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Asymptotic arbitrage in large financial markets*

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Abstract. A large financial market is described by a sequence of standard general models of continuous trading. It turns out that the absence of asymptotic arbitrage of the first kind is equivalent to the contiguity of sequence of objective probabilities with respect to the sequence of upper envelopes of equivalent martingale measures, while absence of asymptotic arbitrage of the second kind is equivalent to the contiguity of the sequence of lower envelopes of equivalent martingale measures with respect to the sequence of objective probabilities. We express criteria of contiguity in terms of the Hellinger processes. As examples, we study a large market with asset prices given by linear stochastic equations which may have random volatilities, the Ross Arbitrage Pricing Model, and a discrete-time model with two assets and infinite horizon. The suggested theory can be considered as a natural extension of Arbirage Pricing Theory covering the continuous as well as the discrete time case.

Key words: Large financial market, continuous trading, asymptotic arbitrage, APM, APT, semimartingale, optional decomposition, contiguity, Hellinger process

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1. Introduction

The main conclusion of the famous Capital Asset Pricing Model (CAPM) invented by Lintner and Sharp is the following: Assume that an asset i has mean excess return μ_i and variance σ_i^2, the market portfolio has mean excess return μ_0 and variance σ_0^2. Let γ_i be the correlation coefficient between the return on the asset i and the market portfolio. Then $\mu_i = \mu_0\beta_i$ where $\beta_i := \sigma_i\gamma_i/\sigma_0$. Though CAPM reveals this remarkable linear relation it has been under strong criticism, in particular, because empirical (β_i, μ_i) values do not follow in the precise manner the security market line. The alternative approach, the Arbitrage Pricing Model (APM), was suggested by Ross in [20]. Based on the idea of asymptotic arbitrage it has attracted considerable attention, see, e.g., [4], [5], [12], [13], and was extended to the Arbitrage Pricing Theory (APT). An important reference is the note by Huberman [11] (also reprinted in the volume "Theory of Valuation" [3][1]) who gave a rigorous definition of the asymptotic arbitrage as well as a short and transparent proof of the fundamental result of Ross.

In a one-factor version the APM is fairly simple. Assume that the discounted returns on assets are described as follows:

$$x^i = \mu_i + \beta_i\epsilon_0 + \eta_i$$

where the random variables ϵ_0 and η_i have zero mean, the η_i are orthogonal and their variances are bounded. Consider a sequence of "economies" or, better to say, "*market models*" such that the n-th model involves only the first n securities. The *arbitrage portfolio* in the n-th model is a vector $\varphi^n \in \mathbf{R}^n$ such that $\varphi^n e^n = 0$ with $e^n = (1, \ldots, 1) \in \mathbf{R}^n$. The return on the portfolio φ is

$$V(\varphi^n) = \varphi^n s^n$$

where $s^n = (x^1, \ldots, x^n)$. *Asymptotic arbitrage* is the existence of a subsequence of arbitrage portfolios $(\varphi^{n'})$ (i.e. portfolios with zero initial endowments) whose discounted returns satisfy the relations: $EV(\varphi^{n'}) \to \infty$, $\sigma^2(V(\varphi^{n'})) \to 0$. If there is no asymptotic arbitrage then there exists a constant μ_0 such that

$$\sum_{i=1}^{\infty}(\mu_i - \mu_0\beta_i)^2 < \infty.$$

This means that between the parameters there is the "approximately linear" relation $\mu_i \approx \mu_0\beta_i$. We shall discuss this model under some further restrictions in Section 6 and show that, in spite of the difference in definitions, the absence of asymptotic arbitrage always implies that the (β_i, μ_i)'s "almost" lay on the security market line.

[1] The reader can find a lot of relevant information in this book, which is a collection of the most significant papers published from 1973 to 1986 accompanied by original essays of experts in the field.

Note that the approach of APT is based on the assumption that agents have some risk-preferences and in the asymptotic setting they may accept the possibility of large losses with small probabilities; the variance is taken as an appropriate measure of risk.

A striking feature of the classic APT is that it completely ignores the problem of the existence of an equivalent martingale measure which is a key point of the Fundamental Theorem of Asset Pricing. In the modern dynamic setting an agent is absolutely risk-averse (at least, "asymptotically"), i.e. he considers as arbitrage opportunities only riskless strategies. This concept seems to be dominant in mathematical finance because of the great success of the Black–Scholes model where the no-arbitrage pricing is such that the option writer avoids any risk.

A problem of extension of APT to the intertemporal setting of continuous time finance was solved in our previous article [16] on the basis of an approach synthesizing ideas of both arbitrage theories; it was shown that the Ross pricing bound has a natural analog in terms of the boundedness of the Hellinger process.

In this paper we continue to study asymptotic arbitrage in the framework of continuous trading (including discrete time multi-stage models as a particular case). On an informal level one can think about a "real-world" financial market with a "large" (unbounded) number of traded securities. An investor is faced with the problem of choosing a "reasonably large" number n of securities to make a self-financing portfolio. Starting from an initial endowment V_0^n, a trading strategy φ leads to the final value $V_T^n(\varphi)$ where the strategy φ and the time horizon T also depend on n. If an "infinitesimally" small endowment gives an "essential" gain with a positive probability (without any losses) we say that there exists an asymptotic arbitrage. To give a precise meaning to the above notions, it seems natural to consider an approximation of a "real-world" market by a sequence of models (i.e. filtered probability spaces with semimartingales describing dynamics of prices of chosen securities) rather than a fixed model as in the traditional theory. Such a device is of common use in mathematical statistics and results of the latter can be applied in a financial context.

In [16] we formalized the concept of a large financial market and introduced the notions of asymptotic arbitrage of the first and second kind. It was shown that under the assumption of completeness of any particular market model the absence of asymptotic arbitrage of the first kind is equivalent to the contiguity of the sequence of the "objective" (reference) probabilities with respect to the sequence of the equivalent martingale measures (which is unique in the complete case). The criterion of the absence of asymptotic arbitrage of the second kind is symmetric: the contiguity of the sequence of the equivalent martingale measures with respect to the sequence of the "objective" probabilities. A theory of contiguity of probability measures on filtered spaces is well-developed (for a nice and complete exposition see [14]); it was applied in [16] to a particular model which can be referred to as a "large Black–Scholes market".

In recent work Klein and Schachermayer [17] made essential progress by extending the no-arbitrage criteria to the incomplete case though under the restriction that the price processes are locally bounded. They discovered that there is no asymptotic arbitrage of the first kind if and only if the sequence of the "objective" probabilities is contiguous with respect to *some* sequence of equivalent martingale measures. They also proved the surprising result that the corresponding criteria for the absence of asymptotic arbitrage of the second kind is not a symmetric version of the latter and involves a certain "ε-δ condition".

Here we continue to develop the theory initiated in [16] starting with some ramifications and extensions of results of Klein and Schachermayer [17] and polishing up simultaneously their original proofs by applications of the minimax theorem. We introduce alternative criteria relating the absence of arbitrage with contiguity of upper and lower envelopes of equivalent martingale measures; these criteria look fairly symmetric, cf. the conditions (b) of Propositions 2 and 3, but, of course, upper and lower envelopes are set functions with radically different properties. We also show that asymptotic arbitrage with probability one ("strong AA") is related to the (entire) asymptotic separation of the sequences of the "objective" probabilities and the envelopes of equivalent martingale measures. The main tool in our analysis is the so called optional decomposition theorem (see [8], [19], [9] for its successive development) which can be useful in the theory of incomplete markets as a source of trading strategies. This theorem allows us easily to get the mentioned criteria without any restrictions on the price processes. However, the equivalence of the new criteria and those of [17] is nontrivial. We established it as a corollary of rather general facts from a "contiguity theory" for sequences of convex sets of probability measures; this refined setting (which does not involve stochastic integration) is studied in Section 3. It should be pointed out that the essential ingredient of our proofs of difficult implications is basically the same as in [17]: we look at the problem in an abstract dual setting and apply some arguments based on a separation theorem. The simplification in our paper comes from a judicious use of the minimax theorem; this replaces some of the direct and bare-hands arguments used in [17]. Criteria of contiguity and asymptotic separation in terms of the Hellinger integrals similar to that of the classic theory are proved.

Section 4 is devoted to an extension of the "contiguity theory on filtered spaces" based on the concept of the Hellinger process which is especially important for use of the general results in the specific context of financial models. As an application, in Section 5 a problem of asymptotic arbitrage is studied for a large market where stock price evolution is given by linear stochastic differential equations which may have random coefficients. Under a certain assumption on a correlation structure of the driving Wiener processes we get effective criteria of absence of asymptotic arbitrage or existence of asymptotic arbitrage with probability one. Further applications are given in Sections 6 and 7 where we treat a one-stage model with an infinite number of assets (which is the one-factor APM with a particular correlation structure when there is a "basic" source of randomness) and a discrete-time model with two assets and infinite horizon. We show that in spite of the difference in the definitions of asymptotic arbitrage our approach gives results which are consistent with the traditional APT.

Notice that in the discrete-time setting a semimartingale is simply an adapted process and there are absolutely no problems with stochastic integrals. Therefore we hope that the major part of the paper concerning financial modeling (especially Sections 2 and 6) will be accessible to the reader with a standard probabilistic background.

2. Asymptotic arbitrage and contiguity of martingale measures

Let $\mathbf{B}^n = (\Omega^n, \mathscr{F}^n, \mathbf{F}^n = (\mathscr{F}_t^n), P^n),\ n \in \mathbf{N}$, be a stochastic basis, i.e. a filtered probability space satisfying the usual assumptions, see, e.g., [14] (this book is also our main reference for contiguity, Hellinger integrals, and Hellinger processes). For simplicity we assume that the initial σ-algebra is trivial (up to P^n-null sets). Asset prices evolve accordingly to a semimartingale $S^n = (S_t^n)_{t \le T^n}$ defined on $\mathbf{B}^n$ and taking values in $\mathbf{R}^d$ for some $d = d(n)$.

We fix a sequence T^n of positive numbers which are interpreted as time horizons. To simplify notation we shall often omit the superscript and write T.

We shall say that the triple $(\mathbf{B}^n, S^n, T^n)$ is a *security market model* and that the sequence $\mathbf{M} = \{(\mathbf{B}^n, S^n, T^n)\}$ is a *large financial market*.

We assume that there exists an asset whose price is constant over time and that all other prices are calculated in units of this asset. Markets are frictionless and admit shortselling.

We denote by $\mathscr{Q}^n$ the set of all probability measures Q^n equivalent to P^n and such that the process $(S_t^n)_{t \le T}$ is a local martingale with respect to Q^n; we refer to $\mathscr{Q}^n$ as the set of local martingale measures. Certainly, it may happen that $\mathscr{Q}^n$ is empty. The existence of a measure $Q^n \in \mathscr{Q}^n$ is closely related to the absence of arbitrage on the market $(\mathbf{B}^n, S^n, T^n)$, while the uniqueness is the property connected with the completeness of the market (see the pioneering paper [10] and, for a modern treatment, [6] and references therein).

Our main assumption is that the sets $\mathscr{Q}^n$ are nonempty for all n.

We define a trading strategy on $(\mathbf{B}^n, S^n, T^n)$ as a predictable process φ^n with values in $\mathbf{R}^d$ such that the stochastic integral with respect to a semimartingale S^n

$$\varphi^n \cdot S_t^n = \int_0^t (\varphi_r^n, dS_r^n)$$

is well-defined on $[0, T]$. Notice that if the process $\varphi^n \cdot S^n$ is bounded from below (or from above) by some constant, it follows from the Emery–Ansel–Stricker theorem [1] that it is a local martingale on $[0, T]$ with respect to any $Q \in \mathscr{Q}^n$.

For a trading strategy φ^n and an initial endowment x^n the value process $V^n(\varphi^n)$ is given by

$$V_t^n(\varphi^n) = x^n + \varphi^n \cdot S_t^n = x^n + \int_0^t (\varphi_r^n, dS_r^n).$$

We shall include a positive number x^n (an initial endowment) in the definition of a trading strategy.

Definition 1 *A sequence of trading strategies* φ^n **realizes the asymptotic arbitrage of the first kind** *if*

1a) $V_t^n(\varphi^n) \geq 0$ *for all* $t \leq T$*;*
1b) $\lim_n V_0^n(\varphi^n) = 0$ *(i.e.* $\lim_n x^n = 0$*);*
1c) $\lim_n P^n(V_T^n(\varphi^n) \geq 1) > 0$.

Definition 2 *A sequence of trading strategies* φ^n **realizes the asymptotic arbitrage of the second kind** *if*

2a) $V_t^n(\varphi^n) \leq 1$ *for all* $t \leq T$*;*
2b) $\lim_n V_0^n(\varphi^n) > 0$*;*
2c) $\lim_n P^n(V_T^n(\varphi^n) \geq \varepsilon) = 0$ *for any* $\varepsilon > 0$.

Definition 3 *A sequence of trading strategies* φ^n **realizes the strong asymptotic arbitrage of the first kind** *(SAA1) if*

3a) $V_t^n(\varphi^n) \geq 0$ *for all* $t \leq T$*;*
3b) $\lim_n V_0^n(\varphi^n) = 0$ *(i.e.* $\lim_n x^n = 0$*);*
3c) $\lim_n P^n(V_T^n(\varphi^n) \geq 1) = 1$.

Notice that 3a) and 3b) are the same as 1a) and 1b).

Definition 4 *A sequence of trading strategies* φ^n **realizes the strong asymptotic arbitrage of the second kind** *(SAA2) if*

4a) $V_t^n(\varphi^n) \leq 1$ *for all* $t \leq T$*;*
4b) $\lim_n V_0^n(\varphi^n) = 1$*;*
4c) $\lim_n P^n(V_T^n(\varphi^n) \geq \varepsilon) = 0$ *for any* $\varepsilon > 0$.

To achieve an "almost non-risk" profit from the arbitrage of the second kind, an investor sells short his portfolio. In the market there is a bound for the total debt value which we take to be equal to 1.

Remark. From a sequence of trading strategies realizing SAA1 it is easy to construct a sequence realizing SAA2 and vice versa. However, there is a slight difference between two concepts related to assumptions on the market regulations. In principle, one may impose a constraint that the total debt value should be equal to zero, or be infinitesimally small, or be bounded by a constant. Certainly, the first and the second variants exclude a game with the asymptotic arbitrage of the second kind.

Definition 5 *A large security market* $\mathbf{M} = \{(\mathbf{B}^n, S^n, T^n)\}$ *has* **no asymptotic arbitrage of the first kind** *(respectively, of the* **second kind***) if for any subsequence* (m) *there are no trading strategies* (φ^m) *realizing the asymptotic arbitrage of the first kind (respectively, of the second kind) for* $\{(\mathbf{B}^m, S^m, T^m)\}$.

To formulate the results we need to extend some notions from measure theory.

Let $\mathscr{Q} = \{Q\}$ be a family of probabilities on a measurable space $(\Omega, \mathscr{F})$. Define the upper and lower envelopes of the measures of $\mathscr{Q}$ as functions on $\mathscr{F}$ with $\overline{\mathbf{Q}}(A) := \sup_{Q \in \mathscr{Q}} Q(A)$ and $\underline{\mathbf{Q}}(A) := \inf_{Q \in \mathscr{Q}} Q(A)$, respectively. We say that

$\mathcal{Q}$ is dominated if any element of $\mathcal{Q}$ is absolutely continuous with respect to some fixed probability measure.

In our setting where for every n a family $\mathcal{Q}^n$ of equivalent local martingale measures is given we shall use the obvious notations $\overline{\mathbf{Q}}^n$ and $\underline{\mathbf{Q}}^n$.

Generalizing in a straightforward way the well-known notions of mathematical statistics (see, e.g., [14], p. 249) we introduce the following definitions:

Definition 6 *The sequence* (P^n) *is* **contiguous** *with respect to* $(\overline{\mathbf{Q}}^n)$ *(notation:* $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$*) when the implication*

$$\lim_{n\to\infty} \overline{\mathbf{Q}}^n(A^n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} P^n(A^n) = 0$$

holds for any sequence $A^n \in \mathcal{F}^n$, $n \geq 1$.

Evidently, $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$ iff the implication

$$\lim_{n\to\infty} \sup_{Q\in\mathcal{Q}^n} E_Q g^n = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} E_{P^n} g^n = 0$$

holds for any uniformly bounded sequence g^n of positive $\mathcal{F}^n$ -measurable functions.

Definition 7 *A sequence* (P^n) *is* **(entirely) asymptotically separable** *from* $(\overline{\mathbf{Q}}^n)$ *(notation:* $(P^n) \bigtriangleup (\overline{\mathbf{Q}}^n)$*) if there exists a subsequence* (m) *with sets* $A^m \in \mathcal{F}^m$ *such that*

$$\lim_{m\to\infty} \overline{\mathbf{Q}}^m(A^m) = 0, \quad \lim_{m\to\infty} P^m(A^m) = 1.$$

The notations $(\underline{\mathbf{Q}}^n) \triangleleft (P^n)$ and $(\underline{\mathbf{Q}}^n) \bigtriangleup (P^n)$ have the obvious meaning. It is clear that $(P^n) \bigtriangleup (\overline{\mathbf{Q}}^n)$ iff $(\underline{\mathbf{Q}}^n) \bigtriangleup (P^n)$.

We shall use the following result, [8], [19]:

Proposition 1 *Let* $\mathcal{Q}$ *be the set of local martingale measures for a semimartingale* S *and let* ξ *be a positive bounded random variable. Then there exists a positive process* X *with regular trajectories which is a supermartingale with respect to any* $Q \in \mathcal{Q}$ *such that*

$$X_t = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} E_Q(\xi \mid \mathcal{F}_t) \quad P\text{-a.s.}$$

Our approach is based on the optional decomposition theorem. This result is due to El Karoui and Quenez [8] in the case of continuous semimartingales and was proved for general locally bounded semimartingales in [19]. We use here a version taken from [9] where an alternative proof allows one to drop the assumption of local boundedness.

Theorem 1 *Let* $\mathcal{Q}$ *be the set of local martingale measures for a semimartingale* S*. Assume that a positive process* X *is a supermartingale with respect to every* $Q \in \mathcal{Q}$*. Then there exists an increasing right-continuous adapted process* C*,* $C_0 = 0$*, and an integrand* φ *such that* $X = X_0 + \varphi \cdot S - C$*.*

Now we formulate and prove the main results of this section.

Proposition 2 *The following conditions are equivalent:*
(a) there is no asymptotic arbitrage of the first kind (NAA1);
(b) $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$;
(c) there exists a sequence $R^n \in \mathscr{Q}^n$ *such that* $(P^n) \triangleleft (R^n)$.

Proof. $(b) \Rightarrow (a)$ Assume that (φ^n) is a sequence of trading strategies realizing the asymptotic arbitrage of the first kind. For any $Q \in \mathscr{Q}^n$ the process $V^n(\varphi^n)$ is a nonnegative local Q-martingale, hence a Q-supermartingale, and

$$\sup_{Q \in \mathscr{Q}^n} E_Q V_T^n(\varphi^n) \leq \sup_{Q \in \mathscr{Q}^n} E_Q V_0^n(\varphi^n) = x^n \to 0$$

by 1b). Thus,

$$\overline{\mathbf{Q}}^n(V_T^n(\varphi^n) \geq 1) := \sup_{Q \in \mathscr{Q}^n} Q(V_T^n(\varphi^n) \geq 1) \to 0$$

and, by virtue of the contiguity $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$, it follows that $P^n(V_T^n(\varphi^n) \geq 1) \to 0$ in contradiction with 1c).
$(a) \Rightarrow (b)$ Assume that (P^n) is not contiguous with respect to $(\overline{\mathbf{Q}}^n)$. Taking a subsequence, if necessary, we can find sets $\Gamma^n \in \mathscr{F}^n$ such that $\overline{\mathbf{Q}}^n(\Gamma^n) \to 0$, $P^n(\Gamma^n) \to \gamma$ as $n \to \infty$ where $\gamma > 0$. According to Proposition 1 there exists a regular process X^n which is a supermartingale with respect to any $Q \in \mathscr{Q}^n$ such that

$$X_t^n = \operatorname{ess\,sup}_{Q \in \mathscr{Q}^n} E_Q(I_{\Gamma^n} \mid \mathscr{F}_t^n) \quad P^n\text{-a.s.}$$

By Theorem 1 it admits a decomposition $X^n = X_0^n + \varphi^n \cdot S^n - C^n$ where φ^n is an integrand for S^n and C^n is an increasing process starting from zero. Let us show that $V^n(\varphi^n) := X_0^n + \varphi^n \cdot S^n$ are the value processes of portfolios realizing AA1. Indeed, $V^n(\varphi^n) = X^n + C^n \geq 0$,

$$V_0^n(\varphi^n) = \sup_{Q \in \mathscr{Q}^n} E_Q I_{\Gamma^n} = \overline{\mathbf{Q}}^n(\Gamma^n) \to 0,$$

and

$$\lim_n P^n(V_T^n(\varphi^n) \geq 1) \geq \lim_n P^n(X_T^n \geq 1) = \lim_n P^n(X_T^n = 1) = \lim_n P^n(\Gamma^n) = \gamma > 0.$$

$(b) \Leftrightarrow (c)$ This relation follows from the convexity of $\mathscr{Q}^n$ and Proposition 5 in Section 3 below. □

To formulate the next result we introduce

Definition 8 *The sequence of sets of probability measures* $(\mathscr{Q}^n)$ *is said to be* **weakly contiguous with respect to** (P^n) *(notation:* $(\mathscr{Q}^n) \triangleleft_w (P^n)$*) if for any* $\varepsilon > 0$ *there are* $\delta > 0$ *and a sequence of measures* $Q^n \in \mathscr{Q}^n$ *such that for any sequence* $A^n \in \mathscr{F}^n$ *with the property* $\limsup_n P^n(A^n) < \delta$ *we have* $\limsup_n Q^n(A^n) < \varepsilon$.

Remark. For the case when the sets $\mathcal{Q}^n$ are singletons containing the only measure Q^n the relation $(\mathcal{Q}^n) \lhd_w (P^n)$ means simply that $(Q^n) \lhd (P^n)$.

Obviously, the property $(\mathcal{Q}^n) \lhd_w (P^n)$ can be reformulated in terms of functions rather than sets:

for any $\varepsilon > 0$ *there are* $\delta > 0$ *and a sequence of measures* $Q^n \in \mathcal{Q}^n$ *such that for any sequence of* $\mathcal{F}^n$*-measurable random variables* g^n, $0 \le g^n \le 1$, *with the property* $\limsup_n E_{P^n} g^n < \delta$, *we have* $\limsup_n E_{Q^n} g^n < \varepsilon$.

Proposition 3 *The following conditions are equivalent:*

(a) there is no asymptotic arbitrage of the second kind (NAA2);

(b) $(\underline{\mathbf{Q}}^n) \lhd (P^n)$*;*

(c) $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$.

Proof. $(b) \Rightarrow (a)$ Assume that (φ^n) is a sequence of trading strategies realizing the asymptotic arbitrage of the second kind. By the contiguity $(\underline{\mathbf{Q}}^n) \lhd (P^n)$ it follows from 2c) that $\underline{\mathbf{Q}}^n(V_T^n(\varphi^n) \ge \varepsilon) \to 0$ or, equivalently, $\underline{\mathbf{Q}}^n([V_T^n(\varphi^n)]^+ \ge \varepsilon) \to 0$. Since $0 \le [V_T^n(\varphi^n)]^+ \le 1$ we have that

$$\inf_{Q \in \mathcal{Q}^n} E_Q [V_T^n(\varphi^n)]^+ \le \varepsilon + \underline{\mathbf{Q}}^n([V_T^n(\varphi^n)]^+ \ge \varepsilon)$$

and hence $\inf_{Q \in \mathcal{Q}^n} E_Q [V_T^n(\varphi^n)]^+ \to 0$ as $n \to \infty$. The process $[V^n(\varphi^n)]^+$ is a submartingale with respect to any $Q \in \mathcal{Q}^n$. Thus,

$$V_0^n(\varphi^n) \le [V_0^n(\varphi^n)]^+ \le \inf_{Q \in \mathcal{Q}^n} E_Q [V_T^n(\varphi^n)]^+ \to 0$$

contradicting 2b).

$(a) \Rightarrow (b)$ Assume that $(\underline{\mathbf{Q}}^n)$ is not contiguous with respect to (P^n). Taking a subsequence, if necessary, we can find sets $\Gamma^n \in \mathcal{F}^n$ such that $P^n(\Gamma^n) \to 0$ while $\underline{\mathbf{Q}}^n(\Gamma^n) \to \gamma > 0$ as $n \to \infty$. According to Proposition 1 (applied with $\xi = -I_{\Gamma^n}$) there exists a regular process X^n which is a submartingale with respect to any $Q^n \in \mathcal{Q}^n$ and

$$X_t^n = \operatorname{ess} \inf_{Q \in \mathcal{Q}^n} E_Q(I_{\Gamma^n} \mid \mathcal{F}_t^n) \quad P^n\text{-a.s.}$$

By Theorem 1 we have the decomposition $X^n = X_0^n + \varphi^n \cdot S^n + C^n$ where φ^n is an integrand for S^n and C^n is an increasing process starting from zero. To show that $V^n(\varphi^n) := X_0^n + \varphi^n \cdot S^n$ are the value processes of portfolios realizing AA2 we notice that $V^n(\varphi^n) = X^n - C^n \le 1$,

$$V_0^n(\varphi^n) = X_0^n = \inf_{Q \in \mathcal{Q}^n} E_Q I_{\Gamma^n} = \underline{\mathbf{Q}}^n(\Gamma^n) \to \gamma > 0,$$

and for any $\varepsilon \in]0, 1]$

$$\limsup_n P^n(V_T^n(\varphi^n) \ge \varepsilon) \le \lim_n P^n(X_T^n \ge \varepsilon) = \lim_n P^n(X_T^n = 1) = \lim_n P^n(\Gamma^n) = 0.$$

$(b) \Leftrightarrow (c)$ This equivalence follows from Proposition 6 in Section 3. □

Remark. The equivalence of (a) and (c) in Propositions 2 and 3 is the main result of [17] where it is proved under the assumption that S is locally bounded. Clearly, for the case where each $\mathcal{Q}^n$ is a singleton the condition $(\mathcal{Q}^n) \triangleleft_w (P^n)$ simply means contiguity. However, in general situation it may happen that $\mathcal{Q}^n$ does not contain a sequence (Q^n) such that $(Q^n) \triangleleft (P^n)$. For an example see [17].

Proposition 4 *The following conditions are equivalent:*

(a) there is a strong asymptotic arbitrage of the first kind (SAA1);

(b) $(P^n) \triangle (\overline{\mathbf{Q}}^n)$*;*

(c) there is a strong asymptotic arbitrage of the second kind (SAA2);

(d) $(\mathbf{Q}^n) \triangle (P^n)$*;*

(e) $(P^n) \triangle (Q^n)$ *for any sequence* $Q^n \in \mathcal{Q}^n$.

Proof. $(a) \Rightarrow (b)$ The existence of SAA1 means that along some subsequence (m) there are trading strategies such that $V_0^m(\varphi^m) \to 0$ but $P^m(V_T^m(\varphi^m) \geq 1) \to 1$. As in the proof of the implication $(b) \Rightarrow (a)$ of Proposition 2 we infer that $\overline{\mathbf{Q}}^m(V_T^m(\varphi^m) \geq 1) \to 0$ and hence the sets $\Gamma^m = \{V_T^m(\varphi^m) \geq 1\}$ form the desired separating subsequence.
$(b) \Rightarrow (a)$ To find a subsequence of trading strategies realizing SAA1 we use the same arguments as those in the proof of the implication $(a) \Rightarrow (b)$ of Proposition 2. The only difference is that in the present case we have $\gamma = 1$.

From any sequence realizing SAA1 it is easy to construct a sequence realizing SAA2 and vice versa. Hence, $(a) \Leftrightarrow (c)$. In Proposition 7 we show that $(b) \Leftrightarrow (e)$. Equivalence of (b) and (d) is clear. □

3. Contiguity and asymptotic separation

We start with a result which gives alternative descriptions of the property $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$.

Our proof uses the minimax theorem, see, e.g., [2]:

Theorem 2 *Let* $f : X \times Y \to \mathbf{R}$ *be a real-valued function, let* X *be a compact convex subset of a vector space, and let* Y *be a convex subset. Assume that*

1) for any $y \in Y$ *the function* $x \mapsto f(x,y)$ *is convex and lower semicontinuous;*

2) for any $x \in X$ *the function* $y \mapsto f(x,y)$ *is concave.*

Then there exists $\bar{x} \in X$ *such that*

$$\sup_{y\in Y} f(\bar{x},y) = \sup_{y\in Y}\inf_{x\in X} f(x,y) = \inf_{x\in X}\sup_{y\in Y} f(x,y).$$

Proposition 5 *Assume that for any* $n \geq 1$ *we are given a probability space* $(\Omega^n, \mathscr{F}^n, P^n)$ *with a dominated family* $\mathcal{Q}^n$ *of probability measures. Then the following conditions are equivalent:*

(a) $(P^n) \triangleleft (\overline{\mathbf{Q}}^n)$*;*

(b) there is a sequence $R^n \in \operatorname{conv}\mathcal{Q}^n$ *such that* $(P^n) \triangleleft (R^n)$*;*

(c) the following equality holds:

$$\lim_{\alpha\downarrow 0}\liminf_{n\to\infty}\sup_{Q\in\operatorname{conv}\mathcal{Q}^n} H(\alpha,Q,P^n)=1,$$

where $H(\alpha,Q,P)=\int(dQ)^{\alpha}(dP)^{1-\alpha}$ *is the Hellinger integral of order* $\alpha\in]0,1[$*;*

(d) the following equality holds:

$$\lim_{K\to\infty}\limsup_{n\to\infty}\inf_{Q\in\operatorname{conv}\mathcal{Q}^n} P^n(dP^n/dQ\geq K)=0.$$

Proof. The implication $(b)\Rightarrow(a)$ is trivial while the implication $(b)\Rightarrow(c)$ is a corollary of the criterion of contiguity $(P^n)\lhd(R^n)$ in terms of the Hellinger integrals, see [14].

$(c)\Rightarrow(d)$ To prove this part we recall some notation concerning the Hellinger integrals. Let $P,\ Q$ be two probabilities on some measurable space, $\nu=(P+Q)/2$, $z_P=dP/d\nu$, $z_Q=dQ/d\nu$. Then $Z=z_P/z_Q$ is the density of the absolutely continuous component of P with respect to Q. For $\alpha\in]0,1[$ put

$$d_H^2(\alpha,Q,P)=E_\nu\varphi_\alpha(z_Q,z_P)$$

where

$$\varphi_\alpha(u,v)=\alpha u+(1-\alpha)v-u^{\alpha}v^{1-\alpha}\geq 0,\quad u,\ v\geq 0.$$

It is usual to omit the parameter $\alpha=1/2$ in notation.

Notice that

$$\alpha(1-\alpha)\varphi(u,v)\leq\varphi_\alpha(u,v)\leq 8\varphi(u,v)=4(\sqrt{u}-\sqrt{v})^2. \tag{1}$$

Obviously, $d_H^2(\alpha,Q,P)=1-H(\alpha,Q,P)$ and (c) can be rewritten (in a more instructive way) as

$$\lim_{\alpha\downarrow 0}\limsup_{n\to\infty}\inf_{Q\in\operatorname{conv}\mathcal{Q}^n} d_H^2(\alpha,Q,P^n)=0. \tag{2}$$

It is clear that for any $\alpha\in]0,1/2]$ there exists $K=K(\alpha)\geq 4$ such that for all $u,\ v\geq 0$ we have

$$\varphi_\alpha(u,v)I_{\{v\geq Ku\}}\geq\varphi_{1/2}(u,v)I_{\{v\geq Ku\}}=(1/2)\Big(\sqrt{u}-\sqrt{v}\Big)^2 I_{\{v\geq Ku\}}.$$

It follows that

$$d_H^2(\alpha,Q,P)\geq E_\nu\varphi_\alpha(z_Q,z_P)I_{\{z_P\geq Kz_Q\}}\geq E_\nu\varphi(z_Q,z_P)I_{\{z_P\geq Kz_Q\}}=$$

$$=(1/2)E_P\left(\sqrt{1/Z}-1\right)^2 I_{\{Z\geq K\}}\geq(1/8)P(Z\geq K).$$

Applying the resulting inequality

$$P(Z\geq K)\leq 8d_H^2(\alpha,Q,P) \tag{3}$$

to the case when $P=P^n$ and Q is an arbitrary element of $\operatorname{conv}\mathcal{Q}^n$ we get that

$$\lim_{K\uparrow\infty}\limsup_{n\to\infty}\inf_{Q\in\operatorname{conv}\mathcal{Q}^n}P^n(Z\geq K)\leq 8\limsup_{n\to\infty}\inf_{Q\in\operatorname{conv}\mathcal{Q}^n}d_H^2(\alpha,Q,P^n). \tag{4}$$

Thus, (2) implies

$$\lim_{K\uparrow\infty}\limsup_{n\to\infty}\inf_{Q\in\operatorname{conv}\mathcal{Q}^n}P^n(Z\geq K)=0.$$

$(d)\Rightarrow(a)$ From the Lebesgue decomposition it follows that

$$P^n(A^n)=E_Q ZI_{\{A^n,\ Z<K\}}+P^n(A^n,\ Z\geq K)\leq KQ(A^n)+P^n(\ Z\geq K)\leq$$

$$\leq K\overline{\mathbf{Q}}^n(A^n)+P^n(\ Z\geq K).$$

Therefore,

$$P^n(A^n)\leq K\overline{\mathbf{Q}}^n(A^n)+\inf_{Q\in\operatorname{conv}\mathcal{Q}^n}P^n(\ Z\geq K).$$

If $\overline{\mathbf{Q}}^n(A^n)\to 0$ then

$$\limsup_{n\to\infty}P^n(A^n)\leq\limsup_{n\to\infty}\inf_{Q\in\operatorname{conv}\mathcal{Q}^n}P^n(\ Z\geq K)\to 0,\quad K\to\infty.$$

$(a)\Rightarrow(b)$ Without loss of generality we can assume that all the measures are defined on a unique measurable space $(\Omega,\mathcal{F})$ and are dominated by a fixed probability μ. For such measures we shall consider the topology induced by $L^1(\mu)$-convergence of their densities with respect to μ.

Put

$$D^{n,\varepsilon}=\{h\in L^\infty(\mu):\ E_{P^n}h\geq\varepsilon,\ 0\leq h\leq 1\}.$$

This set is convex and closed in $\sigma(L^\infty(\mu),L^1(\mu))$.

It is easy to check that

$$(P^n)\lhd(\overline{\mathbf{Q}}^n)\quad\Longleftrightarrow\quad\limsup_{n\to\infty}\inf_{h\in D^{n,\varepsilon}}\sup_{Q\in\operatorname{conv}\mathcal{Q}^n}E_Q h>0\quad\text{for all }\varepsilon>0.$$

By the minimax theorem the condition on the right-hand side is equivalent to

$$\limsup_{n\to\infty}\sup_{Q\in\operatorname{conv}\mathcal{Q}^n}\inf_{h\in D^{n,\varepsilon}}E_Q h>0\quad\text{for all }\varepsilon>0.$$

Hence, for any $\varepsilon_k=1/k$ there is a sequence of probability measures R^{n,ε_k} from $\operatorname{conv}\mathcal{Q}^n$ such that

$$\limsup_{n\to\infty}\inf_{h\in D^{n,\varepsilon}}E_{R^{n,\varepsilon_k}}h=\gamma_k>0.$$

Put

$$R^n=\frac{1}{1-2^{-n-1}}\sum_{k=1}^n 2^{-k}R^{n,\varepsilon_k}.$$

Evidently, for all $\varepsilon=1/k$ (and hence for all ε) we have that

$$\limsup_{n\to\infty}\inf_{h\in D^{n,\varepsilon}}E_{R^n}h>0$$

which is equivalent to contiguity $(P^n)\lhd(R^n)$. $\square$

Remark. The well-known Halmos–Savage lemma asserts that $\mathcal{Q}$ is a dominated family of measures iff it contains an equivalent countable subset. This implies the following qualitative corollary: If $\mathcal{Q}$ is a dominated family and P is a probability on $(\Omega, \mathcal{F})$ such that $P \ll \overline{\mathbf{Q}}$, then there is a countable convex combination R of elements of $\mathcal{Q}$ such that $P \ll R$.

The implication $(a) \Rightarrow (b)$ in Proposition 5 is an asymptotic version of this corollary. Both a quantitative and a dual version of the above corollary are proved in [18], and these general results are then used to derive the no-arbitrage criteria of [17].

Proposition 6 *Assume that for any $n \geq 1$ we are given a probability space $(\Omega^n, \mathcal{F}^n, P^n)$ with a convex dominated set $\mathcal{Q}^n$ of probability measures. Then the following conditions are equivalent:*

(a) $(\underline{\mathbf{Q}}^n) \lhd (P^n)$;

(b) $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$;

(c) the following equality holds:

$$\lim_{\alpha \downarrow 0} \liminf_{n \to \infty} \sup_{Q \in \mathcal{Q}^n} H(\alpha, P^n, Q) = 1;$$

(d) the following equality holds:

$$\lim_{K \to \infty} \limsup_{n \to \infty} \inf_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geq K\right) = 0.$$

Proof. $(a) \Leftrightarrow (b)$ Again we can suppose that all measures are dominated by a unique measure μ. Let us consider the set $B^{n,\delta} = \{g : \ E_{P^n} g \leq \delta, \ 0 \leq g \leq 1\}$ which is convex and closed in $\sigma(L^\infty(\mu), L^1(\mu))$. Since

$$(\underline{\mathbf{Q}}^n) \lhd (P^n) \quad \Longleftrightarrow \quad \limsup_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{g \in B^{n,\delta}} \inf_{Q \in \mathcal{Q}^n} E_Q g = 0,$$

$$(\mathcal{Q}^n) \lhd_w (P^n) \quad \Longleftrightarrow \quad \limsup_{\delta \downarrow 0} \limsup_{n \to \infty} \inf_{Q \in \mathcal{Q}^n} \sup_{g \in B^{n,\delta}} E_Q g = 0,$$

the assertion follows immediately from the minimax theorem.

$(b) \Rightarrow (d)$ By the Chebyshev inequality

$$\sup_{Q \in \mathcal{Q}^n} P^n\left(\frac{dQ}{dP^n} \geq K\right) \leq 1/K.$$

With this remark the assertion follows directly from the definition of weak contiguity.

$(d) \Rightarrow (c)$ From the elementary inequality

$$\varphi_\alpha(u, v) \leq 8\alpha \ln K \varphi(u, v) I_{\{v \leq Ku\}} + 8\varphi(u, v) I_{\{v > Ku\}}$$

which holds when $K \geq e$, we deduce that for any $Q \in \mathcal{Q}^n$

$$d_H^2(\alpha, P^n, Q) \leq 8\alpha \ln K d_H^2(P^n, Q) + 8E_\nu \varphi(z_{P^n}, z_Q) I_{\{z_Q > K z_{P^n}\}} \leq$$

$$\leq 8\alpha \ln K + 4E_Q \left(\sqrt{\frac{dP^n}{dQ}} - 1\right)^2 I_{\{dQ/dP^n \geq K\}} \leq 8\alpha \ln K + 4Q\left(\frac{dQ}{dP^n} \geq K\right). \tag{5}$$

Thus,

$$\lim_{\alpha\downarrow 0} \limsup_{n\to\infty} \inf_{Q\in\mathscr{Q}^n} d_H^2(\alpha, P^n, Q) \leq 4 \limsup_{n\to\infty} \inf_{Q\in\mathscr{Q}^n} Q\left(\frac{dQ}{dP^n} \geq K\right)$$

yielding the result.
$(c) \Rightarrow (d)$ The reasoning follows the same line as in the corresponding implication of Proposition 5. By (3) we have that for any $\alpha \in]0, 1/2]$ there exists $K \geq 4$ such that for any $Q \in \mathscr{Q}^n$

$$Q\left(\frac{dQ}{dP^n} \geq K\right) \leq 8d_H^2(\alpha, P^n, Q).$$

Hence,

$$\limsup_{K\to\infty} \limsup_{n\to\infty} \inf_{Q\in\mathscr{Q}^n} Q\left(\frac{dQ}{dP^n} \geq K\right)$$
$$\leq 8 \limsup_{n\to\infty} \inf_{Q\in\mathscr{Q}^n} d_H^2(\alpha, P^n, Q) \to 0, \quad \alpha \downarrow 0,$$

by the assumption (c).
$(d) \Rightarrow (a)$ Since

$$\underline{\mathbf{Q}}^n(A^n) \leq KP^n(A^n) + \inf_{Q\in\mathscr{Q}^n} Q\left(\frac{dQ}{dP^n} \geq K\right),$$

we see that when $P^n(A^n) \to 0$ we also have $\underline{\mathbf{Q}}^n(A^n) \to 0$. □

Now we prove the criteria for asymptotic separation.

Proposition 7 *Assume that for any $n \geq 1$ the convex family $\mathscr{Q}^n$ of probability measures is dominated. Then the following conditions are equivalent:*

(a) $(P^n) \triangle (\overline{\mathbf{Q}}^n)$;
(b) $(P^n) \triangle (Q^n)$ *for any sequence* $Q^n \in \mathscr{Q}^n$;
(c) for some $\alpha \in]0, 1[$

$$\liminf_{n\to\infty} \sup_{Q\in\mathscr{Q}^n} H(\alpha, Q, P^n) = 0;$$

(d) the above equation holds for all $\alpha \in]0, 1[$;
(e) for all $\varepsilon > 0$

$$\liminf_{n\to\infty} \sup_{Q\in\mathscr{Q}^n} P^n\left(\frac{dQ}{dP^n} \geq \varepsilon\right) = 0$$

(f) $(\underline{\mathbf{Q}}^n) \triangle (P^n)$.

Proof. $(a) \Leftrightarrow (b)$ Let U^n be the unit ball in $L^\infty(\mu^n)$ with center at zero where μ^n is a measure dominating P^n and $\mathcal{Q}^n$. Let d_V be the total variation distance. Notice that

$$\limsup_n d_V(P^n, \mathcal{Q}^n) = 2 \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \inf_{Q\in\mathcal{Q}^n} \sup_{g\in U^n} (E_Q g - E_{P^n} g) = 2,$$

$$(P^n) \triangle (\overline{\mathbf{Q}}^n) \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \sup_{g\in U^n} \inf_{Q\in\mathcal{Q}^n} (E_Q g - E_{P^n} g) = 2.$$

An application of the minimax theorem shows that (a) holds iff $\limsup_n d_V(P^n, \mathcal{Q}^n) = 2$ or, equivalently, iff $\limsup_n d_V(P^n, Q^n) = 2$ for every $Q^n \in \mathcal{Q}^n$; the latter condition is equivalent to (b).

The equivalence of (c), (d), and (e) is because of the following easily verified bounds ([14], V.1.7, V.1.8):

$$\left(\frac{\varepsilon}{2}\right)^\alpha P^n\left(\frac{dQ}{dP^n} \geq \varepsilon\right) \leq H(\alpha, Q, P^n) \leq 2\varepsilon^\alpha + 2\delta^{1-\alpha} + \left(\frac{2}{\delta}\right)^\alpha P^n\left(\frac{dQ}{dP^n} \geq \varepsilon\right).$$

The relation $(b) \Leftrightarrow (c)$ follows from the well-known inequalities (see, e.g., [14], Prop. V.4.4)

$$2(1 - H(Q, P^n)) \leq d_V(P^n, Q) \leq 2\sqrt{1 - H^2(Q, P^n)}.$$

The equivalence of (a) and (f) is obvious. □

4. Contiguity and asymptotic separation on filtered spaces

Now we again consider the situation which interests us most, with a given dominated family $\mathcal{Q}^n$ on a stochastic basis $\mathbf{B}^n = (\Omega^n, \mathcal{F}^n, \mathbf{F}^n = (\mathcal{F}_t^n), P^n)$. We now use the notation z_{P^n}, z_Q for the density processes (or local densities) of P^n and Q with respect to $\nu = (P^n + Q)/2$. Then the process $Z = Z^n_Q = z_{P^n}/z_Q$ is the density process of the absolutely continuous component of P^n with respect to Q. Notice that we can add to the list of equivalent conditions in Proposition 5 the following condition:

$(6.d')$ the following equality holds:

$$\lim_{K\to\infty} \limsup_{n\to\infty} \inf_{Q\in \operatorname{conv}\mathcal{Q}^n} P^n(Z^*_Q \geq K) = 0$$

where $Z^* = \sup_t Z_t$.

We can add to the formulation of Proposition 6 in a similar way.

For $\alpha \in]0,1[$ and a pair of probability measures Q and P given on a filtered space the Hellinger process $h(\alpha, Q, P)$ is defined in the following way, see [14]. Let $Y(\alpha) = z_P^\alpha z_Q^{1-\alpha}$. Obviously, $Y(\alpha)$ is a bounded ν-supermartingale, $\nu = (P+Q)/2$. It admits the multiplicative decomposition $Y(\alpha) = M(\alpha)\mathcal{E}(-h(\alpha))$ where $M(\alpha)$ is a local ν-martingale until the moment σ when $Y(\alpha)$ hits zero,

$h(\alpha)$ is a predictable increasing process uniquely defined until σ, $\mathscr{E}(-h(\alpha))$ denotes the Doléans exponential, i.e. the solution of the linear equation

$$\mathscr{E}(-h(\alpha)) = 1 - \mathscr{E}_-(-h(\alpha)) \circ h(\alpha),$$

$\circ$ denotes integration with respect to an increasing process. Such a process $h(\alpha) = h(\alpha, Q, P)$ is called the Hellinger process of order α (the parameter $\alpha = 1/2$ is usually omitted). The Doob–Meyer additive decomposition of $Y(\alpha)$ can be written in the following specific form:

$$Y(\alpha) = 1 - Y_-(\alpha) \circ h(\alpha) + M(\alpha). \tag{6}$$

It can be shown that

$$E_\nu [Y_-(\alpha) \circ h(\alpha)_\infty]^2 \leq 4. \tag{7}$$

Indeed, let $A := Y_-(\alpha) \circ h(\alpha)$ and $N_t := Y_t(\alpha) - E(Y_\infty(\alpha)|\mathscr{F}_t)$. Then $N_t = E(A_\infty|\mathscr{F}_t) - A_t$, i.e. N is the potential generated by the predictable increasing process A. Clearly, $EA_\infty \leq 1$, $N \leq 2$, and the inequality (7) follows from the energy formula $EA^2_\infty = E(N + N_-) \circ A_\infty$, see [7], VI.94.

The theorems below are generalizations of the Liptser–Shiryaev criteria of contiguity of sequences of probability measures on filtered spaces, [14], Theorem V.2.3.

Theorem 3 *The following conditions are equivalent:*

(a) $(P^n) \lhd (\overline{\mathbf{Q}}^n)$*;*

(b) for all $\varepsilon > 0$

$$\lim_{\alpha \downarrow 0} \limsup_{n \to \infty} \inf_{Q \in \operatorname{conv} \mathscr{Q}^n} P^n(h_\infty(\alpha, Q, P^n) \geq \varepsilon) = 0.$$

Proof. $(a) \Rightarrow (b)$ By Proposition 5 the condition (a) is equivalent to the existence of a sequence $R^n \in \operatorname{conv} \mathscr{Q}^n$ such that (P^n) is contiguous with respect to (R^n). An application of the Liptser–Shiryaev theorem gives the result.

$(b) \Rightarrow (a)$ The desired assertion is an easy consequence of (2) and of the inequality given by the following lemma. □

Lemma 1 *For any* $\alpha \in]0, 1/4[$*,* $\eta \in]0, 1[$*, and* $\varepsilon > 0$

$$d_H^2(\alpha, Q, P) \leq 16\eta^{1/4} + 2\eta^{-\alpha}\varepsilon + 2\sqrt{2}\eta^{-1}\{P(h_\infty(\alpha, Q, P) \geq \varepsilon)\}^{1/2}. \tag{8}$$

Proof. Let $\Gamma = \{z_{P-} \leq \eta\}$ and let $\xi(\alpha) = z^\alpha_{Q-} z^{-\alpha}_{P-} \circ h(\alpha)$ where $h(\alpha) = h(\alpha, Q, P)$. Taking the mathematical expectation with respect to ν of the additive decomposition (6) we deduce that

$$d_H^2(\alpha, Q, P) = E_\nu z^\alpha_{Q-} z^{1-\alpha}_{P-} \circ h(\alpha)_\infty = E_P \xi_\infty(\alpha).$$

On the set Γ

$$z^\alpha_{Q-} z^{-\alpha}_{P-} \leq 2 z_{P-}^{-\alpha - 1/4} \eta^{1/4} \leq 2\eta^{1/4} z_{P-}^{-1/2} \leq 2\eta^{1/4} z^{1/2}_{Q-} z^{-1/2}_{P-}.$$

By the second inequality in (1) the difference $8h - h(\alpha)$ is an increasing process. Hence,

$$E_P I_\Gamma \circ \xi(\alpha)_\infty \le 16\eta^{1/4} E_P \xi_\infty \le 16\eta^{1/4}. \tag{9}$$

Using the bound (7), we get that

$$E_P[I_{\bar\Gamma} \circ \xi(\alpha)_\infty]^2 \le 2E_\nu[I_{\bar\Gamma} z^\alpha_{Q-} z^{-\alpha}_{P-}(z_{P-}/\eta) \circ h(\alpha)_\infty]^2 \le 8\eta^{-2}. \tag{10}$$

Thus,

$$E_P I_{\bar\Gamma} \circ \xi(\alpha)_\infty \le 2\eta^{-\alpha}\varepsilon + E_P I_{\{h_\infty(\alpha)\ge\varepsilon\}} I_{\bar\Gamma} \circ \xi(\alpha)_\infty \le$$
$$\le 2\eta^{-\alpha}\varepsilon + \{E_P[I_{\bar\Gamma} \circ \xi(\alpha)_\infty]^2\}^{1/2}\{P(h_\infty(\alpha) \ge \varepsilon)\}^{1/2}.$$

The bound (8) holds by virtue of (9), (10), and the above inequality. □

Theorem 4 *Assume that the family $\mathcal{Q}^n$ is convex and dominated for any n. Then the following conditions are equivalent:*

(a) $(\underline{\mathbf{Q}}^n) \lhd (P^n)$;

(b) for all $\varepsilon > 0$

$$\lim_{\alpha\downarrow 0} \limsup_{n\to\infty} \inf_{Q\in\mathcal{Q}^n} Q(h_\infty(\alpha, P^n, Q) \ge \varepsilon) = 0.$$

Proof. $(a) \Rightarrow (b)$ Since $d^2_H(\alpha, P^n, Q) = E_Q z^\alpha_{P^n-} z^{-\alpha}_{Q-} \circ h(\alpha, P^n, Q)_\infty$ we have for any $K > 1$ and $\varepsilon > 0$ that

$$d^2_H(\alpha, P^n, Q) \ge \varepsilon \frac{1}{K^\alpha}\left[Q(h_\infty(\alpha, P^n, Q) \ge \varepsilon) - Q\left(\sup_t \frac{dQ_t}{dP^n_t} \ge K\right)\right].$$

From the other hand, by (5) for $K \ge e$

$$d^2_H(\alpha, P^n, Q) \le 8\alpha \ln K + 4Q\left(\frac{dQ_\infty}{dP^n_\infty} \ge K\right).$$

Hence,

$$\begin{aligned} Q(h_\infty(\alpha, P^n, Q) \ge \varepsilon) \quad &\le \quad Q\left(\sup_t \frac{dQ_t}{dP^n_t} \ge K\right) \\ &+ \frac{K^\alpha}{\varepsilon}\left[8\alpha \ln K + 4Q\left(\sup_t \frac{dQ_t}{dP^n_t} \ge K\right)\right]. \end{aligned}$$

Notice that

$$P^n\left(\sup_t \frac{dQ_t}{dP^n_t} \ge K\right) \le 1/K.$$

Let $\eta > 0$ be arbitrary. By (a) and the condition (d) of Proposition 6 there are a sufficiently large K and a sequence $Q^n \in \mathcal{Q}^n$ such that

$$\limsup_n Q^n\left(\sup_t \frac{dQ^n_t}{dP^n_t} \ge K\right) \le \eta$$

Therefore,

$$\limsup_{n\to\infty} \inf_{Q\in\mathcal{Q}^n} Q(h_\infty(\alpha, P^n, Q) \geq \varepsilon) \leq \eta + (K^\alpha/\varepsilon)[8\alpha \ln K + 4\eta]$$

and the condition (b) holds.
$(b) \Rightarrow (a)$ An application of Lemma 1 (with a correspondent adjustment of notations) together with the condition (c) of Proposition 6 gives the result. □

We complete this section by the following result concerning asymptotic separation where we assume that for any $n \geq 1$ the convex family $\mathcal{Q}^n$ of probability measures is dominated.

Theorem 5 *(a) If* $(P^n) \triangle (\overline{\mathbf{Q}}^n)$ *then*

$$\lim_{\eta\downarrow 0} \limsup_{\alpha\downarrow 0} \limsup_{n} \inf_{Q\in\mathcal{Q}^n} P^n(h_\infty(\alpha, Q, P^n) \geq \eta) = 1;$$

(b) if

$$\limsup_{n} \inf_{Q\in\mathcal{Q}^n} P^n(h_\infty(Q, P^n) \geq N) = 1$$

for all $N > 0$ *then* $(P^n) \triangle (\overline{\mathbf{Q}}^n)$.

Proof. (a) For any $\eta > 0$ and $\delta > 0$ the following inequality holds:

$$1 - H(\alpha, Q, P^n) \leq 2\eta + 2\delta^{1-\alpha} + \left(\frac{2}{\delta}\right)^\alpha P^n(h_\infty(\alpha, Q, P^n) \geq \eta),$$

see (V.2.25) in [14]. It implies the desired relation because, by Proposition 7, for all $\alpha \in, \ ]0,1[$

$$\liminf_{n\to\infty} \sup_{Q\in\mathcal{Q}^n} H(\alpha, Q, P^n) = 0.$$

(b) One can use the inequality

$$d_V(P^n, Q) \ \geq 2\Big(1 - \sqrt{E_{P^n} \exp\{-h_\infty(Q, P^n)\}}\Big),$$

see [14], Th. V.4.21. Since

$$\sup_{Q\in\mathcal{Q}^n} E_{P^n} \exp\{-h_\infty(Q, P^n)\} \leq e^{-N} + \sup_{Q\in\mathcal{Q}^n} P(h_\infty(Q, P^n) < N)$$

the assumption implies that $\limsup_n d_V(P^n, \mathcal{Q}^n) = 2$, and the assertion follows from Proposition 7. □

5. Example: the large BS-market

In the paper [16] we considered the problem of asymptotic arbitrage for a "large Black–Scholes market" where the dynamics of discounted asset prices were given by geometric Brownian motions with a certain correlation structure. Here we study a more general setting covering, in particular, a case of stochastic volatilities.

Let $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t), P)$ be a stochastic basis with a countable set of independent one-dimensional Wiener processes $w^i,\ i \in \mathbf{Z}_+,\ \mathbf{w}^n = (w^0, \ldots, w^n)$, and let $\mathbf{F}^n = (\mathscr{F}_t{}^n)$ be a subfiltration of $\mathbf{F}$ such that $(\mathbf{w}^n, \mathbf{F}^n)$ is a Wiener process in the sense that it is a martingale with $\langle \mathbf{w}^n \rangle_t = tI_{n+1}$ where I_{n+1} is the identity matrix. Notice that $\mathbf{F}^n$ may be wider than the filtration generated by $\mathbf{w}^n$.

The behavior of the stock prices is described by the following stochastic differential equations:

$$dX_t^0 = \mu_0 X_t^0 dt + \sigma_0 X_t^0 dw_t^0,$$
$$dX_t^i = \mu_i X_t^i dt + \sigma_i X_t^i (\gamma_i dw_t^0 + \bar{\gamma}_i dw_t^i), \quad i \in \mathbf{N},$$

with deterministic (strictly positive) initial points. The coefficients are $\mathbf{F}^i$-predictable processes,

$$\int_0^t |\mu_i(s)|^2 ds < \infty, \qquad \int_0^t |\sigma_i(s)|^2 ds < \infty$$

for t finite and $\gamma_i^2 + \bar{\gamma}_i^2 = 1$. To avoid degeneracy we shall assume that $\sigma_i > 0$ and $\bar{\gamma}_i > 0$.

Notice that the process ξ^i with

$$d\xi_t^i = \gamma_i dw_t^0 + \bar{\gamma}_i dw_t^i, \quad \xi_0^i = 0,$$

is a Wiener process. The model is designed to reflect the fact that in the market there are two different types of randomness: the first type is proper to each stock while the second one originates from some common source and it is accumulated in a "stock index" (or "market portfolio") whose evolution is described by the first equation.

Set

$$\beta_i := \frac{\gamma_i \sigma_i}{\sigma_0} = \frac{\gamma_i \sigma_i \sigma_0}{\sigma_0^2}.$$

In the case of deterministic coefficients, β_i is a well-known measure of risk which is the covariance between the return on the asset with number i and the return on the index, divided by the variance of the return on the index.

Let us consider the stochastic basis $\mathbf{B}^n = (\Omega, \mathscr{F}, \mathbf{F}^n = (\mathscr{F}_t{}^n)_{t \le T}, P^n)$ with the $(n+1)$-dimensional semimartingale $S^n := (X_t^0, X_t^1, \ldots, X_t^n)$ and $P^n := P|\mathscr{F}_T{}^n$. Assume for simplicity that the time horizon T does not depend on n. The sequence $\mathbf{M} = \{(\mathbf{B}^n, S^n, T)\}$ is a large security market. In our case each $\{(\mathbf{B}^n, S^n, T)\}$ is, in general, a model of an incomplete market as we do not suppose that $\mathbf{F}^n$ is generated by $\mathbf{w}^n$ and the set of equivalent martingale measures $\mathscr{Q}^n$ may have infinitely many points.

Let $\mathbf{b}_n(t) := (b_0(t), b_1(t), \ldots, b_n(t))$ where

$$b_0 := -\frac{\mu_0}{\sigma_0}, \quad b_i := \frac{\beta_i \mu_0 - \mu_i}{\sigma_i \bar{\gamma}_i}.$$

Assume that

$$\int_0^T |\mathbf{b}_n(t)|^2 dt < \infty$$

and $EZ_T(\mathbf{b}) = 1$ where the strictly positive random variable $Z_T(\mathbf{b})$ is the Girsanov exponential

$$Z_T(\mathbf{b}) := \exp\left\{\int_0^T (\mathbf{b}_n(t), d\mathbf{w}_t^n) - \frac{1}{2}\int_0^T |\mathbf{b}_n(t)|^2 dt\right\}$$

(e.g., these conditions are fulfilled for bounded $\mathbf{b}_n$ and finite T). In other words, $Z_T(\mathbf{b}) = dQ^n/dP^n$ where Q^n is a probability measure on $\mathcal{F}_T^n$ equivalent to P^n. By the Girsanov theorem the process

$$\tilde{\mathbf{w}}_t^n := \mathbf{w}_t^n - \int_0^t \mathbf{b}_n(s) ds$$

is Wiener under Q^n and, therefore, Q^n belongs to the set $\mathcal{Q}^n$ of equivalent (local) martingale measures.

Proposition 8 *The following conditions for the large financial market* **M** *are equivalent:*

(a) NAA1;

(b) $U_T < \infty$ P-a.s. where

$$U_T := \int_0^T \left[\left(\frac{\mu_0}{\sigma_0}\right)^2 + \sum_{i=1}^{\infty}\left(\frac{\mu_i - \beta_i \mu_0}{\sigma_i \bar{\gamma}_i}\right)^2\right] ds.$$

Proof. According to Proposition 2 and Theorem 3 the property NAA1 is equivalent to the following condition:

$$\lim_{\alpha \downarrow 0} \limsup_{n \to \infty} \inf_{Q \in \mathcal{Q}^n} P(h_T(\alpha, Q, P) \geq \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.$$

Under an arbitrary measure $Q \in \mathcal{Q}^n$ the process $\tilde{\mathbf{w}}^n$ is a local martingale with $\langle \tilde{\mathbf{w}}^n \rangle_t = tI_{n+1}$, i.e. a Wiener process. Set

$$h_T^{0n}(\alpha) := \frac{\alpha(1-\alpha)}{2} \int_0^T \left[\left(\frac{\mu_0}{\sigma_0}\right)^2 + \sum_{i=1}^{n}\left(\frac{\mu_i - \beta_i \mu_0}{\sigma_i \bar{\gamma}_i}\right)^2\right] ds.$$

By Theorem IV.3.39 in [14] we have the inequality $h_T(\alpha, Q, P^n) \geq h_T^{0n}(\alpha)$. Since $h_T^{0n}(\alpha) = h_T(\alpha, Q^n, P^n)$, the equivalence of (a) and (b) clearly follows. □

Proposition 9 *In the market* **M** *the following properties are equivalent:*

(i) there exists a strong asymptotic arbitrage (of the first and/or the second kind);

(ii) $U_T = \infty$ P-a.s.

Proof. $(ii) \Rightarrow (i)$ For any finite N

$$\limsup_n \inf_{Q\in\mathcal{Q}^n} P^n(h_T(Q,P^n) \geq N) = \limsup_n P^n(h_T^{0n} \geq N) = 1.$$

By Theorem 5 (b) we have $(P^n)\triangle(\overline{\mathbf{Q}}^n)$ and the assertion holds due to Proposition 4.
$(i) \Rightarrow (ii)$ If there is SAA1 then $(P^n) \triangle (\overline{\mathbf{Q}}^n)$ and by Theorem 5 (a)

$$\lim_{\eta\downarrow 0} \limsup_{\alpha\downarrow 0} \limsup_n \inf_{Q\in\mathcal{Q}^n} P^n(h_T(\alpha, Q, P^n) \geq \eta) = 1.$$

But for any $\eta > 0$

$$\inf_{Q\in\mathcal{Q}^n} P^n(h_T(\alpha, Q, P^n) \geq \eta) = P^n(h_T(\alpha, Q^n, P^n) \geq \eta)$$

and

$$\limsup_{\alpha\downarrow 0} \limsup_n P^n(h_T(\alpha, Q^n, P^n) \geq \eta) = P(U_T = \infty).$$

Thus, SAA1 implies that $P(U_T = \infty) = 1$. $\square$

Notice that in the case of deterministic coefficients (when U_T is deterministic) there is the alternative: either the market has the property NAA1 or there exists a strong asymptotic arbitrage. Moreover, the properties NAA1 and NAA2 hold simultaneously.

Remark. In the particular case of constant coefficients and finite T, the condition (b) of Proposition 8 can be written as

$$\sum_{i=1}^{\infty} \left(\frac{\mu_i - \beta_i \mu_0}{\sigma_i \bar{\gamma}_i} \right)^2 < \infty. \tag{11}$$

In the case where $0 < c \leq \sigma_i \bar{\gamma}_i \leq C$ the property NAA1 holds iff

$$\sum_{i=1}^{\infty} (\mu_i - \beta_i \mu_0)^2 < \infty. \tag{12}$$

This assertion has the same form as the famous result in the Ross arbitrage asset pricing theory, see [20]. Qualitatively, in the large financial market with absence of arbitrage the parameters (μ_i, β_i) lay close to the security market line $\mu = \mu_0 \beta$.

6. Example: one-stage APM by Ross

Let $(\epsilon_i)_{i\geq 0}$ be a sequence of independent random variables given on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in a finite interval $[-N, N]$, $E\epsilon_i = 0$, $E\epsilon_i^2 = 1$. At time zero asset prices are positive numbers X_0^i. After a certain period (at time $T = 1$) their discounted values are given by the following relations:

$$\begin{aligned} X_1^0 &= X_0^0(1+\mu_0+\sigma_0\epsilon_0), \\ X_1^i &= X_0^i(1+\mu_i+\sigma_i(\gamma_i\epsilon_0+\bar{\gamma}_i\epsilon_i)), \quad i \in \mathbf{N}. \end{aligned} \tag{13}$$

The coefficients here are deterministic, $\sigma_i > 0$, $\bar{\gamma}_i > 0$ and $\gamma_i^2 + \bar{\gamma}_i^2 = 1$. The asset with number zero is interpreted as a market portfolio, γ_i is the correlation coefficient between the rate of return for the market portfolio and the rate of return for the asset with number i.

For $n \geq 0$ we consider the stochastic basis $\mathbf{B}^n = (\Omega, \mathscr{F}^n, \mathbf{F}^n = (\mathscr{F}_i^n)_{i\in\{0,1\}}, P^n)$ with the $(n+1)$ -dimensional random process $S^n = (X_i^0, X_i^1, \ldots, X_i^n)_{i\in\{0,1\}}$ where $\mathscr{F}_0^n$ is the trivial σ-algebra, $\mathscr{F}_1^n = \mathscr{F}^n := \sigma\{\epsilon_0, ..., \epsilon_n\}$, and $P^n = P|\mathscr{F}^n$. The sequence $\mathbf{M} = \{(\mathbf{B}^n, S^n, 1)\}$ is a large security market by our definition.

Let $\beta_n := \gamma_n\sigma_n/\sigma_0$ and define

$$b_0 := -\frac{\mu_0}{\sigma_0}, \ldots, b_n := \frac{\mu_0\beta_n - \mu_n}{\sigma_n\bar{\gamma}_n}, \quad n \geq 1, \qquad D_n^2 := \sum_{i=0}^n b_i^2.$$

It is convenient to rewrite (13) as follows:

$$\begin{aligned} X_1^0 &= X_0^0(1+\sigma_0(\epsilon_0 - b_0)), \\ X_1^i &= X_0^i(1+\sigma_i\gamma_i(\epsilon_0-b_0)+\sigma_i\bar{\gamma}_i(\epsilon_i-b_i)), \quad i \in \mathbf{N}. \end{aligned}$$

The set $\mathscr{Q}^n$ of equivalent martingale measures has a very simple description: $Q \in \mathscr{Q}^n$ iff $Q \sim P^n$ and

$$E_Q(\epsilon_i - b_i) = 0, \quad 0 \leq i \leq n,$$

i.e. the b_i are mean values of ϵ_i under Q. Obviously, $\mathscr{Q}^n \neq \emptyset$ iff $P(\epsilon_i - b_i > 0) > 0$ and $P(\epsilon_i - b_i < 0) > 0$ for all $i \leq n$. The last conditions has the following equivalent form: there are functions $f_i : [-N, N] \to]0, \infty[$, $i \leq n$, such that

$$E(\epsilon_i - b_i)f_i(\epsilon_i - b_i) = 0.$$

As usual, we shall assume that $\mathscr{Q}^n \neq \emptyset$ for all n; this implies, in particular, that $|b_i| < N$. Without loss of generality we suppose that $N > 1$.

Let F_i be the distribution function of ϵ_i. Put

$$\underline{s}_i := \inf\{t : F_i(t) > 0\}, \quad \bar{s}_i := \inf\{t : F_i(t) = 1\},$$

$\underline{d}_i := b_i - \underline{s}_i$, $\bar{d}_i := \bar{s}_i - b_i$, and $d_i := \underline{d}_i \wedge \bar{d}_i$. In other words, d_i is the distance from b_i to the end points of the interval $[\underline{s}_i, \bar{s}_i]$.

Proposition 10 *The following assertions hold:*
(a) $\inf_i d_i = 0 \;\Leftrightarrow\; SAA \;\Leftrightarrow\; (P^n) \triangle (\overline{\mathbf{Q}}^n)$,
(b) $\inf_i d_i > 0 \;\Leftrightarrow\; NAA1 \;\Leftrightarrow\; (P^n) \triangleleft (\overline{\mathbf{Q}}^n)$,
(c) $\limsup_i |b_i| = 0 \;\Leftrightarrow\; NAA2 \;\Leftrightarrow\; (\underline{\mathbf{Q}}^n) \triangleleft (P^n)$.

Notice that in the proof we can always assume without loss of generality that $b^i = 0$ for $i \le n$ where n is arbitrarily large. Indeed, we can always take as reference probability the measure $\tilde{P} \sim P$ with $\tilde{P} := f_0(\epsilon_0 - b_0)...f_n(\epsilon_n - b_n)P$.

Remark. The hypothesis that the distributions of ϵ_i have finite support is important: it excludes the case when the value of every nontrivial portfolio is negative with positive probability.

Proof. We shall consider here the first parts of each assertion and give direct proofs; the second parts follow from the general theory and we included them in the above formulation only for the reader convenience. Let us start from the simple but important observation: there is a constant $C > 0$ such that $\underline{s}_i \le -C$ and $\overline{s_i} \le C$ (in fact, one can take $C = 1/(8N^2)$). Indeed, if, e.g., $\overline{s_i} \le 1/(8N^2)$ then the condition $E\epsilon_i = 0$ implies that $F(-1/2) - F(-N) \le 1/(4N^2)$ and, hence, $E\epsilon_i^2 \le 1/4 + 1/4 < 1$ in contradiction with the assumption.

In the (one-step) model with number n, a trading strategy is an initial endowment x and a vector $\varphi \in \mathbf{R}^{n+1}$. The value of the corresponding portfolio at $T = 1$ is given by the formula

$$V_1^n = x + \sum_{i=0}^{n} \varphi_i (X_1^i - X_0^i).$$

If we define

$$a_0 := \sum_{i=0}^{n} \varphi_i X_0^i \sigma_i \gamma_i, \quad a_i = \varphi_i X_0^i \sigma_i \overline{\gamma}_i, \quad 1 \le i \le n,$$

the expression for V_1^n can be rewritten in the following more transparent form:

$$V_1^n = x + \sum_{i=0}^{n} a_i (\epsilon_i - b_i).$$

Since φ can be reconstructed from a we shall identify any pair (x, a) with a trading strategy.

Let $\inf_i d_i = 0$. Taking a subsequence we can assume that $d_i \le 2^{-i}$. Then SAA1 is realized by the trading strategies corresponding to the sequence (x^{2n}, a^{2n}) where $x^{2n} := 2^{-n}$, $a_i^{2n} := I_{\bar{\Gamma} \cap \{i \ge n\}} - I_{\Gamma \cap \{i \ge n\}}$, $0 \le i \le n$, $\Gamma := \{i : \overline{d}_i < \underline{d}_i\}$. Indeed,

$$V_1^{2n} = 2^{-n} + \sum_{i=n+1}^{2n} a_i^{2n} (\epsilon_i - b_i) =$$

$$= \sum_{i=n+1}^{2n} \left((\overline{s_i} - \epsilon_i) I_\Gamma + (\epsilon_i - \underline{s}_i) I_{\bar{\Gamma}} \right) + 2^{-n} - \sum_{i=n+1}^{2n} \left(\overline{d}_i I_\Gamma + \underline{d}_i I_{\bar{\Gamma}} \right) \ge$$

$$\geq \sum_{i=n+1}^{2n} \left((\overline{s_i} - \epsilon_i) I_\Gamma + (\epsilon_i - \underline{s}_i) I_{\bar{\Gamma}} \right).$$

The right-hand side of this inequality is non-negative and, moreover, is greater than or equal to

$$n \left(C + \frac{1}{n} \sum_{i=n+1}^{2n} (-1)^{I_\Gamma} \epsilon_i \right).$$

But by the strong law of large numbers this sequence (and hence V^{2n}) tends to infinity with probability one.

Now let $\inf_i d_i = \delta > 0$. From the definitions it follows that for any $\eta > 0$ with strictly positive probability

$$\sum_{i=0}^{n} a_i(\epsilon_i - b_i) \leq -\sum_{i=0}^{n} |a_i| d_i + \eta \leq -\delta \sum_{i=0}^{n} |a_i| + \eta.$$

Thus, if $x^n \geq 0$ then the condition

$$V_1^n := x^n + \sum_{i=0}^{n} a_i^n(\epsilon_i - b_i) \geq 0 \quad \text{a.s.}$$

implies the bound

$$\delta \sum_{i=0}^{n} |a_i^n| \leq x^n$$

and for $x^n \to 0$ we have

$$V_1^n \leq x^n + 2N \sum_{i=0}^{n} |a_i^n| \leq x^n(1 + 2N\delta^{-1}) \to 0.$$

This means that asymptotic arbitrage opportunities of the first kind cannot exist.

Notice that the inverse implications in (a) and (b) follow from the two implications proved above.

Suppose that $\limsup_i |b_i| > 0$. Without loss of generality we may assume that $\nu := \inf_i |b_i| > 0$. Then an asymptotic arbitrage opportunity can be realized by the sequence (x^n, a^n) where $x^n := \nu^2/N^2$ and

$$a_i^n := \frac{\nu^2 b_i}{N^2 D_n^2}, \qquad D_n^2 := \sum_{i=0}^{n} b_i^2.$$

Indeed,

$$V_1^n = \frac{\nu^2}{N^2} + \sum_{i=0}^{n} a_i^n(\epsilon_i - b_i) = \frac{\nu^2}{N^2 D_n^2} \sum_{i=0}^{n} b_i \epsilon_i.$$

Since $D_n^2 \geq Cn$ the strong law of large numbers implies that $V_1^n \to 0$ a.s. when $n \to \infty$. Taking into account that $\nu \leq N$ and

$$\sum_{i=0}^{n} |b_i \epsilon_i| \leq N \sum_{i=0}^{n} |b_i| \leq \frac{ND_n^2}{\nu}$$

we check easily the bound $|V_1^n| \leq 1$.

At last, suppose that $\limsup_i |b_i| = 0$. This implies that $\limsup_i d_i \geq C$ and, hence, $\delta := \inf_i |d_i| > 0$. Fix a number $\gamma \in]0, 1[$. Without loss of generality we can assume that

$$\sup_i |b_i| \leq \frac{\gamma\delta}{2(1-\gamma)}.$$

Let (x^n, a^n) be a sequence such that the first two properties of a strategy realizing AA2 are fulfilled, i.e. $x^n \to x > 0$ and

$$V_1^n := x^n + \sum_{i=0}^{n} a_i^n (\epsilon_i - b_i) \leq 1.$$

It follows that

$$x^n + \delta \sum_{i=0}^{n} |a_i^n| \leq 1.$$

Assume that $x > \gamma$. Then for sufficiently large n

$$\sum_{i=0}^{n} |a_i^n| \leq \frac{1-\gamma}{\delta}$$

and, therefore,

$$V_1^n \geq \gamma - \sum_{i=0}^{n} |a_i^n||b_i| + \sum_{i=0}^{n} a_i^n \epsilon_i \geq \gamma/2 + \sum_{i=0}^{n} a_i^n \epsilon_i .$$

For sufficiently large n

$$P(V_1^n \geq \gamma/4) \geq E(V_1^n - \gamma/4)^+ \wedge 1 \geq E(V_1^n - \gamma/4) \wedge 1 = E(V_1^n - \gamma/4) \geq \gamma/4.$$

Thus, there are no asymptotic arbitrage opportunities of the second kind if $\lim_n x^n > \gamma$. Since γ is arbitrary the property NAA2 holds. □

Remark. If the $\sigma_i \overline{\gamma}_i$'s are bounded away from zero we have again that for a market without asymptotic arbitrage $\mu_i \approx \mu_0 \beta_i$.

7. Example: two-asset model with infinite horizon

We consider here the discrete-time model with only two assets, one of which is taken as a numéraire and its price is constant over time. The price dynamics of the second asset is given by the following relation:

$$X_i = X_{i-1}(1 + \mu_i + \sigma_i \epsilon_i), \quad i \geq 1, \tag{14}$$

where $X_0 > 0$, $(\epsilon_i)_{i\geq 1}$ is a sequence of independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a finite interval $[-N, N]$, $E\epsilon_i = 0$, $E\epsilon_i^2 = 1$. The coefficients here are deterministic, $\sigma_i > 0$ for all i.

For $n \geq 1$ we consider the stochastic basis $\mathbf{B}^n = (\Omega, \mathcal{F}^n, \mathbf{F}^n = (\mathcal{F}_i^n)_{i\leq n}, P^n)$ with the 1-dimensional random process $S^n = (X_i^0)_{i\leq n}$ where $\mathcal{F}_0^n = \mathcal{F}_0$ is the trivial σ-algebra, $\mathcal{F}_i^n = \mathcal{F}_i := \sigma\{\epsilon_0, ..., \epsilon_i\}$, and $P^n = P|\mathcal{F}_n^n$. The sequence $\mathbf{M} = \{(\mathbf{B}^n, S^n, n)\}$ is a large security market according to our definition. Let

$$b_i := -\frac{\mu_i}{\sigma_i}, \qquad D_n^2 := \sum_{i=1}^n b_i^2.$$

Then

$$X_i = X_{i-1}[1 + \sigma_i(\epsilon_i - b_i)], \quad i \geq 1.$$

The set $\mathcal{Q}^n$ of equivalent martingale measures has the following description: $Q \in \mathcal{Q}^n$ iff $Q \sim P^n$ and

$$E_Q(\epsilon_i - b_i \mid \mathcal{F}_{i-1}) = 0, \quad 1 \leq i \leq n.$$

Clearly, $\mathcal{Q}^n \neq \emptyset$ iff $P(\epsilon_i - b_i > 0) > 0$ and $P(\epsilon_i - b_i < 0) > 0$ for all $i \leq n$. The last condition has the following equivalent form: there are functions $f_i : [-N, N] \to]0, \infty[$, $i \leq n$, such that

$$E(\epsilon_i - b_i) f_i(\epsilon_i - b_i) = 0. \tag{15}$$

As usual, we shall assume that $\mathcal{Q}^n \neq \emptyset$ for all n; this implies, in particular, that $|b_i| < N$. Without loss of generality we suppose that $N > 1$.

Proposition 11 *(a) If $D_\infty^2 < \infty$ then $(P^n) \lhd (\overline{\mathbf{Q}}^n)$ and $(\underline{\mathbf{Q}}^n) \lhd (P^n)$ (equivalently, the properties NAA1 and NAA2 hold);*
(b) if $D_\infty^2 = \infty$ then $(P^n) \bigtriangleup (\overline{\mathbf{Q}}^n)$ (equivalently, SAA holds).

In other words, we have the dichotomy: either simultaneously $(P^n) \lhd (\overline{\mathbf{Q}}^n)$ and $(\underline{\mathbf{Q}}^n) \lhd (P^n)$ or $(P^n) \bigtriangleup (\overline{\mathbf{Q}}^n)$ (and $(P^n) \bigtriangleup (\underline{\mathbf{Q}}^n)$), whenever $D_\infty^2 < \infty$ or $D_\infty^2 = \infty$.

Proof. (a) Since $P^n = P|\mathcal{F}^n$, the condition $(P^n) \lhd (\overline{\mathbf{Q}}^n)$ is equivalent to the condition $(\tilde{P}^n) \lhd (\overline{\mathbf{Q}}^n)$ where $\tilde{P}^n := \tilde{P}|\mathcal{F}^n$ and $\tilde{P}$ is any probability measure such that $\tilde{P} \sim P$. If $\tilde{P} := f_0(\epsilon_0 - b_0)...f_n(\epsilon_n - b_n)P$ we get for our model a new specification with $\tilde{b}_i = 0$, $i \leq n$, and $\tilde{b}_i = b_i$, $i > n$. By the assumption, $b_i \to 0$ and the above observation shows that one can suppose without loss of generality that $|b_i| \leq c$ where $c > 0$ is arbitrarily small.

We show that if the $|b_i|$ are bounded by a certain sufficiently small constant then for every n and for every $\alpha \in]0, 1[$ there exists a probability measure $R^n(\alpha) \in \mathcal{Q}^n$ such that

$$\sup_{Q\in\mathcal{Q}^n} H(\alpha; Q, P^n) = H(\alpha, R^n(\alpha), P^n) \tag{16}$$

and

$$H(\alpha, R^n(\alpha), P^n) \geq e^{-C\alpha(1-\alpha)D_n^2} \tag{17}$$

where C is a constant which does not depend on α and n.

It follows from (16) and (17) that

$$\sup_{Q \in \mathcal{Q}^n} H(\alpha; Q, P) \to 0 \text{ as } \alpha \to 0 \text{ or } \alpha \to 1$$

and the assertion (a) holds by virtue of Propositions 5 and 6.

To find $R^n(\alpha)$ let us consider the following optimization problem (corresponding to the case $n = 1$):

$$J(f) := \int f^\alpha(x)\, m(dx) \to \max, \tag{18}$$

$$\int (x - b) f(x)\, m(dx) = 0, \tag{19}$$

$$\int f(x)\, m(dx) = 1, \tag{20}$$

$$f > 0 \quad m\text{-a.s.} \tag{21}$$

where $m(dx)$ is a probability measure on $[-N, N]$ with zero mean and unit variance, $b \in]-N, N[$.

The solution of (18)–(20) can be found with the help of the Kuhn–Tucker theorem which asserts that it is also the solution of the problem

$$\int [\lambda_0 f^\alpha(x) + \lambda_1 (x - b) f(x) + \lambda_2 f(x)]\, m(dx) \to \max$$

with the constraint (21) where $\lambda_0 \geq 0$ and not all λ_i are equal to zero. Simple considerations show that λ_0 is not equal to zero and we can assume that $\lambda_0 = 1$; also $\lambda_2 \neq 0$ and $\lambda_1(x - b) + \lambda_2 \leq 0$. The function $f \mapsto f^\alpha + \lambda_1(x - b)f + \lambda_2 f$ attains its maximum at the point $f^*(x) = C_0(1 + a^*(x - b))^{1/(1-\alpha)}$ where specific expressions for C_0 and a^* are not important. The relation (19) gives an equation determining a^* and we show in Lemma 2 that this equation has a solution at least if $|b|$ is small enough. The normalization constant C_0 is given by (20). The function f^* defined in this way is the solution of (18)–(21) (it follows also from (25)–(27)).

Lemma 2 *There exists a constant $c > 0$ such that for all $\alpha \in]0, 1[$ and $b \in [-c, c]$ the equation*

$$\Psi(a) := \int (x - b)(1 + a(x - b)^{\beta - 1})\, m(dx) = 0 \tag{22}$$

where $\beta = \alpha/(\alpha - 1)$ has the unique root $a^* = a^*_{b,\alpha} \in [-\gamma, \gamma]$, $\gamma^{-1} := 4N(1 + |\beta|)$;

there is a constant C such that for all $a \in [-\gamma, \gamma]$ and $b \in [-c, c]$

$$\int (1 + a(x - b))^\beta m(dx) \geq e^{-C\alpha b^2}. \tag{23}$$

Proof. We first consider the case when $\alpha \in]0, 1/2]$. Let $g(x) := x(1+x)^{\beta-1}$. Since $\beta \in [-1, 0[$ we have

$$g''(x) = (\beta - 1)(1+x)^{\beta-3}(\beta x + 2) \leq -4/9$$

on $[-1/2, 1/2]$ and hence $g(x) \leq x - (2/9)x^2$ on this interval. The function $\Psi(a)$ is continuous and decreasing on $[-1/(4N), 1/(4N)]$. From the last bound it follows that if $|b| \leq 1/(36N)$ then

$$\Psi(-1/(8N)) \geq -b + \frac{1}{36N}(1+b^2) > 0,$$

$$\Psi(1/(8N)) \leq -b - \frac{1}{36N}(1+b^2) < 0,$$

and the existence of the unique root is proved.

On the interval $[-1/2, 1/2]$ we have that $(\partial^2/\partial x^2)(1+x)^\beta \geq \beta(\beta-1)(2/3)^3$, which implies the bound

$$(1+x)^\beta \geq 1 + \beta x + \frac{4}{27}\beta(\beta-1)x^2.$$

It follows that for any $a \in [-1/(4N), 1/(4N)]$

$$\int (1 + a(x+b))^\beta\, m(dx) \geq 1 + \beta ba + \frac{4}{27}\beta(\beta-1)a^2 \geq 1 - \left(\frac{3}{2}\right)^3 \alpha b^2 \geq e^{-C_1\alpha b^2}$$

where the last inequality holds with some sufficiently large constant C_1 when $b^2 \leq (2/3)^3$.

The case $\alpha \in]1/2, 1[$ is similar. There is a constant $c_2 > 0$ such that $(1+x)^{\beta-3} \geq 2c_2$ when $\beta \in]-\infty, -1[$ and $|x| \leq (|\beta|+1)^{-1}$. Thus, $g''(x) \leq -2c_2(|\beta|+1)$ and $g(x) \leq x - c_2(|\beta|+1)x^2$ for such x. From the last bound we get that if $|b| \leq c_2/(4N)$ then

$$\Psi(-\gamma) \geq -b + c_2(|\beta|+1)\gamma(1+b^2) = -b + \frac{c_2}{4N}(1+b^2) > 0,$$

$$\Psi(\gamma) \leq -b - c_2(|\beta|+1)\gamma(1+b^2) = -b - \frac{c_2}{4N}(1+b^2) < 0,$$

and there is a root of Ψ on $[-\gamma, \gamma]$.

For $|x| \leq (|\beta|+1)^{-1}$ we have for some constant $c_3 > 0$ the bound

$$(1+x)^\beta \geq 1 + x + c_3\beta(\beta-1)x^2.$$

Hence for any $a \in [-\gamma, \gamma]$

$$\int (1 + a(x-b))^\beta\, m(dx) \geq 1 - \beta ba + c_3\beta(\beta-1)a^2 \geq 1 - \frac{\alpha b^2}{2c_3} \geq e^{-C_2\alpha b^2}$$

where the last inequality holds with some sufficiently large constant C_2 when $b^2 \leq 2c_3$. The lemma is proved. □

Now we show that the optimal point in (16) is the product of the solutions of one-stage optimization problem (18)–(21) corresponding to $b_1, \dots, b_n$. Assuming that all $|b_i|$ are sufficiently small and applying Lemma 2 with $m(dx)$ equal to the distribution of ϵ_i, we get that for some $a_i \in [-\gamma, \gamma]$

$$E(\epsilon_i - b_i)(1 + a_i(\epsilon_i - b_i))^{\beta-1} = 0 \tag{24}$$

or, equivalently,

$$E(1 + a_i(\epsilon_i - b_i))^{\beta} = E(1 + a_i(\epsilon_i - b_i))^{\beta-1}. \tag{25}$$

The measure $R^n(\alpha)$ given by the density

$$\frac{dR^n(\alpha)}{dP^n} := \prod_{i=1}^{n} \frac{(1 + a_i(\epsilon_i - b_i))^{\beta-1}}{E(1 + a_i(\epsilon_i - b_i))^{\beta-1}}$$

belongs to $\mathcal{Q}^n$,

$$H(\alpha, R^n(\alpha), P^n) = E\left(\frac{dR^n(\alpha)}{dP^n}\right)^{\alpha} = \left(\prod_{i=1}^{n} E(1 + a_i(\epsilon_i - b_i))^{\beta}\right)^{1-\alpha} \geq$$

$$\geq \exp\left\{-C\alpha(1-\alpha)\sum_{i=1}^{n} b_i^2\right\} \tag{26}$$

and (17) holds.

For any $Q \in \mathcal{Q}^n$ we have, using the (inverse) Hölder inequality, that

$$1 = E\frac{dQ}{dP^n}\prod_{i=1}^{n}(1 + a_i(\epsilon_i - b_i)) \geq \left(E\prod_{i=1}^{n}(1 + a_i(\epsilon_i - b_i))^{\beta}\right)^{1/\beta} H^{1/\alpha}(\alpha, Q, P^n) =$$

$$= H^{-1/\alpha}(\alpha, R^n(\alpha), P^n) H^{1/\alpha}(\alpha, Q, P^n). \tag{27}$$

Thus, $H(\alpha, Q, P^n) \leq H(\alpha, R^n(\alpha), P^n)$ and (16) also holds.

(b) Let us consider an arbitrary sequence of measures $Q^n \in \mathcal{Q}^n$. For any n the process $(M_k, \mathcal{F}^k)_{k \leq n}$ with $M_k := \sum_{i=1}^{k} b_i(\epsilon_i - b_i)$ is a Q^n-martingale and

$$E_{Q^n} M_n^2 = \sum_{i=1}^{n} b_i^2 E_{Q^n}(\epsilon_i - b_i)^2 \leq 4N^2 D_n^2.$$

For the sets $A^n := \{D_n^{-3/2} M_n > 1\} \in \mathcal{F}^n$ we have by the Chebyshev inequality that

$$Q^n(A^n) \leq D_n^{-3} E_{Q^n} M_n^2 \leq 4N^2 D_n^{-1} \to 0, \quad n \to \infty.$$

But

$$P^n(\bar{A}^n) = P^n\left(-\sum_{i=1}^{n} b_i\epsilon_i \geq (D_n^2 - D_n^{3/2})\right) \leq \frac{4N^2 D_n^2}{(D_n^2 - D_n^{3/2})^2} \to 0, \quad n \to \infty.$$

Thus, $(P^n) \triangle (Q^n)$ and by Proposition 7 $(P^n) \triangle (\overline{\mathbf{Q}}^n)$. □

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