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ASYMPTOTIC ANALYSIS OF THE LYAPUNOV EXPONENT AND ROTATION NUMBER OF THE RANDOM OSCILLATOR AND APPLICATIONS*

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Abstract. We construct asymptotic expansions for the exponential growth rate (Lyapunov exponent) and rotation number of the random oscillator when the noise is large, small, rapidly varying or slowly varying. We then apply our results to problems in the stability of the random oscillator, the spectrum of the one-dimensional random Schrödinger operator and wave propagation in a one-dimensional random medium.

Key words. Lyapunov exponent, rotation number, random oscillator, random Schrödinger operator, wave propagation

AMS(MOS) subject classifications. primary 60H10; secondary 60J99, 34F05, 70L05, 82A42, 93E15

1. Introduction. The random oscillator equation

$$\ddot{y}+f(t)y=0, \qquad t\in R, \tag{1.1}$$

with $f(t)$ a given random process, arises in many contexts such as solid state theory (cf. Lax and Phillips [20], Frisch and Lloyd [12], Halperin [14], Pastur et al. [28]), vibrations in mechanical and electrical circuits (cf. Stratonovich [29], Van Kampen [30]) and wave propagation in one-dimensional random media (cf. Klyatskin [17], Papanicolaou [26]).

There is also a substantial mathematical theory concerning properties of the stochastic process defined by (1.1), various approximations, the case when $f(t)$ is white noise, large t behavior etc. (cf. Arnold [1], Friedman [10], Khasminskii [15], Arnold and Kliemann [3], Blankenship and Papanicolaou [8], Goldsheid, Molčanov and Pastur [13], Molčanov [23], Wihstutz [31]).

Two quantities associated with solutions of (1.1) are of particular interest. They are the Lyapunov exponent λ and rotation number α. When $f(t)$ is stationary and ergodic (plus other conditions reviewed in § 2) they are defined by

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\log\left(|y(t)|^2+|\dot{y}(t)|^2\right)^{1/2} \tag{1.2}$$

and

$$\alpha=\lim_{t\to\infty}\frac{1}{t}\tan^{-1}\left(\frac{\dot{y}(t)}{y(t)}\right). \tag{1.3}$$

The limits exist with probability one and do not depend on the initial values $y(0)$ and $\dot{y}(0)$ of the solution, provided that the latter do not depend on the random coefficient $f(\cdot)$.

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Roughly, λ determines the growth or decay properties of solutions of (1.1), in particular, their sample stability, while α determines the asymptotic rate of rotation of the unit vector $(y(t), \dot{y}(t))/(y^2(t)+\dot{y}^2(t))^{1/2}$.

Although there are general theorems regarding the existence of the limits (1.2) and (1.3) as well as some general properties (regularity with respect to parameters, strict positivity of λ etc.; cf. Molčanov [23], Kotani [18], Arnold [2], Arnold and Oeljeklaus [4], Kliemann and Arnold [16], for example), λ and α cannot be computed explicitly, except in some very special cases (see Loparo and Blankenship [21], Cohen and Newman [9]). See also Arnold and Wihstutz [5].

Our purpose in this paper is to develop approximations for λ and α under various hypotheses about the random process $f(t)$. For example, we consider the case where $f(t)=f_0+\sigma f_1(t)$ with f_0 a constant and $f_1(t)$ a stationary and ergodic Markov process with mean zero and variance equal to one. We calculate the asymptotic behavior of λ and α when σ tends to zero (§ 4) and when σ tends to infinity (§ 5), plus some other cases.

In § 2 we formulate in detail the problem and introduce the framework in which we carry out the asymptotic analysis. In § 3 we show how suitably constructed formal expansions are in fact correct asymptotic expansions. We do this in a somewhat general way here in order to avoid repetition of details in the various cases we consider.

Section 4 contains the small noise analysis, § 5 the large noise analysis and § 6 several other cases including the white noise limit. In § 7 we prove a central limit theorem for the fluctuations associated with (1.2) and (1.3). We also give asymptotic expansions for the variance of the limit Gaussian law.

In § 8 we interpret our results in the context of stability theory. In § 9 we discuss their implications for wave propagation and spectra.

2. Formulation of the problem. We shall analyze (1.1) in the form

$$-\ddot{y}+\sigma F\left(\xi\left(\frac{t}{\rho}\right)\right)y=\gamma y, \tag{2.1}$$

in which σ, ρ and γ are parameters and the values of y and $\dot{y}$ are given at $t=0$. The noise $\xi(\cdot)$ is assumed to be an ergodic Markov process on a smooth connected Riemannian manifold M (with or without boundary) with invariant probability $\nu(d\xi)$. $F: M\to\mathbb{R}$ is a smooth nonconstant function such that $F(\xi(t))$ has finite mean. We normalize things so that

$$E_\nu F(\xi(t))=\int_M F(\xi)\nu(d\xi)=0. \tag{2.2}$$

Clearly σ measures the strength of the noise and we take it positive (since otherwise we may replace F by $-F$). The parameter γ is real and plays the role of energy or frequency squared. The parameter ρ is real and positive and allows us to change the rate at which the noise varies.

In system form (2.1) becomes

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y}\end{pmatrix}=\begin{pmatrix} 0 & 1 \\ \sigma F\left(\xi\left(\frac{t}{\rho}\right)\right)-\gamma & 0\end{pmatrix}\begin{pmatrix} y \\ \dot{y}\end{pmatrix}.$$

We introduce polar coordinates in a manner that depends on γ. If $\gamma>0$ we write $y=r\cos\varphi$, $\dot{y}=\sqrt{\gamma}\,r\sin\varphi$, while for $\gamma<0$, $y=r\cos\varphi$, $\dot{y}=\sqrt{-\gamma}\,r\sin\varphi$. This leads to the following equations for r and φ.

$$r(t)=r_0 \exp \int_0^t q\left(\varphi(\tau), \xi\left(\frac{\tau}{\rho}\right)\right) d\tau$$

and

$$\dot{\varphi}=h\left(\varphi(t), \xi\left(\frac{t}{\rho}\right)\right),$$

where

$$q(\varphi, \xi)=\begin{cases} \dfrac{\sigma F(\xi)}{2\sqrt{\gamma}} \sin 2\varphi & \text{for } \gamma>0, \\ \left(\sqrt{-\gamma}+\dfrac{\sigma F(\xi)}{2\sqrt{-\gamma}}\right) \sin 2\varphi & \text{for } \gamma<0, \end{cases} \tag{2.3}$$

$$h(\varphi, \xi)=\begin{cases} -\sqrt{\gamma}+\dfrac{\sigma F(\xi)}{\sqrt{\gamma}} \cos^2 \varphi & \text{for } \gamma>0, \\ \sqrt{-\gamma} \cos 2\varphi+\dfrac{\sigma F(\xi)}{\sqrt{-\gamma}} \cos^2 \varphi & \text{for } \gamma<0. \end{cases} \tag{2.4}$$

Let G denote the infinitesimal generator of $\xi(t)$. To streamline our calculations, we will assume the following rather strong conditions. Much of what follows can be done in considerably greater generality.

(H) M is a compact manifold. G is a selfadjoint elliptic diffusion operator on M with zero an isolated, simple eigenvalue.

It follows from $G1=0$, where 1 is the function identically equal to one on M, that the invariant probability has constant density. Without loss of generality we assume that $\nu(d\xi)$ is normalized so that

$$\text{volume } (M)=1, \qquad \text{and we write } \nu(d\xi)=d\xi.$$

The pair $(\varphi(t), \xi(t/\rho))$ is a diffusion process on $S^1\times M$, $S^1=$ unit circle, with generator

$$L=\frac{1}{\rho} G+h(\varphi, \xi) \frac{\partial}{\partial \varphi}. \tag{2.5}$$

The ergodic theory of L has been studied in detail by Kliemann and Arnold [16]. Under assumptions (H), L is hypoelliptic and has a unique (up to π-periodicity) smooth invariant probability $p(\varphi, \xi)\, d\varphi\, d\xi$ on $S^1\times M$ such that its marginal on M is $\nu(d\xi)=d\xi$, and its support is $C\times M$, where

$$C=\begin{cases} S^1 & \text{iff there is a } \xi\in M \text{ with } F(\xi)<\gamma/\sigma, \\ \text{some interval in } [0, \pi/2] & \text{otherwise.} \end{cases}$$

A detailed treatment of the 2×2 system can be found in Arnold and Kliemann [3].

Under the above conditions the Lyapunov exponent λ and the rotation number α defined by (1.2) and (1.3) exist and are given by

$$\lambda=\int_{S^1\times M} q(\varphi, \xi) p(\varphi, \xi)\, d\varphi\, d\xi \tag{2.6}$$

and

$$\alpha = \int_{S^1 \times M} h(\varphi, \xi) p(\varphi, \xi)\, d\varphi\, d\xi. \tag{2.7}$$

Furthermore, λ is the top Lyapunov exponent from Oseledec's multiplicative ergodic theorem (Oseledec [25]) and so $\lambda \geqq 0$. In the present case we actually have $\lambda > 0$ (Kliemann and Arnold [16]).

The invariant density p satisfies the Fokker-Planck equation

$$L^* p = \frac{1}{\rho} Gp - \frac{\partial}{\partial \varphi}(h(\varphi, \xi)p) = 0. \tag{2.8}$$

The asymptotic analysis then reduces to the study of p in various asymptotic limits. For once the expansion for p is known, it can be used in (2.6) and (2.7) to give the expansion for λ and α.

Figures 1 and 2 show level curves of λ and α, resp., on the γ, σ plane. These diagrams are based on formulas (1.2) and (1.3). We have chosen $F(\xi) = \xi$, $\rho = 1$ and $\xi(t) = (2/\pi) \tan^{-1} \eta(t)$, $\eta(t) =$ Ornstein-Uhlenbeck process.

3. Expansions for the invariant probability and their convergence properties. The form of the determining equation for p, eq. (2.8), and the form of the function $h(\varphi, \xi)$ given by (2.4) indicate that in nearly every case of interest, such as $\sigma \to \infty$ or $\sigma \to 0$, we have a singular perturbation problem. In particular, although (2.8) has a unique smooth solution under our hypotheses, it will in general not admit expansions in smooth functions.

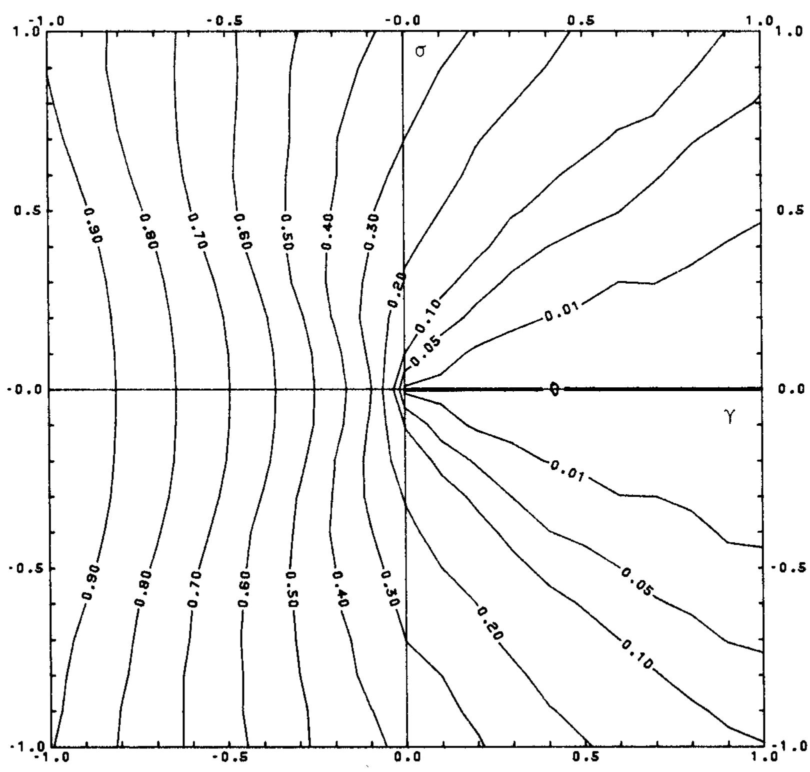

FIG. 1. *Level curves of the Lyapunov exponent λ on the γ, σ plane.*

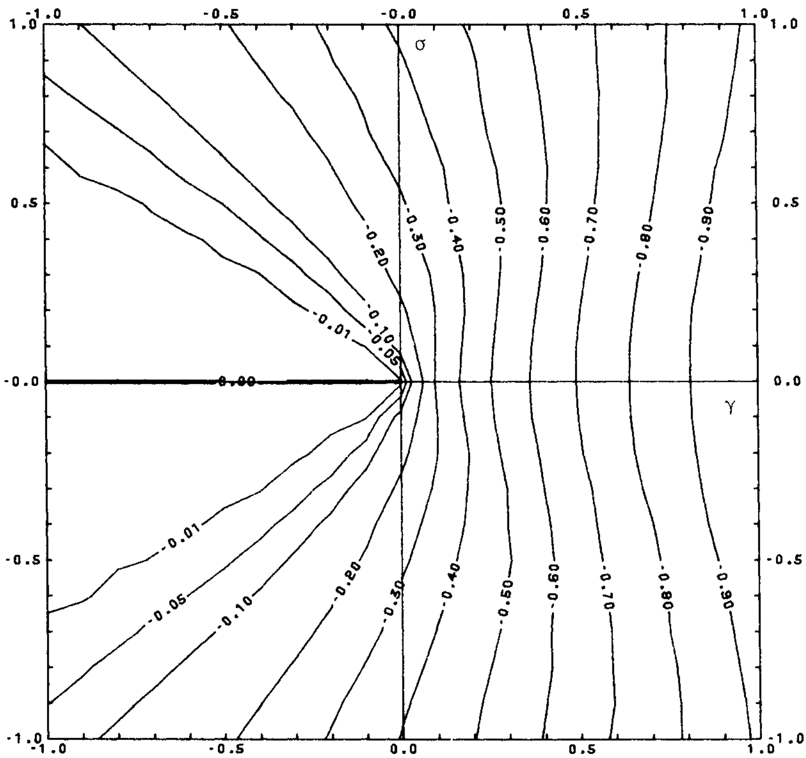

FIG. 2. *Level curves of the rotation number α on the γ, σ plane.*

We develop now the formalism to handle singular expansions in a fairly general way. All our results are obtained by using this formalism. For concreteness, we shall model the general discussion after the case where ρ and γ are fixed and $\sigma = \varepsilon \to 0$.

The operator L defined by (2.5) has the form

$$L_\varepsilon = L_0 + \varepsilon L_1.$$

So we seek a solution of (2.8) which is

$$L_\varepsilon^* p_\varepsilon = 0.$$

If now $f(\varphi, \xi)$ stands for some smooth function such as $q(\varphi, \xi)$ or $h(\varphi, \xi)$ (which may depend on ε), we want to find an expansion for

$$(f, p_\varepsilon) = \int_{S^1 \times M} f(\varphi, \xi) p_\varepsilon(\varphi, \xi)\, d\varphi\, d\xi.$$

Suppose we have constructed a formal expansion of $L_\varepsilon^* p_\varepsilon = 0$ in the form

$$p_\varepsilon = p_0 + \varepsilon p_1 + \cdots + \varepsilon^N p_N + \cdots \tag{3.1}$$

so that

$$L_0^* p_0 = 0, \quad L_0^* p_1 + L_1^* p_0 = 0, \quad \cdots, \quad L_0^* p_N + L_1^* p_{N-1} = 0, \quad \cdots. \tag{3.2}$$

In (3.1) the $p_0, p_1, \cdots$ may be singular as they will be for example in the small noise case for $\gamma < 0$. We want to show that we have the asymptotic expansion

$$(f, p_\varepsilon) = (f, p_0) + \varepsilon (f, p_1) + \cdots + \varepsilon^N (f, p_N) + \cdots. \tag{3.3}$$

To prove that (3.3) is correct we proceed as follows. With f a fixed smooth function we construct an adjoint expansion for

$$L_\varepsilon F_\varepsilon = f$$

with

$$F_\varepsilon = F_0 + \varepsilon F_1 + \cdots + \varepsilon^N F_N.$$

It is not possible in general to construct such an expansion however. What can be done is to find $F_0, F_1, \cdots, F_N$ such that

$$(L_0 + \varepsilon L_1)(F_0 + \cdots + \varepsilon^N F_N) = f - (\tilde{f}_0 + \cdots + \varepsilon^N \tilde{f}_N) + \varepsilon^{N+1} L_1 F_N. \tag{3.4}$$

Here $\tilde{f}_0, \cdots, \tilde{f}_N$ are functions that do not depend on ϕ and are chosen so that the sequence of problems obtained from (3.4)

$$\begin{aligned} L_0 F_0 &= f - \tilde{f}_0, \\ L_0 F_1 + L_1 F_0 &= -\tilde{f}_1, \\ &\cdots \\ L_0 F_N + L_1 F_{N-1} &= -\tilde{f}_N \end{aligned} \tag{3.5}$$

is solvable, i.e. the choice is made according to the Fredholm alternative. Assume that the marginal of $p_0 + \varepsilon p_1 + \cdots + \varepsilon^N p_N$ on M is $\nu(\xi)$. By (3.1) and (3.2),

$$(L_0 + \varepsilon L_1)^*(p_0 + \cdots + \varepsilon^N p_N) = \varepsilon^{n+1} L_1^* p_N, \qquad N = 0, 1, \cdots.$$

Collecting the above and using $(L_\varepsilon F_\varepsilon, p_\varepsilon) = 0$ we arrive at the identity

$$\begin{aligned} &(f, p_\varepsilon) - [(f, p_0) + \cdots + \varepsilon^N (f, p_N)] \\ &\quad = -\varepsilon^{N+1}[(L_1 F_N, p_\varepsilon) + (L_1(F_0 + \cdots + \varepsilon^N F_N), p_N) \\ &\qquad - (L_1 F_N, p_0 + \cdots + \varepsilon^N p_N)]. \end{aligned} \tag{3.6}$$

This identity is valid for $N \geqq 0$ and leads to the following theorem.

THEOREM 3.1. *Choose $N \geqq 0$ fixed. Suppose the formal expansion* (3.1), (3.2) *and* (3.4), (3.5) *have been constructed and that $p_0 + \varepsilon p_1 + \cdots + \varepsilon^N p_N$ has marginal $\nu(\xi)$ on M, and*

$$\sup_{\phi, \xi} |L_1 F_N| \leqq C < \infty. \tag{3.7}$$

Suppose further that $p_0, \cdots, p_N$ and $F_1, \cdots, F_N$ are such that the inner products on the right of (3.6) *are well defined. Then we have the estimate*

$$\begin{aligned} &|(f, p_\varepsilon) - (f, p_0) - \cdots - \varepsilon^N (f, p_N)| \\ &< \varepsilon^{N+1}(C + |(L_1 F_0 + \cdots + \varepsilon^N L_1 F_N), p_N)| + |(L_1 F_N, p_0 + \cdots + \varepsilon^N p_N)|). \end{aligned} \tag{3.8}$$

The proof follows immediately from (3.6) and (3.7). The important point of the theorem is that the estimate on the right of (3.8) contains only objects that are known explicitly from the constructions (3.2) and (3.5).

It is interesting to note that to estimate the error in the formal expansion (3.3) it is not enough to have the expansion of the invariant density (3.1). It is necessary to construct also an adjoint expansion such as (3.4) which leads then to the estimate (3.8).

There is another way of obtaining an expansion for (f, p_ε) that does not involve the expansion of p_ε. We will use this method in § 5 and elsewhere. It works as follows.

Consider the problem

$$L_\varepsilon u_\varepsilon = f - \lambda_\varepsilon.$$

For u_ε to exist it is necessary that λ_ε be defined by

$$\lambda_\varepsilon = (f, p_\varepsilon).$$

Suppose we can construct $u_0, u_1, \cdots, u_N$ and $\mu_0, \mu_1, \cdots, \mu_N$ such that

$$\begin{aligned} &L_0 u_0 &&= f - \mu_0, \\ &L_0 u_1 + L_1 u_0 &&= -\mu_1, \\ &\cdots \\ &L_0 u_N + L_1 u_{N-1} &&= -\mu_N. \end{aligned} \tag{3.9}$$

Suppose further that $\mu_0, \mu_1, \cdots, \mu_N$ do not depend on φ. Then

$$\begin{aligned} &L_\varepsilon(u_\varepsilon - u_0 - \varepsilon u_1 - \cdots - \varepsilon^N u_N) \\ &\quad = f - \lambda_\varepsilon - f + (\mu_0 + \varepsilon\mu_1 + \cdots + \varepsilon^N \mu_N) - \varepsilon^{N+1} L_1 u_N. \end{aligned} \tag{3.10}$$

Taking the inner product of (3.10) with p_ε yields the identity

$$\lambda_\varepsilon = (\mu_0 + \varepsilon\mu_1 + \cdots + \varepsilon^N \mu_N, \nu) + \varepsilon^{N+1}(L_1 u_N, p_\varepsilon). \tag{3.11}$$

Here we have used the fact that $L_\varepsilon^* p_\varepsilon = 0$, that the marginal of p_ε on M is ν and that the μ_i do not depend on φ.

From (3.11) we immediately derive

THEOREM 3.2. *Suppose the sequence of problems* (3.9) *can be solved and that*

$$\sup_{\varphi, \xi} |L_1 u_N| \leqq C < \infty.$$

Then

$$(f, p_\varepsilon) = (\mu_0, \nu) + \varepsilon(\mu_1, \nu) + \cdots + \varepsilon^N(\mu_N, \nu) + O(\varepsilon^{N+1}).$$

4. Small noise analysis. We shall first consider the case $\varepsilon = \sigma \to 0$ and $\gamma > 0$. We let $\rho = 1$ for simplicity. Let

$$C(t) = EF(\xi(t))F(\xi(0)) \tag{4.1}$$

be the covariance function of $F(\xi(t))$ (recall (2.2)). By hypotheses (H) $C(t) \to 0$ as $t \to \infty$ exponentially fast, so the power spectrum (spectral density) of $F(\xi(t))$,

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega t) C(t)\, dt = \frac{1}{\pi} \int_0^\infty \cos \omega t C(t)\, dt, \tag{4.2}$$

is well defined and nonnegative for all $\omega \in \mathbb{R}$.

THEOREM 4.1. *For* $\sigma \to 0$, $\rho = 1$ *and* $\gamma > 0$ *we have*

$$\begin{aligned} \lambda &= \sigma^2 \frac{\pi}{4\gamma} \hat{f}(2\sqrt{\gamma}) + O(\sigma^3), \\ \alpha &= -\sqrt{\gamma} + \frac{\sigma^2}{2\gamma} \int_0^\infty \sin 2\sqrt{\gamma} t\, C(t)\, dt + O(\sigma^3) \end{aligned} \tag{4.3}$$

where $\hat{f}(\omega)$ *is the power spectrum of* $F(\xi(t))$ *defined by* (4.2).

COROLLARY 4.1. *For $\sigma \to 0$, $\rho = 1$, $\gamma = \gamma_1 \sigma$, $\gamma_1 > 0$*

$$\lambda = \sigma \frac{\pi}{4\gamma_1} \hat{f}(0) + O(\sigma^2), \tag{4.4}$$
$$\alpha = -\sqrt{\sigma}\sqrt{\gamma_1} + \sigma^{3/2} \frac{1}{2\gamma_1} \int_0^\infty tC(t)\, dt + O(\sigma^2).$$

Proof. The invariant probability density p_σ satisfies

$$L_\sigma^* p_\sigma = G p_\sigma - \frac{\partial}{\partial \varphi}\left[\left(-\sqrt{\gamma} + \sigma \frac{F(\xi)}{\sqrt{\gamma}} \cos^2 \varphi\right) p_\sigma\right]$$
$$= \left(G + \sqrt{\gamma} \frac{\partial}{\partial \varphi}\right) p_\sigma - \sigma \frac{\partial}{\partial \varphi}\left(\frac{F(\xi)}{\sqrt{\gamma}} \cos^2 \varphi p_\sigma\right) = 0.$$

We seek an expansion $p_\sigma = p_0 + \sigma p_1 + \cdots$. Clearly p_0 satisfies

$$\left(G + \sqrt{\gamma} \frac{\partial}{\partial \varphi}\right) p_0 = 0.$$

Thus, p_0 is the density of the uniform distribution on $S^1 \times M$, i.e.

$$p_0(\varphi, \xi) = \frac{1}{2\pi}. \tag{4.5}$$

For p_1 we have the equation

$$\left(G + \sqrt{\gamma} \frac{\partial}{\partial \varphi}\right) p_1 = -\frac{p_0 F(\xi)}{\sqrt{\gamma}} \sin 2\varphi. \tag{4.6}$$

Let $g(t, \xi, \eta)$ be the transition probability density of $\xi(t)$. It is defined by the equation

$$\frac{\partial g}{\partial t} = Gg, \quad t > 0, \qquad g(0, \xi, \eta) = \delta_\eta(\xi),$$

with $\delta_\eta(\xi)$ the delta function at the point $\eta \in M$. In terms of g we can solve (4.6) to obtain

$$p_1(\varphi, \xi) = \frac{p_0}{\sqrt{\gamma}} \int_0^\infty \sin 2(\varphi + \sqrt{\gamma}\, t) \int_M g(t, \xi, \eta) F(\eta)\, d\eta\, dt.$$

Higher terms in the expansion can be constructed readily but we stop here. With

$$q = \frac{\sigma}{2\sqrt{\gamma}} F(\xi) \sin 2\varphi$$

we see that $(q, p_0) = 0$ and hence

$$\lambda = \frac{\sigma^2}{2\gamma} \int_0^\infty dt \left[\frac{1}{2\pi} \int_0^{2\pi} \sin 2\varphi \sin 2(\varphi + \sqrt{\gamma}\, t)\, d\varphi \right.$$
$$\left. \cdot \int_M \int_M g(t, \xi, \eta) F(\xi) F(\eta)\, d\xi\, d\eta \right] + O(\sigma^3).$$

Taking into account that

$$C(t) = \int_M \int_M g(t, \xi, \eta) F(\xi) F(\eta)\, d\xi\, d\eta$$

is the covariance of $F(\xi(t))$ defined by (4.1), this can be simplified to give

$$\lambda = \frac{\sigma^2}{4\gamma}\int_0^\infty \cos 2\sqrt{\gamma}\, tC(t)\, dt + O(\sigma^3)$$

$$= \sigma^2 \frac{\pi}{4\gamma} f(2\sqrt{\gamma}) + O(\sigma^3).$$

It is easy to see that if γ goes to zero with σ so that $\gamma=\gamma_1\sigma$, $\gamma_1>0$, then it is enough to put $\gamma=\gamma_1\sigma$ in (4.3) to get

$$\lambda = \sigma\frac{\pi}{4\gamma_1}\hat{f}(0) + O(\sigma^2).$$

The details that lead to the verification of the error estimates in (4.3) and (4.4) follow the general pattern of Theorem 3.1, so we omit them.

The expansion for p_σ can also be used to obtain an expansion for the rotation number α. The analogue of (4.3) is

$$\alpha = -\sqrt{\gamma} + \frac{\sigma^2}{4\gamma}\int_0^\infty \sin 2\sqrt{\gamma}\, tC(t)\, dt + O(\sigma^3).$$

This is because $(h, p_0) = -\sqrt{\gamma}$ and $(1, p_2) = 0$. With $\gamma=\gamma_1\sigma$, $\gamma_1>0$, we obtain the analogue of (4.4)

$$\alpha = -\sqrt{\sigma}\sqrt{\gamma_1} + \sigma^{3/2}\frac{1}{2\gamma_1}\int_0^\infty tC(t)\, dt + O(\sigma^2). \qquad \square$$

We pass next to the case $\gamma<0$. With $\rho=1$ again the equation for the invariant probability has the form

$$L_\sigma^* p_\sigma = Gp_\sigma - \frac{\partial}{\partial\varphi}(\sqrt{-\gamma}\cos 2\varphi p_\sigma) - \sigma\frac{F}{\sqrt{-\gamma}}\frac{\partial}{\partial\varphi}(\cos^2\varphi p_\sigma) = 0.$$

In the case $\gamma<0$, (2.4) can vanish. To first order in σ the equilibrium points do not depend on ξ and are $\varphi_0=\pi/4$ and $\varphi_1=3\pi/4$. The first one is stable, the second one unstable. These considerations lead us to the conclusion that the first approximation p_0 of p_σ which satisfies

$$Gp_0 - \frac{\partial}{\partial\varphi}(\sqrt{-\gamma}\cos 2\varphi p_0) = 0$$

should be

$$p_0(\varphi, \xi) = \delta_{\varphi_0}(\varphi).$$

The equation for p_1 in the expansion $p_\sigma = p_0 + \sigma p_1 + \cdots$ takes the form

(4.7) $$Gp_1 - \frac{\partial}{\partial\varphi}(\sqrt{-\gamma}\cos 2\varphi p_1) = \frac{F}{\sqrt{-\gamma}}\frac{\partial}{\partial\varphi}(\cos^2\varphi\,\delta_{\varphi_0}(\varphi)) = \frac{F}{2\sqrt{-\gamma}}\delta'_{\varphi_0}(\varphi).$$

We look for p_1 in the form

(4.8) $$p_1 = \gamma_1(\xi)\delta'_{\varphi_0}(\varphi).$$

Using the fact that for any smooth function $a(\varphi)$ we have the identity

$$(a\delta'_{\varphi_0})' = -a'(\varphi_0)\delta'_{\varphi_0} + a(\varphi_0)\delta''_{\varphi_0},$$

we see that (4.8) solves (4.7) provided that $r_1(\xi)$ satisfies

$$(G-2\sqrt{-\gamma})r_1=\frac{F(\xi)}{2\sqrt{-\gamma}}.$$

Therefore

$$r_1=\frac{1}{2\sqrt{-\gamma}}(G-2\sqrt{-\gamma})^{-1}F.$$

This is well defined since $2\sqrt{-\gamma}>0$ and the inverse operator is the resolvent of G.

It is necessary to construct one more term in the expansion to get nontrivial results. Thus we must find p_2 such that

$$Gp_2-\frac{\partial}{\partial\varphi}(\sqrt{-\gamma}\cos 2\varphi p_2)=\frac{Fr_1}{\sqrt{-\gamma}}\frac{\partial}{\partial\varphi}(\cos^2\varphi\delta'_{\varphi_0}).$$

Using the identity

$$(a\varphi''_{\varphi_0})'=a''(\varphi_0)\delta'_{\varphi_0}-2a'(\varphi_0)\delta''_{\varphi_0}+a(\varphi_0)\delta''_{\varphi_0}$$

we find p_2 in the form

$$p_2(\varphi,\xi)=r_{21}(\xi)\delta'_{\varphi_0}(\varphi)+r_{22}(\xi)\delta''_{\varphi_0}(\varphi),$$

where r_{21} and r_{22} are given by

$$r_{21}=\frac{1}{\sqrt{-\gamma}}(G-2\sqrt{-\gamma})^{-1}Fr_1,$$

$$r_{22}=\frac{1}{2\sqrt{-\gamma}}(G-4\sqrt{-\gamma})^{-1}Fr_1.$$

Collecting the above results and using them in (2.6) with q given by (2.3) for $\gamma<0$, we get the following expansion for $\sigma\to 0$,

$$\lambda=\sqrt{-\gamma}+\frac{\sigma^2}{4\gamma}\int_0^\infty e^{-2\sqrt{-\gamma}\,t}C(t)\,dt+O(\sigma^3). \tag{4.9}$$

Here $C(t)$ is again the covariance of $F(\xi(t))$ defined by (4.1). Note that the coefficient of σ^2 in (4.9) is negative. This is so because γ is negative and the integral is positive. The latter fact follows from its resolvent interpretation and the selfadjointness of G which implies that $C(t)$ is positive.

When $\gamma=\sigma\gamma_1$, $\gamma_1<0$, we obtain the correct expansion by simply setting $\gamma=\sigma\gamma_1$ in (4.9). This gives

$$\begin{aligned}\lambda&=\sqrt{\sigma}\sqrt{-\gamma_1}+\frac{\sigma}{4\gamma_1}\int_0^\infty C(t)\,dt+O(\sigma^{3/2})\\&=\sqrt{\sigma}\sqrt{-\gamma_1}+\sigma\frac{\pi}{4\gamma_1}\hat{f}(0)+O(\sigma^{3/2}).\end{aligned} \tag{4.10}$$

From the behavior of the flow $\varphi(t)$ defined by (2.5) we know that α must be zero for $\gamma<0$.

The proof of the validity of (4.9) and (4.10) is patterned after the Theorem 3.1. In fact, the present expansion with $\gamma<0$ is a model case for the application of the theorem. We have thus obtained the following results.

THEOREM 4.2. *For $\sigma\to 0$, $\rho=1$ and $\gamma<0$ we have*

$$\lambda=\sqrt{-\gamma}+\frac{\sigma^2}{4\gamma}\int_0^\infty e^{-2\sqrt{-\gamma}\,t}C(t)\,dt+O(\sigma^3),$$

$$\alpha\equiv 0,$$

where $C(t)$ is the covariance function of $F(\xi(t))$ defined by (4.1).

COROLLARY 4.2. *For $\sigma\to 0$, $\rho=1$, $\gamma=\gamma_1\sigma$, $\gamma_1<0$*

$$\lambda=\sqrt{\sigma}\sqrt{-\gamma_1}+\sigma\frac{\pi}{4\gamma_1}\hat{f}(0)+O(\sigma^{3/2}),$$

$$\alpha\equiv 0,$$

where $\hat{f}(\omega)$ is the power spectrum of $F(\xi(t))$ defined by (4.2).

5. Large noise analysis. We shall now consider the case $\sigma\to\infty$ and $\gamma=\gamma_0+\gamma_1\sigma$, γ_0 and γ_1 fixed, put again $\rho=1$ for simplicity.

The function $F(\xi)$ is smooth and bounded on M. Let

$$\min F:=\min_{\xi\in M}F(\xi)\leqq F(\xi)\leqq\max_{\xi\in M}F(\xi)=:\max F$$

where $\min F<0<\max F$ since $\int_M F(\xi)\,d\xi=0$.

THEOREM 5.1. *For $\sigma\to\infty$, $\gamma=\gamma_0+\gamma_1\sigma$, $\rho=1$ and*

$$\max F<\gamma_1, \tag{5.1}$$

we have

$$\lambda=\frac{\sqrt{\gamma_1}}{4\pi}\int_0^{2\pi}d\varphi\int_M d\xi\frac{\sqrt{\gamma_1-F(\xi)}}{\gamma_1-F(\xi)\cos^2\varphi}G(\log(\gamma_1-F(\xi)\cos^2\varphi))+O\left(\frac{1}{\sqrt{\sigma}}\right),$$

$$\alpha=-\sqrt{\sigma}\int_M d\xi\,\sqrt{\gamma_1-F(\xi)}+O\left(\frac{1}{\sqrt{\sigma}}\right).$$

Proof. Note first that we can choose $\gamma_0=0$ without loss of generality. We shall follow the procedure described in Theorem 3.2 and shall try to solve the equation

$$Lu=f-\lambda,\qquad \lambda=(p,f), \tag{5.2}$$

where (cf. (2.5))

$$L=G+\sqrt{\sigma}\left(-\sqrt{\gamma_1}+\frac{F(\xi)}{\sqrt{\gamma_1}}\cos^2\varphi\right)\frac{\partial}{\partial\varphi}.$$

We are interested in the cases $f=q$ and $f=h$ given by (2.3) and (2.4), hence

$$f=\sqrt{\sigma}\frac{F(\xi)}{2\sqrt{\gamma_1}}\sin 2\varphi=\sqrt{\sigma}\,q_0$$

and

$$f=\sqrt{\sigma}\left(-\sqrt{\gamma_1}+\frac{F(\xi)}{\sqrt{\gamma_1}}\cos^2\varphi\right)=\sqrt{\sigma}\,h_0.$$

For $\gamma_1>\max F$ we have $h_0(\varphi,\xi)<0$. We will construct the first two terms in the expansion of

$$\left(G+\sqrt{\sigma}\,h_0\frac{\partial}{\partial\varphi}\right)u_\sigma=\sqrt{\sigma}\,q_0-\lambda_\sigma,$$

where

$$u_\sigma = u_0 + \frac{1}{\sqrt{\sigma}} u_1 + O\left(\frac{1}{\sigma}\right), \qquad \lambda_\sigma = \lambda_0 + O\left(\frac{1}{\sqrt{\sigma}}\right).$$

We obtain

$$h_0 \frac{\partial u_1}{\partial \varphi} = q_0, \tag{5.3}$$

$$h_0 \frac{\partial u_1}{\partial \varphi} + G u_0 = -\mu_0, \tag{5.4}$$

and

$$\lambda_0 = (\mu_0, \nu) \tag{5.5}$$

as described in Theorem 3.2.

A simple calculation gives

$$u_0 = -\frac{1}{2} \log\left(1 - \frac{F}{\gamma_1} \cos^2 \varphi\right)$$

which is well defined in view of assumption (5.1). Similarly μ_0 is defined by

$$\mu_0 = -\frac{\int_0^{2\pi} (1/h_0) G u_0 \, d\varphi}{\int_0^{2\pi} (1/h_0) \, d\varphi}$$

which is the solvability condition for (5.4). Now

$$\int_0^{2\pi} \frac{d\varphi}{h_0} = -\frac{2\pi}{\sqrt{\gamma_1 - F(\xi)}},$$

hence

$$\mu_0 = \frac{1}{4\pi} \sqrt{1 - \frac{F}{\gamma_1}} \int_0^{2\pi} \frac{G(\log(1 - (F/\gamma_1)\cos^2 \varphi))}{1 - (F/\gamma_1)\cos^2 \varphi} \, d\varphi.$$

From (5.5) we get the first approximation to λ_σ

$$\lambda_\sigma = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_M d\xi \frac{\sqrt{1 - F(\xi)/\gamma_1}}{1 - (F(\xi)/\gamma_1)\cos^2 \varphi} G\left(\log\left(1 - \frac{F(\xi)}{\gamma_1} \cos^2 \varphi\right)\right) + O\left(\frac{1}{\sqrt{\sigma}}\right).$$

We repeat the analysis for the rotation number after replacing q_0 by h_0 to obtain our result. The proof of the order of magnitude of the error terms follows from Theorem 3.2 by a cumbersome, but elementary calculation.

THEOREM 5.2. *For $\sigma \to \infty$, $\gamma = \gamma_0 + \gamma_1 \sigma$, $\rho = 1$ and*

$$\gamma_1 < \min F$$

we have $\alpha \equiv 0$ and

$$\lambda = \sqrt{\sigma} \int_M d\xi \sqrt{F(\xi) - \gamma_1} + \int_M d\xi \, G(\log \sin(\varphi + \phi(\xi)))\Big|_{\varphi = \phi(\xi)} + O\left(\frac{1}{\sqrt{\sigma}}\right).$$

Here $\phi(\xi) = \tan^{-1}(\sqrt{-\gamma_1 + F(\xi)}/\sqrt{-\gamma_1})$.

Proof. We put again $\gamma_0 = 0$ and follow the procedure of Theorem 3.2. We want to solve equation (5.2) again with L now given by

$$L = G + \sqrt{\sigma}\left(\sqrt{-\gamma_1} \cos 2\varphi + \frac{F}{\sqrt{-\gamma_1}} \cos^2 \varphi\right) \frac{\partial}{\partial \varphi}. \tag{5.6}$$

The two functions f that we want to use in the right-hand side of (5.2) are

$$f=q=\sqrt{\sigma}\left(\sqrt{-\gamma_1}+\frac{F(\xi)}{2\sqrt{-\gamma_1}}\right)\sin 2\varphi=\sqrt{\sigma}\, q_0$$

and

$$f=h=\sqrt{\sigma}\left(\sqrt{-\gamma_1}\cos 2\varphi+\frac{F(\xi)}{\sqrt{-\gamma_1}}\cos^2\varphi\right)=\sqrt{\sigma}\, h_0.$$

Now the rotation field $h_0(\varphi, \xi)$ vanishes at two points in the interval $0\leqq\varphi\leqq\pi$. But since the invariant density p of $(\varphi(t), \xi(t))$ will build-up in $C\times M$ where C is an interval in $(0, \pi/2)$, only the stable zero given by

$$\phi(\xi)=\tan^{-1}\sqrt{\frac{-\gamma_1+F(\xi)}{-\gamma_1}}, \qquad \xi\in M,$$

has to be taken into account.

We now go to the construction of the expansion taking first $f=\sqrt{\sigma}\, q_0$ so that we have

$$\left(G+\sqrt{\sigma}\, h_0\frac{\partial}{\partial\varphi}\right)u_\sigma=\sqrt{\sigma}\, q_0-\lambda_\sigma.$$

This time we let $\lambda_\sigma=\sqrt{\sigma}\,\lambda_0+\lambda_1+(1/\sqrt{\sigma})\lambda_2+\cdots$. The analogue of (5.3), (5.4) is

$$h_0\frac{\partial u_0}{\partial\varphi}=q_0-\mu_0,$$

$$h_0\frac{\partial u_1}{\partial\varphi}+Gu_0=-\mu_1$$

with μ_0 and μ_1 independent of φ. We will then have

$$\lambda_0=(\mu_0, \nu) \quad\text{and}\quad \lambda_1=(\mu_1, \nu).$$

Since

$$\frac{\partial}{\partial\varphi}(h_0\delta_{\phi(\xi)}(\varphi))=0$$

we determine μ_0 by

$$\int_0^{\pi 2}\delta_{\phi(\xi)}(\varphi)q_0(\varphi, \xi)\, d\varphi=\mu_0(\xi).$$

Hence

$$\mu_0(\xi)=q_0(\phi(\xi), \xi)=\sqrt{-\gamma_1+F(\xi)}.$$

Clearly

$$u_0=\int^{\varphi}\frac{1}{h_0}(q_0-\mu_0)\, d\varphi,$$

and the integral is well defined. We have

$$u_0=\int^{\varphi}\frac{\sin 2\varphi-\sin 2\phi}{\cos 2\varphi-\cos 2\phi}\, d\varphi=-\int^{\varphi}\frac{\cos(\varphi+\phi)}{\sin(\varphi+\phi)}\, d\varphi$$
$$=-\log\sin(\varphi+\phi(\xi)).$$

Combining the above we obtain the expression

$$\lambda_\sigma=\sqrt{\sigma}\int_M d\xi\sqrt{-\gamma_1+F(\xi)}+\int_M d\xi\, G(\log\sin(\varphi+\phi(\xi)))\bigg|_{\varphi=\phi}+O\left(\frac{1}{\sqrt{\sigma}}\right).$$

The order of magnitude of the error term follows from a direct application of Theorem 3.2.

Examples. (i) For $\xi(t)=$ Brownian motion on $M=S^1\cong[-\frac{1}{2},\frac{1}{2}]$ we have $G=\frac{1}{2}\,d^2/d\xi^2$, with periodic boundary conditions. The above theorems give for $\sigma\to\infty$, $\gamma=\gamma_0+\gamma_1\sigma$, $\rho=1$ the following estimates:

For $\gamma_1<\min F$

$$\lambda=\sqrt{\sigma}\int_M d\xi\,\sqrt{F(\xi)-\gamma_1}+\frac{1}{32}\int_M d\xi\left(\frac{F'(\xi)}{\gamma_1-F(\xi)}\right)^2+O\left(\frac{1}{\sqrt{\sigma}}\right),$$

$$\alpha\equiv 0,$$

for $\max F<\gamma_1$

$$\lambda=\frac{1}{32}\int_M d\xi\left(\frac{F'(\xi)}{\gamma_1-F(\xi)}\right)^2+O\left(\frac{1}{\sqrt{\sigma}}\right),$$

$$\alpha=-\sqrt{\sigma}\int_M d\xi\,\sqrt{\gamma_1-F(\xi)}+O\left(\frac{1}{\sqrt{\sigma}}\right).$$

(ii) The results in Theorem 5.1 and 5.2 are also true for $\xi(t)$ a stationary diffusion process on an interval $M=(\alpha,\beta)\subset\mathbb{R}$ given by a stochastic differential equation with smooth coefficients

$$d\xi=a(\xi)\,dt+b(\xi)\,dW.$$

We assume $b(\xi)>0$ in M and that the boundaries are natural. The generator is

$$G=a(\xi)\frac{d}{d\xi}+\frac{1}{2}b(\xi)^2\frac{d^2}{d\xi^2}$$

the invariant probability density $\nu(\xi)$ satisfies $G^*\nu=0$, and our function F is chosen so that

$$E_\nu F(\xi(t))=\int_M F(\xi)\nu(\xi)\,d\xi=0.$$

Then for $\gamma_1<\min F$

$$\lambda=\sqrt{\sigma}\int_M d\xi\,\sqrt{F(\xi)-\gamma_1}+\frac{1}{32}\int_M d\xi\left(\frac{F'(\xi)b(\xi)}{\gamma_1-F(\xi)}\right)^2\nu(\xi)+O\left(\frac{1}{\sqrt{\sigma}}\right),$$

$$\alpha\equiv 0,$$

and for $\max F<\gamma_1$

$$\lambda=\frac{1}{32}\int_M d\xi\left(\frac{F'(\xi)b(\xi)}{\gamma_1-F(\xi)}\right)^2\nu(\xi)+O\left(\frac{1}{\sqrt{\sigma}}\right),$$

$$\alpha=-\sqrt{\sigma}\int_M d\xi\,\sqrt{\gamma_1-F(\xi)\nu(\xi)}+O\left(\frac{1}{\sqrt{\sigma}}\right).$$

We should point out that the case $\sigma\to\infty$, $\gamma=\gamma_0+\gamma_1\sigma$ with $\min F\leqq\gamma_1\leqq\max F$ is somewhat more involved and has been treated only formally so far. We have found that for arbitrary $\gamma_0, \gamma_1 \in \mathbb{R}$

$$\lambda = \sqrt{\sigma} \int_M d\xi \sqrt{(F(\xi) - \gamma_1)_+} + o(\sqrt{\sigma}), \tag{5.7}$$

$$\alpha = -\sqrt{\sigma} \int_M d\xi \sqrt{(F(\xi) - \gamma_1)_-} + o(\sqrt{\sigma}), \tag{5.8}$$

where

$$x_+ = \begin{cases} x & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad x_- = \begin{cases} 0 & \text{if } x \geqq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The expansions (5.7) and (5.8) have been confirmed by numerical simulations.

Figures 3 and 4 show λ and α, resp., as functions of σ for the case $\rho = 1$, $\gamma = \gamma_1 \sigma$ and $\gamma_1 = -5, -1, 0, 1, 5$. We have chosen $F(\xi) = \xi$, $\xi(t) = (2/\pi) \tan^{-1} \eta(t)$, $\eta(t) =$ Ornstein–Uhlenbeck process solving $d\eta = -\frac{1}{2}\eta\, dt + dW$.

6. Fast, slow and white noise analysis. We recall from (2.8) that the invariant probability density p in the definitions of the Lyapunov exponent and rotation number

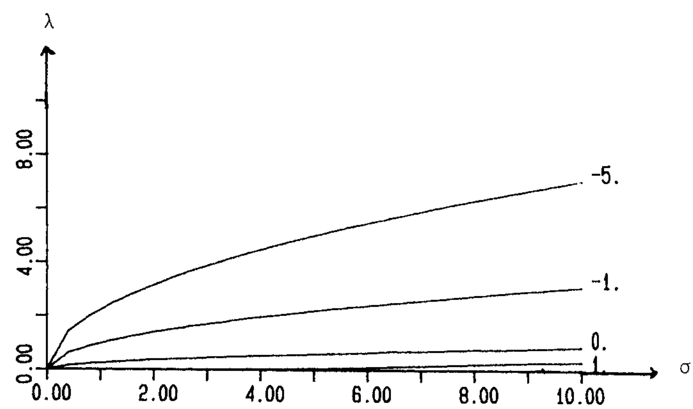

FIG. 3. *Lyapunov exponent λ as a function of σ for $\gamma = \gamma_1 \sigma$, $\lambda_1 = -5, -1, 0, 1, 5$ (the latter not visible).*

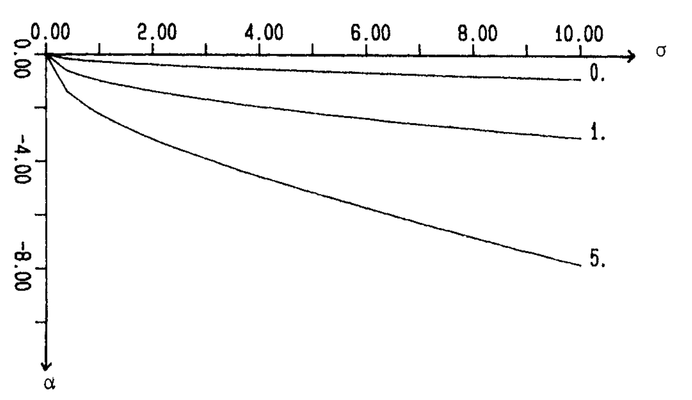

FIG. 4. *Rotation number α as a function of σ for $\gamma = \gamma_1 \sigma$, $\gamma_1 = -5, -1, 0, 1, 5$ (the first two not visible).*

in (2.6) and (2.7) satisfies

$$L^*p=0, \qquad L=\frac{1}{\rho}G+h(\varphi,\xi)\frac{\partial}{\partial\varphi}. \tag{6.1}$$

Therefore p is the solution of

$$Gp-\frac{\partial}{\partial\varphi}(\rho hp)=0$$

with

$$\rho h=\begin{cases}-\rho\sqrt{\gamma}+\rho\sigma\frac{F(\xi)}{\sqrt{\gamma}}\cos^2\varphi, & \gamma>0,\\ \rho\sqrt{-\gamma}\cos 2\varphi+\rho\sigma\frac{F(\xi)}{\sqrt{-\gamma}}\cos^2\varphi, & \gamma<0.\end{cases}$$

The analysis of λ and α as $\rho\to 0$ (fast noise case) or $\rho\to\infty$ (slow noise case) reduces immediately to previously studied cases. Fast noise corresponds to σ small with γ proportional to σ (§ 4). Slow noise corresponds to σ large with γ proportional to σ (§ 5).

For fast noise, the $\rho\to 0$ limit, we simply replace in Corollaries 4.1 and 4.2, $\sqrt{\sigma}$ by ρ, γ_1 by γ, F by σF and λ and α by $\rho\lambda$ and $\rho\alpha$, resp. We obtain, taking into account the form of q in (2.3), the following results.

THEOREM 6.1 (fast noise limit). *For $\rho\to 0$, σ fixed we have*:

(i) *For $\gamma>0$,*

$$\lambda=\rho\frac{\sigma^2\pi}{4\gamma}\hat{f}(0)+O(\rho^2),$$
$$\alpha=-\sqrt{\gamma}+O(\rho^2).$$

(ii) *For $\gamma<0$,*

$$\lambda=\sqrt{-\gamma}+\rho\frac{\sigma^2\pi}{4\gamma}\hat{f}(0)+O(\rho^2),$$
$$\alpha\equiv 0,$$

where $\hat{f}(\omega)$ is again the power-spectrum of $F(\xi(t))$.

For slow noise, the $\rho\to\infty$ limit, we make the same replacements as above in Theorems 5.1 and 5.2. The results are as follows.

THEOREM 6.2 (slow noise limit). *For $\rho\to\infty$, σ fixed we have*:

(i) *For $\gamma>\sigma\max F$,*

$$\lambda=\frac{1}{\rho}\frac{\sqrt{\gamma}}{4\pi}\int_0^{2\pi}d\varphi\int_M d\xi\frac{\sqrt{\gamma-\sigma F(\xi)}}{\gamma-\sigma F(\xi)\cos^2\varphi}G(\log(\gamma-\sigma F(\xi)\cos^2\varphi))+O\left(\frac{1}{\rho^2}\right),$$

$$\alpha=-\int_M d\xi\sqrt{\gamma-\sigma F(\xi)}+O\left(\frac{1}{\rho^2}\right),$$

(ii) *For $\gamma<\sigma\min F$,*

$$\lambda=\int_M d\xi\sqrt{\sigma F(\xi)-\gamma}+\frac{1}{\rho}\int_M d\xi\, G(\log\sin(\varphi+\phi(\xi)))\Big|_{\varphi=\phi(\xi)}+O\left(\frac{1}{\rho^2}\right)$$

$$\alpha\equiv 0,$$

where

$$\phi(\xi)=\tan^{-1}\frac{\sqrt{\sigma F(\xi)-\gamma}}{\sqrt{-\gamma}}.$$

It is clear from the above that no new information about λ and α is obtained by changing the speed of fluctuation of the noise. This is because we are dealing with quantities that are determined by the invariant probability alone.

Another interesting limit is the white noise limit which corresponds to $\rho\to 0$, $\sigma=\delta/\sqrt{\rho}$, $\delta>0$, γ fixed. This time we introduce polar coordinates $y=r\cos\varphi$, $\dot y= r\sin\varphi$ that do not depend on γ. We obtain

$$\dot\varphi = h\left(\varphi(t), \xi\left(\frac{t}{\rho}\right)\right), \qquad r(t)=r_0\exp\int_0^t q\left(\varphi(\tau), \xi\left(\frac{\tau}{\rho}\right)\right)d\tau,$$

where

$$h(\varphi,\xi) = -(\sin^2\varphi+\gamma\cos^2\varphi)+\sigma F(\xi)\cos^2\varphi,$$

$$q(\varphi,\xi)=\tfrac{1}{2}(1-\gamma+\sigma F(\xi))\sin 2\varphi.$$

The infinitesimal generator for $(\varphi(t), \xi(t/\rho))$ is L given in (6.1) with $\rho=\varepsilon^2$ and $\sigma=\delta/\varepsilon$ we have $L_\varepsilon^* p_\varepsilon=0$, where

$$L_\varepsilon=\frac{1}{\varepsilon^2}G+\frac{1}{\varepsilon}\delta F(\xi)\cos^2\varphi\frac{\partial}{\partial\varphi}-(\sin^2\varphi+\gamma\cos^2\varphi)\frac{\partial}{\partial\varphi}.$$

To compute expansions for $\lambda_\varepsilon=(q,p_\varepsilon)$ and $\alpha_\varepsilon=(h,p_\varepsilon)$ for $\varepsilon\to 0$ we follow the expansion scheme of Theorem 3.2 and expand the objects in equation $L_\varepsilon u_\varepsilon=f_\varepsilon-\lambda_\varepsilon$ with $f_\varepsilon=q_\varepsilon$ or h_ε. We omit details of the calculation and immediately state the result.

THEOREM 6.3 (white noise limit). *Let $\rho\to 0$, $\sigma=\delta/\sqrt{\rho}$, $\delta>0$ and γ fixed. Then*

$$\lambda=\bar\lambda+0(\rho), \qquad \alpha=\bar\alpha+0(\rho),$$

where $\bar\lambda$ and $\bar\alpha$ are the Lyapunov exponent and rotation number, resp., of the white noise equation

$$-\ddot y+\delta\sqrt{2\pi\hat f(0)}\,\eta(t)y=\gamma y, \qquad \eta(t)=\text{white noise}.$$

The quantities $\bar\lambda$ and $\bar\alpha$ are given by

$$\bar\lambda=\int_0^{2\pi}\bar q(\varphi)\bar p(\varphi)\,d\varphi, \qquad \bar\alpha=\int_0^{2\pi}\bar h(\varphi)\bar p(\varphi)\,d\varphi.$$

Here

$$\bar q(\varphi)=\delta^2\pi\hat f(0)\cos^2\varphi\cos 2\varphi+\frac{1-\gamma}{2}\sin 2\varphi,$$

$$\bar h(\varphi)=-\delta^2\pi\hat f(0)\cos^2\varphi\sin 2\varphi-(\sin^2\varphi+\cos^2\varphi),$$

$\bar p$ is the unique invariant density of

$$\bar L=\frac{\delta^2\pi\hat f(0)}{2}\cos^2\varphi\frac{d}{d\varphi}\cos^2\varphi\frac{d}{d\varphi}-(\sin^2\varphi+\gamma\cos^2\varphi)\frac{d}{d\varphi},$$

and $\hat f(\omega)$ is the power spectrum of $F(\xi(t))$.

The white noise case was treated by Khasminskii [15]. The number $\bar\lambda$, which depends on γ and δ, was investigated by Kozin and Prodromou [19] and for small δ by Auslender and Mil'shtein [6]. We know that in our case $\bar\lambda>0$ (Kliemann and Arnold [16]). The rotation number $\bar\alpha$ was investigated in detail by Friedman and Pinsky [11].

7. Central limit theorem for λ and α. The Lyapunov exponent λ and rotation number α given by (2.6) and (2.7) are of course the limits

$$\lambda=\lim_{r\to\infty}\frac{1}{t}\int_0^t q\left(\varphi(s),\xi\left(\frac{s}{\rho}\right)\right)ds,$$

$$\alpha=\lim_{t\to\infty}\frac{1}{t}\int_0^t h\left(\varphi(s),\xi\left(\frac{s}{\rho}\right)\right)ds,$$

with probability one. It is of interest to obtain a central limit theorem for the fluctuations

$$\frac{1}{\sqrt{t}}\int_0^t\left(q\left(\varphi(s),\xi\left(\frac{s}{\rho}\right)\right)-\lambda\right)ds$$

and

$$\frac{1}{\sqrt{t}}\int_0^t\left(h\left(\varphi(s),\xi\left(\frac{s}{\rho}\right)\right)-\alpha\right)ds$$

as $t\to\infty$.

More generally, for a smooth function $f(\varphi,\xi)$ such that

$$0=\int_{S^1\times M}f(\varphi,\xi)p(\varphi,\xi)\,d\varphi\,d\xi \tag{7.1}$$

we would like to determine the asymptotic behavior of

$$\frac{1}{\sqrt{t}}\int_0^t f\left(\varphi(s),\xi\left(\frac{s}{\rho}\right)\right)ds. \tag{7.2}$$

Under our hypotheses (H) (see § 2) equation

$$Lu=f \tag{7.3}$$

has a solution $u\in L^2(S^1\times M,\ p\,d\varphi\,d\xi)$ whenever $f\in L^2(S^1\times M,\ p\,d\varphi\,d\xi)$ and (7.1) is satisfied. It is easy to see (see Bhattacharya [7] or [26], for example) that (7.2) converges in distribution to a Gaussian random variable with mean zero and variance V given by

$$V=V(f)=-2(fu,p)=-2\int_{S^1\times M}fup\,d\varphi\,d\xi. \tag{7.4}$$

The asymptotic variance V can also be expressed as

$$V=2\int_0^\infty R(t)\,dt,$$

$R(t)$ being the covariance function of $f(\varphi(t),\xi(t/\rho))$. The latter expression is nonnegative because it is 2π times the power spectrum of $R(t)$ at zero frequency.

We deal briefly with the question of when $V=0$ in our situation.

PROPOSITION 7.1. *Let M be an analytic manifold (such as the torus or the sphere) and let $F(\xi)$ be an analytic function on M with values in $\mathbb{R}$. Then*

(i) $V(q-\lambda)=0$ *if and only if* $F(\xi)\equiv$ *const if and only if* $\lambda=0$.

(ii) $V(h-\alpha)=0$ *if and only if* $F(\xi)\equiv$ *const.*

Proof. (i) Let $V(q-\lambda)=0$ and assume $F(\xi)\not\equiv$ const. Then $(\varphi(t),\xi(t/\rho))$ has a smooth positive density for all $t\geqq T_0$ (see Kliemann and Arnold [16]). A reasoning similar to the one used by Bhattacharya [7], pp. 192–193, yields $q\equiv\lambda$. This can happen if and only if $F(\xi)\equiv(\gamma-1)/\sigma$ which contradicts our assumption $F(\xi)\not\equiv$ const. Thus $V(q-\lambda)=0$ entails $F(\xi)\equiv$ const.

Conversely, if $F(\xi)\equiv\text{const}$ then $p(\varphi,\xi)$ is a product density, and $V(q-\lambda)=0$ can be directly checked.

The fact that $F\not\equiv\text{const}$ if and only if $\lambda>0$ in the analytic situation was proved by Kliemann and Arnold [16].

(ii) is proved analogously. □

Note that in case $F\equiv\text{const}\neq(\gamma-1)/\sigma$ the function $q-\lambda$ is a nonconstant periodic function and yet $V(q-\lambda)=0$. This is an exceptional case of the kind given by Bhattacharya [7, Remark 2.4.2].

We will not try to calculate expansions for $V(q-\lambda)$ and $V(h-\alpha)$ in all cases for which this was done for λ and α. Instead we shall outline one case, the one where $\sigma\to 0$ and $\gamma>0$ (§ 4).

We use the formula (7.4) for V, and the equation (7.3) we must solve is

$$Gu_\sigma-\left(-\sqrt{\gamma}+\sigma\frac{F(\xi)}{\sqrt{\gamma}}\cos^2\varphi\right)\frac{\partial u_\sigma}{\partial\varphi}=\frac{\sigma}{2\sqrt{\gamma}}F(\xi)\sin 2\varphi-\lambda_\sigma,$$

where λ_σ is given by (4.3) in expanded form. Once this is solved asymptotically, then

$$V_\lambda=-2\int_{S^1\times M}u_\sigma\left(\frac{\sigma}{2\sqrt{\gamma}}F(\xi)\sin 2\varphi-\lambda_\sigma\right)p_\sigma\,d\varphi\,d\xi$$

gives the variance of the normal law for the fluctuations in the Lyapunov exponent.

If we look for u_σ in the form $u_\sigma=\sigma u_1+\sigma^2u_2+\cdots$ and use (4.3), we obtain

$$V_\lambda=-2\sigma^2\int_{S^1\times M}u_1(\varphi,\xi)\frac{F(\xi)}{2\sqrt{\gamma}}\sin 2\varphi\frac{d\varphi}{2\pi}\,d\xi+O(\sigma^3),$$

where u_1 solves

$$\left(G-\sqrt{\gamma}\frac{\partial}{\partial\varphi}\right)u_1=\frac{F(\xi)}{2\sqrt{\gamma}}\sin 2\varphi.$$

This gives (compare the proof of Theorem 4.1)

$$V_\lambda=\sigma^2\frac{\pi}{4\gamma}\hat{f}(2\sqrt{\gamma})+O(\sigma^3),$$

with $\hat{f}$ defined by (4.2).

8. Applications to stability. A lot of effort has been devoted to the stability analysis of the damped linear oscillator with random restoring force

$$\ddot{y}+2\beta\dot{y}+(1+\sigma F(\xi(t)))y=0, \tag{8.1}$$

where β and σ are real constants (see e.g. Mitchell and Kozin [22], Arnold and Kliemann [3]). Of course, if we put $y=\bar{y}\exp(-\beta t)$ then $\bar{y}$ satisfies an equation of the form (2.1)

$$-\ddot{\bar{y}}+\sigma\bar{F}(\xi(t))\bar{y}=\gamma\bar{y} \tag{8.2}$$

with $\bar{F}=-F$, $\gamma=1-\beta^2$ and, for simplicity, $\rho=1$. Consequently, the Lyapunov exponents λ of (8.1) and $\bar{\lambda}$ of (8.2) are related by

$$\lambda=\lambda(\beta,\sigma)=-\beta+\bar{\lambda}(\gamma,\sigma). \tag{8.3}$$

(8.1) can serve as a prototypical example since it exhibits a wealth of interesting phenomena such as destabilization and stabilization.

A stability diagram containing the level curves of $\lambda(\beta, \sigma)$ in the (β, σ) parameter plane is shown in Fig. 5. We have used $F(\xi) = \xi$, $\xi(t) =$ Ornstein–Uhlenbeck process. The line $\lambda = 0$ is the dividing line between the parameter region of instability ($\lambda > 0$) and stability ($\lambda < 0$). We will briefly translate and interpret some of our findings for $\bar{\lambda}$ in this context.

Let $C(t)$ and $\hat{f}(\omega)$ be again the covariance function and power spectrum of $F(\xi(t))$, resp.

(i) *Small noise.* For $\sigma \to 0$ and $\beta^2 < 1$ (underdamped case)

$$\lambda = -\beta + \frac{\pi \hat{f}(2\sqrt{1-\beta^2})}{4(1-\beta^2)} \sigma^2 + O(\sigma^3), \tag{8.4}$$

in particular, for $\beta = 0$,

$$\lambda = \frac{\pi}{4} \hat{f}(2) \sigma^2 + O(\sigma^3).$$

For $\sigma \to 0$ and $\beta^2 > 1$ (overdamped case)

$$\lambda = -\beta + \sqrt{\beta^2 - 1} - \frac{\int_0^\infty \exp(-2\sqrt{\beta^2-1}\, t) C(t)\, dt}{4(\beta^2 - 1)} \sigma^2 + O(\sigma^3).$$

This means that in the underdamped case small noise is destabilizing since a positive quantity is added to the deterministic value $\lambda = -\beta$. In contrast to this, small noise is stabilizing in the overdamped case since some positive quantity is subtracted from the deterministic value $\lambda = -\beta + \sqrt{\beta^2 - 1}$.

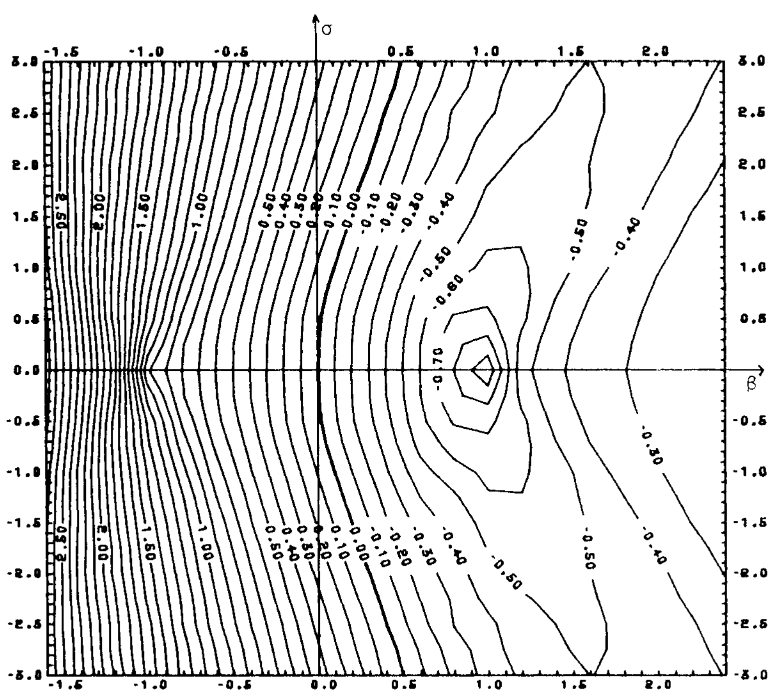

FIG. 5. *Stability diagram for the damped oscillator* (8.1). *The diagram shows level curves of the Lyapunov exponent* λ, *in particular the line* $\lambda = 0$ *which separates the region of stability and instability.*

(ii) *Asymptotic form of the line* $\lambda(\beta, \sigma)=0$. For small σ and β (8.4) yields for $\lambda=0$ approximately

$$\sigma^2=\frac{4}{\pi\hat{f}(2)}\beta.$$

For large σ and β we put $\beta=\sqrt{1-\gamma}\to\infty$ with $\gamma=\gamma_1\sigma$, $\gamma_1<0$, and try to match the leading terms in (8.3) with $\bar{\lambda}$ given by (5.7) which gives

$$\sqrt{-\gamma_1}=E\sqrt{(F(\xi)-\gamma_1)_+}.$$

An elementary calculation shows that

$$g(\gamma_1)=E\sqrt{(F(\xi)-\gamma_1)_+}-\sqrt{-\gamma_1} \tag{8.5}$$

has exactly one zero at $\bar{\gamma}_1$, $\min F<\bar{\gamma}_1<0$, and $g(\gamma_1)$ is negative for $\gamma_1<\bar{\gamma}_1$ and positive for $\gamma_1>\bar{\gamma}_1$. Thus the curve $\lambda=0$ is, for large σ and β, asymptotically described by the parabola

$$\sigma=(-\bar{\gamma}_1)^{-1}\beta^2.$$

(iii) *Asymptotic form of the line* $\lambda(\beta, \sigma)=\min$ *for* β *fixed.* We saw that in the overdamped case ($\beta^2>1$) $\lambda(\beta, \sigma)$ first decreases if σ increases. We ask for the σ which is best possible, i.e. which minimizes the value of $\lambda(\beta, \sigma)$ for given β.

For large σ and β try again $\beta=\sqrt{1-\gamma_1\sigma}$ with $\gamma_1<0$ and find the minimum of (8.3) along this curve, i.e. the minimum of

$$\lambda(\sigma)=-\sqrt{1-\gamma_1\sigma}+\sqrt{\sigma}\,E\sqrt{(F(\xi)-\gamma_1)_+}.$$

This gives

$$\sigma_0=\frac{(E\sqrt{(F(\xi)-\gamma_1)_+})^2}{\gamma_1(\gamma_1+(E\sqrt{(F(\xi)-\gamma_1)_+})^2)}$$

provided $\gamma_1<\bar{\gamma}_1=$ unique zero of function (8.5). In other words for each $\gamma_1<\bar{\gamma}_1$ the parabola $\sigma=(-\gamma_1)^{-1}\beta^2$ intersects the $\lambda=\min$ curve at σ_0. The corresponding β value is uniquely determined from γ_1 by

$$\beta^2=\frac{\gamma_1}{\gamma_1+(E\sqrt{(F(\xi)-\gamma_1)_+})^2}.$$

The minimum value of λ is for large β and σ therefore approximately equal to

$$\lambda_{\min}=(-\beta+\sqrt{\beta^2-1})-\frac{\sqrt{-\gamma_1}-E\sqrt{(F(\xi)-\gamma_1)_+}}{\sqrt{-\gamma_1}}\sqrt{\beta^2-1}.$$

The first term is again the Lyapunov exponent of the undisturbed system, while the second term is the optimal stability gain accomplished by applying noise with intensity $\sigma_0=\sigma_0(\beta)$.

9. Applications to wave propagation and spectra. The primary reason the rotation number is of interest in connection with equation (2.1) is because of its relation to the integrated density of states for (2.1) when it is considered as an operator in $L^2(-\infty, \infty)$

$$Hy=-\ddot{y}+\sigma F\left(\xi\left(\frac{t}{\rho}\right)\right)y=0. \tag{9.1}$$

Consider the operator H on the interval $[-l, l]$ with $y(-l)=y(l)=0$. Call this operator

H_l and let $N_l(\gamma) = 1/2l$ number of eigenvalues of H_l less than γ. It is well known that under our present hypotheses

$$N(\gamma) = \lim_{l\uparrow\infty} N_l(\gamma)$$

exists with probability one. Moreover we have the relation

$$N(\gamma) = -\frac{1}{\pi}\alpha(\gamma) \tag{9.2}$$

where $\alpha(\gamma)$ is the rotation number of (2.1) given by (2.7) (cf. Molčanov [23], [24], Kotani [18]). Relation (9.2) is a consequence of the Sturmian theory for (9.1) that compares different solutions with regard to the location of their zeros.

To the extent that we have analyzed the behavior of α in different asymptotic limits, we have also analyzed the integrated density of (9.2). The expansions given here illustrate primarily the simplicity with which they can be obtained and at the same time proved to be correct.

The Lyapunov exponent plays a very significant role in the follow up of the theorem of Gol'dshied, Molčanov and Pastur [13] which states that the operator H in (9.1) has only point spectrum with probability one. Molčanov [23] has shown that the corresponding eigenfunctions decay exponentially and the decay rate is the Lyapunov exponent. Thus, the Lyapunov exponent is the reciprocal of the localization length in the one-dimensional Schrödinger equation with random potential.

The Lyapunov exponent controls also the exponential decay rate of the energy transmission coefficient of a slab of random medium. We explain briefly this application.

Let $u(x)$ be the wave amplitude at $x \in \mathbb{R}$ of a wave travelling in a one-dimensional random medium. Instead of using the notation (9.1) we write the wave equation in the usual notation

$$u_{xx} + k^2 n^2(x) u = 0 \tag{9.3}$$

in which $n(x)$ is the assured random, refractive index of the medium which occupies the interval $0 < x < L$. We denote by k the wave number of the waves in vacuum. We assume that outside the random slab $u(x)$ is given by

$$u(x) = \begin{cases} e^{ikx} + R\,e^{-ikx}, & x < 0, \\ T\,e^{ik(x-L)}, & x > L. \end{cases}$$

We then require that $u(x)$ and $u_x(x)$ be continuous at $x = 0$ and $x = L$. The complex-valued random variables R and T are the reflection and transmission coefficients, resp. They depend of course on the random refractive index $n(x)$, the wave number k and the slab width L. Since (9.3) is real, we always have conservation of energy flux

$$|T|^2 + |R|^2 = 1.$$

Now it is not difficult to show (Papanicolaou and Keller [27]) that under the hypothesis that

$$n^2(x) = 1 + \sigma F\left(\xi\left(\frac{x}{\rho}\right)\right)$$

with F and ξ as in § 2, we have

$$\lim_{L\uparrow\infty} \frac{1}{L} \log |T|^2 = -2\lambda,$$

where λ is the Lyapunov exponent associated with the random oscillator

$$\ddot{y}+k^2\left(1+\sigma F\left(\xi\left(\frac{t}{\rho}\right)\right)\right)y=0. \tag{9.4}$$

This can be verified by using the fundamental solution matrix of (9.3) to obtain a suitable expression for $|T|^2$.

Clearly $\lambda=\lambda(k)$ in (9.4) and its behavior for both $k\to 0$ and $k\to\infty$ is of interest. The behavior for $k\to 0$ (and $\rho=1$) is given by Corollary 4.1. In the current notation we have that

$$\lambda(k)=k^2\sigma^2\frac{\pi}{4}\hat{f}(0)+O(k^4)\quad \text{for } k\to 0.$$

The large k behavior is given by Theorem 5.1. For $\sigma \min F>-1$ (which can always be achieved for small enough σ) we have for $k\to\infty$

$$\lambda(k)=\lambda_\infty+O\left(\frac{1}{k}\right),$$

where

$$\lambda_\infty=\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_M d\xi\frac{\sqrt{1+\sigma F(\xi)}}{1+\sigma F(\xi)\cos^2\varphi}G(\log(1+\sigma F(\xi)\cos^2\varphi))$$

is a positive constant.

REFERENCES

[1] L. ARNOLD, *Stochastic Differential Equations: Theory and Applications*, Wiley-Interscience, New York, 1974.

[2] ———, *A formula connecting sample and moment stability of linear stochastic systems*, this Journal, 44 (1984), pp. 793-802.

[3] L. ARNOLD AND W. KLIEMANN, *Qualitative theory of stochastic systems*, in Probabilistic Analysis and Related Topics, Vol. 3, A. T. Bharucha-Reid, ed., Academic Press, New York, 1983, pp. 1-79.

[4] L. ARNOLD AND E. OELJEKLAUS, *New results on the stability of linear stochastic systems*, Report No. 130, Forschungsschwerpunkt Dynamische Systeme, Universität Bremen, November, 1984.

[5] L. ARNOLD AND V. WIHSTUTZ (eds.), *Lyapunov exponents*, Proc. Workshop, Bremen, November, 1984, Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[6] E. I. AUSLENDER AND G. N. MIL'SHTEIN, *Asymptotic expansion of the Lyapunov index for linear stochastic systems with small noise*, Probl. Math. Mekh. USSR, 46 (1983), pp. 277-283.

[7] R. N. BHATTACHARYA, *On the functional central limit theorem and the law of the iterated logarithm for Markov processes*, Z. Wahrsch. Verw. Gebiete, 60 (1982), pp. 185-201.

[8] G. BLANKENSHIP AND G. PAPANICOLAOU, *Stability and control of stochastic systems with wide-band noise disturbances* I, this Journal, 34 (1978), pp. 437-476.

[9] J. E. COHEN AND C. M. NEWMAN, *The stability of large random matrices and their products*, Ann. Prob., 12 (1984), pp. 283-310.

[10] A. FRIEDMAN, *Stochastic Differential Equations and Applications*, Academic Press, New York, 1975 (Vol. 1), 1976 (Vol. 2).

[11] A. FRIEDMAN AND M. PINSKY, *Asymptotic stability and spiraling properties for solutions of stochastic equations*, Trans. Amer. Math. Soc., 186 (1973), pp. 331-358.

[12] H. FRISCH AND S. P. LLOYD, *Electron levels in a one-dimensional random lattice*, Phys. Rev., 120 (1960), pp. 1175-1189.

[13] I. J. GOL'DSHEID, S. A. MOLČANOV AND L. A. PASTUR, *A random one-dimensional Schrödinger operator has a pure point spectrum*, Funct. Anal. Appl., 11 (1977), pp. 1-10.

[14] B. I. HALPERIN, *Properties of a particle in a one-dimensional random potential*, Adv. Chem. Phys., 13 (1967), pp. 123-178.

[15] R. Z. KHASMINSKII, *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980 (translation and revision of the Russian edition, Nauka, Moscow, 1969).

[16] W. KLIEMANN AND L. ARNOLD, *Lyapunov exponents of linear stochastic systems*, Report No. 93, Forschungsschwerpunkt Dynamische Systeme, Universität Bremen, June, 1983.

[17] V. I. KLYATSKIN, *Stochastic Equations and Wave Propagation in Random Media*, Nauka, Moscow, 1980. (In Russian.)

[18] S. KOTANI, *Liapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, Proc. Stoch. Analysis, Kyoto, 1982.

[19] F. KOZIN AND S. PRODROMOU, *Necessary and sufficient conditions for almost sure sample stability of linear Itô equations*, this Journal, 21 (1971), pp. 413-424.

[20] M. LAX AND J. C. PHILLIPS, *One-dimensional impurity bands*, Phys. Rev., 110 (1958), pp. 41-49.

[21] K. A. LOPARO AND G. L. BLANKENSHIP, *Almost sure instability of a class of linear stochastic systems with jump process coefficients*, preprint, July 1983.

[22] R. R. MITCHELL AND F. KOZIN, *Sample stability of second order differential equations with wide band noise coefficients*, this Journal, 27 (1974), pp. 571-605.

[23] S. A. MOLČANOV, *The structure of eigenfunctions of one-dimensional unordered structures*, Math. USSR Izvestija, 12 (1978), pp. 69-101. (In Russian.)

[24] ———, *The local structure of the spectrum of the one-dimensional Schrödinger operator*, Comm. Math. Phys., 78 (1981), pp. 429-446.

[25] V. I. OSELEDEC, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., 19 (1968), pp. 197-231.

[26] G. PAPANICOLAOU, *Asymptotic analysis of stochastic equations*, in Studies in Probability Theory, M. Rosenblatt, ed., MAA Studies in Mathematics 18, 1978, pp. 111-179.

[27] G. PAPANICOLAOU AND J. B. KELLER, *Stochastic differential equations with applications to random harmonic oscillators and wave propagation in random media*, this Journal, 21 (1971), pp. 287-305.

[28] L. A. PASTUR, *Spectra of random self-adjoint operators*, Russian Math. Surveys, 28 (1973), pp. 3-64.

[29] R. L. STRATONOVICH, *Topics in the Theory of Random Noise*, Gordon and Breach, New York, 1963.

[30] N. G. VAN KAMPEN, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam, 1981.

[31] V. WIHSTUTZ, *Analytic expansion of the Lyapunov exponent associated to the Schrödinger operator with random potential*, Stoch. Anal. Appl., to appear.