ASYMPTOTIC ANALYSIS OF THE BUCKLING OF EXTERNALLY PRESSURIZED CYLINDERS WITH RANDOM IMPERFECTIONS*

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Abstract. The buckling of finite circular cylindrical shells with random stress-free initial displacements which are subjected to lateral or hydrostatic pressure is studied using a perturbation scheme developed in an earlier paper [1]. A simple approximate asymptotic expression is obtained for the buckling load for small magnitudes of the imperfection. This result is compared with earlier results obtained for localized imperfections and imperfections in the shape of the linear buckling mode.

Introduction. It is generally recognized that the buckling loads of some elastic structures are substantially reduced by the presence of nonuniformities in these structures. These nonuniformities or imperfections may be in the elastic or geometric properties of the structure. In [7, 8], Koiter developed a general theory of post-buckling behavior and derived simple asymptotic formulae for the buckling load of a class of elastic structures with imperfections in the shape of their classical (linear) buckling modes.

In [5] Budiansky and Amazigo applied a reworked version [6] of Koiter's theory in deriving an asymptotic formula for the buckling load of externally pressurized cylinders. Furthermore they derived the range of values of a length parameter Z, introduced by Batdorf [4], for which the cylinder is sensitive to imperfection in the shape of the classical buckling mode. In a more recent study [3], Amazigo and Fraser derive similar results for cylinders with localized or dimple imperfections and obtained the same range of values of Z for imperfection-sensitivity.

It is clear that in general the imperfections in structures are stochastic rather than deterministic. Here we assume that the imperfections are Gaussian and obtain an asymptotic formula for the buckling load. The perturbation scheme used here was developed in [1]. It is found that the range of values of Z for imperfection-sensitivity remains the same and the loss in the buckling load for the three types of imperfections parallels that obtained for columns on nonlinear foundations [1, 2].

Kármán–Donnell equations. A cylindrical shell is characterized by its outward radial displacement $W(X, Y)$ and an Airy stress function $F(X, Y)$ where X and Y are the cartesian coordinates in the axial and circumferential directions. The membrane stress resultants N_X, N_Y, N_{XY} are given by $N_X = F_{,YY}$, $N_Y = F_{,XX}$, and $N_{XY} = -F_{,XY}$ where $(\ )_{,Y} = \partial(\ )/\partial_Y$, etc. Introducing the effect of a stress-free initial outward

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normal displacement $\bar{W}(X, Y)$ into the Kármán–Donnell theory for cylindrical shells leads to the compatibility equation

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{R} W_{,XX} + \tfrac{1}{2}S(W, W) + S(W, \bar{W}) = 0 \tag{1}$$

and the equilibrium equation

$$D\nabla^4 W + \frac{1}{R} F_{,XX} - S(W + \bar{W}, F) + p = 0 \tag{2}$$

where E is Young's modulus, h and R are the shell thickness and radius respectively, p is the external pressure, $D = Eh^3/12(1 - \nu^2)$ is the bending stiffness, ν is Poisson's ratio, ∇^4 is the two-dimensional biharmonic operator, and

$$S(P, Q) = P_{,XX}Q_{,YY} + P_{,YY}Q_{,XX} - 2P_{,XY}Q_{,XY}\,. \tag{3}$$

We assume the usual simply supported boundary conditions, namely zero normal bending moment, zero circumferential displacement; $W = pR^2(1 - \frac{1}{2}\alpha\nu)/Eh$, $N_X = -\alpha pR/2$ at $X = 0$, L where L is the shell length. This leads to

$$W = W_{XX} = F = F_{XX} = 0.$$

The parameter α is introduced for convenience so that lateral and hydrostatic pressures may be analyzed together. $\alpha = 1$ if the pressure contributes to axial stresses through end plates and $\alpha = 0$ if pressure only acts laterally.

It is convenient to introduce the nondimensional quantities:

$$\begin{gathered} x = \pi X/L, \qquad y = nY/R, \qquad \bar{w} = \bar{W}/h, \\ \lambda = PL^2R/\pi^2 D, \qquad A = L^2\sqrt{[12(1 - \nu^2)]}/\pi^2 hR, \qquad \zeta = (nL/\pi R)^2, \\ H = n^2h/R, \qquad K(\zeta) = -A^2(1 + \zeta)^{-2}, \end{gathered} \tag{4}$$

where n is an integer.

Before buckling we assume, as is customary, that the cylinder is in a state of constant membrane stress and that thus w can be approximated by a constant. For thin shells this approximation is good except near the ends of the shell where there is a small boundary layer. Thus

$$\begin{aligned} F &= -(X^2/2 + \alpha Y^2/4)Rp + \frac{Eh^2L^2}{\pi^2 R(1 + \zeta)^2} f, \\ W &= pR^2(1 - \tfrac{1}{2}\alpha\nu)/Eh + hw. \end{aligned} \tag{5}$$

Substituting for F and W in (1) and (2) and using (4) gives

$$\bar{\nabla}^4 f - (1 + \zeta)^2 w_{,xx} + H(1 + \zeta)^2[\tfrac{1}{2}S(w, w) + S(w, \bar{w})] = 0 \tag{6}$$

$$\bar{\nabla}^4 w - K(\zeta)f_{,xx} + \lambda\left(\frac{\alpha}{2} w_{,xx} + \zeta w_{,yy}\right) + HK(\zeta)S(w + \bar{w}, f) = -\lambda\left(\frac{\alpha}{2}\bar{w}_{,xx} + \zeta\bar{w}_{,yy}\right) \tag{7}$$

where $\bar{\nabla}^4 = (\partial^2/\partial x^2 + \zeta\partial^2/\partial y^2)^2$. The simply supported boundary conditions become

$$w = w_{,xx} = f = f_{,xx} = 0 \text{ at } x = 0, \pi. \tag{8}$$

The solution to the linearized version of equations (6) and (7) with $\bar{w} \equiv 0$ obtained by Batdorf [4] is recorded here:

$$w = \sin x \sin y, \quad f = -\sin x \sin y. \tag{9}$$

The buckling load λ_c (called the classical buckling load) is

$$\lambda_c = \left(\frac{\alpha}{2} + \zeta\right)^{-1} [(1 + \zeta)^2 - K(\zeta)] \tag{10}$$

and n in (4) is the integer that minimizes λ_c. Execution of this minimization on the basis of the assumption that ζ varies continuously (see Batdorf [4] for a discussion of the consequences of this assumption) gives

$$\lambda_c = 4(1 + \zeta)^2/(3\zeta + 1 + \alpha), \quad A^2 = (1 + \zeta)^4(\zeta - 1 + \alpha)/(3\zeta + 1 + \alpha). \tag{11}$$

Perturbation scheme. We consider the shell as having an initial stress-free displacement of the form

$$\bar{w}(x, y) = \epsilon w_0(y) \sin x \tag{12}$$

where ϵ is a small parameter characterizing the amplitude of the displacement. This imperfection could be considered as the first term in a Fourier series expansion of an arbitrary imperfection satisfying the boundary conditions (8). This term has the dominant effect in the reduction of the buckling strength for imperfections of the form $u_m(y) \sin mx$ for deterministic $u_m(y)$. Here $w_0(y)$ is assumed to be a sample function from an ensemble of twice-continuously-differentiable zero-mean, stationary Gaussian random functions with known autocorrelation function $R_{00}(z)$. Thus

$$\langle w_0(y) \rangle = 0, \quad \langle w_0(y + z) w_0(y) \rangle = R_{00}(z) \tag{13}$$

where the angular bracket $\langle \cdots \rangle$ denotes ensemble average. We are thus dropping the requirement of periodicity in the circumferential coordinate y and requiring $-\infty < y < \infty$. This is equivalent to the previous assumption that ζ be a continuous variable. The power spectral density $S_{00}(\omega)$ of w_0 is defined by

$$S_{00}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{00}(z) \exp(-i\omega z)\, dz. \tag{14}$$

(Unless otherwise specified the limits of all integrals are $-\infty$, ∞.)

We consider λ to be prescribed and satisfy the inequality $0 < \lambda < \lambda_c$, and expand w and f in powers of ϵ, namely

$$\begin{pmatrix} w \\ f \end{pmatrix} = \sum_{m=1}^{\infty} \epsilon^m \begin{pmatrix} w_m \\ f_m \end{pmatrix}. \tag{15}$$

Substituting for w, f into (6) and (7) using (12) and equating powers of ϵ gives the following sequence of equations:

$$\left.\begin{aligned} L_1(f_1, w_1) &= 0 \\ L_2(f_1, w_1) &= \lambda(\tfrac{1}{2}\alpha w_0 - \zeta w_0'') \sin x \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} L_1(f_2, w_2) &= -(1 + \zeta)^2 H\{\tfrac{1}{2} S(w_1, w_1) + S(w_0 \sin x, w_1)\} \\ L_2(f_2, w_2) &= -K(\zeta) H\{S(w_1, f_1) + S(w_0 \sin x, f_1)\} \end{aligned}\right\} \tag{17}$$

$$L_1(f_3, w_3) = -(1+\zeta)^2 H\{S(w_1, w_2) + S(w_0 \sin x, w_2)\}$$
$$L_2(f_3, w_3) = -K(\zeta) H\{S(w_1, f_2) + S(w_2, f_1) + S(w_0 \sin x, f_2)\} \tag{18}$$

etc., where

$$L_1(f_j, w_j) \equiv \bar{\nabla}^4 f_j - (1+\zeta)^2 w_{j,xx} \qquad j = 1, 2, \cdots$$
$$L_2(f_j, w_j) \equiv \bar{\nabla}^4 w_j - K(\zeta) f_{j,xx} + \lambda(\tfrac{1}{2}\alpha w_{j,xx} + \zeta w_{j,yy}) \tag{19}$$

and prime denotes differentiation with respect to the argument. The boundary conditions (8) become

$$w_j = w_{j,xx} = f_j = f_{j,xx} = 0, \quad j = 1, 2, \cdots . \tag{20}$$

Let Δ^2 be the average of the mean square of the deflection:

$$\Delta^2 = \frac{1}{\pi} \int_0^\pi \langle w^2(x, y) \rangle \, dx. \tag{21}$$

Substituting for w using (15) leads to

$$\Delta^2 = \epsilon^2 \Delta_{11} + 2\epsilon^3 \Delta_{12} + \epsilon^4 (2\Delta_{13} + \Delta_{22}) + O(\epsilon^5) \tag{22}$$

where

$$\Delta_{ij} = \frac{1}{\pi} \int_0^\pi \langle w_i(x, y) w_j(x, y) \rangle \, dx \qquad i, j = 1, 2, \cdots . \tag{23}$$

We anticipate that $\Delta_{12} = 0$ (see Eq. (47)). Since we seek asymptotic formulae valid for $\epsilon \to 0$ and hence $\lambda \to \lambda_c^-$ we also anticipate the result (see Eq. (61))

$$\Delta_{22}/\Delta_{13} \to 0 \quad \text{as} \quad \lambda \to \lambda_c^-. \tag{24}$$

Thus (22) reduces to

$$\Delta^2 \sim \epsilon^2 \Delta_{11} + 2\epsilon^4 \Delta_{13} \quad \text{as} \quad \lambda \to \lambda_c^-. \tag{25}$$

Now the Δ_{ij}s are functions of λ and Eq. (25) gives a relation between Δ^2, λ, and ϵ. The buckling load is thus obtained by maximizing λ with respect to Δ^2. As noted in [1], setting $d\lambda/d\Delta^2 = 0$ in (25) fails to yield the buckling load because the series (25) does not converge for Δ^2 greater than the critical mean square.

The difficulty is overcome by reversing the series (25) to get

$$\epsilon^2 = \alpha_1(\lambda)\Delta^2 + \alpha_2(\lambda)\Delta^4 + O(\Delta^6) \tag{26}$$

where the α_j are obtained by substituting (26) into (25) and equating powers of Δ^2. Performing this elementary operation gives

$$\alpha_1 = 1/\Delta_{11}, \quad \alpha_2 = -2\Delta_{13}/\Delta_{11}^3. \tag{27}$$

We truncate the series (26) at the Δ^4 term to get an approximate load-deflection relationship. Now maximizing λ with respect to Δ^2 using (26) and (27) gives the buckling equation

$$8\epsilon^2 \Delta_{13}(\bar{\lambda})/\Delta_{11}(\bar{\lambda}) \approx 1 \tag{28}$$

as an approximate relation between the buckling load $\bar{\lambda}$ and the imperfection amplitude parameter ϵ. We now seek asymptotic expressions for $\Delta_{11}(\lambda)$ and $\Delta_{13}(\lambda)$ valid for $\epsilon \to 0$ and $\lambda \to \lambda_c^-$.

Solution of first-order perturbation equation. The solution of (16) and (20) can be written in the form

$$w_1(x, y) = u_1(y) \sin x, \quad f_1(x, y) = \phi_1(y) \sin x \tag{29}$$

and substitution of (29) into (16) leads to

$$\begin{aligned} &\left(\zeta \frac{d^2}{dy^2} - 1\right)^2 \phi_1 + (1 + \zeta)^2 u_1 = 0, \\ &\left(\zeta \frac{d^2}{dy^2} - 1\right)^2 u_1 + K\phi_1 + \lambda(-\tfrac{1}{2}\alpha u_1 + \zeta u_1'') = \lambda(\tfrac{1}{2}\alpha w_0 - \zeta w_0''). \end{aligned} \tag{30}$$

Thus ϕ_1 and u_1 are linearly related to w_0 and are therefore stationary Gaussian random functions (see, for example, [10]). It is shown in Appendix A that $\langle u_1(y) \rangle = 0$ and that

$$\begin{aligned} S_u(\omega) &= \lambda^2(\tfrac{1}{2}\alpha + \zeta\omega^2)^2(1 + \omega^2\zeta)^4 Q^2(\omega) S_{00}(\omega), \\ S_{u\phi}(\omega) &= -\lambda^2(\tfrac{1}{2}\alpha + \zeta\omega^2)^2(1 + \zeta)^2(1 + \omega^2\zeta)^2 Q^2(\omega) S_{00}(\omega), \\ S_\phi(\omega) &= \lambda^2(\tfrac{1}{2}\alpha + \zeta\omega^2)^2(1 + \zeta)^4 Q^2(\omega) S_{00}(\omega), \end{aligned} \tag{31}$$

where

$$Q(\omega) = \{(1 + \omega^2\zeta)^2[(1 + \omega^2\zeta)^2 - \lambda(\tfrac{1}{2}\alpha + \zeta\omega^2)] - K(1 + \zeta)^2\}^-, \tag{32}$$

$$S_u(\omega) = \frac{1}{2\pi} \int R_u(z) \exp(-i\omega z)\, dz \tag{33}$$

with

$$R_u(z) = \langle u_1(y + z) u_1(y) \rangle. \tag{34}$$

$S_{u\phi}$ and S_ϕ are defined by expressions similar to (33).

Substitution for w_1 in (23) using (29) gives

$$\Delta_{11} = \tfrac{1}{2}\langle u_1^2(y) \rangle = \tfrac{1}{2} R_u(0) = \frac{1}{2} \int S_u(\omega)\, d\omega. \tag{35}$$

Let

$$B_m \equiv \int F(\omega)[Q(\omega)]^m\, d\omega \qquad m \geq 2, \tag{36}$$

where $F(\omega)$ is any smooth integrable function analytic in the strip $|\mathrm{Im}\ \omega| < a$ for some a with $F(\pm 1) \neq 0$. It is shown in Appendix B that

$$B_m \approx \frac{\pi(m-1)(2m-3)!}{2^{2m-2}[(m-1)!]^2} \cdot \frac{F(1) + F(-1)}{[P(1)]^{1/2}[(\lambda_c - \lambda)g(1)]^{m-1/2}}, \qquad \lambda \to \lambda_c^-, \tag{37}$$

where

$$P(\omega) = \zeta^2 \begin{vmatrix} (\omega^2 - 1)\zeta + \frac{3}{2}(1 + \zeta) & -[(\omega^2 - 1)\zeta + 2(1 + \zeta)] \\ (\omega^2 - 1)\zeta + 2(1 + \zeta) & -\lambda_c \end{vmatrix} \tag{38}$$

and

$$g(\omega) = (1 + \omega^2\zeta)^2(\tfrac{1}{2}\alpha + \zeta\omega^2) \tag{39}$$

Use of this result gives

$$\Delta_{11} \approx \frac{\lambda^2 \pi S_{00}(1)}{4\zeta(\lambda_c - \lambda)^{3/2}} \left[\frac{(\frac{1}{2}\alpha + \zeta)(1 + \zeta)}{4(1 + \zeta) - \frac{3}{2}\lambda_c}\right]^{1/2} \qquad \lambda \to \lambda_c^-. \tag{40}$$

Noting that $\langle w_0^2 \rangle$ is independent of λ and hence $O(1)$ as $\lambda \to \lambda_c^-$ and $\langle u_1^2 \rangle = O[(\lambda_c - \lambda)^{-3/2}]$ by (40), we conclude that $u_1(y) + w_0(y) \sim u_1(y)$ as $\lambda \to \lambda_c^-$.

Second-order perturbation equations. As noted in the above paragraph, we may drop the w_0 terms in (17) and (18) in comparison with u_1 terms. Thus (17) becomes

$$L_1(f_2, w_2) \approx -\tfrac{1}{2}(1 + \zeta)^2 HS(w_1, w_1),$$

$$L_2(f_2, w_2) \approx -K(\zeta)HS(w_1, f_1).$$

Substituting for w_1 and f_1 using (29) gives

$$\begin{aligned} L_1(f_2, w_2) &\approx \tfrac{1}{2}(1 + \zeta)^2 H(u_1''u_1 + u_1'^2) - \tfrac{1}{2}(1 + \zeta)^2 H(u_1''u_1 - u_1^2) \cos 2x \\ L_2(f_2, w_2) &\approx \tfrac{1}{2}KH(u_1\phi_1'' + u_1''\phi_1 + 2u_1'\phi_1') - \tfrac{1}{2}KH(u_1\phi_1'' + u_1''\phi_1 - 2u_1'\phi_1') \cos 2x. \end{aligned} \tag{41}$$

The solutions of these equations with the boundary conditions (20) can be obtained, as shown by Budiansky and Amazigo [5], in the form

$$\begin{pmatrix} w_2(x, y) \\ f_2(x, y) \end{pmatrix} = \sum_{m=1,3,5,}^{\infty} \begin{pmatrix} v_m(y) \\ \psi_m(y) \end{pmatrix} \sin mx. \tag{42*}$$

Substituting this form into (41), and noting that, for p even, $\cos px = -\sum [4m/\pi(p^2 - m^2)] \sin mx$, gives

$$\begin{aligned} M_1^{(m)}(\psi_m, v_m) &= (1 + \zeta^2)(-P_m u_1''u_1 + T_m u_1'^2), \\ M_2^{(m)}(\psi_m, v_m) &= -KP_m(u_1''\phi_1 + u_1\phi_1'') + 2T_m u_1'\phi_1', \end{aligned} \tag{43}$$

where

$$P_m = 8H/[\pi m(m^2 - 4)], \; T_m = 4(m^2 - 2)H/(\pi m(m^2 - 4)], \tag{44}$$

$$\begin{aligned} M_1^{(m)}(\psi, v) &\equiv \left(\zeta \frac{d^2}{dy^2} - m^2\right)^2 \psi(y) + (1 + \zeta)^2 m^2 v(y), \\ M_2^{(m)}(\psi, v) &\equiv Km^2\psi + \left[\left(\zeta \frac{d^2}{dy^2} - m^2\right)^2 - \tfrac{1}{2}\alpha\lambda m^2 + \lambda\zeta \frac{d^2}{dy^2}\right] v. \end{aligned} \tag{45}$$

Now from the definition (23) of Δ_{12} and expressions (29) and (42) for w_1 and w_2 respectively,

$$\Delta_{12} = \tfrac{1}{2}\langle u_1(y) v_1(y) \rangle. \tag{46}$$

It is shown in Appendix C that $\langle u_1(y) v_m(y) \rangle = 0$; hence

$$\Delta_{12} = 0. \tag{47}$$

The use of (42) in the definition (23) of Δ_{22} gives

$$\Delta_{22} = \tfrac{1}{2} \sum \langle v_m^2(y) \rangle. \tag{48}$$

* Unless otherwise specified, all summations are taken over all odd positive integers.

A complete derivation of $\langle v_m^2(y)\rangle$ is lengthy and its presentation would obscure the main trend of this paper. In Appendix C a typical term in Δ_{22} is evaluated asymptotically to show that

$$\Delta_{22} = 0((\lambda_c - \lambda)^{-3}). \tag{49}$$

Third-order perturbation equations. As noted in the derivation of the second-orde equations, we may drop the w_0 terms in (18) to get

$$\begin{aligned} L_1(f_3, w_3) &= -(1+\zeta)^2 HS(w_1, w_2), \\ L_2(f_3, w_3) &= -K(\zeta)H\{S(w_1, f_2) + S(w_2, f_1)\}. \end{aligned} \tag{50}$$

We now substitute for w_1, f_1, w_2, f_2 using (29) and (42). The solution to the resulting equations can be found in the form

$$\begin{pmatrix} w_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} h_1(y) \\ \chi_1(y) \end{pmatrix} \sin x + \sum_{m=3,5,\cdots}^{\infty} \begin{pmatrix} h_m(y) \\ \chi_m(y) \end{pmatrix} \sin mx. \tag{51}$$

The equations for χ_1 and h_1 are

$$\begin{aligned} M_1^{(1)}(\chi_1, h_1) &= -(1+\zeta)^2 \sum \underline{P_m(u_1 v_m''} + m^2 u_1'' v_m) \\ &\quad - 2(1+\zeta)^2 \sum X_m u_1' v_m', \\ M_2^{(1)}(\chi_1, h_1) &= -K(\zeta) \sum P_m(u_1\psi_m'' + m^2 u_1''\psi_m + \phi_1 v_m'' + m^2\phi_1'' v_m) \\ &\quad - 2K(\zeta) \sum X_m(u_1'\psi_m' + \phi_1' v_m'), \end{aligned} \tag{52}$$

where $M_1^{(1)}$ and $M_2^{(1)}$ are defined in Eq. (45) and

$$X_m = 4mH/\pi(m^2 - 4). \tag{53}$$

We have not exhibited the equations for h_m and χ_m, $m > 1$, since our primary interest is in the asymptotic evaluation of Δ_{13} and the use of (51) and (29) in the definition (23) gives

$$\Delta_{13} = \tfrac{1}{2}\langle u_1(y) h_1(y)\rangle. \tag{54}$$

In Appendix D we exhibit the calculations leading to the underlined term in the following expression for Δ_{13} :

$$\Delta_{13} = -\tfrac{1}{2} \sum \iint [I_1(\omega_1, \omega_2; m) + I_2(\omega_1, \omega_2; m) + \cdots + I_7(\omega_1, \omega_2; m)]\, d\omega_1\, d\omega_2, \tag{55}$$

where

$$\begin{aligned} I_1(\omega_1, \omega_2; m) &= K[KQ_1^{(1)}(\omega_1)H_m^{(2)}(\omega_1+\omega_2) + (1+\zeta)^2 Q_1^{(2)}(\omega_1)Q_m^{(1)}(\omega_1+\omega_2)] \\ &\quad \cdot [S_{u\phi}(\omega_1)S_u(\omega_2) + S_u(\omega_1)S_{u\phi}(\omega_2)][-P_m(\omega_1^2 + \omega_2^2) + 2T_m\omega_1\omega_2] \\ &\quad \cdot [m^2 P_m \omega_2^2 + P_m(\omega_1+\omega_2)^2 - 2X_m\omega_2(\omega_1+\omega_2)], \\ I_2(\omega_1, \omega_2; m) &= (1+\zeta)^2[KQ_1^{(1)}(\omega_1)H_m^{(1)}(\omega_1+\omega_2) + \underline{(1-\zeta)^2 Q_1^{(2)}(\omega_1)Q_m^{(2)}(\omega_1+\omega_2)}] \\ &\quad \cdot S_u(\omega_1)S_u(\omega_2)[\underline{-P_m(\omega_1^2 + \omega_2^2)} + 2T_m\omega_1\omega_2] \\ &\quad \cdot [m^2 P_m\omega_2^2 + \underline{P_m(\omega_1+\omega_2)^2} - 2X_m\omega_2(\omega_1+\omega_2)], \end{aligned}$$

$$I_3(\omega_1, \omega_2; m) = -Km^2P_m(P_m + T_m)\omega_1^2\omega_2^2Q_1^{(1)}(\omega_1)S_u(\omega_1)$$
$$\cdot[2KH_m^{(2)}(0)S_{u\phi}(\omega_2) + (1 + \zeta)^2H_m^{(1)}(0)S_u(\omega_2)],$$
$$I_4(\omega_1, \omega_2; m) = -Km^2P_m(P_m + T_m)\omega_1^2\omega_2^2Q_1^{(1)}(\omega_1)S_{u\phi}(\omega_1)$$
$$\cdot[2KQ_m^{(1)}(0)S_{u\phi}(\omega_2) + (1 + \zeta)^2Q_m^{(2)}(0)S_u(\omega_2)],$$
$$I_5(\omega_1, \omega_2; m) = -(1 + \zeta)^2m^2P_m + T_m)\omega_1^2\omega_2^2Q_1^{(2)}(\omega_1)S_u(\omega_1)$$
$$\cdot[2KQ_m^{(1)}(0)S_{u\phi}(\omega_2) + (1 + \zeta)^2Q_m^{(2)}(0)S_u(\omega\ )],$$
$$I_6(\omega_1, \omega_2 : m) = K(1 + \zeta)^2Q_1^{(1)}(\omega_1)Q_m^{(2)}(\omega_1 + \omega_2)S_u(\omega_1)S_{u\phi}(\omega_2)$$
$$\cdot[-P_m(\omega_1^2 + \omega_2^2) + 2T_m\omega_1\omega_2][m^2P_m\omega_2^2 + P_m(\omega_1 + \omega_2)^2 - 2X_m\omega_2(\omega_1 + \omega_2)],$$
$$I_7(\omega_1, \omega_2; m) = K^2Q_1^{(1)}(\omega_1)Q_m^{(1)}(\omega_1 + \omega_2)[S_{u\phi}(\omega_1)S_{u\phi}(\omega_2) + S_u(\omega_1)S_\phi(\omega_2)]$$
$$\cdot[-P_m(\omega_1^2 + \omega_2^2) + 2T_m\omega_1\omega_2][m^2P_m\omega_2^2 + P_m(\omega_1 + \omega_2)^2 - 2X_m\omega_2(\omega_1 + \omega_2)],$$

and

$$\begin{aligned} Q_m^{(1)}(\omega) &= (\omega^2\zeta + m^2)^2Q_m(\omega), \qquad m = 1, 3, 5, \cdots, \\ Q_m^{(2)}(\omega) &= -K(\zeta)m^2Q_m(\omega), \\ H_m^{(1)}(\omega) &= [(\omega^2\zeta + m^2)^2 - \lambda(\tfrac{1}{2}\alpha m^2 + \zeta\omega^2)]Q_m(\omega), \\ H_m^{(2)}(\omega) &= -(1 + \zeta)^2m^2Q_m(\omega), \\ Q_m(\omega) &= \{(\zeta\omega^2 + m^2)^2[(\zeta\omega^2 + m^2)^2 - \lambda(\tfrac{1}{2}\alpha m^2 + \zeta\omega^2)] - (1 + \zeta)^2Km^4\}^{-1}. \end{aligned} \tag{56}$$

P_m, T_m, and X_m are given by (44) and (53). Note, by comparing (32) and (56), that $Q_1(\omega) = Q(\omega)$.

We consider the double integral

$$\begin{aligned} J(r, s) &\equiv \iint F(\omega_1, \omega_2)Q^r(\omega_1)Q^s(\omega_2)\, d\omega\ \ d\omega_2, \qquad r, s \geq 2, \\ &= \int d\omega_2Q^s(\omega_1) \int Q^r(\omega_1)F(\omega_1, \omega_2)\, d\omega_2, \end{aligned} \tag{57}$$

where F is smooth and integrable. Repeated use of the asymptotic result of Appendix B gives

$$J_p(r, s) \approx \frac{\pi^2(2s - 2)!\,(2r - 2)!\,[F(-1, -1) + F(-1, 1) + F(1, -1) + F(1, 1)]}{2^{2r+2s-2}[(r - 1)!\,(s - 1)!]^2P(1)[(\lambda_c - \lambda)g(1)]^{r+s-1}},$$
$$\lambda \to \lambda_c^-. \tag{58}$$

$P(\omega)$ and $g(\omega)$ are defined by (38) and (39) respectively.

The result (58) is used to evaluate the expression for Δ_{13} asymptotically. The lengthy but straightforward calculations give

$$\Delta_{13} \sim \frac{3\pi^2(1 + \zeta)(\frac{1}{2}\alpha + \zeta)\lambda^4\lambda_cS_{00}^2(1)}{8\zeta^2[4(1 + \zeta) - \frac{3}{2}\lambda_c](\lambda_c - \lambda)^4}(-b) \tag{59}$$

where the imperfection parameter b is defined as in references [5] (note that Eq. (58) of [5] contains misprints) and [3] by

$$\frac{b}{1-\nu^2} = \frac{24\zeta^2}{\lambda_c(\frac{1}{2}\alpha + \zeta)}\left\{\frac{3}{32} - \frac{8A^2}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\frac{[1+2m^2(1+\zeta)^{-2}]^2}{m^2(m^2-4)^2(m^4 - \frac{1}{2}\alpha\lambda_c m^2 + A^2)}\right.$$
$$\left. - \frac{4}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\frac{(m^2+4\zeta)^2[4A^2(1+\zeta)^{-4} - 1] + 4A^2m^2(1+\zeta)^{-2} + \frac{1}{2}\alpha\lambda_c m^2 + 4\zeta\lambda c}{m^2\{(m^2+4\zeta)^2[(m^2+4\zeta)^2 - \frac{1}{2}\alpha\lambda_c m^2 - 4\zeta\lambda_c] + A^2m^4\}}\right\}. \tag{60}$$

Comparison of (49) and (59) leads to

$$\Delta_{22}/\Delta_{13} = O((\lambda_c - \lambda)) \quad \text{as} \quad \lambda \to \lambda_c^-, \tag{61}$$

which confirms the anticipated result.

We substitute for Δ_{11} and Δ_{13} using (40) and (59) into the buckling equation (28) to get

$$(1 - \bar{\lambda}/\lambda_c)^{5/4} \approx 2\left[\frac{\lambda_c(1+\zeta)(\frac{1}{2}\alpha + \zeta)}{4(1+\zeta) - \frac{3}{2}\lambda_c}\right]^{1/4}\left[\frac{3\pi S_{00}(1)}{\zeta}\right]^{1/2}(-b)^{1/2}\epsilon\bar{\lambda}/\lambda_c \tag{62}$$

for $b < 0$. The shell is thus imperfection-sensitive (i.e. $\bar{\lambda} < \lambda_c$) for $b < 0$. This was found to be the case for modal imperfections [5] and localized dimple imperfection [3].

Concluding remarks. We exhibit the asymptotic results found for various kinds of imperfections. In each case, the imperfection is in the form

$$\bar{W}(x, y) = \epsilon w_0(y) \sin x.$$

The classical buckling load λ_c is

$$\lambda_c = 4(1+\zeta)^2/(3\zeta + 1 + \alpha)$$

and the relations between the buckling load $\bar{\lambda}$ and the imperfection amplitude parameter ϵ for sufficiently small ϵ are as follows:

(i) Modal imperfection [5]: $w_0(y) = \sin y$:

$$(1 - \bar{\lambda}/\lambda_c)^{3/2} = \frac{3\sqrt{3}}{2}(-b)^{1/2}\epsilon\bar{\lambda}/\lambda_c\ . \tag{63}$$

(ii) Dimple imperfection [3]: $|\bar{w}_0(y)| < M \exp(-a|y|)$, $M, a > 0$:

$$1 - \bar{\lambda}/\lambda_c = \frac{1}{\zeta}\left[\left(\frac{\alpha}{2} + \zeta\right)(1+\zeta)\right]^{1/2}(-b)^{1/2}\epsilon\ |\bar{w}_0(1)|\ \bar{\lambda}/\lambda_c\ , \tag{64}$$

where $\bar{w}_0(1) = \int w_0(y) \exp(iy)\, dy$.

(iii) Random imperfection (Eq. (62)): $w_0(y)$ random, stationary, Gaussian:

$$(1 - \bar{\lambda}/\lambda)^{5/4} \approx 2\left[\frac{\lambda_c(1+\zeta)(\alpha/_2 + \zeta)}{4(1+\zeta) - \frac{3}{2}\lambda_c}\right]^{1/4}\left(\frac{3\pi}{\zeta}\right)^{1/2}(-b)^{1/2}\epsilon[S_{00}(1)]^{1/2}\bar{\lambda}/\lambda c\ , \tag{65}$$

where $S_{00}(1) = (1/2\pi)\int R_{00}(z)\exp(-iz)\,dz$ and $R_{00}(z) = \langle w_0(y+z)w_0(y)\rangle$.

It should be noted that the result (65) breaks down if $S_{00}(1) \ll S_{00}(\omega)$ for $\omega \approx 1$. Under this circumstance buckling may no longer be provoked by $S_{00}(1)$. Higher-order perturbations are thus necessary.

Formulas (63)–(65) are fairly simple expressions for determining the buckling load $\bar{\lambda}$ for a large class of small-amplitude imperfections. Numerical values are obtained for $\bar{\lambda}$ by assigning values to ζ, calculating λ_c by (11), Batdorf's length parameter $Z = \pi^2 A/\sqrt{12}$ by (11) and b by (60). Graphs of $b/(1 - \nu^2)$ vs. Z are given in [5].

We observe that the structure is imperfection-sensitive ($b < 0$) to modal, dimple, and random imperfections for the same range of values of Z. The loss in the buckling strength is of order $\epsilon^{2/3}$ for modal imperfections, ϵ for dimple imperfections and $\epsilon^{4/5}$ for random imperfections.

Appendix A: Power spectral density of u_1. The coupled equations for ϕ_1 and u_1 are

$$\begin{aligned} M_1^{(1)}(\phi_1, u_1) &= 0, \qquad -\infty < y < \infty, \\ M_2^{(1)}(\phi_1, u_1) &= \lambda(\tfrac{1}{2}\alpha w_0 - \zeta w_0''), \end{aligned} \tag{66}$$

where $M_1^{(1)}$ and $M_2^{(1)}$ are defined by (45). Let the Green's functions $G(y - y_1)$ and $T(y - y_1)$ satisfy the equations

$$M_1^{(1)}(T, G) = 0, \quad M_2^{(1)}(T, G) = \delta(y - y_1).$$

Then taking Fourier transforms leads to

$$\tilde{G}(\omega) = \int G(z) \exp(i\omega z)\, dz = (1 + \omega^2\zeta)^2 Q(\omega). \tag{67}$$

The expressions for the Green's functions are omitted since they are irrelevant to the analysis. The solution to (66) may be written as

$$u_1(y) = \lambda \int G(y - y_1)[\tfrac{1}{2}\alpha w_0(y_1) - \zeta w_0''(y_1)]\, dy_1 . \tag{68}$$

We use this result in (37) to get

$$R_u(z) = \lambda^2 \iint G(y + z - y_1)G(y - y_2)\langle[\tfrac{1}{2}\alpha w_0(y_1) - \zeta w_0''(y_1)]\cdot[\tfrac{1}{2}\alpha w_0(y_2)]\rangle\, dy_1\, dy_2 . \tag{69}$$

Now $R_{00}(y_1 - y_2) = \langle w_0(y_1)w_0(y_2)\rangle$ and by appropriate differentiation

$$\langle w_0''(y_1)w_0(y_2)\rangle = \langle w_0(y_1)w_0''(y_2)\rangle = R_{00}''(y_1 - y_2)$$

etc. Thus

$$\begin{aligned} R_u(z) = \lambda^2 \iint & G(y + z - y_1)G(y_1 - y_2) \\ & \cdot[\tfrac{1}{4}\alpha^2 R_{00}(y_1 - y_2) - \alpha\zeta R_{00}''(y_1 - y_2) + \zeta^2 R_0^{\mathrm{IV}}(y_1 - y_2)]\, dy_1\, dy_2 . \end{aligned}$$

By introducing the power spectral density $S_{00}(\omega)$ defined by (33) and using properties of Fourier transforms we obtain

$$R_u(z) = \lambda^2 \int (\tfrac{1}{2}\alpha + \zeta)^2 \tilde{G}^2(\omega) S_{00}(\omega) \exp(i\omega z)\, d\omega.$$

Thus $S_u(\omega) = \lambda^2(\frac{1}{2}\alpha + \zeta)^2\tilde{G}^2(\omega)S_{00}(\omega)$. Substituting for $\tilde{G}$ using (67) leads to

$$S_u(\omega) = \lambda^2(\tfrac{1}{2}\alpha + \zeta)^2(1 + \omega^2\zeta)^4 Q^2(\omega) S_{00}(\omega). \tag{70}$$

We take the ensemble average of Eq. (68), interchanging averaging and integration, to get

$$\langle u_1(y)\rangle = \lambda \int G(y - y_1)[\tfrac{1}{2}\alpha\langle w_0(y_1)\rangle - \zeta\langle w_0''(y)\rangle]\, dy.$$

Since w_0 is a zero-mean stationary Gaussian random function, $\langle w_0(y)\rangle = 0$, $\langle w_0''(y)\rangle = 0$; hence $\langle u_1(y)\rangle = 0$. A similar calculation gives $\langle \phi_1(y)\rangle = 0$ and expressions (32) for $S_{u\phi}(\omega)$ and $S_\phi(\omega)$.

Appendix B: Asymptotic evaluation of an integral. Let $B_m \equiv \int F(\omega)Q^m(\omega)\, d\omega$, $m \geq 2$, where F is any smooth integrable function analytic in the strip $|\mathrm{Im}\ \omega| < a$ for some a with $F(\pm 1) \neq 0$ and

$$Q(\omega) = \{(1 + \omega^2\zeta)^2[(1 + \omega^2\zeta)^2 - \lambda(\tfrac{1}{2}\alpha + \zeta\omega^2)] - K(1 + \zeta)^2\}^{-1}.$$

It can be shown by using (11), and (4) for $K(\zeta)$, that

$$\frac{1}{Q(\omega)} = (\omega^2 - 1)^2 P(\omega) + (\lambda_c - \lambda)g(\omega)$$

where $P(\omega)$ and $g(\omega)$ are defined by (38) and (39). Thus

$$B_m = \int \frac{F(\omega)}{P^m(\omega)}\left[(\omega^2 - 1)^2 + (\lambda_c - \lambda)\frac{g(\omega)}{P(\omega)}\right]^{-m} d\omega.$$

There are poles of order m of $[(\omega^2 - 1)^2 + (\lambda_c - \lambda)(g(\omega)/P(\omega))]^{-m}$ in the upper half-plane given by

$$\omega_{1,2} = \pm 1 + (i/2)[(\lambda_c - \lambda)g(1)/P(1)]^{1/2} + 0((\lambda_c - \lambda)).$$

Note that g and P are even functions. Since $F(\omega)/P^m(\omega)$ is analytic for $|\mathrm{Im}\ (\omega)| < a$ for some a, the integral can be shifted in the complex ω plane to give

$$B_m = \int_{-\infty + ia_1}^{\infty + ia_1} \frac{F(\omega)}{P^m(\omega)}\left[(\omega^2 - 1)^2 + (\lambda_c - \lambda)\frac{g(\omega)}{P(\omega)}\right]^{-m} d\omega$$

$+\ 2\pi i$ [residue of integrand at ω_1, ω_2].
where $\frac{1}{2}(\lambda_c - \lambda)g(1)/P(1) < a_1 < a$. With a_1 fixed, the integral is bounded and hence $O(1)$ as $\lambda \to \lambda_c^-$. Evaluating the residues yields

$$B_m \approx \frac{\pi(m-1)(2m-3)!}{2^{2m-2}[(m-1)!]^2}\cdot\frac{F(-1) + F(1)}{[g(1)]^{m-1/2}[P(1)]^{1/2}(\lambda_c - \lambda)^{m-1/2}} \quad \text{as} \quad \lambda \to \lambda_c^-. \tag{71}$$

Appendix C: Solution of second-order perturbation equations. The second-order perturbation equations as given in (43) are

$$\begin{aligned} M_1^{(m)}(\psi_m, v_m) &= \Phi_m(y), \qquad -\infty < \zeta < \infty \\ M_2^{(m)}(\psi_m, v_m) &= \Phi_m(y) \end{aligned} \tag{72}$$

where $M_1^{(m)}$ and $M_2^{(m)}$ are defined by Eqs. (45) and

$$\begin{aligned} \Phi_m(y) &= (1 + \zeta)^2(T_m u_1'^2), \\ \Psi_m(y) &= -KP_m(u_1''\phi_1 + u_1\phi_1'') + 2T_m u_1'\phi_1'. \end{aligned} \tag{73}$$

The solution for v_m in (72) may be written in terms of Green's functions $G_m^{(1)}(y - y_1)$ and $G_m^{(2)}(y - y_1)$ as

$$v_m(y) = \int G_m^{(1)}(y - y_1)\Psi_m(y_1)\,dy_1 + \int G_m^{(2)}(y - y_1)\Phi_m(y_1)\,dy_1 \tag{74}$$

where the Fourier transforms $Q_m^{(1)}(\omega)$ and $Q_m^{(2)}(\omega)$ of $G_m^{(1)}$ and $G_m^{(2)}$ are given by Eq. (56). Similar integrals can be written for $\psi_m(y)$. Here the Green's functions $G_m^{(1)}(y - y_1)$, $G_m^{(2)}(y - y_1)$, $\Omega_m^{(1)}(y - y_1)$, $\Omega_m^{(2)}(y - y_2)$ satisfy the pairs of equations

$$M_1^{(m)}(\Omega_m^{(1)}, G_m^{(1)}) = 0, \quad M_2^{(m)}(\Omega_m^{(1)}, G_m^{(1)}) = \delta(y - y_1)$$

and

$$M_1^{(m)}(\Omega_m^{(2)}, G_m^{(2)}) = \delta(y - y_1), \quad M_2^{(m)}(\Omega_m^{(2)}, G_m^{(2)}) = 0$$

with the condition $G_m^{(i)}, \Omega_m^{(i)}, G_m^{(i)\prime}, \Omega_m^{(i)\prime} \to 0$ for $|y| \to \infty$.

Recalling that u_1 and ϕ_1 are linear functions of a zero-mean Gaussian random function w_0, we note that

$$\langle u_1(y_1)u_1(y_2)u_1(y_3)\rangle = 0, \quad \langle u_1(y_1)u_1(y_2)\phi_1(y_3)\rangle = 0$$

for any values of y_1, y_2, y_3 (see, for example, [9]). Appropriate differentiation of these equations leads to

$$\langle u_1''(y_1)u_2(y_2)u_1(y_3)\rangle = 0, \quad \langle u_1(y_1)u_1'(u_2)\phi_1'(y_3)\rangle = 0.$$

Thus multiplying Eq. (74) by $u_1(y)$ and taking ensemble average gives

$$\langle u_1(y)v_m(y)\rangle = 0.$$

We exhibit the calculation of a typical term in Δ_{22} given by (48). Consider the contribution, $\bar{\Delta}$ say, to Δ_{22} obtained by the multiplication of the underlined term in (73) by itself:

$$\bar{\Delta} = \tfrac{1}{2} \sum (1 + \zeta)^4 P_m^2 \left\langle \left[\int G_m^{(2)}(y - y_1)u_1''(y_1)u_1(y_1)\,dy_1\right]^2 \right\rangle. \tag{75}$$

Now

$$\left\langle \left[\int G_m^{(2)}(y - y_1)u_1''(y_1)u_1(y_1)\,dy_1\right]^2 \right\rangle$$
$$= \iint G_m^{(2)}(y - y_1)G_m^{(2)}(y - y_2)\langle u_1''(y_1)u_1(y_1)u_1''(y_2)u_1(y_2)\rangle\,dy_1\,dy_2$$
$$= \iint G_m^{(2)}(y - y_1)G_m^{(2)}(y - y_2)$$
$$\cdot\{[R_u''(0)]^2 + R_u(y_1 - y_2)R_u^{\mathrm{IV}}(y_1 - y_2) + [R_u''(y_1 - y_2)]^2\}\,dy_1\,dy_2,$$

since u_1 is a Gaussian random function, and hence

$$\begin{aligned}\langle u_1''(y_1)u_1(y_1)u_1''(y_2)u_1(y_2)\rangle &= \langle u_1''(y_1)u_1(y_1)\rangle\langle u_1''(y_2)u_1(y_2)\rangle \\ &\quad + \langle u_1''(y_1)u_1''(y_2)\rangle\langle u_1(y_1)u_1(y_2)\rangle \\ &\quad + \langle u_1''(y_1)u_1(y_2)\rangle\langle u_1(y_1)u_1''(y_2)\rangle.\end{aligned} \tag{76}$$

By introducing the power spectral density and using properties of Fourier transforms, the last double integral can be reduced to

$$[Q_m^{(2)}(0)]^2\left[\int \omega^2 S_u(\omega)\, d\omega\right]^2 + \iint \omega_2^2(\omega_1^2 + \omega_2^2)S_u(\omega_1)S_u(\omega_2)[Q_m^{(2)}(\omega_1 + \omega_2)]^2\, d\omega_1\, d\omega_2$$

where S_u and $Q_m^{(2)}$ are defined in Eqs. (31) and (56) respectively. These integrals are special cases of the integrals of (36) and (57). The use of the asymptotic results (37) and (58) gives each integral as $O((\lambda_c - \lambda)^{-3})$. Thus, by (75), $\bar{\Delta} = O((\lambda_c - \lambda)^{-3})$. Similar calculations, made for all other terms in the expression for Δ_{22}, lead to

$$\Delta_{22} = O((\lambda_c - \lambda)^{-3}). \tag{77}$$

Appendix D: Derivation of a typical term in Δ_{13}. Eqs. (52) are the differential equations for $\chi_1(y)$ and $h_1(y)$. These equations have the same differential operators as in (72) with $m = 1$; hence the solution for h_1 can be written in terms of the Green's functions of Eq. (74). We shall exhibit the derivation of only one term in the expression for Δ_{13} since the calculations are lengthy and repetitious. The underlined term in (52) gives rise to the following term in the expression for $h_1(y)$:

$$-(1 + \zeta)^2 \sum P_m \int G_1^{(2)}(y - y_1)u_1(y_1)v_m''(y_2)\, dy_2\ .$$

From Eqs. (74) and (73) for v_m, we consider only the contribution to $h_1(y)$ from the underlined term in (73). This contribution is

$$(1 + \zeta)^4 \sum P_m^2 \iint G_1^{(2)}(y - y_1)G_m^{(2)\prime\prime}(y_1 - y_2)u_1(y_1)u_1''(y_2)u_1(y_2)\, dy_1\, dy_2\ .$$

Since by (54) $\Delta_{13} = \frac{1}{2}\langle u_1(y)h_1(y)\rangle$, the above expression contributes a term, $\bar{\Delta}_{13}$ say, to Δ_{13} given by

$$\bar{\Delta}_{13} = \tfrac{1}{2}(1 + \zeta)^4 \sum P_m^2 \iint G_1^{(2)}(y - y_2)G_m^{(2)\prime\prime}(y_1 - y_2)\langle u_1(y)u_1(y_1)u_1''(y_2)u_1(y_2)\rangle\, dy_1\, dy_2.$$

The use of a result similar to (76) leads to

$$\bar{\Delta}_{13} = \tfrac{1}{2}(1 + \zeta)^4 \sum P_m^2 \iint G_1^{(2)}(y - y_2)G_m^{(2)\prime\prime}(y_1 - y_2)$$
$$\cdot |R_u(y - y_1)R_u''(0) + R_u''(y - y_2)R_u(y_1 - y_2) + R_u(y - y_2)R_u''(y_1 - y_2)|\, dy_1\, dy_2\ .$$

The double integral can be expressed in terms of the power spectral density $S_u(\omega)$ defined by (33) and the Fourier transform $Q_m^{(2)}(\omega)$ of $G_m^{(2)}(\omega)$. Thus

$$\bar{\Delta}_{13} = \tfrac{1}{2}(1 + \zeta)^4$$
$$\cdot \sum P_m^2 \iint (\omega_1 + \omega_2)^2(\omega_1^2 + \omega_2^2)Q_1^{(2)}(\omega_1)Q_1^{(2)}(\omega_1 + \omega_2)S_u(\omega_1)S_u(\omega_2)\, d\omega_1\, d\omega_2\ .$$

This is the term in (55) which is underlined in the expression for I_2 following Eq. (55).

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