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L^2 ASYMPTOTES FOR FOURIER TRANSFORMS OF SURFACE-CARRIED MEASURES

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ABSTRACT. W. Littman has shown how to obtain asymptotic approximations for Fourier transforms of surface-carried measures of the form $\mu(X)dA$ where dA represents the area measure for the surface as a subset of Euclidean space and $\mu(X)$ is a compactly supported C^∞ function. Here we extend to the case where $\mu(X)$ is an L^2 function.

Let M be the surface in R^{n+1} defined by

$$X_{n+1} = \phi(X_1, \cdots, X_n) = \phi(X'), \qquad \phi \in C^\infty(M'),$$

where M' is an open subset of R^n. For complex valued functions μ on M define

$$\hat{\mu}(Y) = \int_{X \in M} e^{iY\cdot X}\mu(X)dA(X), \qquad Y \in R^{n+1}. \tag{1}$$

Define a normal to M at $X \in M$ by $\nu(X) = N(X')/|N(X')|$ where $N(X') = (-\text{grad } \phi(X'), 1)$. Let $\lambda_1(X), \cdots, \lambda_n(X)$ be the principle values of the curvature at $X \in M$ with signs chosen so that $X + [\lambda_k(X)]^{-1}\nu(X)$ gives the center of curvature for the kth direction. Using Littman [1] one can establish

THEOREM 1. *Assume $\mu(X)$ is C^∞ and has compact support. Assume M has nowhere zero Gaussian curvature $K(X) = \prod_{k=1}^n \lambda_k(X)$. Let S^n be the unit sphere in R^{n+1} and assume that the normal $\nu: M \to S^n$ is a one-to-one map. Define $a(Y)$ for $Y \in \{Y: Y_{n+1} > 0\}$ as follows: if $\pi(Y) \equiv Y/|Y|$ is not in $\nu(M)$ put $a(Y) = 0$; if $\pi(Y) = \nu(X)$, $X \in M$, put*

$$a(Y) = e^{iY\cdot X} e^{i\sigma\pi/4}(2\pi)^{n/2}\mu(X)\,|Y|^{-n/2}\,|K(X)|^{-1/2}$$

where $\sigma = \sum_{k=1}^n \lambda_k/|\lambda_k|$. Then there is a constant B such that $|\hat{\mu}(Y) - a(Y)| \leqq B|Y|^{-(n+1)/2}$ for all Y such that $|Y| \geqq 1$ and $Y_{n+1} > 0$.

A corollary of this theorem is that for $t > 1$,

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$$\begin{aligned} \|\hat{\mu} - a:t\|^2 &\equiv \int |\hat{\mu}(Y', t) - a(Y', t)|^2 dY' \\ &\leqq \int B^2(t^2 + |Y'|^2)^{-(n+1)/2} dY' \qquad (2) \\ &= t^{-1} \int_0^\infty B^2(1 + r^2)^{-(n+1)/2} \omega_n r^{n-1} dr \end{aligned}$$

where ω_n is the area of S^{n-1}. For μ in L^2 instead of C_c^∞ we have

THEOREM 2. *Assume that $K(X)$ is never zero and that $\nu: M \rightarrow S^n$ is one-to-one. If*

$$\|\mu\|_2^2 = \int_{M'} |\mu(X', \phi(X'))|^2 |N(X')|^2 dX' < \infty$$

then $\|\hat{\mu} - a:t\| \rightarrow 0$ as $t \rightarrow \infty$.

PROOF. We will show
(i) $\|\chi a:t\|^2 = (2\pi)^n \|\mu\|_2^2 = \|\hat{\mu}:t\|^2$ for $t > 0$;
(ii) $\|\chi(\hat{\mu} - a):t\|^2 \rightarrow 0$ as $t \rightarrow \infty$;
(iii) $\|\hat{\mu}:t\|^2 - \|\chi\hat{\mu}:t\|^2 \rightarrow 0$ as $t \rightarrow \infty$;
where

$$\begin{aligned} \chi(Y) &\equiv 1 \qquad \text{if } \pi(Y) \in \nu(M), \\ &\equiv 0 \qquad \text{otherwise.} \end{aligned}$$

Since $(1 - \chi)a = 0$, (ii) and (iii) are equivalent to the conclusion of the theorem. By the triangle inequality

$$\|\chi a:t\| \leqq \|\chi(a - \hat{\mu}):t\| + \|\chi\hat{\mu}:t\|$$

so that (iii) follows from (i) and (ii).

Using $dA(X) = |N(X')| dX'$ to rewrite (1) as

$$\hat{\mu}(Y) = \int_{M'} e^{iY' \cdot X'} e^{iY_{n+1}\phi(X')} \mu(X', \phi(X')) |N(X')| dX',$$

we recognize the right-hand side of (i) as Parseval's equality.

To prove the other half of (i) let

$$Q_t = \{ Y \in R^{n+1} : Y_{n+1} = t \text{ and } \pi(Y) \in \nu(M) \}$$

and let π_t be the restriction of π to Q_t. Then

$$\int_{Y\in Q_t} f(Y)\,dY' = \int_{v\in\nu(M)} f(\pi_t^{-1}(v))\,\frac{|\pi_t^{-1}(v)|^n}{v_{n+1}}\,d\omega(v)$$

$$(3) \qquad = \int_{X\in M} f(\pi_t^{-1}\circ\nu(X))\,\frac{|\pi_t^{-1}\circ\nu(X)|^n}{\nu_{n+1}(X)}\,|K(X)|\,dA(X)$$

$$= \int_{X\in M} f(tN(X'))\,|tN(X')|^n\,|N(X')|\,|K(X)|\,dA(X)$$

where $d\omega$ denotes surface area on S^n. Since

$$|a(tN(X'))|^2 = (2\pi)^n|\mu(X)|^2|tN(X')|^{-n}|K(X)|^{-1}$$

we see that the left-hand side of (i) follows from (3).

To prove (ii) we use (i) to reduce to the case $\mu\in C_c^\infty(M)$. Let $\epsilon>0$ and choose $\mu_1\in C_c^\infty(M)$ such that $\|\mu-\mu_1\|_2<\epsilon(2\pi)^{-n/2}$. Since $\mu\to a$ is linear, if $\mu_2=\mu-\mu_1$ then $a_2=a-a_1$. Applying (i) to μ_2 we have

$$\|\chi(\hat\mu-\hat\mu_1):t\| \leqq \|(\hat\mu-\hat\mu_1):t\| = \|\chi(a-a_1):t\| = (2\pi)^{n/2}\|\mu-\mu_1\|_2 < \epsilon.$$

Thus

$$\|\chi(\hat\mu-a):t\| \leqq \|\chi(\hat\mu-\hat\mu_1):t\| + \|\chi(\hat\mu_1-a_1):t\| + \|\chi(a_1-a):t\|$$
$$< \epsilon + \|\chi(\hat\mu_1-a_1):t\| + \epsilon.$$

Since $\mu_1\in C_c^\infty(M)$, (2) shows that there exists $\tau>0$ such that the last line is less than 3ϵ for all $t>\tau$.

For a special case of Theorem 2 with an application see [**2**].

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