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ASYMPTOTIC ANALYSIS OF THE NAVIER–STOKES EQUATIONS IN THIN DOMAINS

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Dedicated to O. A. Ladyzhenskaya

0. Introduction

We are interested in this article with the Navier–Stokes equations of viscous incompressible fluids in three dimensional thin domains. Let Ω_ε be the thin domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$, where ω is a suitable domain in $\mathbb{R}^2$ and $0 < \varepsilon < 1$.

Our aim is to derive an asymptotic expansion of the strong solution u^ε of the Navier–Stokes equations in the thin domain Ω_ε when ε is small, which is valid uniformly in time. This study should give a better understanding of the global existence results in thin domains obtained previously; see [15]–[17] and [23], [22]. We consider in this work two types of boundary conditions: the Dirichlet-periodic boundary condition and the purely periodic condition. For the first type of boundary condition we derive an asymptotic expansion of the solution u^ε in terms of the solution of the associated Stokes problem. More precisely, we prove that the solution can be written, for ε small, as

$$u^\varepsilon(t) = w^\varepsilon + \overline{u}^\varepsilon \exp\bigg(-\frac{\nu t}{2\varepsilon^2}\bigg), \quad \forall t > 0,$$

where w^ε is the solution of the associated Stokes problem and $\overline{u}^\varepsilon$ is a bounded (in time) function depending on the initial data. We also give a new proof and an improvement of the global existence result obtained in [23].

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For the purely periodic boundary condition case, the asymptotic expansion involves the solution of the 2D-Navier–Stokes equations and a solution of an auxilary Stokes problem with exterior force

$$f^\varepsilon - \frac{1}{\varepsilon}\int_0^\varepsilon f(x_1,x_2,x_3)\,dx_3.$$

More precisely, we prove that the solution can be written, as:

$$u^\varepsilon(t) = w^\varepsilon + u^\varepsilon_{2D}(t) + \overline{u}^\varepsilon \exp\bigg(-\frac{\nu t}{2\varepsilon^2}\bigg), \quad \forall t > 0 \text{ and } \varepsilon \text{ small},$$

where w^ε is the solution of the auxilary Stokes problem, $u^\varepsilon_{2D}(t)$ is the solution of the 2D-Navier–Stokes equations with three components and $\overline{u}^\varepsilon$ is a bounded (in time) function depending on the initial data. The nondimensionalized form of the Navier–Stokes equations (NSE) reads

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega_\varepsilon\times(0,\infty), \tag{0.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega_\varepsilon\times(0,\infty), \tag{0.2}$$

$$u(\,\cdot\,,0) = u_0(\,\cdot\,) \quad \text{in } \Omega_\varepsilon. \tag{0.3}$$

Here $u = (u_1,u_2,u_3)$ is the velocity vector at point x and time t, and $p = p(x,t)$ is the pressure.

Equations (0.1)–(0.3) are supplemented with boundary conditions. We denote the boundary of Ω_ε by $\partial\Omega_\varepsilon = \Gamma_t\cup\Gamma_b\cup\Gamma_l$, where

$$\Gamma_t = \omega\times\{\varepsilon\}, \quad \Gamma_b = \omega\times\{0\} \quad\text{and}\quad \Gamma_l = \partial\omega\times(0,\varepsilon). \tag{0.4}$$

The boundary conditions of interest to us are the mixed Dirichlet-periodic condition, i.e. the Dirichlet boundary condition on $\Gamma_t\cup\Gamma_b$ and the periodic condition on Γ_l, and the purely periodic boundary condition on $\partial\Omega_\varepsilon$, in which case $\omega = (0,l_1)\times(0,l_2)$ and u and p are Ω_ε-periodic, and, for the data

$$\int_{\Omega_\varepsilon} u_0\,dx = \int_{\Omega_\varepsilon} f\,dx = 0.$$

We denote by $H^s(\Omega_\varepsilon)$, $s\in\mathbb{R}$, the Sobolev space constructed on $L^2(\Omega_\varepsilon)$ and $\mathbb{L}^2(\Omega_\varepsilon) = (L^2(\Omega_\varepsilon))^3$, $\mathbb{H}^s(\Omega_\varepsilon) = (H^s(\Omega_\varepsilon))^3$. We also denote by $H^s_0(\Omega_\varepsilon)$ the closure in the space $H^s(\Omega_\varepsilon)$ of $\mathcal{C}^\infty_0(\Omega_\varepsilon)$, the space of infinitely differentiable functions with compact support in Ω_ε.

We need also the following spaces:

$$\dot{\mathbb{H}}^m(\Omega_\varepsilon) = \bigg\{u\in\mathbb{H}^m(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} u\,dx = 0\bigg\}, \tag{0.5}$$

and the spaces $H^m_{\rm per}(\Omega_\varepsilon)$, which are defined with the help of Fourier series; we write

$$u(x) = \sum_{k\in\mathbb{Z}^3} u_k \exp\Big(2ik\cdot\frac{x}{L}\Big), \tag{0.6}$$

with $\overline{u}_k = u_{-k}$ (so that u is real valued) and

$$\frac{x}{L} = \Big(\frac{x_1}{l_1}, \frac{x_2}{l_2}, \frac{x_3}{\varepsilon}\Big), \quad k\cdot\frac{x}{L} = k_1\frac{x_1}{l_1} + k_2\frac{x_2}{l_2} + k_3\frac{x_3}{\varepsilon}.$$

Then, u is in $L^2(\Omega_\varepsilon)$ if and only if

$$|u|^2_{L^2(\Omega_\varepsilon)} = \varepsilon l_1 l_2 \sum_{k\in\mathbb{Z}^3} |u_k|^2 < \infty,$$

and u is said to be in $H^s_{\rm per}(\Omega_\varepsilon)$, $s\in\mathbb{R}_+$, if and only if

$$\sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |u_k|^2 < \infty.$$

For the mathematical setting of the Navier–Stokes equations, we classically consider a Hilbert space H_ε, which is a closed subspace of $\mathbb{L}^2(\Omega_\varepsilon)$. Depending on the boundary condition, we define the following:

$$H_P = H_P^\varepsilon = \Big\{u\in\mathbb{L}^2(\Omega_\varepsilon):\ \operatorname{div} u = 0,\ \int_{\Omega_\varepsilon} u\,dx = 0,\ u_j \text{ is periodic in the direction } x_j,\ j=1,2,3\Big\}$$

in the case of the purely periodic boundary condition, and

$$H_{DP} = H_{DP}^\varepsilon = \Big\{u\in\mathbb{L}^2(\Omega_\varepsilon):\ \operatorname{div} u = 0,\ u_3 = 0 \text{ on } \Gamma_t\cup\Gamma_b,\ \int_{\Omega_\varepsilon} u_\alpha\,dx = 0, \text{ and } u_\alpha \text{ is periodic in the direction } x_\alpha,\ \alpha=1,2\Big\},$$

in the case of the mixed Dirichlet-periodic boundary condition.

Another useful space is V_ε, a closed subspace of $\mathbb{H}^1(\Omega_\varepsilon)$, which is defined as follows depending on the boundary condition:

$$\begin{aligned} V_P = V_P^\varepsilon &= \big\{u\in\dot{\mathbb{H}}^1_{\rm per}(\Omega_\varepsilon):\ \operatorname{div} u = 0\big\},\\ V_{DP} = V_{DP}^\varepsilon &= \big\{u\in\mathbb{H}^1(\Omega_\varepsilon)\cap H_{DP}:\ u = 0 \text{ on } \Gamma_t\cup\Gamma_b \\ &\qquad \text{and } u \text{ is periodic in the directions } x_1 \text{ and } x_2\big\}, \end{aligned}$$

The scalar product on H_ε is denoted by $(\,\cdot\,,\,\cdot\,)_\varepsilon$, the one on V_ε is denoted by $((\,\cdot\,,\,\cdot\,))_\varepsilon$, and the associated norms are denoted by $|\cdot|_\varepsilon$ and $||\cdot||_\varepsilon$ respectively. We denote by A_ε the Stokes operator defined as an isomorphism from V_ε onto the dual V'_ε of V_ε, by

$$\langle A_\varepsilon u, v\rangle_{V'_\varepsilon, V_\varepsilon} = ((u,v))_\varepsilon, \quad \forall v\in V_\varepsilon. \tag{0.7}$$

The operator A_ε is extended to H_ε as a linear unbounded operator. The domain of A_ε in H_ε is denoted by $D(A_\varepsilon)$. The space $D(A_\varepsilon)$ can be fully characterized using the regularity theory. We refer for the study of the regularity of the Stokes operator to [2], [6], [10], [12], [18]–[20] and [24].

Let b_ε be the continuous trilinear form on V_ε defined by

$$b_\varepsilon(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega_\varepsilon} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u,v,w \in V_\varepsilon. \tag{0.8}$$

We denote by B_ε the bilinear form on V_ε defined for $(u,v) \in V_\varepsilon \times V_\varepsilon$ by

$$\langle B_\varepsilon(u,v), w\rangle_{V'_\varepsilon, V_\varepsilon} = b_\varepsilon(u,v,w), \quad \forall w \in V_\varepsilon,$$

and we set $B_\varepsilon(u) = B_\varepsilon(u,u)$.

We assume in this article that the data ν, u_0 and f satisfy

$$\nu > 0, \quad u_0 \in H_\varepsilon \text{ (or } V_\varepsilon), \quad f \in L^\infty(0,\infty; H_\varepsilon). \tag{0.9}$$

The system of equations (0.1)–(0.3), with one of the boundary conditions listed above, can be written as a differential equation in V'_ε

$$\begin{cases} u' + \nu A_\varepsilon u + B_\varepsilon(u) = f, \\ u(0) = u_0, \end{cases} \tag{0.10}$$

where u' denotes the derivative (in the distribution sense) of the function u with respect to time. We recall now the classical result of existence of solutions to problem (0.10). See e.g. [4], [9], [10], [14], [19], [20].

Theorem 0.1. *For $u_0 \in H_\varepsilon$, there exists a solution* (*not necessarily unique*) *$u = u_\varepsilon$ to problem* (0.10) *such that*

$$u_\varepsilon \in L^2(0,T;V_\varepsilon) \cap L^\infty(0,T;H_\varepsilon), \quad \forall T > 0. \tag{0.11}$$

Moreover, if $u_0 \in V_\varepsilon$, then there exists $T_\varepsilon = T_\varepsilon(\Omega_\varepsilon, \nu, u_0, f) > 0$ and a unique solution u_ε to problem (0.10) *such that*

$$u_\varepsilon \in L^2(0,T_\varepsilon; D(A_\varepsilon)) \cap L^\infty(0,T_\varepsilon; V_\varepsilon). \tag{0.12}$$

The solution u_ε which satisfies (0.12) is called the strong solution of (0.10).

1. Functional inequalities in thin domains

In this section we present some functional inequalities in thin domains. We will only state the inequalities without proofs and we refer the reader to [23] for a detailed discussion. The functional inequalities considered here are Sobolev-type inequalities and the Cattabriga–Solonnikov regularity inequality for the Stokes operator. We should mention that in the classical Sobolev inequalities, the constants are dilation invariant but do, however, depend on the shape of the domain, i.e., in our case the thickness ε. The significance of the inequalities given below lies in the exact dependence of the constants on ε .

First we introduce some notations. For a scalar function $\varphi \in L^2(\Omega_\varepsilon)$, we define its average in the thin direction as follows

$$(M_\varepsilon \varphi)(x_1, x_2) = \frac{1}{\varepsilon}\int_0^\varepsilon \varphi(x_1, x_2, s)\, ds, \tag{1.1}$$

and we set

$$N_\varepsilon \varphi = \varphi - M_\varepsilon \varphi, \quad \text{i.e. } M_\varepsilon + N_\varepsilon = I_{L^2(\Omega_\varepsilon)}, \tag{1.2}$$

where $I_{L^2(\Omega_\varepsilon)}$ is the identity operator on $L^2(\Omega_\varepsilon)$. For $u = (u_1, u_2, u_3) \in \mathbb{L}^2(\Omega_\varepsilon)$, we write $M_\varepsilon u = (M_\varepsilon u_1, M_\varepsilon u_2, M_\varepsilon u_3)$ and we set

$$N_\varepsilon u = u - M_\varepsilon u, \quad \text{i.e. } M_\varepsilon + N_\varepsilon = I_{\mathbb{L}^2(\Omega_\varepsilon)}. \tag{1.3}$$

- The Poincaré inequalities:

$$\begin{cases} |u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon \left| \dfrac{\partial u}{\partial x_3} \right|_{L^2(\Omega_\varepsilon)} & \forall u \in V^\varepsilon_{DP}, \\ |u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon^2 |A_\varepsilon u| & \forall u \in D(A_{\varepsilon DP}), \end{cases} \tag{1.4}$$

$$\begin{cases} |N_\varepsilon u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon \left| \dfrac{\partial N_\varepsilon u}{\partial x_3} \right|_{L^2(\Omega_\varepsilon)} & \forall u \in V^\varepsilon_{P}, \\ |N_\varepsilon u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon^2 |A_\varepsilon N_\varepsilon u|_\varepsilon & \forall u \in D(A_{\varepsilon P}). \end{cases} \tag{1.5}$$

- Ladyzhenskaya's inequalities: There exists a positive constant c_0, independent of ε, such that

$$\big|u\big|^2_{L^6(\Omega_\varepsilon)} \leq c_0 ||u||^2_\varepsilon \qquad \forall u \in V^\varepsilon_{DP}, \tag{1.6}$$

$$\big|N_\varepsilon u\big|^2_{L^6(\Omega_\varepsilon)} \leq c_0 \|N_\varepsilon u\|^2_\varepsilon \quad \forall u \in V^\varepsilon_{P}. \tag{1.7}$$

For $2 \leq q \leq 6$, there exists a positive constant $c(q)$, independent of ε, such that

$$\big|u\big|^2_{L^q(\Omega_\varepsilon)} \leq c(q)\varepsilon^{(6-q)/q} ||u||^2_\varepsilon \qquad \forall u \in V^\varepsilon_{DP}. \tag{1.8}$$

$$\big|N_\varepsilon u\big|^2_{L^q(\Omega_\varepsilon)} \leq c(q)\varepsilon^{(6-q)/q} \|N_\varepsilon u\|^2_\varepsilon \quad \forall u \in V^\varepsilon_{P}. \tag{1.9}$$

- Agmon's inequality: There exists a positive constant $c_0(\omega)$, independent of ε, such that

$$\big|u\big|_{L^\infty(\Omega_\varepsilon)} \leq c_0 \big|u\big|^{1/4}_{L^2(\Omega_\varepsilon)} \left(\sum_{i,j=1}^{3} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{L^2(\Omega_\varepsilon)} \right)^{3/4} \quad \forall u \in D(A_{\varepsilon DP}), \tag{1.10}$$

$$\big|N_\varepsilon u\big|_{L^\infty(\Omega_\varepsilon)} \leq c_0 \big|N_\varepsilon u\big|^{1/4}_{L^2(\Omega_\varepsilon)} \left(\sum_{i,j=1}^{3} \left| \frac{\partial^2 N_\varepsilon u}{\partial x_i \partial x_j} \right|_{L^2(\Omega_\varepsilon)} \right)^{3/4} \quad \forall u \in D(A_{\varepsilon P}). \tag{1.11}$$

- Cattabriga–Solonnikov inequality: There exists a positive constant $c_0(\omega)$, independent of ε, such that

$$\sum_{i,j=1}^{3}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^2_{L^2(\Omega_\varepsilon)} \le c_0 \left|A_\varepsilon u\right|^2_{L^2(\Omega_\varepsilon)}, \quad \forall u \in D(A_\varepsilon).$$

2. The Dirichlet-periodic boundary condition

In this section we derive an asymptotic expansion of the solution u_ε of the Navier–Stokes equations in the thin domains Ω_ε, when ε goes to zero. The boundary condition under consideration is the mixed Dirichlet-periodic condition. It is shown in [23] that the H^1-norm of u_ε converges to zero when ε goes to zero. Hence, one expects, in this case, a slow motion of the fluid. Our purpose in this section is to establish rigourously that the fluid has slow motion and to find the leading term. For this purpose, we first compare the solution of the nonlinear stationary problem to the solution of the Stokes problem (the linear problem). Then, we compare the solution of the evolutionary problem to the solution of the nonlinear stationary problem. This yields an asymptotic expression of the solution u_ε when ε is small.

2.1. Comparison between the nonlinear stationary problem and the Stokes problem. Consider the steady state Navier–Stokes equations in the thin domain Ω_ε

$$-\nu\Delta v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon + \nabla q^\varepsilon = f^\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{2.1}$$

$$\operatorname{div} v^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \tag{2.2}$$

$$v^\varepsilon = 0 \quad \text{on } \omega \times \{0, \varepsilon\}, \tag{2.3}$$

$$v^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2. \tag{2.4}$$

First, note using (1.4), that

$$\nu\left|A_\varepsilon^{1/2} v^\varepsilon\right|^2_\varepsilon = (f^\varepsilon, v^\varepsilon) \le |f^\varepsilon|_\varepsilon |v^\varepsilon|_\varepsilon \le \varepsilon |f^\varepsilon|_\varepsilon \left|A_\varepsilon^{1/2} v^\varepsilon\right|_\varepsilon. \tag{2.5}$$

Hence

$$\left|A_\varepsilon^{1/2} v^\varepsilon\right|^2_\varepsilon \le \varepsilon^2 |f^\varepsilon|^2_\varepsilon / \nu^2. \tag{2.6}$$

Let w^ε be the unique solution of the Stokes problem:

$$-\nu\Delta w^\varepsilon + \nabla \overline{q}^\varepsilon = f^\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{2.7}$$

$$\operatorname{div} w^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \tag{2.8}$$

$$w^\varepsilon = 0 \quad \text{on } \omega \times \{0, \varepsilon\}, \tag{2.9}$$

$$w^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2. \tag{2.10}$$

We note that

$$\big|A_\varepsilon^{1/2} w^\varepsilon\big|_\varepsilon^2 \leq \varepsilon^2 |f^\varepsilon|_\varepsilon^2/\nu^2. \tag{2.11}$$

Now we write the equations satisfied by $V^\varepsilon = v^\varepsilon - w^\varepsilon$ and $Q^\varepsilon = q^\varepsilon - \overline{q}^\varepsilon$. We have

$$\begin{cases} -\nu\Delta V^\varepsilon + (V^\varepsilon\cdot\nabla)V^\varepsilon + \nabla Q^\varepsilon \\ \qquad\qquad = -(w^\varepsilon\cdot\nabla)V^\varepsilon - (V^\varepsilon\cdot\nabla)w^\varepsilon - (w^\varepsilon\cdot\nabla)w^\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} V^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{2.12}$$

and the boundary condition

$$\begin{cases} V^\varepsilon = 0 \quad \text{on } \omega\times\{0,\varepsilon\}, \\ V^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2. \end{cases} \tag{2.13}$$

We multiply (2.12) with V^ε, integrate over Ω_ε and obtain

$$\nu\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2 = -\int_{\Omega_\varepsilon}(V^\varepsilon\cdot\nabla)w^\varepsilon\cdot V^\varepsilon\,dx - \int_{\Omega_\varepsilon}(w^\varepsilon\cdot\nabla)w^\varepsilon\cdot V^\varepsilon\,dx, \tag{2.14}$$

and with

$$\begin{aligned} \left|\int_{\Omega_\varepsilon}(V^\varepsilon\cdot\nabla)w^\varepsilon\cdot V^\varepsilon\,dx\right| &\leq |V^\varepsilon|_{\mathbb{L}^4(\Omega_\varepsilon)}^2 |A_\varepsilon^{1/2}w^\varepsilon|_\varepsilon \\ &\leq c_0\varepsilon^{1/2}|A_\varepsilon^{1/2}V^\varepsilon|_\varepsilon^2|A_\varepsilon^{1/2}w^\varepsilon|_\varepsilon \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \left|\int_{\Omega_\varepsilon}(w^\varepsilon\cdot\nabla)w^\varepsilon\cdot V^\varepsilon dx\right| &\leq \big|w^\varepsilon\big|_{\mathbb{L}^4(\Omega_\varepsilon)}\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon\big|V^\varepsilon\big|_{\mathbb{L}^4(\Omega_\varepsilon)} \\ &\leq c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon^2\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon, \end{aligned} \tag{2.16}$$

we have

$$\begin{aligned} \nu\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2 \leq\, & c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon \\ & + c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon^2\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon \\ \leq\, & c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon + \nu\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2/2 \\ & + c_0^2\,\varepsilon\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon^4/2\nu. \end{aligned} \tag{2.17}$$

Let R_0 be a positive function defined on $\mathbb{R}_+$ and satisfying

$$\lim_{\varepsilon\to 0}\varepsilon R_0^2(\varepsilon) = 0 \tag{2.18}$$

and choose ε_1 such that, for $0<\varepsilon\leq\varepsilon_1$

$$c_0\varepsilon^{1/2}R_0(\varepsilon) \leq \nu/16. \tag{2.19}$$

Assume also (see (2.36)) that

$$\varepsilon^2|f^\varepsilon|_\varepsilon^2 \leq R_0^2(\varepsilon)/\nu^2. \tag{2.20}$$

We then infer from (2.11) and (2.17) that

$$\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2 \le 2c_0^2\varepsilon R_0^2(\varepsilon)\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon^2/\nu^2. \tag{2.21}$$

Thanks to (2.18) and (2.21), $\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2$ is negligeable compared to $\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon^2$ for ε small. We have proved the

Lemma 2.1. *Let w^ε (resp. v^ε) be the solution of the Stokes problem (resp. the nonlinear stationary Navier–Stokes equations) in the thin domain Ω_ε. Assume that* (2.18)–(2.20) *hold. Then we can write $v^\varepsilon = w^\varepsilon + V^\varepsilon$, with V^ε small compared to w^ε, i.e.,*

$$\lim_{\varepsilon\to 0}\frac{\big|A_\varepsilon^{1/2}V^\varepsilon\big|_\varepsilon^2}{\big|A_\varepsilon^{1/2}w^\varepsilon\big|_\varepsilon^2} = 0. \tag{2.22}$$

2.2. Comparison between the evolutionary and the stationary problems. In this subsection we prove the global existence of the strong solution $u^\varepsilon(t)$ for ε small and show that up to a time boundary layer near $t = 0$, the solution converges exponentially (in time) to a stationary solution of the Navier-Stokes equations. We also show that the convergence, when ε goes to zero, is exponential as long as the initial data belongs to a ball in H^1 with radius less than $\nu/(16c_0\varepsilon^{1/2})$ and center v^ε, a solution of the stationary problem.

Let $U^\varepsilon(t) = u^\varepsilon(t) - v^\varepsilon$. The equations satisfied by $U^\varepsilon(t)$ are:

$$\begin{cases} \dfrac{\partial U^\varepsilon}{\partial t} - \nu\Delta U^\varepsilon + (U^\varepsilon\cdot\nabla)U^\varepsilon + (U^\varepsilon\cdot\nabla)v^\varepsilon & \\ \qquad + (v^\varepsilon\cdot\nabla)U^\varepsilon + \nabla(p^\varepsilon - q^\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ \operatorname{div} U^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ U^\varepsilon = 0 & \text{on } \omega\times\{0,\varepsilon\}, \\ U^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2, & \end{cases} \tag{2.23}$$

and the initial condition reads

$$U^\varepsilon(0) = u_0^\varepsilon - v^\varepsilon. \tag{2.24}$$

Using equations (2.23), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 + \nu\big|A_\varepsilon U^\varepsilon(t)\big|_\varepsilon^2 \\ &\qquad \le \big|b(U^\varepsilon,U^\varepsilon,A_\varepsilon U^\varepsilon)\big| + \big|b(U^\varepsilon,v^\varepsilon,A_\varepsilon U^\varepsilon)\big| + \big|b(v^\varepsilon,U^\varepsilon,A_\varepsilon U^\varepsilon)\big|, \end{aligned} \tag{2.25}$$

and with inequalities (1.4), (1.8) and (1.10), we can write

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 + \frac{\nu}{2}\big|A_\varepsilon U^\varepsilon(t)\big|_\varepsilon^2 \\ &\qquad \le c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon\big|A_\varepsilon U^\varepsilon(t)\big|_\varepsilon^2 + c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}v^\varepsilon\big|_\varepsilon\big|A_\varepsilon U^\varepsilon(t)\big|_\varepsilon^2. \end{aligned} \tag{2.26}$$

Hence,

$$\frac{d}{dt}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 + \big[\nu - 2c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon - 2c_0\varepsilon^{1/2}\big|A_\varepsilon^{1/2}v^\varepsilon\big|_\varepsilon\big]\big|A_\varepsilon U^\varepsilon(t)\big|_\varepsilon^2 \le 0. \tag{2.27}$$

With R_0 defined as in (2.18), (2.19), we supplement (2.20) by assuming that

$$\big|A_\varepsilon^{1/2}U_0^\varepsilon\big|_\varepsilon^2 + \varepsilon^2|f^\varepsilon|_\varepsilon^2/\nu^2 \le R_0^2(\varepsilon). \tag{2.28}$$

Then there exists $T(\varepsilon) > 0$ such that

$$\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 \le 4R_0^2(\varepsilon) \quad \text{for } 0 \le t \le T(\varepsilon). \tag{2.29}$$

Let $\big[0, T(\varepsilon)\big)$ denote the maximal interval on which (2.29) holds. Note that if $T(\varepsilon) < \infty$, then

$$\big|A_\varepsilon^{1/2}U^\varepsilon(T(\varepsilon))\big|_\varepsilon^2 = 4\,R_0^2(\varepsilon). \tag{2.30}$$

We infer from (2.6) and (2.29) that

$$\frac{d}{dt}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 + \big[\nu - 16c_0\varepsilon^{1/2}R_0(\varepsilon)\big]\big|A_\varepsilon U^\varepsilon(t)\big|_\varepsilon^2 \le 0, \quad 0 \le t \le T(\varepsilon). \tag{2.31}$$

Using (2.19) we see that for $0 < \varepsilon \le \varepsilon_1$ and $0 \le t \le T(\varepsilon)$, we have by the Poincaré inequality

$$\frac{d}{dt}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 + \frac{\nu}{2\varepsilon^2}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 \le 0, \tag{2.32}$$

which implies that $\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2$ is decreasing as a function of t and therefore $T(\varepsilon) = +\infty$, for $\varepsilon \le \varepsilon_1$. Moreover, we have

$$\begin{aligned}\big|A_\varepsilon^{1/2}U^\varepsilon(t)\big|_\varepsilon^2 &\le \big|A_\varepsilon^{1/2}U_0^\varepsilon\big|_\varepsilon^2 \exp\Big(-\frac{\nu t}{2\varepsilon^2}\Big)\\ &\le \big|A_\varepsilon^{1/2}u_0^\varepsilon - A_\varepsilon^{1/2}v^\varepsilon\big|_\varepsilon^2 \exp\Big(-\frac{\nu t}{2\varepsilon^2}\Big).\end{aligned} \tag{2.33}$$

Finally, we write

$$u^\varepsilon(t) = v^\varepsilon + U^\varepsilon(t) = w^\varepsilon + V^\varepsilon + U^\varepsilon(t), \tag{2.34}$$

where w^ε is the unique solution of the Stokes problem with exterior force f^ε, and V^ε and $U^\varepsilon(t)$ satisfy

$$\big|\big|V^\varepsilon\big|\big|_\varepsilon^2 \le 2c_0^2\varepsilon\big|\big|w^\varepsilon\big|\big|_\varepsilon^4/\nu^2 \tag{2.35}$$

and

$$\big|\big|U^\varepsilon(t)\big|\big|_\varepsilon^2 \le \big|\big|u_0^\varepsilon - w^\varepsilon - V^\varepsilon\big|\big|_\varepsilon^2 \exp\Big(-\frac{\nu t}{2\varepsilon^2}\Big), \quad t \ge 0$$

Theorem 2.2. *Let $R_0(\varepsilon)$ be a monotone positive function satisfying condition $\lim_{\varepsilon\to 0} \varepsilon R_0^2(\varepsilon) = 0$. Assume that v^ε is a solution of the stationary Navier–Stokes equations with exterior force f^ε in the domain Ω_ε, and*

$$\big|\big|u_0^\varepsilon - v^\varepsilon\big|\big|_\varepsilon^2 + \varepsilon^2\big|f^\varepsilon\big|_\varepsilon^2/\nu^2 \le R_0^2(\varepsilon). \tag{2.36}$$

Then there exists $\varepsilon_1 = \varepsilon_1(\nu)$ such that for $0 < \varepsilon \le \varepsilon_1$, the maximal time $T(\varepsilon)$ of existence of the strong solution $u_\varepsilon(t)$ of the 3D*-Navier–Stokes equations in Ω_ε satisfies $T(\varepsilon) = \infty$, and for all $t \ge 0$*

$$\big|\big|u^\varepsilon(t) - v^\varepsilon\big|\big|_\varepsilon^2 \le \big|\big|u_0^\varepsilon - v^\varepsilon\big|\big|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right). \tag{2.37}$$

Moreover, if w^ε is the unique solution of the Stokes problem with exterior force f^ε, then

$$u^\varepsilon(t) = w^\varepsilon + V^\varepsilon + U^\varepsilon(t), \quad \forall t \ge 0, \tag{2.38}$$

with

$$\big|\big|V^\varepsilon\big|\big|_\varepsilon^2 \le \frac{c_0^2}{2\nu}\,\varepsilon\, R_0^2(\varepsilon)\big|\big|w^\varepsilon\big|\big|_\varepsilon^2 \tag{2.39}$$

and for all $t \ge 0$

$$\big|\big|U^\varepsilon(t)\big|\big|_\varepsilon^2 \le \big|\big|u_0^\varepsilon - v^\varepsilon\big|\big|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right).$$

Remark 2.1. (i) We obtained in Theorem 2.1 an improvement for the global regularity result obtained in [23]. Note that the conditions on the data are given in (2.36); in particular, due to (2.19) u_0^ε can belong to a ball in $\mathbb{H}^1(\Omega_\varepsilon)$ of center v^ε and radius $\nu/(16c_0\varepsilon^{1/2})$.

(ii) We also obtained an asymptotic expansion for the solution $u^\varepsilon(t)$ for ε small which is uniformly valid in time. This asymptotic expansion suggests that the attractor of the dynamical system associated with the Navier–Stokes equation with Dirichlet-periodic boundary condition in the thin domain Ω_ε reduces to the set of stationary solutions, when ε is small enough.

(iii) The solution w^ε to the stationary problem (2.7)–(2.10) which approximates v^ε and hence u^ε, can be itself approximated by a simpler expression, possibly an explicit one. For example, in the case of a pressure driven flow,

$$f^\varepsilon = Pe_1, \tag{2.40}$$

where P is constant (the pressure gradient), then $w^\varepsilon \approx \varphi^\varepsilon e_1$, with

$$\varphi^\varepsilon = Px_3(\varepsilon - x_3)/2\nu. \tag{2.41}$$

Note that since $0 < x_3 < \varepsilon$, φ^ε is of order of ε^2.

3. The purely periodic boundary condition

This section is devoted to the asymptotic study of the solutions $u^\varepsilon(t)$ of the 3D-Navier–Stokes equations, with the purely periodic boundary condition in the thin domains Ω_ε, when the thickness ε goes to zero. We have shown in [23] that the average $M_\varepsilon u^\varepsilon(t)$ converges to the strong solution of the 2D-Navier–Stokes equations. Therefore, one cannot expect to see the slow motion obtained in the case of the Dirichlet–Periodic condition (see Section 2).

The idea here is to establish some a priori estimates for $N_\varepsilon u^\varepsilon(t) = u^\varepsilon(t) - M_\varepsilon u^\varepsilon(t)$, which are similar to those obtained for $u^\varepsilon(t)$ in the case of the Dirichlet-periodic condition, and to show that the dynamics of the 3D-Navier–Stokes equations is roughly carried by the orbits of a 2D-Navier–Stokes system up to the translation by a 3D-vector function which is independent of time, namely the solution of the Stokes problem with exterior force $N_\varepsilon f^\varepsilon = f^\varepsilon - M_\varepsilon f^\varepsilon$.

We recall from [23] the following result: we consider the problem (0.1)–(0.3) with periodic boundary conditions, and we assume that for arbitrary fixed constants K_1 and K_2,

$$a_0^2(\varepsilon) + \alpha^2(\varepsilon) \le K_1 \varepsilon \ln|\ln\varepsilon|, \quad b_0^2(\varepsilon) + \beta^2(\varepsilon) \le K_2 \ln|\ln\varepsilon|, \tag{3.1}$$

where we have set

$$\begin{aligned} a_0(\varepsilon) &= |A_\varepsilon^{1/2} M_\varepsilon u_0^\varepsilon|_\varepsilon, & b_0(\varepsilon) &= |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon, \\ \alpha(\varepsilon) &= |M_\varepsilon f^\varepsilon|_\varepsilon, & \beta(\varepsilon) &= |N_\varepsilon f^\varepsilon|_\varepsilon. \end{aligned}$$

Then there exists $\varepsilon_0 = \varepsilon_0(\nu, K_1, K_2, \omega) > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the maximal time of existence $T(\varepsilon)$ of the strong solution u^ε of the 3D-Navier–Stokes equations with periodic boundary conditions satisfies $T(\varepsilon) = +\infty$, and

$$u^\varepsilon \in \mathcal{C}([0,\infty); V_P^\varepsilon) \cap L^2(0,T; D(A_{\varepsilon P})) \quad \forall T > 0.$$

Moreover, considering a suitable constant $K_3(\nu) > K_1 + K_2$ and setting

$$R_0^2(\varepsilon) = K_3 \ln|\ln\varepsilon| \tag{3.2}$$

we have for all $t \ge 0$

$$|A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 \le \sigma R_0^2(\varepsilon), \tag{3.3}$$

where σ is constant (depending possibly on ν) such that $\sigma > 2$.

3.1. An auxiliary pseudo-stationary problem. We consider $\overline{w}^\varepsilon = N_\varepsilon \overline{w}^\varepsilon$ solution of the following problem

$$\nu A_\varepsilon \overline{w}^\varepsilon + N_\varepsilon B_\varepsilon(N_\varepsilon \overline{w}^\varepsilon + M_\varepsilon u^\varepsilon, N_\varepsilon \overline{w}^\varepsilon + M_\varepsilon u^\varepsilon) - N_\varepsilon B_\varepsilon(M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon) = N_\varepsilon f^\varepsilon. \tag{3.4}$$

Equivalently for all $v \in V_p$, $\overline{w}^\varepsilon \in N_\varepsilon V_P$ satisfies

$$\text{(3.5)}\quad \nu(A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon, A_\varepsilon^{1/2}N_\varepsilon v)_\varepsilon + b_\varepsilon(N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v) + b_\varepsilon(M_\varepsilon u^\varepsilon, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v) + b_\varepsilon(N_\varepsilon\overline{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon v) = (N_\varepsilon f^\varepsilon, N_\varepsilon v)_\varepsilon.$$

Since $u^\varepsilon = u^\varepsilon(t)$, we shall consider the time as a parameter.

The proof of the existence and uniqueness of $\overline{w}^\varepsilon$ is standard (note that the uniqueness holds if we consider ε small enough). We omit the details and will only derive the estimates for $\overline{\omega}^\varepsilon$. For this purpose and throughout this section, we use extensively the following estimates on the trilinear form b_ε

Lemma 3.1. *Let $q \in (0,1/2)$. There exists a positive constant $c_1(q)$, independent of ε, such that:*

$$|b_\varepsilon(M_\varepsilon u, N_\varepsilon v, w)| \le c_1\varepsilon^q|A_\varepsilon^{1/2}M_\varepsilon u|_\varepsilon|A_\varepsilon N_\varepsilon v|_\varepsilon|w|_\varepsilon$$
$$|b_\varepsilon(N_\varepsilon v, M_\varepsilon u, w)| \le c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}M_\varepsilon u|_\varepsilon|A_\varepsilon N_\varepsilon v|_\varepsilon|w|_\varepsilon$$

for all $u \in D(A_\varepsilon^{1/2})$, $v \in D(A_\varepsilon)$, $w \in \mathbb{L}^2(\Omega_\varepsilon)$,

$$\begin{aligned}|b_\varepsilon(N_\varepsilon u, N_\varepsilon v, w)| &\le c_1|A_\varepsilon^{1/2}N_\varepsilon u|_\varepsilon^{1/2}|A_\varepsilon N_\varepsilon u|_\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon v|_\varepsilon|w|_\varepsilon\\ &\le c_1\varepsilon^{1/2}|A_\varepsilon N_\varepsilon u|_\varepsilon|A_\varepsilon^{1/2}N_\varepsilon v|_\varepsilon|w|_\varepsilon\end{aligned}$$

for all $u \in D(A_\varepsilon)$, $v \in D(A_\varepsilon^{1/2})$, $w \in \mathbb{L}^2(\Omega_\varepsilon)$,

$$|b_\varepsilon(N_\varepsilon u, N_\varepsilon v, w)| \le c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon u|_\varepsilon|A_\varepsilon N_\varepsilon v|_\varepsilon|w|_\varepsilon$$

for all $u \in D(A_\varepsilon^{1/2})$, $v \in D(A_\varepsilon)$, $w \in \mathbb{L}^2(\Omega_\varepsilon)$.

This lemma is a slight generalization of Lemma 2.7 in [23]; we omit the details of the proof, which essentially relies on the functional inequalities (1.5), (1.7), (1.9) and (1.11).

Estimates for $\overline{w}^\varepsilon$. We set $N_\varepsilon v = N_\varepsilon\overline{w}^\varepsilon$ in (3.5) and obtain

$$\text{(3.6)}\qquad \nu|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon^2 + b_\varepsilon(N_\varepsilon\overline{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon\overline{w}^\varepsilon) = (N_\varepsilon f^\varepsilon, N_\varepsilon\overline{w}^\varepsilon)_\varepsilon,$$

which by (1.9) leads to

$$\text{(3.7)}\qquad \begin{aligned}&\nu|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon^2\\ &\le |N_\varepsilon f^\varepsilon|_\varepsilon|N_\varepsilon\overline{w}^\varepsilon|_\varepsilon + c|N_\varepsilon\overline{w}^\varepsilon|_{L^4(\Omega_\varepsilon)}^2|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon\\ &\le \frac{\nu}{4}|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon^2 + \frac{2\varepsilon^2}{\nu}|N_\varepsilon f^\varepsilon|_\varepsilon^2 + c\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon^2|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon,\end{aligned}$$

where c is a constant independent of ε.

Now we take into account (3.2) and (3.3) and we obtain the existence of $\varepsilon_1 = \varepsilon_1(\nu, \omega, K_1, K_2)$ such that for $0 < \varepsilon \le \varepsilon_1$,

$$\text{(3.8)}\qquad |A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon^2 \le 4\varepsilon^2|N_\varepsilon f^\varepsilon|_e^2/\nu^2$$

We observe that $|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon$ is small in a sense that we make precise now. Indeed, we differentiate (3.5) with respect to t and we obtain

$$\begin{aligned}(3.9)\qquad &\nu\Big(\frac{d}{dt}A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon, A_\varepsilon^{1/2}N_\varepsilon v\Big)_\varepsilon + b_\varepsilon\Big(\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\Big)\\ &+ b_\varepsilon\Big(N_\varepsilon\overline{w}^\varepsilon, \frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\Big) + b_\varepsilon\Big(\frac{d}{dt}M_\varepsilon u^\varepsilon, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\Big)\\ &+ b_\varepsilon\Big(M_\varepsilon u^\varepsilon, \frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\Big) + b_\varepsilon\Big(\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon v\Big)\\ &+ b_\varepsilon\Big(N_\varepsilon\overline{w}^\varepsilon, \frac{d}{dt}M_\varepsilon u^\varepsilon, N_\varepsilon v\Big) = 0.\end{aligned}$$

For $t>0$ fixed, we set $v = A_\varepsilon^{-1}\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon$ in (3.9) and we obtain

$$\begin{aligned}(3.10)\qquad &\nu\Big|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon\Big|_\varepsilon^2 + b_\varepsilon\Big(A_\varepsilon N_\varepsilon v, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\Big) + b_\varepsilon\Big(N_\varepsilon\overline{w}^\varepsilon, A_\varepsilon N_\varepsilon v, N_\varepsilon v\Big)\\ &+ b_\varepsilon\Big(\frac{d}{dt}M_\varepsilon u^\varepsilon, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\Big) + b_\varepsilon\Big(M_\varepsilon u^\varepsilon, A_\varepsilon N_\varepsilon v, N_\varepsilon v\Big)\\ &+ b_\varepsilon\Big(A_\varepsilon N_\varepsilon v, M_\varepsilon u^\varepsilon, N_\varepsilon v\Big) + b_\varepsilon\Big(N_\varepsilon\overline{w}^\varepsilon, \frac{d}{dt}M_\varepsilon u^\varepsilon, N_\varepsilon v\Big) = 0,\end{aligned}$$

so that, by Lemma 3.1, we obtain

$$\begin{aligned}(3.11)\qquad \nu\left|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon\right|_\varepsilon^2 &\le 2c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon v|_\varepsilon^2\\ &+ 2c_1\varepsilon^{1/2}\left|\frac{d}{dt}M_\varepsilon u^\varepsilon\right|_\varepsilon|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon v|_\varepsilon\\ &+ 2c_1\varepsilon^{q}|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon v|_\varepsilon^2\end{aligned}$$

Using (3.8), (3.2) and (3.3), we deduce that there exists $\varepsilon_2 = \varepsilon_2(\nu,\omega,K_1,K_2)$ such that if $0<\varepsilon\le\varepsilon_2$, then by (3.8)

$$\begin{aligned}(3.12)\qquad \left|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon\right|_\varepsilon &\le \frac{c}{\nu}\varepsilon^{1/2}\left|\frac{d}{dt}M_\varepsilon u^\varepsilon\right|_\varepsilon|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon\\ &\le c(\nu)\varepsilon^{3/2}\left|\frac{d}{dt}M_\varepsilon u^\varepsilon\right|_\varepsilon|N_\varepsilon f^\varepsilon|_\varepsilon.\end{aligned}$$

Now we need to bound $\left|\frac{d}{dt}M_\varepsilon u^\varepsilon\right|_\varepsilon$ in terms of $R_0^2(\varepsilon)$. We have

$$(3.13)\qquad \frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2}u^\varepsilon|_\varepsilon^2 + \nu|A_\varepsilon u^\varepsilon|_\varepsilon^2 + b_\varepsilon(u^\varepsilon,u^\varepsilon,A_\varepsilon u^\varepsilon) = (f^\varepsilon, A_\varepsilon u^\varepsilon)_\varepsilon,$$

and therefore

$$(3.14)\qquad \frac{d}{dt}|A_\varepsilon^{1/2}u^\varepsilon|_\varepsilon^2 + \nu|A_\varepsilon u^\varepsilon|_\varepsilon^2 \le \frac{c}{\nu}|f^\varepsilon|_\varepsilon^2 + \frac{c}{\nu^3}|A_\varepsilon^{1/2}u^\varepsilon|_\varepsilon^6,$$

c being a numerical constant (independent of ε). Let $t_0 > 0$ be an arbitrarly small time. We deduce from (3.1), (3.3) and (3.14) that

$$\nu \int_t^{t+t_0} |A_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \, ds \le c(\nu) \left[R_0^2(\varepsilon) + R_0^6(\varepsilon)\right] t_0, \quad \forall t \ge 0. \tag{3.15}$$

Since $|du^\varepsilon/dt|_\varepsilon \le \nu |A_\varepsilon u^\varepsilon|_\varepsilon + |B_\varepsilon(u^\varepsilon, u^\varepsilon)|_\varepsilon + |f^\varepsilon|_\varepsilon$, a simple computation yields

$$\int_t^{t+t_0} \left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^2 \le c(\nu) R_0^2(\varepsilon)(1 + R_0^2(\varepsilon))^2 t_0, \quad \forall t \ge 0. \tag{3.16}$$

Now we differentiate (0.10) with respect to t and we obtain

$$\frac{d^2 u^\varepsilon}{dt^2} + \nu \frac{d}{dt} A_\varepsilon u^\varepsilon + B_\varepsilon\left(\frac{du^\varepsilon}{dt}, u^\varepsilon\right) + B_\varepsilon\left(u^\varepsilon, \frac{du^\varepsilon}{dt}\right) = 0, \tag{3.17}$$

which leads then to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^2 + \nu \left|A_\varepsilon^{1/2}\left(\frac{du^\varepsilon}{dt}\right)\right|_\varepsilon^2 &\le \left|b_\varepsilon\left(\frac{du^\varepsilon}{dt}, u^\varepsilon, \frac{du^\varepsilon}{dt}\right)\right| \\ &\le c \left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^{1/2} \left|A_\varepsilon^{1/2}\left(\frac{du^\varepsilon}{dt}\right)\right|_\varepsilon^{3/2} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon, \end{aligned} \tag{3.18}$$

c being a numerical constant (independent of ε). We infer from (3.18) that

$$\frac{d}{dt}\left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^2 \le \frac{c}{\nu^3} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^4 \left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^2. \tag{3.19}$$

We apply the uniform Gronwall lemma recalled below (see Lemma 3.2) with

$$y = \left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^2, \quad g = \frac{c}{\nu^3}|A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^4, \quad h = 0.$$

From (3.16) and (3.3), we infer the following estimates (say $t_0 \le 1$)

$$\begin{aligned} \int_t^{t+t_0} g(s)\, ds &\le c(\nu) R_0^4(\varepsilon) t_0 \le c(\nu) R_0^4(\varepsilon), \\ \int_t^{t+t_0} y(s)\, ds &\le c(\nu) R_0^2(\varepsilon)[1 + R_0^2(\varepsilon)]^2 t_0, \end{aligned}$$

so that

$$\left|\frac{du^\varepsilon}{dt}\right|_\varepsilon^2 \le c(\nu) R_0^2(\varepsilon)[1 + R_0^2(\varepsilon)]^2 \exp(c(\nu) R_0^4(\varepsilon)) \tag{3.20}$$

holds for every $t \ge t_0 > 0$. Since $t_0 > 0$ is arbitrarily small and the right hand side of (3.20) is independent of t_0, (3.20) holds for (almost) every $t > 0$. We use (3.20) in (3.12) and we obtain

$$\left|\frac{d}{dt} N_\varepsilon \overline{w}^\varepsilon\right|_\varepsilon \le c(\nu) \varepsilon^{3/2} R_0^2(\varepsilon)[1 + R_0^2(\varepsilon)] \exp(c(\nu) R_0^4(\varepsilon)). \tag{3.21}$$

Taking into account the expression of $R_0^2(\varepsilon)$ given by (3.2) we conclude that, for any arbitrarily small $\gamma > 0$, there exists $c = c(\nu, q, \gamma)$ such that

$$\left|\frac{d}{dt} N_\varepsilon \overline{w}^\varepsilon\right|_\varepsilon \leq c(\nu, q, \gamma)\varepsilon^{3/2-\gamma}, \quad \forall t > 0. \tag{3.22}$$

For the convenience of the reader we recall the uniform Gronwall lemma

Lemma 3.2. *Let g, h, y be three positive locally integrable functions on (t_0, ∞) such that y' is locally integrable on (t_0, ∞), and which satisfy for $t \geq t_0$*

$$\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+r} g(s)\,ds \leq a_1, \quad \int_t^{t+r} h(s)\,ds \leq a_2, \quad \int_t^{t+r} y(s)\,ds \leq a_3,$$

where a_1, a_2, a_3 and r are positive constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1), \quad \forall t \geq t_0.$$

3.2. An auxiliary two dimensional problem. We consider first the following evolutionary Navier–Stokes problem in Ω_ε:

$$\frac{\partial \overline{u}^\varepsilon}{\partial t} - \nu\Delta\overline{u}^\varepsilon + (\overline{u}^\varepsilon \cdot \nabla)\overline{u}^\varepsilon + \nabla\overline{p}_\varepsilon = M_\varepsilon f^\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{3.23}$$

$$\operatorname{div}\overline{u}^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \tag{3.24}$$

$$u^\varepsilon \text{ is periodic in the directions } x_1, x_2 \text{ and } x_3, \tag{3.25}$$

with the initial condition

$$\overline{u}^\varepsilon|_{t=0} = M_\varepsilon u_0^\varepsilon. \tag{3.26}$$

Since the forcing term $M_\varepsilon f^\varepsilon$ and the initial data $M_\varepsilon u_0^\varepsilon$ are independent of x_3, we can show that there exists a unique global strong solution $\overline{u}^\varepsilon(t)$ of this three dimensional problem which is independent of x_3, i.e. $\overline{u}^\varepsilon = M_\varepsilon\overline{u}^\varepsilon$. For that purpose we look for $\overline{u}^\varepsilon = \overline{u}^\varepsilon_{2D} + \overline{u}^\varepsilon_v$, where $\overline{u}^\varepsilon_{2D} = (\overline{u}^\varepsilon_1, \overline{u}^\varepsilon_2, 0)$, $\overline{u}^\varepsilon_v = (0, 0, \overline{u}^\varepsilon_3)$, and $\overline{u}^\varepsilon_{2D}$ is first defined by the following two dimensional problem:

$$\frac{\partial \overline{u}^\varepsilon_{2D}}{\partial t} - \nu\Delta'\overline{u}^\varepsilon_{2D} + (\overline{u}^\varepsilon_{2D} \cdot \nabla')\overline{u}^\varepsilon_{2D} + \nabla'\overline{p}_\varepsilon = M_\varepsilon f^\varepsilon_{2D} \quad \text{in } \omega, \tag{3.27}$$

$$\operatorname{div}'\overline{u}^\varepsilon_{2D} = 0 \quad \text{in } \omega, \tag{3.28}$$

$$\overline{u}^\varepsilon_{2D} \text{ is periodic in the directions } x_1 \text{ and } x_2, \tag{3.29}$$

with the initial condition

$$\overline{u}^\varepsilon_{2D}|_{t=0} = M_\varepsilon(u^\varepsilon_{01}, u^\varepsilon_{02}, 0), \tag{3.30}$$

where Δ', ∇', div' are two-dimensional operators, $f^\varepsilon_{2D} = (f^\varepsilon_1, f^\varepsilon_2, 0)$. Note that $\overline{u}^\varepsilon_{2D}$ depends on ε only because f^ε_{2D} and $M_\varepsilon(u^\varepsilon_{01}, u^\varepsilon_{02}, 0)$ depend on ε. We then define $\overline{u}^\varepsilon_v$ as the solution of the two-dimensional problem

$$\frac{\partial \overline{u}^\varepsilon_v}{\partial t} - \nu\Delta'\overline{u}^\varepsilon_v + (\overline{u}^\varepsilon_{2D}\cdot\nabla')\overline{u}^\varepsilon_v = M_\varepsilon f^\varepsilon_3 \vec{e}_3 \qquad \text{in } \omega, \tag{3.31}$$

$$\int_\omega \overline{u}^\varepsilon_v \, dx' = 0, \tag{3.32}$$

$$\overline{u}^\varepsilon_v \text{ is periodic in the directions } x_1 \text{ and } x_2, \tag{3.33}$$

with the initial condition

$$\overline{u}^\varepsilon_v|_{t=0} = M_\varepsilon u^\varepsilon_{03}\vec{e}_3. \tag{3.34}$$

The proof of the existence and uniqueness of $\overline{u}^\varepsilon_{2D}$ is classical, $\overline{u}^\varepsilon_{2D}$ is the global strong solution of a 2D-Navier–Stokes problem [9], [10]. Then we solve the linear problem for $\overline{u}^\varepsilon_v$; it is then easy to verify that $\overline{u}^\varepsilon = \overline{u}^\varepsilon_{2D} + \overline{u}^\varepsilon_v$ is a strong global solution of (3.23)–(3.26).

Estimates for $\overline{u}^\varepsilon$ in $L^2(\omega)$. First we multiply (3.27) by $\overline{u}^\varepsilon$, integrate over ω and obtain

$$\frac{1}{2}\frac{d}{dt}|\overline{u}^\varepsilon|^2_{L^2(\omega)} + \nu|\widetilde{A}^{1/2}\overline{u}^\varepsilon|^2_{L^2(\omega)} = (M_\varepsilon f^\varepsilon, \overline{u}^\varepsilon)_{L^2(\omega)}, \tag{3.35}$$

where $\widetilde{A}$ is the 2D-Stokes operator in ω. Thus

$$\frac{d}{dt}|\overline{u}^\varepsilon|^2_{L^2(\omega)} + \nu|\widetilde{A}^{1/2}\overline{u}^\varepsilon|^2_{L^2(\omega)} \le \frac{1}{\nu\lambda_1}|M_\varepsilon f^\varepsilon|^2_{L^2(\omega)}, \tag{3.36}$$

λ_1 being the first eigenvalue of $\widetilde{A}$. We deduce that for all $t \ge 0$

$$\begin{aligned}\int_t^{t+1} |\widetilde{A}^{1/2}\overline{u}^\varepsilon(s)|^2_{L^2(\omega)}\, ds \le \frac{1}{\nu^2\lambda_1}|M_\varepsilon f^\varepsilon|^2_{L^2(\omega)} + \frac{1}{\nu\lambda_1}|M_\varepsilon f^\varepsilon|^2_{L^2(\omega)} \\ + |\overline{u}^\varepsilon(0)|^2_{L^2(\omega)}\exp(-\nu\lambda_1 t),\end{aligned} \tag{3.37}$$

and taking into account (3.1), we obtain

$$\begin{aligned}\int_t^{t+1} |\widetilde{A}^{1/2}\overline{u}^\varepsilon(s)|^2_{L^2(\omega)}\, ds &\le c(\nu)[|\widetilde{A}^{1/2}M_\varepsilon u^\varepsilon_0|^2_{L^2(\omega)} + |M_\varepsilon f^\varepsilon|^2_{L^2(\omega)}] \\ &\le c(\nu)K_1 \ln|\ln\varepsilon|.\end{aligned} \tag{3.38}$$

Estimates for $\overline{u}^\varepsilon_{2D}$ in $H^1(\omega)$. We multiply (3.27) by $\widetilde{A}\overline{u}^\varepsilon_{2D}$, integrate over ω and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)} + \nu|\widetilde{A}\overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)} + \widetilde{b}(\overline{u}^\varepsilon_{2D}, \overline{u}^\varepsilon_{2D}, \widetilde{A}\overline{u}^\varepsilon_{2D}) \\ = (M_\varepsilon f^\varepsilon_{2D}, \widetilde{A}\overline{u}^\varepsilon_{2D})_{L^2(\omega)}\end{aligned} \tag{3.39}$$

Note that $\widetilde{b}(\overline{u}^\varepsilon_{2D}, \overline{u}^\varepsilon_{2D}, \widetilde{A}\overline{u}^\varepsilon_{2D}) = 0$ (space periodic case). Thus we deduce:

$$\frac{d}{dt}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)} + \nu|\widetilde{A}\overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)} \le \frac{1}{\nu}|M_\varepsilon f^\varepsilon_{2D}|^2_{L^2(\omega)}, \tag{3.40}$$

and consequently for all $t \ge 0$

$$|\widetilde{A}^{1/2}\overline{u}^\varepsilon_{2D}(t)|^2_{L^2(\omega)} \le |\widetilde{A}^{1/2}\overline{u}^\varepsilon_{2D}(0)|^2_{L^2(\omega)} \exp(-\nu\lambda_1 t) + \frac{1}{\nu^2\lambda_1}|M_\varepsilon f^\varepsilon_{2D}|^2_{L^2(\omega)}, \tag{3.41}$$

and also

$$\begin{aligned}\nu\int_t^{t+t_0} |\widetilde{A}\overline{u}^\varepsilon_{2D}(s)|^2_{L^2(\omega)}\,ds \le |\widetilde{A}^{1/2}\overline{u}^\varepsilon_{2D}(0)|^2_{L^2(\omega)} \exp(-\nu\lambda_1 t)& \\ + \frac{1}{\nu^2\lambda_1}|M_\varepsilon f^\varepsilon_{2D}|^2_{L^2(\omega)} + \frac{1}{\nu}|M_\varepsilon f^\varepsilon_{2D}|^2_{L^2(\omega)}&.\end{aligned} \tag{3.42}$$

Taking into account the hypothesis (3.1), for all $t \ge 0$ we obtain from (3.40) and (3.42)

$$\begin{aligned}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_{2D}(t)|^2_{L^2(\omega)} &\le c(\nu)[|\widetilde{A}^{1/2}M_\varepsilon u^\varepsilon_0|^2_{L^2(\omega)} + |M_\varepsilon f^\varepsilon|^2_{L^2(\omega)}] \\ &\le c(\nu)K_1 \ln|\ln\varepsilon|,\end{aligned} \tag{3.43}$$

and also

$$\int_t^{t+t_0} |\widetilde{A}\overline{u}^\varepsilon_{2D}(s)|^2_{L^2(\omega)}\,ds \le c(\nu)K_1 \ln|\ln\varepsilon|. \tag{3.44}$$

Estimates for $\overline{u}^\varepsilon_v$ *in* $H^1(\omega)$. Multiply (3.31) by $\widetilde{A}\overline{u}^\varepsilon_v$ and integrate over ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_v|^2_{L^2(\omega)} + \nu|\widetilde{A}\overline{u}^\varepsilon_v|^2_{L^2(\omega)} + \widetilde{b}(\overline{u}^\varepsilon_{2D}, \overline{u}^\varepsilon_v, \widetilde{A}\overline{u}^\varepsilon_v)& \\ = (M_\varepsilon f^\varepsilon_3 \vec{e}_3, \widetilde{A}\overline{u}^\varepsilon_v)_{L^2(\omega)}&,\end{aligned} \tag{3.45}$$

and therefore with Agmon's inequality,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_v|^2_{L^2(\omega)} + \nu|\widetilde{A}\overline{u}^\varepsilon_v|^2_{L^2(\omega)} \le |M_\varepsilon f^\varepsilon_3|_{L^2(\omega)}|\widetilde{A}\overline{u}^\varepsilon_v|_{L^2(\omega)}& \\ + c(\omega)|\overline{u}^\varepsilon_{2D}|^{1/2}_{L^2(\omega)}|\widetilde{A}\overline{u}^\varepsilon_{2D}|^{1/2}_{L^2(\omega)}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_v|_{L^2(\omega)}|\widetilde{A}\overline{u}^\varepsilon_v|_{L^2(\omega)}&.\end{aligned} \tag{3.46}$$

We infer from (3.46) that

$$\frac{d}{dt}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_v|^2_{L^2(\omega)} \le \frac{c}{\nu}|M_\varepsilon f^\varepsilon_3|^2_{L^2(\omega)} + \frac{c}{\nu\lambda_1}|\widetilde{A}\overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)}|\widetilde{A}^{1/2}\overline{u}^\varepsilon_v|^2_{L^2(\omega)}. \tag{3.47}$$

We apply the uniform Gronwall lemma with

$$y = |\widetilde{A}^{1/2}\overline{u}^\varepsilon_v|^2_{L^2(\omega)}, \quad g = c|\widetilde{A}\overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)}/\nu\lambda_1, \quad h = c|M_\varepsilon f^\varepsilon|^2_{L^2(\omega)}/\nu.$$

We use (3.44), (3.49) and (3.1) and for all $t \geq 0$ we deduce

$$\int_t^{t+1} g(s)\,ds \leq c(\nu)K_1 \ln|\ln\varepsilon| = a_1,$$
$$\int_t^{t+1} h(s)\,ds \leq c(\nu)K_1 \ln|\ln\varepsilon| = a_2,$$
$$\int_t^{t+1} y(s)\,ds \leq c(\nu)K_1 \ln|\ln\varepsilon| = a_3,$$

so that

$$\begin{aligned}(3.48)\qquad |\widetilde{A}^{1/2}\overline{u}_v^\varepsilon(t)|^2_{L^2(\omega)} &\leq \big[|\widetilde{A}^{1/2}\overline{u}_v^\varepsilon(0)|^2_{L^2(\omega)} + a_2 + a_3\big]\exp(a_1)\\ &\leq c(\nu)K_1 \ln|\ln\varepsilon| \exp(c(\nu)K_1\ln|\ln\varepsilon|)\end{aligned}$$

for all $t \geq 0$. We infer from (3.43) and (3.48) that for all $t \geq 0$

$$(3.49)\qquad |\widetilde{A}^{1/2}\overline{u}^\varepsilon(t)|^2_{L^2(\omega)} \leq c(\nu)K_1\ln|\ln\varepsilon|\,(1+\exp(c(\nu)K_1\ln|\ln\varepsilon|)).$$

Note furthermore, that

$$(3.50)\qquad |\widetilde{A}^{1/2}\overline{u}^\varepsilon(t)|^2_\varepsilon = \varepsilon|\widetilde{A}^{1/2}\overline{u}^\varepsilon(t)|^2_{L^2(\omega)} \leq c(\nu,\gamma)\varepsilon^{1-\gamma}$$

for all $t \geq 0$ and any arbitrarily small $\gamma > 0$.

3.3. The comparison theorem. Our first result stated at the end of section 3.3 (Theorem 3.3) gives a comparison between u^ε and $\overline{u}^\varepsilon + \overline{w}^\varepsilon$. We set $U^\varepsilon = u^\varepsilon - \overline{u}^\varepsilon - \overline{w}^\varepsilon$ and we aim to estimate the N_ε and the M_ε components of U^ε.

Estimates for $N_\varepsilon U^\varepsilon = N_\varepsilon u^\varepsilon - N_\varepsilon\overline{w}^\varepsilon$. Starting from the weak formulation for the equations defining u^ε, $\overline{u}^\varepsilon$ and $\overline{w}^\varepsilon$, for all $v \in V_p^\varepsilon$ we obtain

$$\begin{aligned}(3.51)\qquad \frac{d}{dt}(N_\varepsilon U^\varepsilon, N_\varepsilon v)_\varepsilon &+ \nu(A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon, A_\varepsilon^{1/2}N_\varepsilon v)_\varepsilon + b_\varepsilon(N_\varepsilon U^\varepsilon, N_\varepsilon U^\varepsilon, N_\varepsilon v)\\ &+ b_\varepsilon(N_\varepsilon U^\varepsilon, N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v) + b_\varepsilon(N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon U^\varepsilon, N_\varepsilon v)\\ &+ b_\varepsilon(N_\varepsilon U^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon v) + b_\varepsilon(M_\varepsilon u^\varepsilon, N_\varepsilon U^\varepsilon, N_\varepsilon v)\\ &+ \left(\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon, N_\varepsilon v\right)_\varepsilon = 0,\end{aligned}$$

$$(3.52)\qquad N_\varepsilon U^\varepsilon|_{t=0} = N_\varepsilon u_0^\varepsilon - N_\varepsilon\overline{w}^\varepsilon(0).$$

We choose $v = A_\varepsilon U^\varepsilon(t)$ and we obtain, using Lemma 3.1

$$\begin{aligned}(3.53)\qquad &\frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon|^2_\varepsilon + \nu|A_\varepsilon N_\varepsilon U^\varepsilon|^2_\varepsilon\\ &\leq c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon U^\varepsilon|^2_\varepsilon + 2c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon U^\varepsilon|^2_\varepsilon\\ &\quad + c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon U^\varepsilon|^2_\varepsilon + c_1\varepsilon^q|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon|A_\varepsilon N_\varepsilon U^\varepsilon|^2_\varepsilon\\ &\quad + \left|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon\right|_\varepsilon|A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon;\end{aligned}$$

since $0 < q < 1/2$ and $0 < \varepsilon < 1$, we deduce from (3.53)

$$\frac{d}{dt}|A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon|_\varepsilon^2 + [\nu - 4c_1\varepsilon^q|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon - 2c_1\varepsilon^q|A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon|_\varepsilon - 4c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon]|A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \le \frac{1}{\nu}\left|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon\right|_\varepsilon^2. \tag{3.54}$$

Now using (3.3), (3.8) and (3.2), we deduce that there exists $\varepsilon_3 = \varepsilon_3(\nu,\omega,K_1,K_2)$ such that if $0 < \varepsilon \le \varepsilon_3$, then by (3.22)

$$\frac{d}{dt}|A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon|_\varepsilon^2 + \frac{\nu}{2}|A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \le \frac{1}{\nu}\left|\frac{d}{dt}N_\varepsilon\overline{w}^\varepsilon\right|_\varepsilon^2 \le c(\nu,q,\gamma)\varepsilon^{3-\gamma} \tag{3.55}$$

(γ being an arbitrarily small positive number). By the Cauchy–Schwarz inequality we have

$$|A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon|_\varepsilon \le \varepsilon|A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon,$$

which gives together with (3.55)

$$|A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon(t)|_\varepsilon^2 \le |A_\varepsilon^{1/2}N_\varepsilon U^\varepsilon(0)|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + c(\nu,q,\gamma)\varepsilon^{5-\gamma} \tag{3.56}$$

for all $t \ge 0$ (we recall that $q \in (0,1/2)$ is an arbitrary number and $\gamma > 0$ is an arbitrarily small number).

Estimates for $M_\varepsilon U^\varepsilon = M_\varepsilon u^\varepsilon - M_\varepsilon\overline{u}^\varepsilon$. The weak formulation for $M_\varepsilon U^\varepsilon$ reads:

$$\begin{aligned}\frac{d}{dt}(M_\varepsilon U^\varepsilon, M_\varepsilon v)_\varepsilon &+ \nu(A_\varepsilon^{1/2}M_\varepsilon U^\varepsilon, A_\varepsilon^{1/2}M_\varepsilon v) + b_\varepsilon(M_\varepsilon U^\varepsilon, M_\varepsilon U^\varepsilon, M_\varepsilon v)\\ &+ b_\varepsilon(M_\varepsilon U^\varepsilon, M_\varepsilon\overline{u}^\varepsilon, M_\varepsilon v) + b_\varepsilon(M_\varepsilon\overline{u}^\varepsilon, M_\varepsilon U^\varepsilon, M_\varepsilon v)\\ &+ b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, M_\varepsilon v) = 0,\end{aligned} \tag{3.57}$$

for all $v \in V_\varepsilon^\varepsilon$ with the initial condition

$$M_\varepsilon U^\varepsilon|_{t=0} = 0. \tag{3.58}$$

Estimates for $M_\varepsilon U_{2D}^\varepsilon$ *in* H^1. We choose $v = A_\varepsilon U_{2D}^\varepsilon$ in (3.57), where $U_{2D}^\varepsilon = (U_1^\varepsilon, U_2^\varepsilon, 0)$ and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2}M_\varepsilon U_{2D}^\varepsilon|_\varepsilon^2 + \nu|A_\varepsilon M_\varepsilon U_{2D}^\varepsilon|_\varepsilon^2 + b_\varepsilon(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon)\\ + b_\varepsilon(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon\overline{u}_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) + b_\varepsilon(M_\varepsilon\overline{u}_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon)\\ + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) = 0.\end{aligned} \tag{3.59}$$

Note that $b_\varepsilon(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) = \varepsilon\,\widetilde{b}(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, \widetilde{A}M_\varepsilon U_{2D}^\varepsilon) = 0$.

Using the L^2-scalar product and the L^2-norm on ω we rewrite (3.59) as:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{1/2}M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 &+ \nu|\widetilde{A}M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2\\ &= -\,\widetilde{b}(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon\overline{u}_{2D}^\varepsilon, \widetilde{A}M_\varepsilon U_{2D}^\varepsilon) - \widetilde{b}(M_\varepsilon\overline{u}_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, \widetilde{A}M_\varepsilon U_{2D}^\varepsilon)\\ &\quad - b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon)/\varepsilon.\end{aligned} \tag{3.60}$$

We estimate the nonlinear terms as follows:

$$|\widetilde{b}(M_\varepsilon U^\varepsilon_{2D}, M_\varepsilon \overline{u}^\varepsilon_{2D}, \widetilde{A} M_\varepsilon U^\varepsilon_{2D})| + |\widetilde{b}(M_\varepsilon \overline{u}^\varepsilon_{2D}, M_\varepsilon U^\varepsilon_{2D}, \widetilde{A} M_\varepsilon U^\varepsilon_{2D})| \\ \leq c\lambda_1^{-1/2} |\widetilde{A} M_\varepsilon \overline{u}^\varepsilon_{2D}|_{L^2(\omega)} |\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_{2D}|_{L^2(\omega)} |\widetilde{A} M_\varepsilon U^\varepsilon_{2D}|_{L^2(\omega)},$$

$$\frac{1}{\varepsilon}|b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U^\varepsilon)| \leq c_1 \varepsilon^{-1/2} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon |A_\varepsilon M_\varepsilon U^\varepsilon|_\varepsilon \\ = c_1 |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon |\widetilde{A} M_\varepsilon U^\varepsilon|_{L^2(\omega)}.$$

We deduce then from (3.60)

$$\text{(3.61)} \quad \frac{d}{dt}|\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_{2D}|^2_{L^2(\omega)} + \nu |\widetilde{A} M_\varepsilon U^\varepsilon_{2D}|^2_{L^2(\omega)} \\ \leq \frac{c}{\nu\lambda_1} |\widetilde{A} M_\varepsilon \overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)} |\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_{2D}|^2_{L^2(\omega)} + \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|^2_\varepsilon |A_\varepsilon N_\varepsilon u^\varepsilon|^2_e.$$

Then we apply the uniform Gronwall lemma with

$$\text{(3.62)} \quad y = |\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_{2D}|^2_{L^2(\omega)},$$

$$\text{(3.63)} \quad g = \frac{c}{\nu\lambda_1} |\widetilde{A} M_\varepsilon \overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)},$$

$$\text{(3.64)} \quad h = \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|^2_\varepsilon |A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon.$$

We have the following estimate, using (3.44):

$$\text{(3.65)} \quad \int_t^{t+1} g(s)\,ds \leq c(\nu) K_1 \ln|\ln\varepsilon|, \quad \forall t \geq 0.$$

We recall from [23] (formula (3.13)) the following relation

$$\frac{d}{dt}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|^2_\varepsilon + \frac{\nu}{2}|A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon \leq \frac{1}{\nu}|N_\varepsilon f^\varepsilon|^2_\varepsilon,$$

so that for all $t \geq 0$ we have

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|^2_\varepsilon \leq |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|^2_\varepsilon \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2}|N_\varepsilon f^\varepsilon|^2_\varepsilon,$$

and

$$\text{(3.66)} \quad \frac{\nu}{2}\int_t^{t+1} |A_\varepsilon N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\,ds \leq |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|^2_\varepsilon \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) \\ + \frac{2\varepsilon^2}{\nu^2}|N_\varepsilon f^\varepsilon|^2_\varepsilon + \frac{1}{\nu}|N_\varepsilon f^\varepsilon|^2_\varepsilon,$$

which yield to

$$\text{(3.67)} \quad \int_t^{t+1} h(s)\,ds \leq c(\nu) R_0^4(\varepsilon), \quad \forall t \geq 0,$$

and finally,

$$\int_t^{t+1} y(s)\,ds = \int_t^{t+1} |\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_{2D}|^2_{L^2(\omega)}\,ds$$
$$\leq 2\int_t^{t+1} [|\widetilde{A}^{1/2} M_\varepsilon u^\varepsilon_{2D}|^2_{L^2(\omega)} + |\widetilde{A}^{1/2} M_\varepsilon \overline{u}^\varepsilon_{2D}|^2_{L^2(\omega)}|]\,ds.$$

We recall also from [23] (formula (3.27)) the following inequality:

$$\frac{d}{dt}|M_\varepsilon u^\varepsilon|^2_{L^2(\omega)} + \nu|\widetilde{A}^{1/2} M_\varepsilon u^\varepsilon|^2_{L^2(\omega)} \leq \frac{1}{\nu\lambda_1}|M_\varepsilon f^\varepsilon|^2_{L^2(\omega)} + \frac{c}{\nu}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|^2_\varepsilon |A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon,$$

which for all $t \geq 0$ gives

$$\nu\int_t^{t+1} |\widetilde{A}^{1/2} M_\varepsilon u^\varepsilon(s)|^2_{L^2(\omega)}\,ds \leq |M_\varepsilon u^\varepsilon(t)|^2_{L^2(\omega)} + \frac{1}{\nu\lambda_1}|M_\varepsilon f^\varepsilon|^2_{L^2(\omega)}$$
$$+ \frac{c}{\nu}\int_t^{t+1} |\widetilde{A}^{1/2} N_\varepsilon u^\varepsilon|^2_\varepsilon |A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon\,ds$$
$$\leq c(\nu)\left[R_0^2(\varepsilon) + R_0^4(\varepsilon)\right].$$

Taking into account (3.39), we conclude

$$\int_t^{t+1} y(s)\,ds \leq c(\nu)\left[R_0^2(\varepsilon) + R_0^4(\varepsilon)\right], \quad \forall t \geq 0. \tag{3.68}$$

Using the usual and uniform Gronwall lemmas, we infer from (3.62), (3.64) and (3.65) that

$$|\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_{2D}(t)|^2_{L^2(\omega)} \leq c(\nu)\left[R_0^2(\varepsilon) + R_0^4(\varepsilon)\right]\exp(c(\nu)K_1 \ln|\ln\varepsilon|), \quad \forall t \geq 0.$$

Estimates for $M_\varepsilon U^\varepsilon_v$ *in* H^1. We write $v = A_\varepsilon U^\varepsilon_v$ in (3.57), where $U^\varepsilon_v = (0,0,U^\varepsilon_3)$, and we find:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2} M_\varepsilon U^\varepsilon_v|^2_\varepsilon + \nu|A_\varepsilon M_\varepsilon U^\varepsilon_v|^2_\varepsilon + b_\varepsilon(M_\varepsilon U^\varepsilon_{2D}, M_\varepsilon U^\varepsilon_v, A_\varepsilon M_\varepsilon U^\varepsilon_v)\\ + b_\varepsilon(M_\varepsilon U^\varepsilon_{2D}, M_\varepsilon \overline{u}^\varepsilon_v, A_\varepsilon M_\varepsilon U^\varepsilon_v) + b_\varepsilon(M_\varepsilon \overline{u}^\varepsilon_{2D}, M_\varepsilon U^\varepsilon_v, A_\varepsilon M_\varepsilon U^\varepsilon_v)\\ + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U^\varepsilon_v) = 0.\end{aligned} \tag{3.70}$$

Using the L^2 scalar product and the L^2 norm on ω we rewrite (3.70) as:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{1/2} M_\varepsilon U^\varepsilon_v|^2_{L^2(\omega)} + \nu|\widetilde{A} M_\varepsilon U^\varepsilon_v|^2_{L^2(\omega)} + \widetilde{b}(M_\varepsilon U^\varepsilon_{2D}, M_\varepsilon U^\varepsilon_v, \widetilde{A} M_\varepsilon U^\varepsilon_v)\\ + \widetilde{b}(M_\varepsilon U^\varepsilon_{2D}, M_\varepsilon \overline{u}^\varepsilon_v, \widetilde{A} M_\varepsilon U^\varepsilon_v) + \widetilde{b}(M_\varepsilon \overline{u}^\varepsilon_{2D}, M_\varepsilon U^\varepsilon_v, \widetilde{A} M_\varepsilon U^\varepsilon_v)\\ + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon, M_\varepsilon U^\varepsilon_v)/\varepsilon = 0,\end{aligned}$$

and then

$$\begin{aligned}&\frac{d}{dt}|\widetilde{A}^{1/2}M_\varepsilon U_v^\varepsilon|^2_{L^2(\omega)}+\nu|\widetilde{A}M_\varepsilon U_v^\varepsilon|^2_{L^2(\omega)}\\&\qquad\le\frac{c}{\nu}|M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}^{1/2}M_\varepsilon U_v^\varepsilon|^2_{L^2(\omega)}\\&\qquad\quad+\frac{c}{\nu}|M_\varepsilon\overline{u}_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}M_\varepsilon\overline{u}_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}^{1/2}M_\varepsilon U_v^\varepsilon|^2_{L^2(\omega)}\\&\qquad\quad+\frac{c}{\nu}|M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}^{1/2}M_\varepsilon \overline{u}_v^\varepsilon|^2_{L^2(\omega)}\\&\qquad\quad+\frac{c}{\nu}|A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon|^2_\varepsilon|A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon\end{aligned}$$

We apply again the uniform Gronwall lemma with

$$g=\frac{c}{\nu\lambda_1}[|\widetilde{A}M_\varepsilon U_{2D}^\varepsilon|^2_{L^2(\omega)}+|\widetilde{A}M_\varepsilon\overline{u}_{2D}^\varepsilon|^2_{L^2(\omega)}],\tag{3.71}$$

$$\begin{aligned}h=&\frac{c}{\nu}|A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon|^2_\varepsilon|A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon\\&+\frac{c}{\nu}|M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}|\widetilde{A}^{1/2}M_\varepsilon\overline{u}_v^\varepsilon|^2_{L^2(\omega)},\end{aligned}\tag{3.72}$$

$$y=|\widetilde{A}^{1/2}M_\varepsilon U_v^\varepsilon|^2_{L^2(\omega)}.\tag{3.73}$$

For all $t\ge 0$ we have the following estimates:

$$\begin{gathered}\int_t^{t+1}g(s)\,ds,\quad\int_t^{t+1}y(s)\,ds\ \le c(\nu)\left[R_0^2(\varepsilon)+R_0^4(\varepsilon)\right],\\\int_t^{t+1}h(s)\,ds\le c(\nu)\left[R_0^2(\varepsilon)+R_0^4(\varepsilon)\right]\exp(c(\nu)(R_0^2(\varepsilon)+R_0^4(\varepsilon)))\end{gathered}$$

so that, taking into account the fact that $M_\varepsilon U^\varepsilon(0)=0$,

$$|\widetilde{A}^{1/2}M_\varepsilon U_v^\varepsilon(t)|^2_{L^2(\omega)}\le c(\nu)\left[R_0^2(\varepsilon)+R_0^4(\varepsilon)\right]\exp(c(\nu)(R_0^2(\varepsilon)+R_0^4(\varepsilon))).\tag{3.74}$$

We infer from (3.66) and (3.68) that

$$|\widetilde{A}^{1/2}M_\varepsilon U^\varepsilon(t)|^2_{L^2(\omega)}\le c(\nu)\left[R_0^2(\varepsilon)+R_0^4(\varepsilon)\right]\exp(c(\nu)(R_0^2(\varepsilon)+R_0^4(\varepsilon))),\quad\forall t\ge 0.$$

Note that for all $t\ge 0$

$$\begin{aligned}|A_\varepsilon^{1/2}M_\varepsilon U^\varepsilon(t)|^2_{L^2(\omega)}&=\varepsilon|\widetilde{A}^{1/2}M_\varepsilon U^\varepsilon(t)|^2_{L^2(\omega)}\\&\le c(\nu)\varepsilon\left[R_0^2(\varepsilon)+R_0^4(\varepsilon)\right]\exp(c(\nu)(R_0^2(\varepsilon)+R_0^4(\varepsilon))),\\&\le c(\nu,\gamma)\varepsilon^{1-\gamma}.\end{aligned}\tag{3.75}$$

We summarize the previous result in the following Theorem comparing u^ε to $\overline{u}^\varepsilon+\overline{w}^\varepsilon$.

Theorem 3.3. *In the fully periodical case, we assume that* (3.1) *holds so that* $u = u^\varepsilon$*, the solution to the Navier–Stokes equations* (0.1)–(0.3) *is defined and regular for all* $t > 0$*, for* $0 < \varepsilon \le \varepsilon_1$*, for some* ε_1*. Let* $\overline{w}^\varepsilon$ *and* $\overline{u}^\varepsilon$ *be the solutions of* (3.5) *and of the* 2D*-like Navier–Stokes problem* (3.23)–(3.26) (*see also* (3.27)–(3.34)). *Then for* $0 < \varepsilon \le \varepsilon_3 \le \varepsilon_1$*, where* ε_3 *depends only on the data,* $U^\varepsilon = u^\varepsilon - \overline{u}^\varepsilon - \overline{w}^\varepsilon$ *is small in the following sense:*

$$\|M_\varepsilon U^\varepsilon(t)\|_\varepsilon^2 \le c(\nu,\gamma)\,\varepsilon^{1-\gamma}, \tag{3.76}$$

$$\|N_\varepsilon U^\varepsilon(t)\|_\varepsilon^2 \le \|N_\varepsilon U^\varepsilon(0)\|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + c(\nu,q,\gamma)\varepsilon^{5-\gamma}, \tag{3.77}$$

for all $t \ge 0$*, some* $q \in (0,1/2)$ *and any* $\gamma > 0$ *small.*

Remark 3.1. (i) In section 3.4 we will approximate $\overline{w}^\varepsilon$ by a function w^ε, solution of a problem simpler than (3.5) which does not involve u^ε. Hence w^ε will be "explicit", and this will make the approximation results above more useful.

(ii) We could also approximate $\overline{u}^\varepsilon$ by a function $\overline{u}$ independent of ε, solution of a problem similar to (3.23)–(3.26), where $M_\varepsilon f^\varepsilon$ and $M_\varepsilon u_0^\varepsilon$ are replaced by their limit as $\varepsilon \to 0$. The estimates on the rest of the expansion depend then of the differences between $M_\varepsilon f^\varepsilon$ and $M_\varepsilon u_0^\varepsilon$ and their limit; the details are left to the reader.

3.4. Comparison between $N_\varepsilon \overline{w}^\varepsilon$ and w^ε. Let w^ε be the unique solution of the Stokes problem:

$$\begin{cases} -\nu\Delta w^\varepsilon + \nabla q = N_\varepsilon f^\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} w^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ w^\varepsilon \text{ is periodic in the directions } x_1,\ x_2 \text{ and } x_3. \end{cases} \tag{3.78}$$

We note that $w^\varepsilon = N_\varepsilon w^\varepsilon$. Using (1.5) we easily find the following estimates for w^ε:

$$|A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon \le \frac{\varepsilon}{\nu}|N_\varepsilon f^\varepsilon|_\varepsilon, \tag{3.79}$$

$$|A_\varepsilon w^\varepsilon|_\varepsilon \le \frac{1}{\nu}|N_\varepsilon f^\varepsilon|_\varepsilon. \tag{3.80}$$

Remark also that if $f^\varepsilon \in H_p$, then $\nabla q^\varepsilon = N_\varepsilon f^\varepsilon + \nu\Delta w^\varepsilon \in H_p^\varepsilon$, which implies $\nabla q^\varepsilon = 0$. Consider now the difference

$$N_\varepsilon W^\varepsilon = N_\varepsilon \overline{w}^\varepsilon - N_\varepsilon w^\varepsilon.$$

Using the weak formulation (3.52) for $N_\varepsilon \overline{w}^\varepsilon$, we find for $N_\varepsilon W^\varepsilon$

$$\begin{aligned} \nu|A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon^2 &+ b_\varepsilon(N_\varepsilon \overline{w}^\varepsilon, N_\varepsilon \overline{w}^\varepsilon, N_\varepsilon W^\varepsilon) \\ &+ b_\varepsilon(N_\varepsilon \overline{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon W^\varepsilon) + b_\varepsilon(M_\varepsilon u^\varepsilon, N_\varepsilon \overline{w}^\varepsilon, N_\varepsilon W^\varepsilon) = 0, \end{aligned} \tag{3.81}$$

and thus

$$\nu|A_\varepsilon^{1/2}N_\varepsilon W^\varepsilon|_\varepsilon^2 \le c_1\varepsilon^{1/2}|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon^2|A_\varepsilon^{1/2}N_\varepsilon W^\varepsilon|_\varepsilon + 2c_1\varepsilon^q|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon|_\varepsilon|A_\varepsilon^{1/2}N_\varepsilon\overline{w}^\varepsilon|_\varepsilon|A_\varepsilon^{1/2}N_\varepsilon W^\varepsilon|_\varepsilon, \tag{3.82}$$

which implies, using (3.3) and (3.8),

$$\begin{aligned}\nu|A_\varepsilon^{1/2}N_\varepsilon W^\varepsilon|_\varepsilon &\le c(\nu)\varepsilon^{5/2}|N_\varepsilon f^\varepsilon|_\varepsilon^2 + c(\nu)\varepsilon^{1+q}R_0(\varepsilon)|N_\varepsilon f^\varepsilon|_\varepsilon \\ &\le c(\nu)\varepsilon^{1+q}R_0^2(\varepsilon).\end{aligned} \tag{3.83}$$

Taking into account (3.2), this then leads to

$$|A_\varepsilon^{1/2}N_\varepsilon W^\varepsilon|_\varepsilon \le c(\nu,q,\gamma)\varepsilon^{1+q-\gamma}, \tag{3.84}$$

for any arbitrarily small $\gamma > 0$. Combining (3.84) with Theorem 3.1 we see that (3.70) and (3.71) still hold for $U^\varepsilon = u^\varepsilon - \overline{u}^\varepsilon - w^\varepsilon$.

Corollary 3.4. *Under the hypothesis of Theorem* 3.3*, w^ε being the solution of the Stokes problem* (3.72)*, then for $0 < \varepsilon \le \varepsilon_3$, $U^\varepsilon = u^\varepsilon - \overline{u}^\varepsilon - w^\varepsilon$ is small in the sense of* (3.70) *and* ((3.71)*.*

Remark 3.2. It is easy to see that w^ε can be itself approximated by $\widetilde{w}^\varepsilon$:

$$\widetilde{w}^\varepsilon = \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \widehat{w}_k e^{ikx},$$

$$\widehat{w}_k = \frac{1}{\nu(k_1^2+k_2^2)}\widehat{g}_k \quad \text{if } k_3 = 0, \qquad \widehat{w}_k = \frac{\varepsilon^2}{\nu k_3^2}\widehat{g}_k \quad \text{if } k_3 \neq 0,$$

where $\widehat{g}_k$ are the Fourier coefficients of $N_\varepsilon f$.

4. Complements in the space periodic case

In this section we give some complements concerning the purely periodic case. We show how the results of [15], [16] and [23] can be improved, namely that one can obtain, for thin domains, the existence for all time of a smooth solution for a larger set of initial data u_0 and volume forces f. These results can also be used to improve those of Section 3, but this will not be developed here.

We consider the problem (0.1)–(0.3) with periodic boundary conditions. Let $R_0(\varepsilon)$ be a positive function satisfying for some $q \in (0,1)$

$$\lim_{\varepsilon\to 0}\varepsilon^q R_0^2(\varepsilon) = 0. \tag{4.1}$$

We set

$$\begin{cases} R_n^2(\varepsilon) = g_n^2(\varepsilon)R_0^2(\varepsilon), \\ R_m^2(\varepsilon) = g_m^2(\varepsilon)R_0^2(\varepsilon), \end{cases} \tag{4.2}$$

where

$$\text{(4.3)}\qquad \begin{cases} g_n^2(\varepsilon) = \dfrac{\varepsilon^{(5q-1)/6}}{|\ln\varepsilon|}, \\[2mm] g_m^2(\varepsilon) = \dfrac{\varepsilon^{2(q+1)/3}}{|\ln\varepsilon|}. \end{cases}$$

We assume that the data $u_0^\varepsilon \in V_p^\varepsilon$ and $f^\varepsilon \in H_p^\varepsilon$ satisfy:

$$\text{(4.4)}\qquad \begin{cases} |A_\varepsilon^{1/2} M_\varepsilon u_0^\varepsilon|_\varepsilon^2 + |M_\varepsilon f^\varepsilon|_\varepsilon^2 \le R_m^2(\varepsilon), \\ |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 + |N_\varepsilon f^\varepsilon|_\varepsilon^2 \le R_n^2(\varepsilon), \end{cases}$$

and let $T^\sigma(\varepsilon)$ be the maximal time such that

$$\text{(4.5)}\qquad |A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 \le \sigma R_0^2(\varepsilon), \quad 0 \le t < T^\sigma(\varepsilon).$$

Here $\sigma > 2$ is a fixed number which will be chosen later on (see (4.46)). Note that if $T^\sigma(\varepsilon) < \infty$, then

$$\text{(4.6)}\qquad |A_\varepsilon^{1/2} u^\varepsilon(T^\sigma(\varepsilon))|_\varepsilon^2 = \sigma R_0^2(\varepsilon).$$

Since $\lim_{\varepsilon\to 0} \varepsilon^q R_0^2(\varepsilon) = 0$, there exists $\varepsilon_1 = \varepsilon_1(\nu, q)$ (depending also on the function R_0) such that

$$\text{(4.7)}\qquad \varepsilon^q R_0^2(\varepsilon) \le \nu^2/4 \quad \text{for } 0 < \varepsilon \le \varepsilon_1.$$

In what follows we restrict ourselves to $\varepsilon \le \varepsilon_1$, and we aim first to derive a number of a priori estimates.

A priori estimates.

Estimates for $N_\varepsilon u^\varepsilon$. We multiply (0.1) with $A_\varepsilon N_\varepsilon u^\varepsilon$ and we integrate over Ω_ε. We estimate the nonlinear terms using Lemma 3.1, then we take into account (4.5) and (4.7) to obtain for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$

$$\text{(4.8)}\qquad \frac{d}{dt}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 + \frac{\nu}{2}|A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2 \le \frac{|N_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu},$$

and since by the Cauchy–Schwarz inequality

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 \le \varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2,$$

we deduce from (4.8) that

$$\text{(4.9)}\qquad \frac{d}{dt}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 + \frac{\nu}{2\varepsilon^2}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 \le \frac{|N_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu}.$$

Thus by the Gronwall lemma

$$\text{(4.10)}\qquad |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2}|N_\varepsilon f^\varepsilon|_\varepsilon^2,$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$. Taking into account (4.4), we deduce

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le R_n^2 \left[\exp\left(-\frac{\nu t}{2\varepsilon^2} \right) + \frac{2\varepsilon^2}{\nu^2} \right]. \tag{4.11}$$

We also infer from (4.8) that

$$\frac{\nu}{2} \int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \, ds \le |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 + \frac{|N_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu} t,$$

so that

$$\int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \, ds \le c(\nu) R_n^2(\varepsilon)\,(1+t), \tag{4.12}$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$.

Estimates for $M_\varepsilon u^\varepsilon$. We first multiply (0.1) with $M_\varepsilon u^\varepsilon$ and we integrate over Ω_ε. A simple computation taking into account (4.5) and (4.7) yields:

$$\frac{d}{dt}|M_\varepsilon u^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon^2 \le \frac{|M_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu \lambda_1} + \frac{c\,\varepsilon}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^4, \tag{4.13}$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$, where λ_1 is the first eigenvalue of the two-dimensional Stokes operator defined on ω. Then (4.13) implies

$$\nu \int_0^t |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \, ds \le \frac{|M_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu \lambda_1} t + \frac{c\,\varepsilon}{\nu} \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^4 \, ds, \tag{4.14}$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$.
Using (4.11), (4.2) and (4.7) we estimate

$$\begin{aligned}
\frac{c\,\varepsilon}{\nu} \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^4 \, ds &\le \frac{c\,\varepsilon}{\nu} \big[\sup_{0 \le s \le t} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \big] \left(\int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \, ds \right) \\
&\le c(\nu)\, \varepsilon\, R_n^4(\varepsilon) \left[\exp(-\frac{\nu t}{2\varepsilon^2}) + \frac{2\varepsilon^2}{\nu^2} \right] \left[\frac{2\varepsilon^2}{\nu} + \frac{2\varepsilon^2}{\nu^2}\, t \right] \\
&\le c(\nu) \varepsilon^3 R_n^4(\varepsilon)(1+t) = c(\nu)\varepsilon^3 g_n^4(\varepsilon) R_0^4(\varepsilon)(1+t) \\
&\le c(\nu) \varepsilon^{3-q} g_n^4(\varepsilon) R_0^2(\varepsilon)(1+t),
\end{aligned}$$

so that we deduce from (4.14)

$$\begin{aligned}
\int_0^t |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \, ds &\le c(\nu,\lambda_1) R_m^2(\varepsilon)(1+t) \\
&\quad + c(\nu) \varepsilon^{3-q} g_n^4(\varepsilon) R_0^2(\varepsilon)(1+t) \\
&= c(\nu,\lambda_1) g_m^2(\varepsilon) R_0^2(\varepsilon) \left[1 + \frac{\varepsilon^{3-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} \right] (1+t),
\end{aligned} \tag{4.15}$$

for $0 < \varepsilon \le \varepsilon_1, 0 \le t < T^\sigma(\varepsilon)$. We set $u^\varepsilon_{2D} = (u^\varepsilon_1, u^\varepsilon_2, 0)$. We multiply (0.1) with $A_\varepsilon M_\varepsilon u^\varepsilon_{2D}$ and integrate over Ω_ε to obtain:

$$(4.16)\quad \frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon + b_\varepsilon(M_\varepsilon u^\varepsilon_{2D}, M_\varepsilon u^\varepsilon_{2D}, A_\varepsilon M_\varepsilon u^\varepsilon_{2D}) + \nu|A_\varepsilon M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon u^\varepsilon_{2D}) = (M_\varepsilon f^\varepsilon, A_\varepsilon M_\varepsilon u^\varepsilon_{2D})_\varepsilon.$$

Note that

$$b_\varepsilon(M_\varepsilon u^\varepsilon_{2D}, M_\varepsilon u^\varepsilon_{2D}, A_\varepsilon M_\varepsilon u^\varepsilon_{2D}) = \varepsilon\, \widetilde{b}(M_\varepsilon u^\varepsilon_{2D}, M_\varepsilon u^\varepsilon_{2D}, \widetilde{A} M_\varepsilon u^\varepsilon_{2D}) = 0,$$

due to a well-known orthognality property in the periodic boundary conditions case; therefore (4.16) becomes

$$(4.17)\quad \frac{d}{dt}|A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon + \nu|A_\varepsilon M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon \le \frac{c|M_\varepsilon f^\varepsilon|^2_\varepsilon}{\nu} + \frac{c\,\varepsilon}{\nu}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|^2_\varepsilon\,|A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$. Hence, with the Gronwall lemma we have:

$$(4.18)\quad |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon_{2D}(t)|^2_\varepsilon \le |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon_{2D}(0)|^2_\varepsilon \exp(-\nu\lambda_1 t) + \frac{c\,\varepsilon}{\nu}\int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\,|A_\varepsilon N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\,ds + \frac{c|M_\varepsilon f^\varepsilon|^2_\varepsilon}{\nu^2\lambda_1}$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$.
Using (4.11), (4.12), (4.2) and (4.7) we estimate

$$\begin{aligned}
&\frac{c\,\varepsilon}{\nu}\int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\,|A_\varepsilon N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\,ds \\
&\quad\le \frac{c\,\varepsilon}{\nu}\left[\sup_{0\le s\le t}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\right]\left[\int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\,ds\right] \\
&\quad\le c(\nu)\,\varepsilon\, R_n^4(\varepsilon)(1+t) = c(\nu)\,\varepsilon\, g_n^4 R_n^4(\varepsilon)(1+t) \\
&\quad\le c(\nu)\varepsilon^{1-q} g_n^4(\varepsilon) R_0^2(\varepsilon)(1+t).
\end{aligned}$$

We deduce from (4.18), (4.2) and the previous estimate that

$$\begin{aligned}
(4.19)\quad |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon_{2D}(t)|^2_\varepsilon &\le c(\nu,\lambda_1) R_m^2(\varepsilon) + c(\nu)\,\varepsilon^{1-q}\, g_n^4(\varepsilon)\, R_0^2(\varepsilon)(1+t) \\
&= c(\nu,\lambda_1) g_m^2(\varepsilon)\left[1 + \frac{\varepsilon^{1-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)}(1+t)\right] R_0^2(\varepsilon),
\end{aligned}$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$. Now we set $v^\varepsilon = (0, 0, M_\varepsilon u^\varepsilon_3)$. We multiply (0.1) with $A_\varepsilon M_\varepsilon v^\varepsilon$ and we integrate over Ω_ε to obtain

$$(4.20)\quad \frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2} v^\varepsilon|^2_\varepsilon + \nu|A_\varepsilon v^\varepsilon|^2_\varepsilon + b_\varepsilon(M_\varepsilon u^\varepsilon_{2D}, v^\varepsilon, A_\varepsilon v^\varepsilon) + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon v^\varepsilon) = (M_\varepsilon f^\varepsilon_3, A_\varepsilon v^\varepsilon).$$

Note that

$$(4.21)\quad \begin{aligned}|b_\varepsilon(M_\varepsilon u^\varepsilon_{2D}, v^\varepsilon, A_\varepsilon v^\varepsilon)| &\le c\,\varepsilon\,|M_\varepsilon u^\varepsilon_{2D}|_{L^4(w)}|\nabla' v^\varepsilon|_{L^4(w)}|\widetilde{A} v^\varepsilon|_{L^2(\omega)}\\ &\le \frac{1}{8}\nu|A_\varepsilon v^\varepsilon|^2_\varepsilon + \frac{c}{\nu^3\varepsilon^2}|M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon\,|A^{1/2}_\varepsilon M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon\,|A^{1/2}_\varepsilon v^\varepsilon|^2_\varepsilon\end{aligned}$$

and

$$(4.22)\quad |b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon v^\varepsilon)| \le \nu|A_\varepsilon v^\varepsilon|^2_\varepsilon/8 + c\varepsilon|A^{1/2}_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon|A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon/\nu.$$

Hence (4.20)–(4.22) for $0<\varepsilon\le\varepsilon_1,\ 0\le t<T^\sigma(\varepsilon)$ give

$$(4.23)\quad \begin{aligned}\frac{d}{dt}|A^{1/2}_\varepsilon v^\varepsilon|^2_\varepsilon + \nu|A_\varepsilon v^\varepsilon|^2_\varepsilon &\le \frac{c\,|M_\varepsilon f^\varepsilon_3|^2_\varepsilon}{\nu} + \frac{c\,\varepsilon}{\nu}|A^{1/2}_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon\,|A_\varepsilon N_\varepsilon u^\varepsilon|^2_\varepsilon\\ &\quad + \frac{c}{\nu^3\varepsilon^2}|M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon\,|A^{1/2}_\varepsilon M_\varepsilon u^\varepsilon_{2D}|^2_\varepsilon\,|A^{1/2}_\varepsilon v^\varepsilon|^2_\varepsilon,\end{aligned}$$

and Gronwall's lemma for $0<\varepsilon\le\varepsilon_1,\ 0\le t<T^\sigma(\varepsilon)$. yields

$$(4.24)\quad \begin{aligned}|A^{1/2}_\varepsilon v^\varepsilon(t)|^2_\varepsilon &\le |A^{1/2}_\varepsilon v^\varepsilon(0)|^2_\varepsilon\exp(-\nu\lambda_1 t) + \frac{c|M_\varepsilon f^\varepsilon_3|^2_\varepsilon}{\nu^2\lambda_1}\\ &\quad + c(\nu)\varepsilon\big[\sup_{0\le s\le t}|A^{1/2}_\varepsilon N_\varepsilon u^\varepsilon(s)|^2_\varepsilon\big]\bigg(\int_0^t|A_\varepsilon N_\varepsilon u^\varepsilon(s)|^2\,ds\bigg)\\ &\quad + \frac{c(\nu)}{\varepsilon^2}\big[\sup_{0\le s\le t}\lambda_1^{-1}|A^{1/2}_\varepsilon M_\varepsilon u^\varepsilon_{2D}(s)|^4_\varepsilon\big]\bigg(\int_0^t|A^{1/2}_\varepsilon v^\varepsilon(s)|^2_\varepsilon\,ds\bigg),\end{aligned}$$

We use (4.11), (4.12),(4.19) and (4.15) in (4.24) and we obtain

$$\begin{aligned}|A^{1/2}_\varepsilon v^\varepsilon(t)|^2_\varepsilon &\le c(\nu,\lambda_1)R^2_m(\varepsilon) + c(\nu)\,\varepsilon\,R^4_n(\varepsilon)(1+t) + \frac{c(\nu,\lambda_1)}{\varepsilon^2}g^6_m(\varepsilon)\\ &\quad\cdot\bigg[1+\frac{\varepsilon^{1-q}g^4_n(\varepsilon)}{g^2_m(\varepsilon)}(1+t)^2\bigg]^2\bigg[1+\frac{\varepsilon^{3-q}g^4_n(\varepsilon)}{g^2_m(\varepsilon)}\bigg]R^6_0(\varepsilon)(1+t).\end{aligned}$$

We use (4.7) and we obtain

$$(4.25)\quad \begin{aligned}|A^{1/2}_\varepsilon v^\varepsilon(t)|^2_\varepsilon &\le c(\nu,\lambda_1)\bigg\{g^2_m(\varepsilon) + \varepsilon^{1-q}g^4_n(\varepsilon)(1+t) + \varepsilon^{-2-2q}g^6_m(\varepsilon)\\ &\quad\cdot\bigg[1+\frac{\varepsilon^{1-q}g^4_n(\varepsilon)}{g^2_m(\varepsilon)}(1+t)^2\bigg]^2\bigg[1+\frac{\varepsilon^{3-q}g^4_n(\varepsilon)}{g^2_m(\varepsilon)}\bigg](1+t)\bigg\}R^2_0(\varepsilon)\end{aligned}$$

for $0<\varepsilon\le\varepsilon_1,\ 0\le t<T^\sigma(\varepsilon)$.

Now we take into account the expressions of g_m and g_n given by (4.3) and we rewrite (4.19) and (4.25) as

$$(4.26)\quad |A^{1/2}_\varepsilon M_\varepsilon u^\varepsilon_{2D}(t)|^2_\varepsilon \le c(\nu,\lambda_1)\frac{\varepsilon^{2(q+1)/3}}{|\ln\varepsilon|}\bigg[1+\frac{1+t}{|\ln\varepsilon|}\bigg]R^2_0(\varepsilon),$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$ and

$$(4.27)\ |A_\varepsilon^{1/2} v^\varepsilon(t)|^2\varepsilon \le c(\nu,\lambda_1)\frac{\varepsilon^{2(q+1)/3}}{|\ln\varepsilon|}\left(1+\frac{1+t}{|\ln\varepsilon|}\right)R_0^2(\varepsilon) + c(\nu,\lambda_1)\frac{1+t}{|\ln\varepsilon|^3}\left(1+\frac{(1+t)^2}{|\ln\varepsilon|}\right)^2\left(1+\frac{\varepsilon^2}{|\ln\varepsilon|}\right)R_0^2(\varepsilon)$$
$$\le c(\nu,\lambda_1)\frac{\varepsilon^{2(q+1)/3}}{|\ln\varepsilon|}\left(1+\frac{1+t}{|\ln\varepsilon|}\right)R_0^2(\varepsilon) + c(\nu,\lambda_1)\left[\frac{(1+t)^3}{|\ln\varepsilon|^5}+\frac{(1+t)^2}{|\ln\varepsilon|^4}+\frac{(1+t)}{|\ln\varepsilon|^3}\right]R_0^2(\varepsilon).$$

for $0 < \varepsilon \le \varepsilon_1,\ 0 \le t < T^\sigma(\varepsilon)$. At this stage we are able to prove that

$$\lim_{\varepsilon\to 0}\frac{T^\sigma(\varepsilon)}{|\ln\varepsilon|^{1/2}} = \infty. \tag{4.28}$$

If this were not true, we would have ($T^\sigma(\varepsilon) < \infty$):

$$\big(|A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon|_\varepsilon^2 + |A_\varepsilon^{1/2}M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 + |A_\varepsilon^{1/2}v^\varepsilon|_\varepsilon^2\big)(T^\sigma(\varepsilon)) = \sigma R_0^2(\varepsilon), \tag{4.29}$$

so that, using (4.11), (4.26) and (4.27) we obtain

$$\sigma \le c(\nu)\frac{\varepsilon^{(5q-1)/6}}{|\ln\varepsilon|} + c(\nu,\lambda_1)\frac{\varepsilon^{2(q+1)/3}}{|\ln\varepsilon|}\left[1+\frac{1+T^\sigma(\varepsilon)}{|\ln\varepsilon|}\right] + c(\nu,\lambda_1)\left[\frac{(1+T^\sigma(\varepsilon))^3}{|\ln\varepsilon|^5}+\frac{(1+T^\sigma(\varepsilon))^2}{|\ln\varepsilon|^4}+\frac{1+T^\sigma(\varepsilon)}{|\ln\varepsilon|^3}\right] \tag{4.30}$$

Since the right-hand side of the inequality (4.30) goes to zero as ε goes to zero, we find $\sigma = 0$, a contradiction. Hence we have proved (4.28).

Now we will prove that $T^\sigma(\varepsilon) = \infty$. We use the same notation as in (3.1), namely we set

$$a_0(\varepsilon) = |A_\varepsilon^{1/2}M_\varepsilon u_0^\varepsilon|_\varepsilon, \qquad b_0(\varepsilon) = |A_\varepsilon^{1/2}N_\varepsilon u_0^\varepsilon|_\varepsilon,$$
$$\alpha(\varepsilon) = |M_\varepsilon f^\varepsilon|_\varepsilon, \qquad \beta(\varepsilon) = |N_\varepsilon f^\varepsilon|_\varepsilon,$$

We also consider:

$$K_\varepsilon^2 = |A_\varepsilon^{1/2}M_\varepsilon u_0^\varepsilon|_\varepsilon^2 + \frac{64}{\nu^2\lambda_1}|M_\varepsilon f^\varepsilon|_\varepsilon^2 + B_\varepsilon^2, \tag{4.31}$$

where

$$B_\varepsilon^2 = |A_\varepsilon^{1/2}N_\varepsilon u_0^\varepsilon|_\varepsilon^2 + |N_\varepsilon f^\varepsilon|_\varepsilon^2 \tag{4.32}$$

Note that B_ε and K_ε are both bounded by $c\,R_0$ and therefore due to (4.1),

$$\lim_{\varepsilon\to 0}\varepsilon^q B_\varepsilon^2 = \lim_{\varepsilon\to 0}\varepsilon^q K_\varepsilon^2 = 0. \tag{4.33}$$

We choose $\varepsilon_4 = \varepsilon_4(\nu, \lambda_1, q) > 0$ satisfying the following conditions, where $c_{10}(\nu)$ is defined below in (4.37)

$$
\begin{cases}
\text{(i)} & 0 < \varepsilon_4 \le 1 \\
\text{(ii)} & c_{10}(\nu)\varepsilon^q B_\varepsilon^2 \le 1/32,\ \varepsilon^{1-q}(1 + |\ln\varepsilon|^{1/2}) \le 2 \ \text{ for } 0 < \varepsilon \le \varepsilon_4, \\
\text{(iii)} & 2\varepsilon^2/\nu^2 \le 1/8, \exp\big(-\nu|\ln\varepsilon|^{1/2}/2\varepsilon^2\big) \le 1/4, \\
 & \exp(-\nu\lambda_1|\ln\varepsilon|^{1/2}) \le 1/8 \ \text{ for } 0 < \varepsilon \le \varepsilon_4, \\
\text{(iv)} & T^\sigma(\varepsilon)/|\ln\varepsilon|^{1/2} > 4 \ \text{ for } 0 < \varepsilon \le \varepsilon_4.
\end{cases}
\tag{4.34}
$$

The existence of ε_4 is obvious, since the left-hand side of the inequalities (ii) and (iii) go to zero as ε goes to zero; and by (4.28) the left-hand side of (iv) goes to infinity as ε goes to zero.

Using (4.8), (4.10), (4.34) and $|A_\varepsilon^{1/2} N_\varepsilon^\varepsilon| \le \varepsilon\, |A_\varepsilon^1 N_\varepsilon^\varepsilon|$, we easily find

$$
\int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^3\, |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon\, ds \le \frac{\nu}{4}\max(1, 1/\nu^3) B_\varepsilon^2(1+t)
\tag{4.35}
$$

for $0 \le t \le T^\sigma(\varepsilon)$. Hence

$$
|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le b_0^2(\varepsilon)\exp\bigg(-\frac{\nu t}{2\varepsilon^2}\bigg) + \frac{2\varepsilon^2}{\nu^2}\beta^2(\varepsilon)
\tag{4.36}
$$

and for a suitable constant $c(\nu)$ which we denote $c_{10}(\nu)$ and for $0 \le t \le T^\sigma(\varepsilon)$

$$
|A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le a_0^2(\varepsilon)\exp(-\nu\lambda_1 t) + \frac{2}{\nu^2\lambda_1}|M_\varepsilon f^\varepsilon|_\varepsilon^2 + c_{10}(\nu)\varepsilon B_\varepsilon^4(1+t).
\tag{4.37}
$$

We set

$$
t_\varepsilon = |\ln\varepsilon|^{1/2} \quad \text{for } 0 < \varepsilon \le \varepsilon_4.
\tag{4.38}
$$

Observe that by (4.34)(iv), $t_\varepsilon \le T^\sigma(\varepsilon)/4$. According to (4.36) and (4.37), we have

$$
\begin{aligned}
|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t_\varepsilon)|_\varepsilon^2 &\le b_0^2(\varepsilon)\exp\bigg(-\frac{\nu t_\varepsilon}{2\varepsilon^2}\bigg) + \frac{2\varepsilon^2}{\nu^2}\beta^2(\varepsilon) \\
&\le b_0^2(\varepsilon)\exp\bigg(-\frac{\nu|\ln\varepsilon|^{1/2}}{2\varepsilon^2}\bigg) + \frac{1}{8}\beta^2(\varepsilon) \\
&\le \frac{1}{4}b_0^2(\varepsilon) + \frac{1}{8}\beta^2(\varepsilon) \le \frac{1}{4}B_\varepsilon^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(t_\varepsilon)|_\varepsilon^2 &\le a_0^2(\varepsilon)\exp(-\nu\lambda_1 t_\varepsilon) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} + c_{10}(\nu)(\varepsilon^q B_\varepsilon^2)\varepsilon^{1-q}(1+t_\varepsilon)B_\varepsilon^2 \\
&\le a_0^2(\varepsilon)\exp(-\nu\lambda_1|\ln\varepsilon|^{1/2}) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} + \frac{\varepsilon^{1-q}}{32}(1 + |\ln\varepsilon|^{1/2})B_\varepsilon^2 \\
&\le \frac{1}{8}\bigg(a_0^2(\varepsilon) + \frac{16\alpha^2(\varepsilon)}{\nu^2\lambda_1}\bigg) + \frac{1}{16}B_\varepsilon^2 \le \frac{1}{4}K_\varepsilon^2.
\end{aligned}
$$

Hence, adding the last two relations,

$$|A_\varepsilon^{1/2}u^\varepsilon(t_\varepsilon)|_\varepsilon^2 \le B_\varepsilon^2/4 + K_\varepsilon^2/4 \le K_\varepsilon^2/2. \tag{4.39}$$

We claim that for any $n \ge 1$

$$\begin{cases} nt_\varepsilon \le T^\sigma(\varepsilon) \\ \text{and} \\ |A_\varepsilon^{1/2}N_\varepsilon(nt_\varepsilon)|_\varepsilon^2 \le B_\varepsilon^2/4, \quad |A_\varepsilon^{1/2}M_\varepsilon(nt_\varepsilon)|_\varepsilon^2 \le K_\varepsilon^2/4. \end{cases} \tag{4.40}$$

We have shown that the claim holds for $n = 1$. Suppose now that the claim holds for some n. We want to prove the induction step. For $nt_\varepsilon \le t \le T^\sigma(\varepsilon)$ we obtain the following estimates

$$|A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le |A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(nt_\varepsilon)|_\varepsilon^2 \exp\left(-\frac{\nu(t-nt_\varepsilon)}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2}\beta^2(\varepsilon)$$

and using the induction hypothesis we find

$$\begin{aligned} |A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 &\le \frac{1}{4}B_\varepsilon^2 \exp\left(-\frac{\nu(t-nt_\varepsilon)}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2}\beta^2(\varepsilon) \\ &\le \frac{1}{4}B_\varepsilon^2 \exp\left(-\frac{\nu(t-nt_\varepsilon)}{2\varepsilon^2}\right) + \frac{1}{8}\beta^2(\varepsilon). \end{aligned} \tag{4.41}$$

Similarly, we have

$$\begin{aligned} |A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le\, &|A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon(nt_\varepsilon)|_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + 2\alpha^2(\varepsilon)/\nu^2\lambda_1 \\ &+ c_{10}(\nu)\varepsilon[|A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(nt_\varepsilon)|_\varepsilon^2 + |M_\varepsilon f^\varepsilon|_\varepsilon^2]^2(1+(t-nt_\varepsilon)) \end{aligned}$$

for $nt_\varepsilon \le t < T^\sigma(\varepsilon)$ and using the induction hypothesis we obtain

$$\begin{aligned} |A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le\, &\frac{1}{4}K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &+ c_{10}(\nu)\,\varepsilon\left[\frac{1}{4}B_\varepsilon^2 + \beta^2(\varepsilon)\right]^2(1+(t-nt_\varepsilon)) \\ \le\, &\frac{1}{4}K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &+ c_{10}(\nu)\,\varepsilon\left(\frac{5}{4}\right)^2 B_\varepsilon^4(1+(t-nt_\varepsilon)) \\ =\, &\frac{1}{4}K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &+ c_{10}(\nu)(\varepsilon^q B_\varepsilon^2)\left(\frac{5}{4}\right)^2 B_\varepsilon^2\varepsilon^{1-q}[1+(t-nt_\varepsilon)] \\ \le\, &\frac{1}{4}K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &+ \frac{1}{32}\left(\frac{5}{4}\right)^2 B_\varepsilon^2\varepsilon^{1-q}[1+(t-nt_\varepsilon)]. \end{aligned} \tag{4.42}$$

Now, if $nt_\varepsilon \le t \le (n+1)t_\varepsilon$, we obtain from (4.41) and (4.42)

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \le \frac{1}{4}B_\varepsilon^2 + \frac{1}{8}\beta^2(\varepsilon) \le \frac{1}{8}3B_\varepsilon^2 \tag{4.43}$$

$$\begin{aligned}|A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 &\le \frac{1}{4}K_\varepsilon^2 + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &\quad + \frac{1}{32}\left(\frac{5}{4}\right)^2 B_\varepsilon^2\,\varepsilon^{1-q}(1+|\ln\varepsilon|^{1/2}) \\ &\le \frac{1}{4}K_\varepsilon^2 + \frac{1}{32}K_\varepsilon^2 + \frac{1}{32}\left(\frac{5}{4}\right)^2 2B_\varepsilon^2 \le \frac{1}{8}3K_\varepsilon^2\end{aligned} \tag{4.44}$$

Hence

$$|A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 \le 3B_\varepsilon^2/8 + 3K_\varepsilon^2/8 \le K_\varepsilon^2 \quad \text{for } nt_\varepsilon \le t \le (n+1)t_\varepsilon \tag{4.45}$$

and if

$$\sigma > \max(1, 16/\nu^2\lambda_1), \tag{4.46}$$

we obtain

$$|A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 < \sigma R_0^2(\varepsilon) \quad \text{for } nt_\varepsilon \le t \le (n+1)t_\varepsilon. \tag{4.47}$$

In addition, taking $t = (n+1)t_\varepsilon$ in (4.45) and (4.46), we obtain

$$\begin{aligned}|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon((n+1)t_\varepsilon)|_\varepsilon^2 &\le \frac{1}{4}B_\varepsilon^2 \exp\left(-\frac{\nu t_\varepsilon}{2\varepsilon^2}\right) + \frac{1}{8}\beta^2(\varepsilon) \\ &\le \frac{1}{16}B_\varepsilon^2 + \frac{1}{8}\beta^2(\varepsilon) \le \frac{1}{4}B_\varepsilon^2, \\ |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon((n+1)t_\varepsilon)|_\varepsilon^2 &\le \frac{1}{4}K_\varepsilon^2 \exp(-\nu\lambda_1 t_\varepsilon) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &\quad + \frac{1}{32}\left(\frac{5}{4}\right)^2 B_\varepsilon^2\varepsilon^{1-q}(1+t_\varepsilon) \\ &\le \frac{1}{32}K_\varepsilon^2 + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} + \frac{1}{16}\left(\frac{5}{4}\right)^2 B_\varepsilon^2 \\ &\le \frac{1}{32}K_\varepsilon^2 + \frac{1}{8}B_\varepsilon^2 \le \frac{1}{4}K_\varepsilon^2.\end{aligned}$$

This proves the claim for $n+1$ and proves that $T^\sigma(\varepsilon) > nt_\varepsilon$ for all n provided (4.46) is satisfied. Hence $T^\sigma(\varepsilon) = \infty$ for $0 < \varepsilon \le \varepsilon_4$. We can state the following result

Theorem 4.1. *There exists $\varepsilon_4 = \varepsilon_4(\nu, q, \sigma)$ such that if u_0, f are given, $u_0 \in V_p^\varepsilon$, $f \in H_p^\varepsilon$, u_0, f satisfying* (4.1)–(4.4), *and $0 < \varepsilon \le \varepsilon_4$, then the strong solution u of* (0.1)–(0.3) *with periodic boundary conditions exists for all times, i.e. for all $T > 0$*

$$u^\varepsilon \in C([0,\infty), V_p^\varepsilon) \cap L^2(0,T; D(A_\varepsilon)).$$

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