2$).[7] To see this notice that given this continuity of the bid functions in the parameters, a discrete change in the profits requires both an atom in the distribution of equilibrium bids and that this atom transverses the minimum bid. An atom in the distribution of equilibrium bids can only occur if $\min\{\overline{v}_1, \overline{v}_2\} < m < \max\{\overline{v}_1, \overline{v}_2\}$. In such a case, the atom is always at m and thus the atom doesn't transverse m.

[7] The reason why we need $\underline{v}_1 < \overline{v}_1$ and $\underline{v}_2 < \overline{v}_2$ is, for instance, if $\overline{v}_1 < \underline{v}_2 = \overline{v}_2$, there will be a discrete jump in profits as we lower m from above $\overline{v}_2$ to below $\overline{v}_2$.

5 Some New Examples

In this section, we provide a few examples of interest that were not solved analytically before. In looking at these examples, we note the minimum bid m provides a way to model distributions of values with atoms at the lower end of the intervals. In fact, when $V_i \sim U[\underline{v}_i, \overline{v}_i]$ and m is in $(\underline{v}_i, \overline{v}_i)$, then this is equivalent to a distribution with an atom $\delta_i = \frac{(m-\underline{v}_i)}{(\overline{v}_i-\underline{v}_i)}$ at m and a uniform distribution on $[m, \overline{v}_i]$ with the remaining probability.[8]

Thus, our analytic solution for the general uniform case with a minimum bid also covers the case of two buyers with distributions that are uniform on intervals with either (or both) having an atom at the lower end of the interval.

In this section, we generate examples using the solution with a minimum bid given by equations (23) and (24).

Example 1 $\underline{v}_1 = 0,\ \underline{v}_2 = 1,\ m = 2,\ \overline{v}_2 = 3,\ \overline{v}_1 = 4.$ *Here, we have*

$$v_1(b) = \frac{2}{b - 1 + (b-2)^{\frac{2}{3}}(b+1)^{\frac{1}{3}}c_1}, \quad c_1 = \frac{(10)^{\frac{2}{3}}}{(-4)},$$

$$v_2(b) = \frac{2}{b + (b-2)^{\frac{1}{3}}(b+1)^{\frac{2}{3}}c_2} + 1, \quad c_2 = \frac{2\,(10)^{\frac{1}{3}}}{(-5)}.$$

[8] In the distribution with atoms, we have to relax the assumption that a buyer bids his value when he has zero probability of winning.

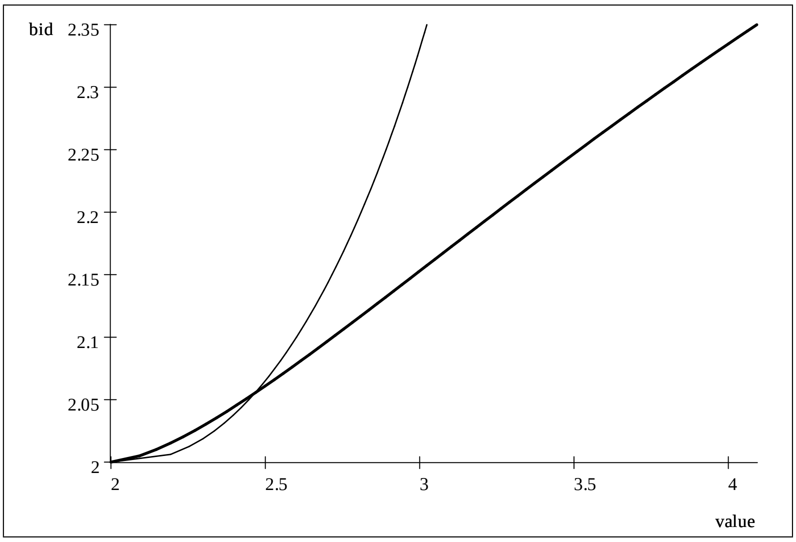

Figure 1: Solution when $\underline{v}_1 = 0, \underline{v}_2 = 1, m = 2, \overline{v}_2 = 3, \overline{v}_1 = 4$. The thick line is $v_1(b)$.

We note that the conditional distribution of V_1 above the minimum bid $m = 2$ stochastically dominates that of V_2. Nevertheless, there is no dominance of the bid functions in this region (see Figure 1). As a matter fact, this is the first case of intersecting bid functions that we are aware of.

It is interesting to compare this with the same conditional value distributions above 2 (without the atoms at $m = 2$), namely, $V_1 \sim U[2, 4]$ and $V_2 \sim U[2, 3]$. This is given in Figure 2 (and it is a shift of the Griesmer et al.,1967, result).

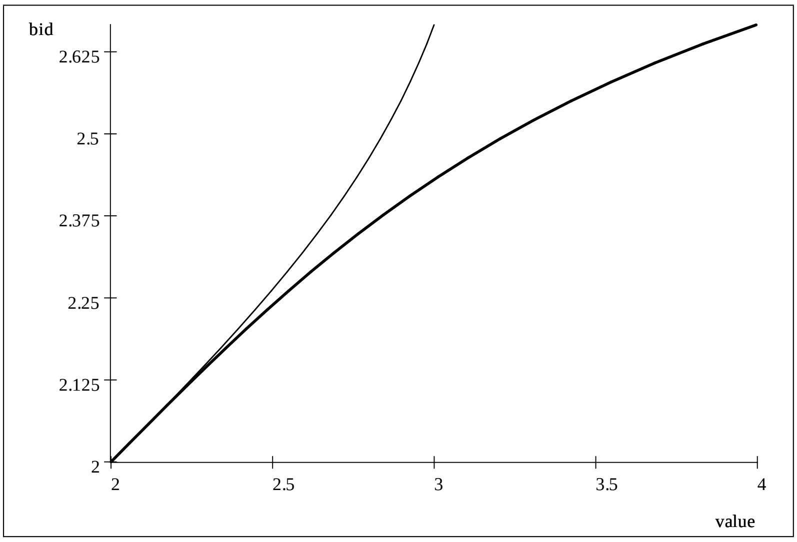

Figure 2: Solution when $\underline{v}_1 = 2, \underline{v}_2 = 2, \overline{v}_2 = 3, \overline{v}_1 = 4$. The thicker line is $v_1(b)$.

As we see, the presence of a minimum bid, even though it is at the center of both distributions, changes the equilibrium qualitatively by introducing the crossing of the bid functions. This example generalizes to the whole range of minimum bids.

Example 2 $\underline{v}_1 = 0,\ \underline{v}_2 = 1,\ 1/2 < m < 3,\ \overline{v}_2 = 3,\ \overline{v}_1 = 4.$
By (23) and (24), we have

$$\begin{aligned} v_1(b) &= \frac{m(m-1)}{b-1+(b-m)^{\frac{m}{2m-1}}(b+m-1)^{\frac{m-1}{2m-1}}c_1}, \\ c_1 &= -\frac{(4-m)(\frac{(m+3)(m+2)}{(3-m)(4-m)})^{\frac{m}{2m-1}}}{2(m+2)}, \\ \overline{b} &= \frac{\overline{v}_1 \cdot \overline{v}_2 + m^2 - m(\underline{v}_1 + \underline{v}_2)}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)} = \frac{12 + m^2 - m}{6}, \\ v_2(b) &= 1 + \frac{m(m-1)}{b + (b-m)^{\frac{m-1}{2m-1}}(b+m-1)^{\frac{m}{2m-1}}c_2}, \\ c_2 &= -\frac{2(3-m)\left(\frac{(m+2)(m+3)}{(3-m)(4-m)}\right)^{\frac{m}{2m-1}}}{m+3}. \end{aligned}$$

We have found by numerical computation of the solution that the crossing occurs for all values of m in the range.

In the following example we characterize a family of auctions with uniform distributions with linear equilibrium bid functions.

Example 3 $\underline{v}_1 = 0,\ \overline{v}_1 = m + z,\ \underline{v}_2 = 3m/2,\ \overline{v}_2 = 3m/2 + z$ *(where $z > 0$).*
Here we obtain from (23) and (24) that

$$\begin{aligned} v_1(b) &= 2(b-m) + m = 2b - m, \\ v_2(b) &= 2(b-m) + 3m/2 = 2b - m/2, \end{aligned}$$

$$b_1(v) = \frac{v}{2} + \frac{m}{2},\ \ b_2(v) = \frac{v}{2} + \frac{m}{4}.$$

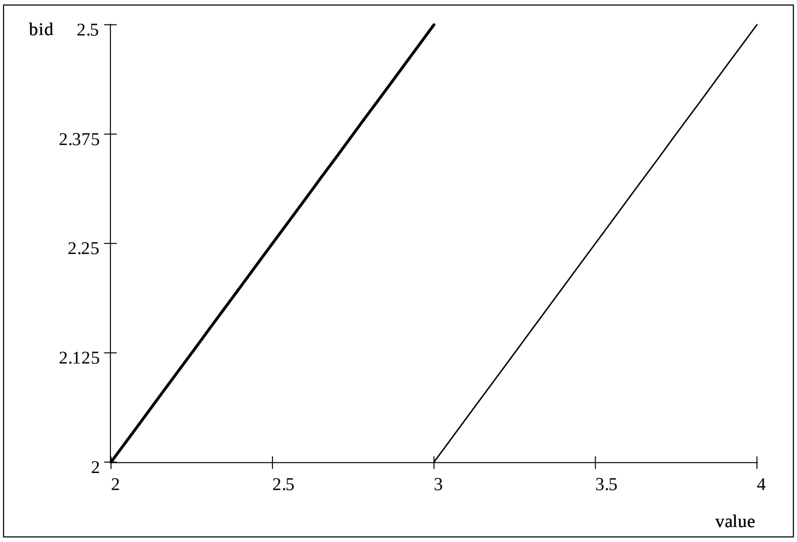

Figure 3: Solution when $\underline{v}_1 = 0, \underline{v}_2 = 3, m = 2, \overline{v}_1 = 3, \overline{v}_2 = 4$. The thicker line is $v_1(b)$.

Notice that these bid functions are independent of z and linear. Furthermore, the measure of values where a bid is submitted above the minimum is the same for both buyers, namely, z. Also notice that when $m \to 0$, this goes to the standard symmetric uniform case of uniformly distributed values on $[0, z]$.

It turns out that linear bid functions appear only in this special case, as we see in the following proposition.

Proposition 7 *The bid functions are linear if and only if* $m = (2\underline{v}_2 + \underline{v}_1)/3$ *(the minimum bid is two-thirds of the way from the lower end of the support of buyer 1's values to the lower end of the support of buyer 2's values) and* $\overline{v}_1 - m = \overline{v}_2 - \underline{v}_2$ *(the range of values above the minimum bid is the same length for both buyers).*

Proof. See the Appendix A.5. ■

We note that in this class of auctions, the revenue for the first-price auction is

$$R_{FP} = \frac{12m^2 + 15mz + 4z^2}{12(m+z)},$$

and the revenue for the second-price auction is

$$R_{SP} = \begin{cases} \frac{m^3+42m^2z+60mz^2+16z^3}{48z(m+z)} & \text{if } z > m/2, \\ \frac{2m^2+2mz+z^2}{2(m+z)} & \text{if } z \leq m/2. \end{cases}$$

In both cases, the first-price auction has higher revenue (it is higher by $\frac{m^2(6z-m)}{48z(m+z)}$ when $z > m/2$ and by $\frac{(3m-2z)z}{12(m+z)}$ when $z \leq m/2$).

The following example helps illustrate the Proposition 7 by demonstrating that linearity is lost by stretching the upper range.

Example 4 $\underline{v}_1 = 0,\ \overline{v}_1 = 3,\ \underline{v}_2 = 4,\ \overline{v}_2 = 6,\ m = 2.$ *Here we obtain*

$$\begin{aligned} v_1(b) &= \frac{8(b-1)}{(8+b(b-4))}, \\ v_2(b) &= 3 + \frac{10(b-2)}{(4+2b-b^2)}. \end{aligned}$$

By inverting the functions, we get the following non-linear bid functions (see Figure 4):

$$\begin{aligned} b_1(v) &= \frac{2\left(2+v-\sqrt{4+2v-v^2}\right)}{v}, \\ b_2(v) &= \frac{v-8+\sqrt{5}\sqrt{8-4v+v^2}}{(v-3)}. \end{aligned}$$

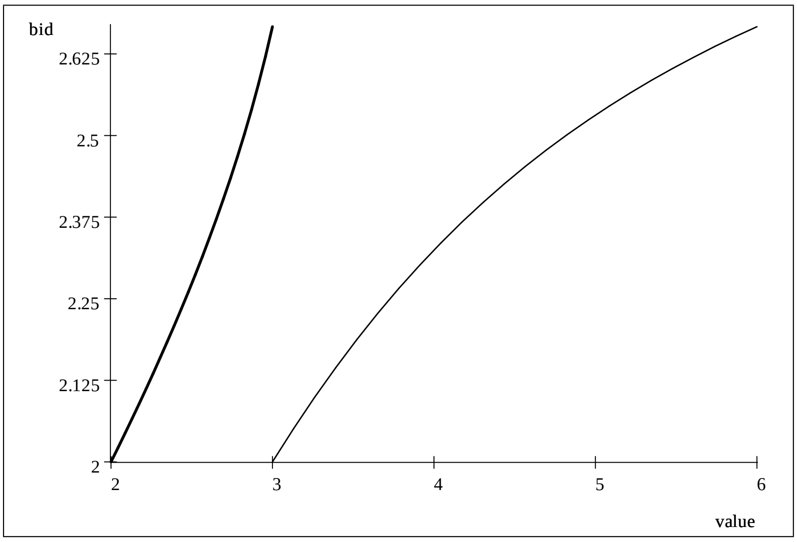

Figure 4: Solution when $\underline{v}_1 = 0, \underline{v}_2 = 3, m = 2, \overline{v}_1 = 3, \overline{v}_2 = 6$. The thicker line is $v_1(b)$.

6 Concluding Remarks

In this paper, we have analytically solved the general uniform case for two bidders. The uniform distribution is one of the simplest and, in addition to knowing of the existence of the equilibrium, have an explicit analytic expression of the bid functions. In particular, it is useful in comparative statics as well as in detecting interesting features of asymmetric auctions. A future direction of research would be to search for analytic solutions for other environments, such as extending our solution to N bidders. Another direction of research, would be to find environments where simple solutions exist: the simplest being of course the linear solution. We have work in progress that shows that a linear solution exists when the values are drawn from power distributions (not necessarily the same) and any risk aversions (also not necessarily the same). This expands recent independent derived results of Cheng (2005) and Kirkegaard (2006). Together these should provide a useful set of examples for researchers and students as well as suggest a set of parameters for additional experiments (see Güth et al., 2005) on asymmetric auctions.

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A Appendix

A.1 Proof that second order conditions are satisfied

Here we show that second-order conditions are satisfied for our solution. (This is adapted from Wolfstetter, 1996.) Buyer j with value v and bid b has probability of winning $Pwin^j(b)$ and expected profit $\pi^j(v,b)$, where

$$\pi^j(v,b) = Pwin^j(b)(v-b).$$

Define $b^j(v)$ as a bid function that is monotonic and solves the first-order conditions, namely, $\pi^j_b(v,b) = 0$. Assume that these bid functions are monotonic. Then, second-order conditions are satisfied. Since $\pi^j_b(v,b) = Pwin^{j\prime}(b)(v-b) - Pwin^j(b)$, we have

$$\pi^j_{bv}(v,b) = Pwin^{j\prime}(b) > 0. \tag{30}$$

Take $b^* = b^j(v^*)$. If $\widehat{b} < b^*$, then by monotonicity of the bid function, we $\widehat{v} \equiv \left(b^j\right)^{-1}\left(\widehat{b}\right) < v^*$. Hence, by (30) we have $\pi^j_b(v^*,b) > \pi^j_b(\widehat{v},b)$ for all b. This includes $\pi^j_b(v^*,\widehat{b}) > \pi^j_b(\widehat{v},\widehat{b}) = 0$. Thus, $\pi^j_b(v,\widehat{b}) > 0$ for all $\widehat{b} < b^j(v)$. Likewise, $\pi^j_b(v,\widehat{b}) < 0$ for all $\widehat{b} > b^j(v)$. Hence, second-order conditions are satisfied (as long as our solution is monotonic).

A.2 Proof of Proposition 3: solution with minimum bids

The solution that we presented with minimum bids is

$$v_1(b) = \underline{v}_1 + \frac{(m-\underline{v}_1)(m-\underline{v}_2)}{b - \underline{v}_2 + (b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{m-\underline{v}_2}{(m-\underline{v}_1)+(m-\underline{v}_2)}}c_1},$$

$$c_1 = -\frac{(\overline{v}_1-m)(\overline{v}_2-\underline{v}_2)\left(\frac{(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}\right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}{(\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}.$$

To derive this solution we divide both sides of equation (22) by

$$(v_1(b)-\underline{v}_1)^2(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{m-\underline{v}_2}{(m-\underline{v}_1)+(m-\underline{v}_2)}}$$

to obtain

$$\frac{v_1'(b)}{(v_1(b)-\underline{v}_1)^2(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}=$$

$$\frac{(v_1(b)-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}. \tag{31}$$

The RHS can be broken into two expressions:

$$\frac{1}{(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}+$$
$$\frac{(\underline{v}_1-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}.$$

Observe that

$$\int\frac{1}{(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}db=$$
$$\frac{1}{(b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}\cdot\frac{\underline{v}_2-b}{(m-\underline{v}_1)(m-\underline{v}_2)}+C$$

and

$$\int\left[\begin{array}{c}\frac{v_1'(b)}{(v_1(b)-\underline{v}_1)^2(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}-\\ \frac{(\underline{v}_1-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{1+\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}\end{array}\right]db=$$
$$-\frac{1}{(b-m)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}(b+m-\underline{v}_1-\underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}}}\cdot\frac{1}{v_1(b)-\underline{v}_1}+C.$$

Hence, we can integrate (31). From this we can obtain $v_1(b)$ as in (23), and the expression for c_1 is obtained by the boundary condition B3.

A.3 Proof of Proposition 4: solution when $m = \underline{v}_2$

The differential equation for this case is obtained by substituting $m = \underline{v}_2$ in equation (22):

$$v_1'(b) \cdot (b - \underline{v}_2)(b - \underline{v}_1) = (v_1(b) - \underline{v}_1)(v_1(b) - b).$$

Dividing both sides by $(b - \underline{v}_2)^2(b - \underline{v}_1)(v_1(b) - \underline{v}_1)^2$ and rewriting yields

$$\frac{v_1'(b)}{(b - \underline{v}_2)(v_1(b) - \underline{v}_1)^2} - \frac{(v_1(b) - b)}{(b - \underline{v}_2)^2(b - \underline{v}_1)(v_1(b) - \underline{v}_1)} = 0.$$

By further rewriting, we have

$$\left(\frac{1}{\underline{v}_2 - \underline{v}_1}\right)^2 \left(\frac{1}{b - \underline{v}_2} - \frac{1}{b - \underline{v}_1}\right) + \left(\frac{1}{\underline{v}_2 - \underline{v}_1}\right)\left(-\frac{1}{(b - \underline{v}_2)^2}\right) + \left(\frac{v_1'(b)(b - \underline{v}_2) + (v_1(b) - \underline{v}_1)}{(v_1(b) - \underline{v}_1)^2(b - \underline{v}_2)^2}\right) = 0.$$

Now by integration, we derive the solution:

$$\left(\frac{1}{\underline{v}_2 - \underline{v}_1}\right)^2 (\log(b - \underline{v}_2) - \log(b - \underline{v}_1)) + \frac{1}{(\underline{v}_2 - \underline{v}_1)(b - \underline{v}_2)} - \frac{1}{(v_1(b) - \underline{v}_1)(b - \underline{v}_2)} = c_1.$$

Rewriting this yields

$$\frac{1}{\underline{v}_2 - \underline{v}_1}\left[-\left(\frac{b - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\log\left(\frac{b - \underline{v}_1}{b - \underline{v}_2}\right) + 1 - \frac{c}{(\underline{v}_2 - \underline{v}_1)}(b - \underline{v}_2)\right] = \frac{1}{(v_1(b) - \underline{v}_1)},$$

where $c = c_1 \cdot (\underline{v}_2 - \underline{v}_1)^2$. Rearranging yields

$$v_1(b) = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{b - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\left[c + \log\left(\frac{b - \underline{v}_1}{b - \underline{v}_2}\right)\right]}.$$

Using boundary condition B3, $v_1(\bar{b}) = \bar{v}_1$, we have

$$\bar{v}_1 = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{\bar{b} - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\left[c + \log\left(\frac{\bar{b} - \underline{v}_1}{\bar{b} - \underline{v}_2}\right)\right]},$$

which implies

$$c = \frac{(\overline{v}_1 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{b} - \underline{v}_2)} - \log\left(\frac{\overline{b} - \underline{v}_1}{\overline{b} - \underline{v}_2}\right).$$

Substituting $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$ yields

$$c = \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_2)} - \log\left(\frac{(\overline{v}_2 - \underline{v}_1)(\overline{v}_1 - \underline{v}_1)}{(\overline{v}_2 - \underline{v}_2)(\overline{v}_1 - \underline{v}_2)}\right).$$

Thus, we have $v_1(b)$ and c equivalent to those in equations (26) and (25).

A.4 Proof of Continuity when $m \to \underline{v}_2$

Starting with equations (23) and (24), denote

$$\begin{aligned}
A(m) &= (m - \underline{v}_1)(m - \underline{v}_2), \\
B(m) &= (b + m - \underline{v}_1 - \underline{v}_2)^{\frac{(m - \underline{v}_2)}{(m - \underline{v}_1) + (m - \underline{v}_2)}}, \\
C(m) &= -\frac{(\overline{v}_1 - m)(\overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}, \\
D(m) &= \left(\frac{(b - m)(m - \underline{v}_2 + \overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - m)(\overline{v}_2 - m)}\right)^{\frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}}.
\end{aligned}$$

Thus,

$$v_1(b) = \underline{v}_1 + \frac{A(m)}{b - \underline{v}_2 + B(m)C(m)D(m)}.$$

Since $B(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2) = -(b - \underline{v}_2)$ as $m \to \underline{v}_2$, we get $\frac{A(m)}{b - \underline{v}_2 + B(m)C(m)D(m)} \to \frac{0}{0}$, so we need to use L'Hopital's rule, which yields

$$\lim_{m \to \underline{v}_2} v_1(b) = \underline{v}_1 + \left.\frac{A'(m)}{B'(m)C(m)D(m) + B(m)C'(m)D(m) + B(m)C(m)D'(m)}\right|_{m = \underline{v}_2}. \tag{32}$$

Step 1. Finding $A'(\underline{v}_2)$.

$A'(m) = (m - \underline{v}_1) + (m - \underline{v}_2)$, which implies that

$$A'(\underline{v}_2) = \underline{v}_2 - \underline{v}_1. \tag{33}$$

Step 2. Finding $B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2)$.

$B(m) = (b + m - \underline{v}_1 - \underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}} \equiv f(m)^{g(m)}$. By this definition, we have $f(\underline{v}_2) = b - \underline{v}_1$, $g(\underline{v}_2) = 0$, $f'(m) = 1$, and $g'(\underline{v}_2) = \frac{1}{\underline{v}_2 - \underline{v}_1}$. Recall that $\left(f(m)^{g(m)}\right)' = f(m)^{g(m)}\left[\log\left(f(m)\right)g'(m) + g(m)\frac{f'(m)}{f(m)}\right]$; hence, $B'(\underline{v}_2) = \frac{\log(b-\underline{v}_1)}{\underline{v}_2-\underline{v}_1}$.

We also have $C(\underline{v}_2) = -\frac{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}$ and $D(\underline{v}_2) = \frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}$. Thus,

$$B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2) = -\frac{\log\left(b - \underline{v}_1\right)}{\underline{v}_2 - \underline{v}_1}\left(b - \underline{v}_2\right). \tag{34}$$

Step 3. Finding $B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2)$.

We have $C'(\underline{v}_2) = \frac{(\overline{v}_2-\underline{v}_2)(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)^2}$ and $B(\underline{v}_2) = 1$. Thus,

$$B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2) = \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_2 - \underline{v}_1)}\left(\frac{b - \underline{v}_2}{\overline{v}_1 - \underline{v}_2}\right). \tag{35}$$

Step 4. Finding $B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2)$.

Similarly to Step 2, $D(m) = \left(\frac{(b-m)(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}\right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}} \equiv f(m)^{g(m)}$. By simple substitution, we have $f(\underline{v}_2) = \left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)$, $g(\underline{v}_2) = 1$, and $g'(\underline{v}_2) = -\frac{1}{\underline{v}_2-\underline{v}_1}$.

Since $f(m) = \frac{(b-m)(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}$, we can take the log of both sides and then the derivative w.r.t. m. This evaluated at $m = \underline{v}_2$ yields

$$\frac{f'(\underline{v}_2)}{f(\underline{v}_2)} = \frac{1}{\overline{v}_1 - \underline{v}_1} + \frac{1}{\overline{v}_2 - \underline{v}_1} + \frac{1}{\overline{v}_1 - \underline{v}_2} + \frac{1}{\overline{v}_2 - \underline{v}_2} - \frac{1}{b - \underline{v}_2}.$$

Again, recall that $\left(f(m)^{g(m)}\right)' = f(m)^{g(m)}\left[\log\left(f(m)\right)g'(m) + g(m)\frac{f'(m)}{f(m)}\right]$;

hence, we have

$$D'(\underline{v}_2) = \left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)\left[\begin{array}{c} -\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)\frac{1}{\underline{v}_2-\underline{v}_1}+\frac{1}{\overline{v}_1-\underline{v}_1}+ \\ \frac{1}{\overline{v}_2-\underline{v}_1}+\frac{1}{\overline{v}_1-\underline{v}_2}+\frac{1}{\overline{v}_2-\underline{v}_2}-\frac{1}{b-\underline{v}_2} \end{array}\right].$$

This implies that

$$B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2) = 1+(b-\underline{v}_2)\left[\begin{array}{c} \log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)\frac{1}{\underline{v}_2-\underline{v}_1} \\ -\frac{1}{\overline{v}_1-\underline{v}_1}-\frac{1}{\overline{v}_2-\underline{v}_1}-\frac{1}{\overline{v}_1-\underline{v}_2}-\frac{1}{\overline{v}_2-\underline{v}_2} \end{array}\right]. \quad (36)$$

Step 5. Finding $B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2)+B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2)+B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2)$.

By (34), (35), and (36), we now have

$$B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2)+B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2)+B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2) =$$
$$-\frac{\log(b-\underline{v}_1)}{\underline{v}_2-\underline{v}_1}(b-\underline{v}_2)+\frac{(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_2-\underline{v}_1)}\left(\frac{b-\underline{v}_2}{\overline{v}_1-\underline{v}_2}\right)+$$
$$1+(b-\underline{v}_2)\left[\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)\frac{1}{\underline{v}_2-\underline{v}_1}-\frac{1}{\overline{v}_1-\underline{v}_1}-\frac{1}{\overline{v}_2-\underline{v}_1}-\frac{1}{\overline{v}_1-\underline{v}_2}-\frac{1}{\overline{v}_2-\underline{v}_2}\right]$$
$$1+(b-\underline{v}_2)\left[\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(b-\underline{v}_1)(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)\frac{1}{\underline{v}_2-\underline{v}_1}-\frac{1}{\overline{v}_1-\underline{v}_1}-\frac{1}{\overline{v}_2-\underline{v}_2}\right]. \quad (37)$$

Step 6. Finding $v_1(b)$.

By substituting of (33) and (37) into (32), we have

$$\lim_{m\to\underline{v}_2} v_1(b) = \underline{v}_1+\frac{\underline{v}_2-\underline{v}_1}{1+(b-\underline{v}_2)\left[\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(b-\underline{v}_1)(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}\right)\frac{1}{\underline{v}_2-\underline{v}_1}-\frac{1}{\overline{v}_1-\underline{v}_1}-\frac{1}{\overline{v}_2-\underline{v}_2}\right]}.$$

This is equivalent to equation (25) after substituting the expression for c by equation (26).

A.5 Proof of Proposition 7: Linear Solutions

We know in the symmetric case that linear bid functions are possible for the uniform distribution. Here we ask what conditions are necessary for linear solutions to exist in general (for the uniform asymmetric case)?

Recall our two differential equations from the first-order conditions (6):

$$\begin{aligned} v_1'(b)(v_2(b) - b) &= v_1(b) - \underline{v}_1, \\ v_2'(b)(v_1(b) - b) &= v_2(b) - \underline{v}_2. \end{aligned}$$

Assume that there is a linear solution for both inverse bid functions:

$$v_i(b) = \alpha_i b + \beta_i \text{ where } \alpha_i > 0.$$

This implies that

$$v_i'(b) = \alpha_i.$$

Substituting this into the above two differential equations yields

$$\begin{aligned} \alpha_1(\alpha_2 b + \beta_2 - b) &= \alpha_1 b + \beta_1 - \underline{v}_1, \\ \alpha_2(\alpha_1 b + \beta_1 - b) &= \alpha_2 b + \beta_2 - \underline{v}_2. \end{aligned}$$

Since this is true for all b, the derivative of both sides must also be equal. Hence,

$$\alpha_1(\alpha_2 - 1) = \alpha_1, \quad \alpha_2(\alpha_1 - 1) = \alpha_2.$$

This implies that $\alpha_1 = \alpha_2 = 2$. Substituting this into the equations yields

$$2\beta_2 = \beta_1 - \underline{v}_1, \quad 2\beta_1 = \beta_2 - \underline{v}_2.$$

Combining these equations shows that

$$\beta_1 = -\frac{1}{3}\underline{v}_1 - \frac{2}{3}\underline{v}_2.$$

By boundary condition B1, $v_1(\underline{b}) = \underline{b}$, we have $\underline{b} = 2\underline{b} + \beta_1$. This implies that $\beta_1 = -\underline{b}$ and $\underline{b} = \frac{1}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$. Since $\underline{b} > (\underline{v}_1 + \underline{v}_2)/2$, it must be, by (2), that there is a binding minimum bid $m = \underline{b}$.

Now rewriting, $m = \frac{1}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$ yields $m - \underline{v}_1 = 2(\underline{v}_2 - m)$ (or $\underline{v}_2 = \frac{3}{2}m - \frac{1}{2}\underline{v}_1$). Finally, we use the upper boundary conditions in B3 to find that

$$\overline{v}_1 = 2\overline{b} - m,$$

$$\overline{v}_2 = 2\overline{b} - m/2 - \underline{v}_1/2.$$

Elimination of $\overline{b}$ implies that $\overline{v}_1 = \overline{v}_2 + \underline{v}_1/2 - m/2$ (or $\overline{v}_1 - m = \overline{v}_2 - \underline{v}_2$). Thus, if we define z such that $\overline{v}_1 = m + z$, we have $\overline{v}_2 = \frac{3}{2}m + z - \underline{v}_1/2$.