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Asymptotic behavior for non-autonomous stochastic plate equation on unbounded domains

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Abstract: We study the asymptotic behavior of solutions to the non-autonomous stochastic plate equation driven by additive noise defined on unbounded domains. We first prove the uniform estimates of solutions, and then establish the existence and upper semicontinuity of random attractors.

Keywords: attractors, plate equations, unbounded domains, upper semicontinuity, tail-estimates

MSC: 35B40, 35B41

1 Introduction

Consider the following non-autonomous stochastic plate equation with additive noise defined in the entire space \mathbb{R}^n :

$$u_{tt} + \alpha u_t + \Delta^2 u_t + \Delta^2 u + \lambda u + f(x, u) = g(x, t) + \beta h(x) \frac{dW}{dt}, \quad (1.1)$$

with the initial value conditions

$$u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x), \quad (1.2)$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$ and β are constants, f is a nonlinearity satisfying certain growth and dissipative conditions, $g(x, \cdot)$ and h are given functions in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and $H^2(\mathbb{R}^n)$, respectively, $W(t)$ is a two-sided real-valued Wiener process on a probability space.

Plate equations have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theories of solid mechanics. The study of the long-time dynamics of plate equations has become an outstanding area in the field of the infinite-dimensional dynamical system. While the attractors are regarded as a proper notation to describe the long-time dynamics of solutions. To the best of our knowledge, there have been many works on the investigation of the attractors for the plate equations over the last few years. For instance, if the random term is vanished and $g(x, t) = g(x)$, then (1.1)-(1.2) change into a deterministic autonomous plate equation. The existence and uniqueness of the global attractor of the corresponding dynamical system was studied in [1-10]; besides, the uniform attractor of the dynamical system generated by the non-autonomous plate equation was established in [11].

For the stochastic plate equations, if the forcing term $g(x, t) = g(x)$, then the existence of a random attractor of (1.1)-(1.2) on bounded domain have been proved in [12-14]. However, it is not yet considered by any predecessors to the stochastic plate equation on unbounded domain. In recent years, the existence of

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random attractors for stochastic dynamical system on unbounded domains have been investigated by several authors, such as Reaction-diffusion equations with additive noise [15], Reaction-diffusion equations with multiplicative noise [16], FitzHugh-Nagumo equations with additive noise [17, 18], Navier-Stokes equations with additive noise [19], wave equations with additive noise [20-22], wave equations with multiplicative noise [23].

Motivated by above literatures, the goal of the present paper is to study random attractors and its upper semicontinuity of non-autonomous stochastic equation (1.1). By applying the abstract results in [24-26], we will first prove the stochastic plate equation (1.1) has tempered random attractors in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then establish the upper semicontinuity of the random attractors.

In general, the existence of global random attractor depends on some kind compactness (see, e.g., [27-30]). To prove the existence of random attractors for (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we must establish the pullback asymptotic compactness of solutions. Since Sobolev embeddings are not compact on unbounded domain, we cannot get the desired asymptotic compactness directly from the regularity of solutions. We here overcome the difficulty by using the uniform estimates on the tails of solutions outside a bounded ball in \mathbb{R}^n and the splitting technique, see [20, 31] for details.

The framework of this paper is as follows. In the next Section, we recall a sufficient and necessary criterion for existence of pullback attractors for cocycle or nonautonomous random dynamical systems. In Section 3, we define a continuous cocycle for Eq. (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then we derive all necessary uniform estimates of solutions in Section 4. In Section 5, we prove the existence and uniqueness of tempered random attractor for the non-autonomous stochastic plate equation. Finally, in Section 6, we prove the upper semicontinuity of random attractors as β to zero.

Throughout the paper, the letters c and c_i ($i = 1, 2, \dots$) are generic positive constants which may change their values from line to line or even in the same line.

2 Preliminaries

In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems.

Let X be a separable Banach space and $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, \mathcal{F} is the Borel σ -algebra induced by the compact open topology of Ω , and \mathcal{P} is the Wiener measure on (Ω, \mathcal{F}) . There is a classical group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega, t \in \mathbb{R}. \tag{2.1}$$

We often say that $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a parametric dynamical system.

The following four definitions and one proposition are from [24].

Definition 2.1. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$ is called a continuous cocycle on X over \mathbb{R} and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)-(4) are satisfied:

- (1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \rightarrow X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable;
- (2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X ;
- (3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
- (4) $\Phi(t, \tau, \omega, \cdot) : X \rightarrow X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over \mathbb{R} and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$, and \mathcal{D} is the collection of all tempered families of nonempty bounded subsets of X parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}\}.$$

\mathcal{D} is said to be tempered if there exists $x_0 \in X$ such that for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the following holds:

$$\lim_{t \rightarrow -\infty} e^{ct} d(D(\tau + t, \theta_t \omega), x_0) = 0. \tag{2.2}$$

Given $D \in \mathcal{D}$, the family $\Omega(D) = \{\Omega(D, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω -limit set of D where

$$\Omega(D, \tau, \omega) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \overline{\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega))}. \quad (2.3)$$

The cocycle Φ is said to be \mathcal{D} -pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \quad (2.4)$$

whenever $t_n \rightarrow \infty$, and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$ with $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.2. A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a \mathcal{D} -pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \text{ for all } t \geq T. \quad (2.5)$$

If, in addition, $K(\tau, \omega)$ is closed in X and is measurable in ω with respect to \mathcal{F} , then K is called a closed measurable \mathcal{D} -pullback absorbing set for Φ .

Definition 2.3. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a \mathcal{D} -pullback attractor for Φ if the following conditions (1)-(3) are fulfilled: for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

- (1) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to \mathcal{F} .
- (2) \mathcal{A} is invariant, that is,

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega). \quad (2.6)$$

- (3) For every $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t \rightarrow \infty} d_H(\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0, \quad (2.7)$$

where d_H is the Hausdorff semi-distance given by $d_H(F, G) = \sup_{u \in F} \inf_{v \in G} \|u - v\|_X$, for any $F, G \subset X$.

As in the deterministic case, random complete solutions can be used to characterize the structure of a \mathcal{D} -pullback attractor. The definition of such solutions are given below.

Definition 2.4. A mapping $\Psi : \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow X$ is called a random complete solution of Φ if for every $t \in \mathbb{R}^+$, $s, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau + s, \theta_s\omega, \Psi(s, \tau, \omega)) = \Psi(t + s, \tau, \omega). \quad (2.8)$$

If, in addition, there exists a tempered family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\Psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t\omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then Ψ is called a tempered random complete solution of Φ .

Proposition 2.1. Suppose Φ is \mathcal{D} -pullback asymptotically compact in X and has a closed measurable \mathcal{D} -pullback absorbing set K in \mathcal{D} . Then Φ has a unique \mathcal{D} -pullback attractor \mathcal{A} in \mathcal{D} which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{D \in \mathcal{D}} \Omega(D, \tau, \omega) \quad (2.9)$$

$$= \{\Psi(0, \tau, \omega) : \Psi \text{ is a tempered random complete solution of } \Phi\}. \quad (2.10)$$

3 Cocycles for stochastic plate equation

In this section, we outline some basic settings about (1.1)-(1.2) and show that it generates a continuous cocycle in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Let $-\Delta$ denote the Laplace operator in \mathbb{R}^n , $A = \Delta^2$ with the domain $D(A) = H^4(\mathbb{R}^n)$. We can define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_\nu = D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm

$$(u, v)_\nu = (A^{\frac{\nu}{4}}u, A^{\frac{\nu}{4}}v), \quad \|\cdot\|_\nu = \|A^{\frac{\nu}{4}}\cdot\|.$$

For brevity, the notation (\cdot, \cdot) for L^2 -inner product will also be used for the notation of duality pairing between dual spaces.

We denote $\mathcal{H} = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

We define a new norm $\|\cdot\|_{\mathcal{H}}$ by

$$\|Y\|_{\mathcal{H}} = (\|v\|^2 + (\delta^2 + \lambda - \delta\alpha)\|u\|^2 + (1 - \delta)\|\Delta u\|^2)^{\frac{1}{2}}, \tag{3.1}$$

for $Y = (u, v)^\top \in \mathcal{H}$, where \top stands for the transposition.

Let $\xi = u_t + \delta u$, where δ is a small positive constant whose value will be determined later, then (1.1)-(1.2) can be rewritten as the equivalent system

$$\begin{cases} \frac{du}{dt} = \xi - \delta u, \\ \frac{d\xi}{dt} = [\delta(\alpha + A - \delta) - A]u - (\alpha + A - \delta)\xi - \lambda u - f(x, u) + g(x, t) + \beta h(x) \frac{dW}{dt}, \end{cases} \tag{3.2}$$

with the initial value conditions

$$u(x, \tau) = u_0(x), \quad \xi(x, \tau) = \xi_0(x), \tag{3.3}$$

where $\xi_0(x) = u_1(x) + \delta u_0(x)$, $x \in \mathbb{R}^n$.

Let $F(x, u) = \int_0^u f(x, s)ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. The function f will be assumed to satisfy the following conditions:

- (F1) $|f(x, u)| \leq c_1|u|^k + \eta_1(x)$,
- (F2) $f(x, u)u - c_2F(x, u) \geq \eta_2(x)$,
- (F3) $F(x, u) \geq c_3|u|^{k+1} - \eta_3(x)$,
- (F4) $|f_u(x, u)| \leq \varpi$,

where $\varpi > 0$, $1 \leq k \leq \frac{n+4}{n-4}$, $\eta_1 \in L^2(\mathbb{R}^n)$, $\eta_2 \in L^1(\mathbb{R}^n)$, $\eta_3 \in L^1(\mathbb{R}^n)$.

Note that (F1) and (F2) imply

$$F(x, u) \leq c(|u|^2 + |u|^{k+1} + \eta_1^2 + \eta_2). \tag{3.4}$$

We also need the following condition on g : there exists a positive constants σ such that

$$\int_{-\infty}^{\tau} e^{\sigma s} \|g(\cdot, s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \tag{3.5}$$

which implies that

$$\lim_{r \rightarrow \infty} \int_{-\infty}^{\tau} \int_{|x| \geq r} e^{\sigma s} |g(\cdot, s)|^2 dx ds = 0, \quad \forall \tau \in \mathbb{R}, \tag{3.6}$$

where $|\cdot|$ denotes the absolute value of real number in \mathbb{R} .

For our purpose, it is convenient to convert the problem (1.1)-(1.2) (or (3.2)- (3.3)) into a deterministic system with a random parameter, and then show that it generates a cocycle over \mathbb{R} and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

We identify $\omega(t)$ with $W(t)$, i.e., $\omega(t) = W(t) = W(t, x)$, $t \in \mathbb{R}$. Consider Ornstein-Uhlenbeck equation $dy + ydt = dW(t)$, and Ornstein-Uhlenbeck process

$$y(\theta_t \omega) = - \int_{-\infty}^0 e^s (\theta_t \omega)(s) ds, \quad t \in \mathbb{R}.$$

From [32], it is known that the random variable $|y(\omega)|$ is tempered, and there is a θ_t -invariant set $\tilde{\Omega} \subset \Omega$ of full \mathcal{P} measure such that $y(\theta_t \omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$. Put

$$z(\theta_t \omega) = z(x, \theta_t \omega) = h(x)y(\theta_t \omega), \tag{3.7}$$

which solves

$$dz + zdt = hdW. \tag{3.8}$$

Lemma 3.1 [33] *For any $\varepsilon > 0$, there exists a tempered random variable $\gamma : \Omega \rightarrow \mathbb{R}^+$, such that for all $t \in \mathbb{R}$, $\omega \in \Omega$,*

$$\begin{aligned} \|z(\theta_t \omega)\| &\leq e^{\varepsilon|t|} \gamma(\omega) \|h\|, \\ \|\nabla z(\theta_t \omega)\| &\leq e^{\varepsilon|t|} \gamma(\omega) \|\nabla h\|, \\ \|\Delta z(\theta_t \omega)\| &\leq e^{\varepsilon|t|} \gamma(\omega) \|\Delta h\|, \end{aligned}$$

where $\gamma(\omega)$ satisfies

$$e^{-\varepsilon|t|} \gamma(\omega) \leq \gamma(\theta_t \omega) \leq e^{\varepsilon|t|} \gamma(\omega).$$

Now, let $v(t, \tau, \omega) = \xi(t, \tau, \omega) - \beta z(\theta_t \omega)$, we obtain the equivalent system of (3.2)-(3.3),

$$\begin{cases} \frac{du}{dt} = v - \delta u + \beta z(\theta_t \omega), \\ \frac{dv}{dt} = (\delta - \alpha - A)v + [\delta(-\delta + \alpha + A) - \lambda - A]u + \beta[1 - (\alpha + A - \delta)]z(\theta_t \omega) - f(x, u) + g(x, t), \end{cases} \quad (3.9)$$

with the initial value conditions

$$u(x, \tau, \tau) = u_0(x), \quad v(x, \tau, \tau) = v_0(x), \quad (3.10)$$

where $v_0(x) = \xi_0(x) - z(\theta_\tau \omega)$, $x \in \mathbb{R}^n$. We will consider (3.9)-(3.10) for $\omega \in \tilde{\Omega}$ and write $\tilde{\Omega}$ as Ω from now on.

The well-posedness of the deterministic problem (3.9)-(3.10) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ can be established by standard methods as in [34, 35]. If (F1)–(F4) are fulfilled, let $\varphi^{(\beta)}(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_0^{(\beta)}) = (u(t + \tau, \tau, \theta_{-\tau} \omega, u_0), v(t + \tau, \tau, \theta_{-\tau} \omega, v_0))^\top$, where $\varphi_0^{(\beta)} = (u_0, v_0)^\top$. Then for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $\varphi_0^{(\beta)} \in \mathcal{H}(\mathbb{R}^n)$, problem (3.9)-(3.10) has a unique $(\mathcal{F}, \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$ -measurable solution $\varphi^{(\beta)}(\cdot, \tau, \omega, \varphi_0^{(\beta)}) \in C([\tau, \infty), \mathcal{H}(\mathbb{R}^n))$ with $\varphi^{(\beta)}(\tau, \tau, \omega, \varphi_0^{(\beta)}) = \varphi_0^{(\beta)}$, $\varphi^{(\beta)}(t, \tau, \omega, \varphi_0^{(\beta)}) \in \mathcal{H}_0(\mathbb{R}^n)$ being continuous in $\varphi_0^{(\beta)}$ for each $t > \tau$. Moreover, for every $(t, \tau, \omega, \varphi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{H}(\mathbb{R}^n)$, the mapping

$$\Phi_\beta(t, \tau, \omega, \varphi_0^{(\beta)}) = \varphi^{(\beta)}(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_0^{(\beta)}) \quad (3.11)$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{H}(\mathbb{R}^n)$ to $\mathcal{H}(\mathbb{R}^n)$ over \mathbb{R} and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

Introducing the homeomorphism $P(\theta_t \omega)(u, v)^\top = (u, v + z(\theta_t \omega))^\top$, $(u, v)^\top \in \mathcal{H}(\mathbb{R}^n)$ with an inverse homeomorphism $P^{-1}(\theta_t \omega)(u, v)^\top = (u, v - z(\theta_t \omega))^\top$. Then, the transformation

$$\tilde{\Phi}_\beta(t, \tau, \omega, (u_0, \xi_0)) = P(\theta_t \omega) \Phi_\beta(t, \tau, \omega, (u_0, v_0)) P^{-1}(\theta_t \omega) \quad (3.12)$$

generates a continuous cocycle with (3.2)-(3.3) over \mathbb{R} and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

Note that these two continuous cocycles are equivalent. By (3.12), it is easy to check that $\tilde{\Phi}_\beta$ has a random attractor provided Φ_β possesses a random attractor. Then, we only need to consider the continuous cocycle Φ_β .

One can prove Φ_β is measurable by using the same method as in the following paper: H. Cui, J.A. Langa, Y. Li, Measurability of random attractors for quasi strong-to-weak continuous random dynamical systems, J. Dynam. Differ. Equ. 30 (2018) 1873-1898.

4 Uniform estimates of solutions

In this subsection, we derive uniform estimates on the solutions of the stochastic plate equations (3.9)-(3.10) defined on \mathbb{R}^n when $t \rightarrow \infty$. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the random dynamical system associated with the equations. In particular, we will show that the tails of the solutions for large space variables are uniformly small when time is sufficiently large.

Let $\delta \in (0, 1)$ be small enough such that

$$\delta^2 + \lambda - \delta\alpha > 0, \quad 1 - \delta > 0,$$

and define σ appearing in (3.5) by

$$\sigma = \min\{\alpha - \delta, \delta, \frac{c_2 \delta}{2}\}, \tag{4.1}$$

where c_2 is the positive constant in (F2).

Lemma 4.1 *Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,*

$$\begin{aligned} & \|\varphi^{(\beta)}(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_{\mathcal{J}_t}^2 + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 ds \\ & + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 ds + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 ds \\ & + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|\Delta v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 ds \\ & < c + c\beta^2 \int_{-\infty}^0 e^{\sigma s} (1 + \|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds, \end{aligned} \tag{4.2}$$

where $\varphi_0^{(\beta)} = (u_0, v_0)^\top \in D(\tau - t, \theta_{-\tau}\omega)$ and c is a positive constant depending on λ, σ, α and δ , but independent of τ, ω and D .

Proof. Taking the inner product of the second equation of (3.9) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 &= -(\alpha - \delta)(v, v) - (\lambda + \delta^2 - \delta\alpha)(u, v) - (1 - \delta)(Au, v) - (Av, v) \\ &+ \beta(1 - \alpha + \delta)(z(\theta_t \omega), v) - \beta(Az(\theta_t \omega), v) + (g(x, t), v) - (f(x, u), v). \end{aligned} \tag{4.3}$$

By the first equation of (3.9), we have

$$v = \frac{du}{dt} + \delta u - \beta z(\theta_t \omega). \tag{4.4}$$

Then substituting the above v into the second, third and last terms on the right-hand side of (4.3), we find that

$$\begin{aligned} (u, v) &= (u, \frac{du}{dt} + \delta u - \beta z(\theta_t \omega)) \\ &= \frac{1}{2} \frac{d}{dt} \|u\|^2 + \delta \|u\|^2 - \beta(u, z(\theta_t \omega)) \\ &\geq \frac{1}{2} \frac{d}{dt} \|u\|^2 + \delta \|u\|^2 - |\beta| \|z(\theta_t \omega)\| \|u\| \\ &\geq \frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{3\delta}{4} \|u\|^2 - \frac{\beta^2}{3\delta} \|z(\theta_t \omega)\|^2, \end{aligned} \tag{4.5}$$

$$\begin{aligned} -(Au, v) &= -(\Delta^2 u, \frac{du}{dt} + \delta u - \beta z(\theta_t \omega)) \\ &= -\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 - \delta \|\Delta u\|^2 + \beta(\Delta u, \Delta z(\theta_t \omega)) \\ &\leq -\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 - \delta \|\Delta u\|^2 + |\beta| \|\Delta z(\theta_t \omega)\| \|\Delta u\| \\ &\leq -\frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 - \frac{3\delta}{4} \|\Delta u\|^2 + \frac{\beta^2}{3\delta} \|\Delta z(\theta_t \omega)\|^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} (f(x, u), v) &= (f(x, u), \frac{du}{dt} + \delta u - \beta z(\theta_t \omega)) \\ &= \frac{d}{dt} \int_{\mathbb{R}^n} F(x, u) dx + \delta (f(x, u), u) - \beta (f(x, u), z(\theta_t \omega)). \end{aligned} \tag{4.7}$$

From condition (F2) we get

$$f(x, u, u) \geq c_2 \int_{\mathbb{R}^n} F(x, u) dx + \int_{\mathbb{R}^n} \eta_2(x) dx. \quad (4.8)$$

By conditions (F1) and (F3), it yields

$$\begin{aligned} \beta(f(x, u), z(\theta_t \omega)) &\leq |\beta| \int_{\mathbb{R}^n} (c_1 |u|^k + \eta_1(x)) |z(\theta_t \omega)| dx \\ &\leq |\beta| \|\eta_1(x)\| \|z(\theta_t \omega)\| + c_1 |\beta| \left(\int_{\mathbb{R}^n} |u|^{k+1} dx \right)^{\frac{k}{k+1}} \|z(\theta_t \omega)\|_{k+1} \\ &\leq |\beta| \|\eta_1(x)\| \|z(\theta_t \omega)\| + c_1 |\beta| \left(\int_{\mathbb{R}^n} (F(x, u) + \eta_3(x)) dx \right)^{\frac{k}{k+1}} \|z(\theta_t \omega)\|_{k+1} \\ &\leq \frac{1}{2} \|\eta_1(x)\|^2 + \frac{\beta^2}{2} \|z(\theta_t \omega)\|^2 + \frac{\delta c_2}{2} \int_{\mathbb{R}^n} F(x, u) dx + \frac{\delta c_2}{2} \int_{\mathbb{R}^n} \eta_3(x) dx + c\beta^2 \|z(\theta_t \omega)\|_{H^2}^{k+1}. \end{aligned} \quad (4.9)$$

Using the Cauchy-Schwartz inequality and the Young inequality, there holds

$$\beta(1 - \alpha + \delta)(z(\theta_t \omega), v) \leq \frac{2(1 - \alpha + \delta)^2 \beta^2}{\alpha - \delta} \|z(\theta_t \omega)\|^2 + \frac{\alpha - \delta}{8} \|v\|^2, \quad (4.10)$$

$$-\beta(Az(\theta_t \omega), v) = -\beta(\Delta z(\theta_t \omega), \Delta v) \leq \frac{\beta^2}{2} \|\Delta z(\theta_t \omega)\|^2 + \frac{1}{2} \|\Delta v\|^2, \quad (4.11)$$

$$(g(x, t), v) \leq \|g(x, t)\| \|v\| \leq \frac{2}{\alpha - \delta} \|g(x, t)\|^2 + \frac{\alpha - \delta}{8} \|v\|^2. \quad (4.12)$$

By (4.5)-(4.12), it follows from (4.3) that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|v\|^2 + (\delta^2 + \lambda - \delta\alpha)\|u\|^2 + (1 - \delta)\|\Delta u\|^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx) \\ &\leq -\frac{3}{4}(\alpha - \delta)\|v\|^2 - \frac{3}{4}\delta(\delta^2 + \lambda - \delta\alpha)\|u\|^2 - \frac{3}{4}\delta(1 - \delta)\|\Delta u\|^2 \\ &\quad - \frac{\delta c_2}{2} \int_{\mathbb{R}^n} F(x, u) dx - \frac{1}{2}\|\Delta v\|^2 + c\beta^2(1 + \|\Delta z(\theta_t \omega)\|^2 \\ &\quad + \|z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|_{H^2}^{k+1}) + \frac{2}{\alpha - \delta} \|g(x, t)\|^2. \end{aligned} \quad (4.13)$$

Recalling the norm $\|\cdot\|_{\mathcal{J}_t}$ in (3.1). By (4.1) we obtain from (4.13) that

$$\begin{aligned} &\frac{d}{dt} (\|\varphi\|_{\mathcal{J}_t}^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx) + \sigma (\|\varphi\|_{\mathcal{J}_t}^2 + 2 \int_{\mathbb{R}^n} F(x, u) dx) \\ &\quad + \frac{1}{2}(\alpha - \delta)\|v\|^2 + \frac{1}{2}\delta(\delta^2 + \lambda - \delta\alpha)\|u\|^2 + \frac{1}{2}\delta(1 - \delta)\|\Delta u\|^2 + \|\Delta v\|^2 \\ &\leq c\beta^2(1 + \|\Delta z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|_{H^2}^{k+1}) + \frac{4}{\alpha - \delta} \|g(x, t)\|^2. \end{aligned} \quad (4.14)$$

Multiplying (4.14) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we have

$$\begin{aligned} &e^{\sigma \tau} (\|\varphi(\tau, \tau - t, \omega, \varphi_0)\|_{\mathcal{J}_t}^2 + 2 \int_{\mathbb{R}^n} F(x, u(\tau, \tau - t, \omega, u_0)) dx) \\ &\quad + \frac{1}{2}(\alpha - \delta) \int_{\tau-t}^{\tau} e^{\sigma s} \|v(s, \tau - t, \omega, v_0)\|^2 ds + \frac{1}{2}\delta(\delta^2 + \lambda - \delta\alpha) \int_{\tau-t}^{\tau} e^{\sigma s} \|u(s, \tau - t, \omega, u_0)\|^2 ds \\ &\quad + \frac{1}{2}\delta(1 - \delta) \int_{\tau-t}^{\tau} e^{\sigma s} \|\Delta u(s, \tau - t, \omega, u_0)\|^2 ds + \int_{\tau-t}^{\tau} e^{\sigma s} \|\Delta v(s, \tau - t, \omega, v_0)\|^2 ds \end{aligned}$$

$$\begin{aligned} &\leq e^{\sigma(\tau-t)}(\|\varphi_0\|_{\mathfrak{H}^c}^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx) + c\beta^2 \int_{\tau-t}^{\tau} e^{\sigma s} (1 + \|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 \\ &+ \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds + \frac{4}{\alpha - \delta} \int_{\tau-t}^{\tau} e^{\sigma s} \|g(x, s)\|^2. \end{aligned} \tag{4.15}$$

Substituting ω by $\theta_{-\tau}\omega$, then we have from (4.15) that

$$\begin{aligned} &(\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_{\mathfrak{H}^c}^2 + 2 \int_{\mathbb{R}^n} F(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)) dx) \\ &+ \frac{1}{2}(\alpha - \delta) \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 ds \\ &+ \frac{1}{2} \delta(\delta^2 + \lambda - \delta\alpha) \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 ds \\ &+ \frac{1}{2} \delta(1 - \delta) \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 ds + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|\Delta v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 ds \\ &\leq e^{-\sigma t}(\|\varphi_0\|_{\mathfrak{H}^c}^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx) + c \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (1 + \|\Delta z(\theta_{s-\tau}\omega)\|^2 + \|z(\theta_{s-\tau}\omega)\|^2 \\ &+ \|z(\theta_{s-\tau}\omega)\|_{H^2}^{k+1}) ds + \frac{4}{\alpha - \delta} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|g(x, s)\|^2 \\ &\leq e^{-\sigma t}(\|\varphi_0\|_{\mathfrak{H}^c}^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx) + c\beta^2 \int_{-t}^0 e^{\sigma s} (1 + \|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 \\ &+ \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds + \frac{4}{\alpha - \delta} \int_{-\infty}^{\tau} e^{\sigma(s-\tau)} \|g(x, s)\|^2. \end{aligned} \tag{4.16}$$

Thanks to Lemma 3.1, it follows that

$$\begin{aligned} &\int_{-t}^0 e^{\sigma s} (\|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds \\ &\leq \int_{-\infty}^0 e^{\sigma s} (\|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds \\ &\leq \int_{-\infty}^0 e^{\frac{\sigma s}{2}} (\gamma^2(\omega)(\|\Delta h\|^2 + \|h\|^2) + \gamma^{k+1}(\omega)(\|\Delta h\|^{k+1} + \|\nabla h\|^{k+1} + \|h\|^{k+1})) ds \\ &< +\infty. \end{aligned} \tag{4.17}$$

From (3.4) yields

$$\int_{\mathbb{R}^n} F(x, u_0) dx \leq c(1 + \|u_0\|^2 + \|u_0\|_{H^2}^{k+1}). \tag{4.18}$$

Due to $\varphi_0 = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, we get from (4.18) that

$$\lim_{t \rightarrow +\infty} e^{-\sigma t} (\|\varphi_0\|_{\mathfrak{H}^c}^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx) = 0. \tag{4.19}$$

Therefore, there exists $T = T(\tau, \omega, D) > 0$ such that $e^{-\sigma t}(\|\varphi_0\|_{\mathcal{H}^1}^2 + 2 \int_{\mathbb{R}^n} F(x, u_0) dx) \leq 1$ for all $t \geq T$. Thus the Lemma follows from (3.5), (4.16) and (4.17). \square

Lemma 4.2 Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then there exists a random ball $\{E_\beta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ centered at 0 with random radius

$$\varrho(\tau, \omega) = c + c\beta^2 \int_{-\infty}^0 e^{\sigma s} (1 + \|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds,$$

such that $\{E_\beta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a closed measurable \mathcal{D} -pullback absorbing set for the continuous cocycle associated with problem (3.9)-(3.10) in \mathcal{D} , that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$, such that for all $t \geq T$,

$$\Phi_\beta(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq A(\tau, \omega). \tag{4.20}$$

Proof. This is an immediate consequence of (3.11) and Lemma 4.1. \square

Choose a smooth function ρ , such that $0 \leq \rho \leq 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & 0 \leq |s| \leq 1, \\ 1, & |s| \geq 2, \end{cases} \tag{4.21}$$

and there exist constants $\mu_1, \mu_2, \mu_3, \mu_4$ such that $|\rho'(s)| \leq \mu_1, |\rho''(s)| \leq \mu_2, |\rho'''(s)| \leq \mu_3, |\rho''''(s)| \leq \mu_4$ for $s \in \mathbb{R}$.

Given $r \geq 1$, denote $\mathbb{H}_r = \{x \in \mathbb{R}^n : |x| < r\}$ and $\mathbb{R}^n \setminus \mathbb{H}_r$ the complement of \mathbb{H}_r . To prove asymptotic compactness of solution on \mathbb{R}^n , we prove the following lemma.

Lemma 4.3 Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \varepsilon) > 0$ and $\tilde{R} = \tilde{R}(\tau, \omega, \varepsilon) \geq 1$, such that for all $t \geq T, r \geq \tilde{R}$,

$$\|\varphi^{(\beta)}(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_{\mathcal{H}^1(\mathbb{R}^n \setminus \mathbb{H}_r)}^2 \leq \varepsilon, \tag{4.22}$$

where $\varphi_0^{(\beta)} = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$.

Proof. We first consider the random equations (3.9)-(3.10). Taking the inner product of the second equation of (3.9) with $\rho(\frac{|x|^2}{r^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |v|^2 dx &= -(\alpha - \delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |v|^2 dx - (\lambda + \delta^2 - \delta\alpha) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) uv dx \\ &\quad - (1 - \delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (Au)v dx - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (Av)v dx + \beta(1 - \alpha + \delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) z(\theta_t \omega)v dx \\ &\quad - \beta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (Az(\theta_t \omega))v dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) g(x, t)v dx - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u)v dx. \end{aligned} \tag{4.23}$$

Substituting v in (4.4) into the second, third and last terms on the right-hand side of (4.23), using Young inequality and the Sobolev interpolation inequality

$$\|\nabla v\| \leq \zeta \|v\| + C_\zeta \|\Delta v\|, \quad \forall \zeta > 0,$$

we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) uv dx &= \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) u \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right) dx \\ &= \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) \left(\frac{1}{2} \frac{d}{dt} u^2 + \delta u^2 - \beta z(\theta_t \omega)u\right) dx \end{aligned}$$

$$\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |u|^2 dx - \frac{\beta^2}{2\delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |z(\theta_t \omega)|^2 dx, \tag{4.24}$$

$$\begin{aligned} & - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (\Delta u) v dx \\ & = - \int_{\mathbb{R}^n} (\Delta^2 u) \rho\left(\frac{|x|^2}{r^2}\right) \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right) dx \\ & = - \int_{\mathbb{R}^n} (\Delta u) \Delta \left(\rho\left(\frac{|x|^2}{r^2}\right) \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right)\right) dx \\ & = - \int_{\mathbb{R}^n} (\Delta u) \left(\frac{2}{r^2} \rho'\left(\frac{|x|^2}{r^2}\right) + \frac{4x^2}{r^4} \rho''\left(\frac{|x|^2}{r^2}\right)\right) \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right) \\ & \quad + 2 \cdot \frac{2|x|}{r^2} \rho'\left(\frac{|x|^2}{r^2}\right) \nabla \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right) + \rho\left(\frac{|x|^2}{r^2}\right) \Delta \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right) dx \\ & \leq \int_{r < x < \sqrt{2}r} \left(\frac{2\mu_1}{r^2} + \frac{4\mu_2 x^2}{r^4}\right) |(\Delta u) v| dx + \int_{r < x < \sqrt{2}r} \frac{4\mu_1 x}{r^2} |(\Delta u)(\nabla v)| dx \\ & \quad - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx - \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \beta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u| |\Delta z(\theta_t \omega)| dx \\ & \leq \int_{\mathbb{R}^n} \left(\frac{2\mu_1 + 8\mu_2}{r^2}\right) |(\Delta u) v| dx + \int_{\mathbb{R}^n} \frac{4\sqrt{2}\mu_1}{r} |(\Delta u)(\nabla v)| dx - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx \\ & \quad - \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \frac{\beta^2}{2\delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)|^2 dx \\ & \leq \frac{\mu_1 + 4\mu_2}{r^2} (\|\Delta u\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{r} \|\Delta u\| \|\nabla v\| - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx \\ & \quad - \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \frac{\beta^2}{2\delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)|^2 dx \\ & \leq \frac{\mu_1 + 4\mu_2}{r^2} (\|\Delta u\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{r} \|\Delta u\| (\zeta \|v\| + C_\zeta \|\Delta v\|) - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx \\ & \quad - \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx + \frac{\beta^2}{2\delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)|^2 dx \\ & \leq \frac{\mu_1 + 4\mu_2}{r^2} (\|\Delta u\|^2 + \|v\|^2) + \frac{2\sqrt{2}\mu_1}{r} (\|\Delta u\|^2 + 2\zeta^2 \|v\|^2 + 2C_\zeta^2 \|\Delta v\|^2) \\ & \quad - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx - \frac{\delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx \\ & \quad + \frac{\beta^2}{2\delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)|^2 dx, \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u) v dx \tag{4.25} \end{aligned}$$

$$\begin{aligned} & = \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u) \left(\frac{du}{dt} + \delta u - \beta z(\theta_t \omega)\right) dx \\ & = \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) F(x, u) dx + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u) u dx - \beta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u) z(\theta_t \omega) dx. \tag{4.26} \end{aligned}$$

From condition (F2), we find

$$\delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u) u dx \geq c_2 \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) F(x, u) dx + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) \eta_2(x) dx. \tag{4.27}$$

In line with conditions (F1) and (F3), it leads to

$$\begin{aligned} \beta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) f(x, u) z(\theta_t \omega) dx &\leq |\beta| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (c_1 |u|^k + \eta_1(x)) |z(\theta_t \omega)| dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\eta_1(x)|^2 dx + \frac{\beta^2}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |z(\theta_t \omega)|^2 dx \\ &\quad + c\beta^2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |z(\theta_t \omega)|^{k+1} dx + \frac{c_2 \delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (F(x, u) + \eta_3(x)) dx. \end{aligned} \tag{4.28}$$

By means of the Cauchy-Schwartz inequality and the Young inequality, we obtain

$$\begin{aligned} - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (\Delta v) v dx &= - \int_{\mathbb{R}^n} (\Delta^2 v) \rho\left(\frac{|x|^2}{r^2}\right) v dx \\ &= - \int_{\mathbb{R}^n} (\Delta v) \Delta(\rho\left(\frac{|x|^2}{r^2}\right) v) dx \\ &= - \int_{\mathbb{R}^n} (\Delta v) \left(\left(\frac{2}{r^2} \rho' \left(\frac{|x|^2}{r^2} \right) + \frac{4x^2}{r^4} \rho'' \left(\frac{|x|^2}{r^2} \right) \right) v + 2 \cdot \frac{2|x|}{r^2} \rho' \left(\frac{|x|^2}{r^2} \right) \nabla v + \rho \left(\frac{|x|^2}{r^2} \right) \Delta v \right) dx \\ &\leq \int_{r < x < \sqrt{2}r} \left(\frac{2\mu_1}{r^2} + \frac{4\mu_2 x^2}{r^4} \right) |(\Delta v) v| dx + \int_{r < x < \sqrt{2}r} \frac{4\mu_1 x}{r^2} |(\Delta v)(\nabla v)| dx - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta v|^2 dx \\ &\leq \int_{\mathbb{R}^n} \left(\frac{2\mu_1 + 8\mu_2}{r^2} \right) |(\Delta v) v| dx + \int_{\mathbb{R}^n} \frac{4\sqrt{2}\mu_1}{r} |(\Delta v)(\nabla v)| dx - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta v|^2 dx \\ &\leq \frac{\mu_1 + 4\mu_2}{r^2} (\|\Delta v\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{r} \|\Delta v\| \|\nabla v\| - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta v|^2 dx \\ &\leq \frac{\mu_1 + 4\mu_2}{r^2} (\|\Delta v\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1}{r} \|\Delta v\| (\zeta \|v\| + C_\zeta \|\Delta v\|) - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta v|^2 dx \\ &\leq \frac{\mu_1 + 4\mu_2}{r^2} (\|\Delta v\|^2 + \|v\|^2) + \frac{2\sqrt{2}\mu_1}{r} (\|\Delta v\|^2 + 2\zeta^2 \|v\|^2 + 2C_\zeta^2 \|\Delta v\|^2) \\ &\quad - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta v|^2 dx, \tag{4.29} \\ &\quad - \beta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (\Delta z(\theta_t \omega)) v dx \\ &= - \beta \int_{\mathbb{R}^n} (\Delta^2 z(\theta_t \omega)) \rho\left(\frac{|x|^2}{r^2}\right) v dx \\ &= - \beta \int_{\mathbb{R}^n} (\Delta z(\theta_t \omega)) \Delta(\rho\left(\frac{|x|^2}{r^2}\right) v) dx \\ &= - \beta \int_{\mathbb{R}^n} (\Delta z(\theta_t \omega)) \left(\left(\frac{2}{r^2} \rho' \left(\frac{|x|^2}{r^2} \right) + \frac{4x^2}{r^4} \rho'' \left(\frac{|x|^2}{r^2} \right) \right) v + 2 \cdot \frac{2|x|}{r^2} \rho' \left(\frac{|x|^2}{r^2} \right) \nabla v + \rho \left(\frac{|x|^2}{r^2} \right) \Delta v \right) dx \\ &\leq |\beta| \int_{r < x < \sqrt{2}r} \left(\frac{2\mu_1}{r^2} + \frac{4\mu_2 x^2}{r^4} \right) |(\Delta z(\theta_t \omega)) v| dx + |\beta| \int_{r < x < \sqrt{2}r} \frac{4\mu_1 x}{r^2} |(\Delta z(\theta_t \omega))(\nabla v)| dx \end{aligned}$$

$$\begin{aligned}
 & - |\beta| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)| |\Delta v| dx \\
 \leq & |\beta| \int_{\mathbb{R}^n} \left(\frac{2\mu_1 + 8\mu_2}{r^2}\right) |(\Delta z(\theta_t \omega))v| dx + |\beta| \int_{\mathbb{R}^n} \frac{4\sqrt{2}\mu_1}{r} |(\Delta z(\theta_t \omega))(\nabla v)| dx \\
 & - |\beta| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)| |\Delta v| dx \\
 \leq & \frac{\mu_1 + 4\mu_2}{r^2} (\beta^2 \|\Delta z(\theta_t \omega)\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1|\beta|}{r} \|\Delta z(\theta_t \omega)\| \|\nabla v\| - |\beta| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)| |\Delta v| dx \\
 \leq & \frac{\mu_1 + 4\mu_2}{r^2} (\beta^2 \|\Delta z(\theta_t \omega)\|^2 + \|v\|^2) + \frac{4\sqrt{2}\mu_1|\beta|}{r} \|\Delta z(\theta_t \omega)\| (\zeta \|v\| + C_\zeta \|\Delta v\|) \\
 & - |\beta| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)| |\Delta v| dx \\
 \leq & \frac{\mu_1 + 4\mu_2}{r^2} (\beta^2 \|\Delta z(\theta_t \omega)\|^2 + \|v\|^2) + \frac{2\sqrt{2}\mu_1}{r} (\beta^2 \|\Delta z(\theta_t \omega)\|^2 + 2\zeta^2 \|v\|^2 + 2C_\zeta^2 \|\Delta v\|^2) \\
 & + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta v|^2 dx + \frac{\beta^2}{4} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta z(\theta_t \omega)|^2 dx, \tag{4.30}
 \end{aligned}$$

$$\begin{aligned}
 & \beta(1 - \alpha + \delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) z(\theta_t \omega) v dx \\
 \leq & \frac{(1 - \alpha + \delta)^2 \beta^2}{\alpha - \delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |z(\theta_t \omega)|^2 dx + \frac{\alpha - \delta}{4} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |v|^2 dx, \tag{4.31}
 \end{aligned}$$

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) g(x, t) v dx \leq \frac{1}{\alpha - \delta} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |g(x, t)|^2 dx + \frac{\alpha - \delta}{4} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |v|^2 dx. \tag{4.32}$$

Then it follows from (4.24)-(4.32)

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (|v|^2 + (\delta^2 + \lambda - \delta\alpha)|u|^2 + (1 - \delta)|\Delta u|^2 + 2F(x, u)) dx \\
 \leq & \frac{c}{r^2} (\|\Delta v\|^2 + \|v\|^2 + \|\Delta u\|^2 + \|\Delta z(\theta_t \omega)\|^2) - \frac{\alpha - \delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |v|^2 dx - \frac{\delta(\delta^2 + \lambda - \delta\alpha)}{2} \\
 & \times \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |u|^2 dx - \frac{\delta(1 - \delta)}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) |\Delta u|^2 dx - \frac{c_2 \delta}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) F(x, u) dx \\
 & + c\beta^2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (1 + |\Delta z(\theta_t \omega)|^2 + |z(\theta_t \omega)|^2 + |z(\theta_t \omega)|^{k+1} + |g(x, t)|^2) dx. \tag{4.33}
 \end{aligned}$$

Let

$$X = |v|^2 + (\delta^2 + \lambda - \delta\alpha)|u|^2 + (1 - \delta)|\Delta u|^2. \tag{4.34}$$

Then, by (4.1) we show from (4.33) and (4.34) that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X + 2F(x, u)) dx + \sigma \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X + 2F(x, u)) dx \\
 \leq & \frac{c}{r^2} (\|\Delta v\|^2 + \|v\|^2 + \|\Delta u\|^2 + \|\Delta z(\theta_t \omega)\|^2) + c\beta^2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (1 + |\Delta z(\theta_t \omega)|^2 \\
 & + |z(\theta_t \omega)|^2 + |z(\theta_t \omega)|^{k+1} + |g(x, t)|^2) dx. \tag{4.35}
 \end{aligned}$$

Multiplying (4.35) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we deduce

$$\begin{aligned}
 & \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X(\tau, \tau - t, \omega, X_0) + 2F(x, u(\tau, \tau - t, \omega, u_0))) dx \\
 & \leq e^{-\sigma t} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X_0 + 2F(x, u_0)) dx + \frac{c}{r^2} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta v(s, \tau - t, \omega, v_0)\|^2 \\
 & \quad + \|v(s, \tau - t, \omega, v_0)\|^2 + \|\Delta u(s, \tau - t, \omega, u_0)\|^2 + \|\Delta z(\theta_s \omega)\|^2) ds \\
 & \quad + c\beta^2 \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (1 + |\Delta z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^2 \\
 & \quad + |z(\theta_s \omega)|^{k+1} + |g(x, s)|^2) dx ds. \tag{4.36}
 \end{aligned}$$

By replacing ω by $\theta_{-\tau}\omega$, it then follows from (4.36) that

$$\begin{aligned}
 & \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X(\tau, \tau - t, \theta_{-\tau}\omega, X_0) + 2F(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_0))) dx \\
 & \leq e^{-\sigma t} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X_0 + 2F(x, u_0)) dx + \frac{c}{r^2} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 \\
 & \quad + \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 + \|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|\Delta z(\theta_{s-\tau}\omega)\|^2) ds \\
 & \quad + c\beta^2 \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (1 + |\Delta z(\theta_{s-\tau}\omega)|^2 + |z(\theta_{s-\tau}\omega)|^2 \\
 & \quad + |z(\theta_{s-\tau}\omega)|^{k+1} + |g(x, s)|^2) dx ds \\
 & \leq e^{-\sigma t} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X_0 + 2F(x, u_0)) dx + \frac{c}{r^2} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 \\
 & \quad + \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 + \|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2) ds + \frac{c}{r^2} \int_{-t}^0 e^{\sigma s} \|\Delta z(\theta_s \omega)\|^2 ds \\
 & \quad + c \int_{\tau-t}^{\tau} e^{\sigma s} \int_{|x| \geq r} |g(x, s)|^2 dx ds + c\beta^2 \int_{-t}^0 e^{\sigma s} \int_{|x| \geq r} (1 + |\Delta z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^2 \\
 & \quad + |z(\theta_s \omega)|^{k+1}) dx ds \\
 & \leq e^{-\sigma t} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X_0 + 2F(x, u_0)) dx + \frac{c}{r^2} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 \\
 & \quad + \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 + \|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2) ds + \frac{c}{r^2} \int_{-\infty}^0 e^{\sigma s} \|\Delta z(\theta_s \omega)\|^2 ds \\
 & \quad + c \int_{-\infty}^{\tau} e^{\sigma s} \int_{|x| \geq r} |g(x, s)|^2 dx ds + c\beta^2 \int_{-\infty}^0 e^{\sigma s} \int_{|x| \geq r} (1 + |\Delta z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^2 \\
 & \quad + |z(\theta_s \omega)|^{k+1}) dx ds \tag{4.37}
 \end{aligned}$$

In what follows, we estimate the terms on the right-hand side of (4.37). Due to $\varphi_0^{(\beta)} \in D(\tau - t, \theta_{-t}\omega) \in \mathcal{D}$ and (4.18), it's easy to obtain that, there exists $\widetilde{T}_1 = \widetilde{T}_1(\tau, \varepsilon, \omega, D) > 0$, such that for all $t > \widetilde{T}_1$,

$$e^{-\sigma t} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X_0 + 2F(x, u_0)) dx \leq \varepsilon. \tag{4.38}$$

By Lemma 4.1, there are $\widetilde{T}_2 = \widetilde{T}_2(\tau, \varepsilon, \omega, D) > 0$ and $\widetilde{R}_1 = \widetilde{R}_1(\varepsilon, \omega, D) > 1$, such that for all $t > \widetilde{T}_2$ and $r > \widetilde{R}_1$,

$$\frac{c}{r^2} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta v(s, \tau-t, \theta_{-\tau}\omega, v_0)\|^2 + \|v(s, \tau-t, \theta_{-\tau}\omega, v_0)\|^2 + \|\Delta u(s, \tau-t, \theta_{-\tau}\omega, u_0)\|^2) ds \leq \varepsilon. \tag{4.39}$$

By Lemma 3.1, there are $\widetilde{T}_3 = \widetilde{T}_3(\varepsilon, \omega) > 0$ and $\widetilde{R}_2 = \widetilde{R}_2(\varepsilon, \omega) > 1$, such that for all $t > \widetilde{T}_3$ and $r > \widetilde{R}_2$,

$$c\beta^2 \int_{-\infty}^0 e^{\sigma s} \int_{|x| \geq r} (1 + |\Delta z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^{k+1}) dx ds + \frac{c}{r^2} \int_{-\infty}^0 e^{\sigma s} \|\Delta z(\theta_s \omega)\|^2 ds \leq \varepsilon. \tag{4.40}$$

According to condition (3.6), there is $\widetilde{R}_3 = \widetilde{R}_3(\tau, \varepsilon) > 1$, such that for all $r > \widetilde{R}_3$,

$$c \int_{-\infty}^{\tau} e^{\sigma s} \int_{|x| \geq r} |g(x, s)|^2 dx ds \leq \varepsilon. \tag{4.41}$$

Letting $\widetilde{T} = \max\{\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3\}$, $\widetilde{R} = \max\{\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3\}$, then combing with (4.38)-(4.41), we have for all $t > \widetilde{T}$ and $r > \widetilde{R}$,

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{r^2}\right) (X(\tau, \tau-t, \theta_{-\tau}\omega, X_0) + 2F(x, u(\tau, \tau-t, \theta_{-\tau}\omega, u_0))) dx \leq 4\varepsilon, \tag{4.42}$$

which implies

$$\|\varphi^{(\beta)}(\tau, \tau-t, \theta_{-\tau}\omega, \varphi_0^{(\beta)})\|_{\mathcal{H}(\mathbb{R}^n; \mathbb{H}_r)}^2 \leq 4\varepsilon, \tag{4.43}$$

Then we complete the proof. □

Let $\widehat{\rho} = 1 - \rho$ with ρ given by (4.21). Fix $r \geq 1$ and set

$$\begin{cases} \widehat{u}(t, \tau, \omega, \widehat{u}_0) = \widehat{\rho}\left(\frac{|x|^2}{r^2}\right)u(t, \tau, \omega, u_0), \\ \widehat{v}(t, \tau, \omega, \widehat{v}_0) = \widehat{\rho}\left(\frac{|x|^2}{r^2}\right)v(t, \tau, \omega, v_0), \end{cases} \tag{4.44}$$

then $\widehat{\varphi}^{(\beta)}(t, \tau, \omega, \widehat{\varphi}_0^{(\beta)}) = (\widehat{u}(t, \tau, \omega, \widehat{u}_0), \widehat{v}(t, \tau, \omega, \widehat{v}_0))^T$ is the solution of problem (3.9)- (3.10) on the bounded domain \mathbb{H}_{2r} , where $\widehat{\varphi}_0^{(\beta)} = \widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\varphi_0^{(\beta)} \in \mathcal{H}(\mathbb{H}_{2r})$.

Multiplying (3.9) by $\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)$ and using (4.44) we find that

$$\begin{cases} \frac{d\widehat{u}}{dt} = \widehat{v} - \delta\widehat{u} + \beta\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)z(\theta_t\omega), \\ \frac{d\widehat{v}}{dt} = -(\alpha - \delta)\widehat{v} - (\lambda + \delta^2 - \delta\alpha)\widehat{u} - (1 - \delta)A\widehat{u} - A\widehat{v} + \beta(1 - \alpha + \delta)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)z(\theta_t\omega) \\ \quad - \beta\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)Az(\theta_t\omega) + \widehat{\rho}\left(\frac{|x|^2}{r^2}\right)g(x, t) - \widehat{\rho}\left(\frac{|x|^2}{r^2}\right)f(x, u) + 4(1 - \delta)\Delta\nabla\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\nabla u \\ \quad + 6(1 - \delta)\Delta\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\Delta u + 4(1 - \delta)\nabla\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\Delta\nabla u + (1 - \delta)uA\widehat{\rho}\left(\frac{|x|^2}{r^2}\right) \\ \quad + 4\Delta\nabla\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\nabla v + 6\Delta\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\Delta v + 4\nabla\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)\Delta\nabla v + vA\widehat{\rho}\left(\frac{|x|^2}{r^2}\right). \end{cases} \tag{4.45}$$

Considering the eigenvalue problem

$$A\widehat{u} = \lambda\widehat{u} \text{ in } \mathbb{H}_{2r}, \text{ with } \widehat{u} = \frac{\partial\widehat{u}}{\partial n} = 0 \text{ on } \partial\mathbb{H}_{2r}. \tag{4.46}$$

The problem (4.46) has a family of eigenfunctions $\{e_i\}_{i \in \mathbb{N}}$ with the eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$:

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots, \lambda_i \rightarrow +\infty (i \rightarrow +\infty),$$

such that $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{H}_{2r})$. Given n , let $X_n = \text{span}\{e_1, \dots, e_n\}$ and $P_n : L^2(\mathbb{H}_{2r}) \rightarrow X_n$ be the projection operator.

Lemma 4.4 Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $\widehat{T} = \widehat{T}(\tau, \omega, D, \varepsilon) > 0$ and $\widehat{R} = \widehat{R}(\tau, \omega, \varepsilon) \geq 1$ and $N = N(\tau, \omega, \varepsilon) > 0$, such that for all $t \geq \widehat{T}$, $r \geq \widehat{R}$ and $n \geq N$,

$$\|(I - P_n)\widehat{\varphi}^{(\beta)}(\tau, \tau-t, \theta_{-\tau}\omega, \widehat{\varphi}_0^{(\beta)})\|_{\mathcal{H}(\mathbb{H}_{2r})}^2 \leq \varepsilon, \tag{4.47}$$

where $\widehat{\varphi}_0^{(\beta)} = \widehat{\rho}(\frac{|x|^2}{r^2})\varphi_0^{(\beta)}$, $\varphi_0^{(\beta)} = (u_0, v_0)^\top \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Let $\widehat{u}_{n,1} = P_n\widehat{u}$, $\widehat{u}_{n,2} = (I - P_n)\widehat{u}$, $\widehat{v}_{n,1} = P_n\widehat{v}$, $\widehat{v}_{n,2} = (I - P_n)\widehat{v}$. Applying $I - P_n$ to the first equation of (4.45), we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \beta(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega). \quad (4.48)$$

Then applying $I - P_n$ to the second equation of (4.45) and taking the inner product of the resulting equation with $\widehat{v}_{n,2}$ in $L^2(\mathbb{H}_{2r})$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widehat{v}_{n,2}\|^2 &= -(\alpha - \delta)\|\widehat{v}_{n,2}\|^2 - (\lambda + \delta^2 - \delta\alpha)(\widehat{u}_{n,2}, \widehat{v}_{n,2}) - (1 - \delta)(A\widehat{u}_{n,2}, \widehat{v}_{n,2}) - (A\widehat{v}_{n,2}, \widehat{v}_{n,2}) \\ &+ \beta(1 - \alpha + \delta)(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega), \widehat{v}_{n,2}) - \beta(\widehat{\rho}(\frac{|x|^2}{r^2})Az(\theta_t\omega), \widehat{v}_{n,2}) + (\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t), \widehat{v}_{n,2}) \\ &- (\widehat{\rho}(\frac{|x|^2}{r^2})f(x, u), \widehat{v}_{n,2}) + (4(1 - \delta)\Delta\nabla\widehat{\rho}(\frac{|x|^2}{r^2})\nabla u + 6(1 - \delta)\Delta\widehat{\rho}(\frac{|x|^2}{r^2})\Delta u \\ &+ 4(1 - \delta)\nabla\widehat{\rho}(\frac{|x|^2}{r^2})\Delta\nabla u + (1 - \delta)uA\widehat{\rho}(\frac{|x|^2}{r^2}), \widehat{v}_{n,2}) + (4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{r^2})\nabla v \\ &+ 6\Delta\widehat{\rho}(\frac{|x|^2}{r^2})\Delta v + 4\nabla\widehat{\rho}(\frac{|x|^2}{r^2})\Delta\nabla v + vA\widehat{\rho}(\frac{|x|^2}{r^2}), \widehat{v}_{n,2}). \end{aligned} \quad (4.49)$$

Substituting $\widehat{v}_{n,2}$ in (4.48) into the second, third and eighth terms on the right-hand side of (4.49), we obtain

$$\begin{aligned} (\widehat{u}_{n,2}, \widehat{v}_{n,2}) &= (\widehat{u}_{n,2}, \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \beta(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)) \\ &\geq \frac{1}{2} \frac{d}{dt} \|\widehat{u}_{n,2}\|^2 + \delta\|\widehat{u}_{n,2}\|^2 - \beta\|\widehat{u}_{n,2}\| \cdot \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\| \\ &\geq \frac{1}{2} \frac{d}{dt} \|\widehat{u}_{n,2}\|^2 + \frac{\delta}{2}\|\widehat{u}_{n,2}\|^2 - \frac{\beta^2}{2\delta}\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|^2, -(A\widehat{u}_{n,2}, \widehat{v}_{n,2}) \quad (4.50) \\ &= -(\Delta\widehat{u}_{n,2}, \Delta(\frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - \beta(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega))) \\ &\leq -\frac{1}{2} \frac{d}{dt} \|\Delta\widehat{u}_{n,2}\|^2 - \delta\|\Delta\widehat{u}_{n,2}\|^2 + \beta\|\Delta\widehat{u}_{n,2}\| \cdot \|(I - P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega))\| \\ &\leq -\frac{1}{2} \frac{d}{dt} \|\Delta\widehat{u}_{n,2}\|^2 - \frac{\delta}{2}\|\Delta\widehat{u}_{n,2}\|^2 + \frac{\beta^2}{2\delta}\|(I - P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega))\|^2. \end{aligned} \quad (4.51)$$

By (F1) and Gagliardo-Nirenberg interpolation inequality, we set $\theta = \frac{n(k-1)}{4(k+1)}$, then the eighth term on the right hand of (4.49) is bounded by

$$\begin{aligned} (\widehat{\rho}(\frac{|x|^2}{r^2})f(x, u), \widehat{v}_{n,2}) &\leq c_1 \int_{\mathbb{R}^n} \widehat{\rho}(\frac{|x|^2}{r^2})|u|^k |\widehat{v}_{n,2}| dx + \int_{\mathbb{R}^n} \widehat{\rho}(\frac{|x|^2}{r^2})|\eta_1(x)| |\widehat{v}_{n,2}| dx \\ &\leq c_1 \|u\|_{k+1}^k \|\widehat{v}_{n,2}\|_{k+1} + \|\eta_1\| \|\widehat{v}_{n,2}\| \\ &\leq c_1 \|u\|_{k+1}^k \|\Delta\widehat{v}_{n,2}\|^\theta \|\widehat{v}_{n,2}\|^{1-\theta} + \lambda_{n+1}^{-\frac{1}{2}} \|\eta_1\| \|\Delta\widehat{v}_{n,2}\| \\ &\leq c_1 \lambda_{n+1}^{\frac{\theta-1}{2}} \|u\|_{H^2}^k \|\Delta\widehat{v}_{n,2}\| + \lambda_{n+1}^{-\frac{1}{2}} \|\eta_1\| \|\Delta\widehat{v}_{n,2}\| \\ &\leq \lambda_{n+1}^{-\frac{1}{2}} \|\Delta\widehat{v}_{n,2}\| (c_1 \lambda_{n+1}^{\frac{\theta}{2}} \|u\|_{H^2}^k + \|\eta_1\|) \\ &\leq \frac{1}{6} \|\Delta\widehat{v}_{n,2}\|^2 + \frac{3}{2} \lambda_{n+1}^{-1} (c_1 \lambda_{n+1}^{\frac{\theta}{2}} \|u\|_{H^2}^k + \|\eta_1\|)^2. \end{aligned} \quad (4.52)$$

Applying the Cauchy-Schwartz inequality and the Young inequality, we have

$$\begin{aligned} -\beta(\widehat{\rho}(\frac{|x|^2}{r^2})Az(\theta_t\omega), \widehat{v}_{n,2}) &\leq \beta\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\| \cdot \|\Delta\widehat{v}_{n,2}\| \\ &\leq \frac{3\beta^2}{2}\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|^2 + \frac{1}{6}\|\Delta\widehat{v}_{n,2}\|^2, \quad (4.53) \\ &\beta(1 - \alpha + \delta)(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega), \widehat{v}_{n,2}) \end{aligned}$$

$$\begin{aligned} &\leq |\beta|(1 - \alpha + \delta) \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\| \cdot \|\widehat{v}_{n,2}\| \\ &\leq \frac{7(1 - \alpha + \delta)^2\beta^2}{2(\alpha - \delta)} \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|^2 + \frac{\alpha - \delta}{14} \|\widehat{v}_{n,2}\|^2, \end{aligned} \tag{4.54}$$

$$\begin{aligned} &(\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t), \widehat{v}_{n,2}) \\ &\leq \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t)\| \cdot \|\widehat{v}_{n,2}\| \\ &\leq \frac{7}{2(\alpha - \delta)} \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t)\|^2 + \frac{\alpha - \delta}{14} \|\widehat{v}_{n,2}\|^2, \end{aligned} \tag{4.55}$$

$$\begin{aligned} &(1 - \delta)(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{r^2}) \cdot \nabla u + 6\Delta\widehat{\rho}(\frac{|x|^2}{r^2}) \cdot \Delta u + 4\nabla\widehat{\rho}(\frac{|x|^2}{r^2}) \cdot \Delta\nabla u + uA\widehat{\rho}(\frac{|x|^2}{r^2}), \widehat{v}_{n,2}) \\ &= (1 - \delta)(4\nabla u \cdot (\frac{12|x|}{r^4}\widehat{\rho}''(\frac{|x|^2}{r^2}) + \frac{8|x|^3}{r^6}\widehat{\rho}'''(\frac{|x|^2}{r^2}))) + 6\Delta u \cdot (\frac{2}{r^2}\widehat{\rho}'(\frac{|x|^2}{r^2}) \\ &\quad + \frac{4x^2}{r^4}\widehat{\rho}''(\frac{|x|^2}{r^2})) + \frac{8|x|}{r^2}\Delta\nabla u \cdot \widehat{\rho}'(\frac{|x|^2}{r^2}) + u(\frac{12}{r^4}\widehat{\rho}''(\frac{|x|^2}{r^2}) + \frac{48x^2}{r^6}\widehat{\rho}'''(\frac{|x|^2}{r^2}) \\ &\quad + \frac{16x^4}{r^8}\widehat{\rho}''''(\frac{|x|^2}{r^2})), \widehat{v}_{n,2}) \\ &\leq \frac{16\sqrt{2}(1 - \delta)(3\mu_2 + 4\mu_3)}{r^3} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta u\| \cdot \|\widehat{v}_{n,2}\| + \frac{12(1 - \delta)(\mu_1 + 4\mu_2)}{r^2} \|\Delta u\| \cdot \|\widehat{v}_{n,2}\| \\ &\quad + \frac{8\sqrt{2}(1 - \delta)\mu_1}{r} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta u\| \cdot \|\Delta\widehat{v}_{n,2}\| + \frac{4(1 - \delta)(3\mu_2 + 24\mu_3 + 16\mu_4)}{r^4} \|u\| \cdot \|\widehat{v}_{n,2}\| \\ &\leq (1 - \delta)^2 \lambda_{n+1}^{-\frac{1}{2}} (\frac{42(48\mu_2 + 64\mu_3)^2}{5(\alpha - \delta)r^6} + \frac{96\mu_1^2}{r^2}) \|\Delta u\|^2 + \frac{1}{3} \|\Delta\widehat{v}_{n,2}\|^2 + \frac{21(1 - \delta)^2}{5(\alpha - \delta)} \\ &\quad (\frac{(12\mu_2 + 96\mu_3 + 64\mu_4)^2}{r^8} \|u\|^2 + \frac{(12\mu_1 + 48\mu_2)^2}{r^4} \|\Delta u\|^2) + \frac{5(\alpha - \delta)}{28} \|\widehat{v}_{n,2}\|^2, \end{aligned} \tag{4.56}$$

$$\begin{aligned} &(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{r^2}) \cdot \nabla v + 6\Delta\widehat{\rho}(\frac{|x|^2}{r^2}) \cdot \Delta v + 4\nabla\widehat{\rho}(\frac{|x|^2}{r^2}) \cdot \Delta\nabla v + vA\widehat{\rho}(\frac{|x|^2}{r^2}), \widehat{v}_{n,2}) \\ &= (4\nabla v \cdot (\frac{12|x|}{r^4}\widehat{\rho}''(\frac{|x|^2}{r^2}) + \frac{8|x|^3}{r^6}\widehat{\rho}'''(\frac{|x|^2}{r^2}))) + 6\Delta v \cdot (\frac{2}{r^2}\widehat{\rho}'(\frac{|x|^2}{r^2}) \\ &\quad + \frac{4x^2}{r^4}\widehat{\rho}''(\frac{|x|^2}{r^2})) + \frac{8|x|}{r^2}\Delta\nabla v \cdot \widehat{\rho}'(\frac{|x|^2}{r^2}) + v(\frac{12}{r^4}\widehat{\rho}''(\frac{|x|^2}{r^2}) + \frac{48x^2}{r^6}\widehat{\rho}'''(\frac{|x|^2}{r^2}) \\ &\quad + \frac{16x^4}{r^8}\widehat{\rho}''''(\frac{|x|^2}{r^2})), \widehat{v}_{n,2}) \\ &\leq \frac{16\sqrt{2}(3\mu_2 + 4\mu_3)}{r^3} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta v\| \cdot \|\widehat{v}_{n,2}\| + \frac{12(\mu_1 + 4\mu_2)}{r^2} \|\Delta v\| \cdot \|\widehat{v}_{n,2}\| \\ &\quad + \frac{8\sqrt{2}\mu_1}{r} \lambda_{n+1}^{-\frac{1}{4}} \|\Delta v\| \cdot \|\Delta\widehat{v}_{n,2}\| + \frac{4(3\mu_2 + 24\mu_3 + 16\mu_4)}{r^4} \|v\| \cdot \|\widehat{v}_{n,2}\| \\ &\leq \lambda_{n+1}^{-\frac{1}{2}} (\frac{42(48\mu_2 + 64\mu_3)^2}{5(\alpha - \delta)r^6} + \frac{96\mu_1^2}{r^2}) \|\Delta v\|^2 + \frac{1}{3} \|\Delta\widehat{v}_{n,2}\|^2 + \frac{21}{5(\alpha - \delta)} \\ &\quad (\frac{(12\mu_2 + 96\mu_3 + 64\mu_4)^2}{r^8} \|v\|^2 + \frac{(12\mu_1 + 48\mu_2)^2}{r^4} \|\Delta v\|^2) + \frac{5(\alpha - \delta)}{28} \|\widehat{v}_{n,2}\|^2. \end{aligned} \tag{4.57}$$

From (4.50)-(4.57) yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\widehat{v}_{n,2}\|^2 + (\delta^2 + \lambda - \delta\alpha)\|\widehat{u}_{n,2}\|^2 + (1 - \delta)\|\Delta\widehat{u}_{n,2}\|^2) \\ &\leq -\frac{\alpha - \delta}{2} \|\widehat{v}_{n,2}\|^2 - \frac{\delta}{2} (\delta^2 + \lambda - \delta\alpha)\|\widehat{u}_{n,2}\|^2 - \frac{\delta}{2} (1 - \delta)\|\Delta\widehat{u}_{n,2}\|^2 + c\beta^2 \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\|^2 \\ &\quad + \|(I - P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega))\|^2 + \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|^2 \\ &\quad + c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t)\|^2 + (1 - \delta)^2 \lambda_{n+1}^{-\frac{1}{2}} (\frac{42(48\mu_2 + 64\mu_3)^2}{5(\alpha - \delta)r^6} + \frac{96\mu_1^2}{r^2}) \|\Delta u\|^2 \\ &\quad + \lambda_{n+1}^{-\frac{1}{2}} (\frac{42(48\mu_2 + 64\mu_3)^2}{5(\alpha - \delta)r^6} + \frac{96\mu_1^2}{r^2}) \|\Delta v\|^2 + \frac{c}{r^8} (\|u\|^2 + \|v\|^2) + \frac{c}{r^4} (\|\Delta u\|^2 + \|\Delta v\|^2) \end{aligned}$$

$$+ \frac{3}{2} \lambda_{n+1}^{-1} (c_1 \lambda_{n+1}^{\frac{\theta}{2}} \|u\|_{H^2}^k + \|\eta_1\|)^2. \quad (4.58)$$

Recalling the norm $\|\cdot\|_{\mathcal{J}_C}$ in (3.1), from (4.1) and (4.58) we conclude that

$$\begin{aligned} \frac{d}{dt} (\|\widehat{\varphi}_{n,2}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2) &\leq -\sigma \|\widehat{\varphi}_{n,2}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 + c\beta^2 (\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega)\| \\ &\quad + \|(I - P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{r^2})z(\theta_t\omega))\|^2 + \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})\Delta z(\theta_t\omega)\|^2) \\ &\quad + c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t)\|^2 + 2(1 - \delta)^2 \lambda_{n+1}^{-\frac{1}{2}} (\frac{42(48\mu_2 + 64\mu_3)^2}{5(\alpha - \delta)r^6} + \frac{96\mu_1^2}{r^2}) \|\Delta u\|^2 \\ &\quad + 2\lambda_{n+1}^{-\frac{1}{2}} (\frac{42(48\mu_2 + 64\mu_3)^2}{5(\alpha - \delta)r^6} + \frac{96\mu_1^2}{r^2}) \|\Delta v\|^2 + \frac{c}{r^8} (\|u\|^2 + \|v\|^2) + \frac{c}{r^4} (\|\Delta u\|^2 + \|\Delta v\|^2) \\ &\quad + 3\lambda_{n+1}^{-1} (c_1 \lambda_{n+1}^{\frac{\theta}{2}} \|u\|_{H^2}^k + \|\eta_1\|)^2. \end{aligned} \quad (4.59)$$

Recall that $\eta_1 \in L^2(\mathbb{R}^n)$, $\lambda_n \rightarrow \infty$ and (3.7), there exist $\widehat{N}_1 = \widehat{N}_1(\varepsilon) > 0$ and $\widehat{R}_1 = \widehat{R}_1(\varepsilon) > 0$ such that for all $n > \widehat{N}_1$ and $r > \widehat{R}_1$,

$$\begin{aligned} \frac{d}{dt} (\|\widehat{\varphi}_{n,2}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2) &\leq -\sigma \|\widehat{\varphi}_{n,2}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 + c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, t)\|^2 + \frac{c}{r^8} (\|u\|^2 + \|v\|^2) + (\frac{c}{r^4} + \varepsilon) (\|\Delta u\|^2 + \|\Delta v\|^2) \\ &\quad + \varepsilon(1 + \|u\|_{H^2}^{2k} + |y(\theta_t\omega)|^2). \end{aligned} \quad (4.60)$$

Multiplying (4.60) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we have for all $n > \widehat{N}_1$ and $r > \widehat{R}_1$,

$$\begin{aligned} \|\widehat{\varphi}_{n,2}^{(\beta)}(\tau, \tau - t, \omega, \widehat{\varphi}_{n,2,0}^{(\beta)})\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 &\leq e^{-\sigma t} \|\widehat{\varphi}_{n,2,0}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 + c \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, s)\|^2 ds \\ &\quad + \frac{c}{r^8} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|u(s, \tau - t, \omega, u_0)\|^2 + \|v(s, \tau - t, \omega, v_0)\|^2) ds \\ &\quad + (\frac{c}{r^4} + \varepsilon) \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta u(s, \tau - t, \omega, u_0)\|^2 + \|\Delta v(s, \tau - t, \omega, v_0)\|^2) ds \\ &\quad + \varepsilon \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (1 + \|u(s, \tau - t, \omega, u_0)\|_{H^2}^{2k} + |y(\theta_s\omega)|^2) ds. \end{aligned} \quad (4.61)$$

By substituting ω by $\theta_{-\tau}\omega$, we can get from (4.61) that,

$$\begin{aligned} \|\widehat{\varphi}_{n,2}^{(\beta)}(\tau, \tau - t, \theta_{-\tau}\omega, \widehat{\varphi}_{n,2,0}^{(\beta)})\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 &\leq e^{-\sigma t} \|\widehat{\varphi}_{n,2,0}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 + c \int_{-\infty}^{\tau} e^{\sigma(s-\tau)} \|(I - P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x, s)\|^2 ds \\ &\quad + \frac{c}{r^8} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2) ds \\ &\quad + (\frac{c}{r^4} + \varepsilon) \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|\Delta v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2) ds \\ &\quad + \varepsilon \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (1 + \|u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|_{H^2}^{2k} + |y(\theta_{s-\tau}\omega)|^2) ds. \end{aligned} \quad (4.62)$$

We next estimate each term on the right-hand side of (4.62). Thanks to condition (F1), $\varphi_0^{(\beta)} \in D(\tau - t, \theta_{-t}\omega)$ and $D(\tau - t, \theta_{-t}\omega) \in \mathcal{D}$, there exists $\widehat{T}_1 = \widehat{T}_1(\tau, \varepsilon, D, \omega) > 0$ and $\widehat{R}_1 = \widehat{R}_1(\tau, \varepsilon, \omega) > 1$, such that if $t > \widehat{T}_1$ and $r > \widehat{R}_1$, then

$$e^{-\sigma t} \|\widehat{\varphi}_{n,2,0}^{(\beta)}\|_{\mathcal{J}_C(\mathbb{H}_{2r})}^2 \leq \varepsilon. \quad (4.63)$$

For the second term on the right-hand side of (4.62), due to condition (3.5), there is $\widehat{N} = \widehat{N}(\tau, \varepsilon, \omega) > 0$, such that for all $n > \widehat{N}$, then

$$c \int_{-\infty}^{\tau} e^{\sigma(s-\tau)} \|(I - P_n)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)g(x, s)\|^2 ds \leq \varepsilon. \tag{4.64}$$

For the third and fourth terms on the right-hand side of (4.62), taking advantage of Lemma 4.1, there exist $\widehat{T}_2 = \widehat{T}_2(\tau, \varepsilon, D, \omega) > 0$ and $\widehat{R}_2(\tau, \varepsilon, \omega) > 1$, such that for all $t > \widehat{T}_2$ and $r > \widehat{R}_2$, there holds

$$\begin{aligned} & \frac{c}{r^8} \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|u(s, \tau-t, \theta_{-\tau}\omega, u_0)\|^2 + \|v(s, \tau-t, \theta_{-\tau}\omega, v_0)\|^2) ds \\ & + \left(\frac{c}{r^4} + \varepsilon\right) \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (\|\Delta u(s, \tau-t, \theta_{-\tau}\omega, u_0)\|^2 + \|\Delta v(s, \tau-t, \theta_{-\tau}\omega, v_0)\|^2) ds \leq \varepsilon. \end{aligned} \tag{4.65}$$

For the last term on the right-hand side of (4.62), due to Lemma 4.1, there is $\widehat{T}_3 = \widehat{T}_3(\tau, \varepsilon, D, \omega) > 0$, such that for all $t > \widehat{T}_3$, it follows that

$$\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} (1 + \|u(s, \tau-t, \theta_{-\tau}\omega, u_0)\|_{H^2}^{2k} + |y(\theta_{s-\tau}\omega)|^2) ds < \infty. \tag{4.66}$$

Let $\widehat{T} = \max\{\widehat{T}_1, \widehat{T}_2, \widehat{T}_3\}$, and $\widehat{R} = \max\{\widehat{R}_1, \widehat{R}_2\}$. Then, collecting all (4.63), (4.64), (4.65) and (4.66), for all $t > \widehat{T}$, $r > \widehat{R}$ and $n > \widehat{N}$, we arrive at

$$\|\widehat{\varphi}_{n,2}^{(\beta)}(\tau, \tau-t, \theta_{-\tau}\omega, \widehat{\varphi}_{n,2,0}^{(\beta)})\|_{\mathcal{H}(\mathbb{H}_{2r})}^2 \leq c\varepsilon, \tag{4.67}$$

which completes the proof. □

5 Random attractors

In this section, we prove the existence of \mathcal{D} -pullback attractors for the stochastic problem (3.9)-(3.10) in $\mathcal{H}(\mathbb{R}^n)$. We are now ready to apply the Lemmas in Section 4 to prove the asymptotic compactness of solutions in $\mathcal{H}(\mathbb{R}^n)$.

Lemma 5.1 *Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then the solution of problem (3.9)-(3.10) is asymptotic compactness in $\mathcal{H}(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence $\{\varphi^{(\beta)}(\tau, \tau-t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\}$ has a convergent subsequence in $\mathcal{H}(\mathbb{R}^n)$ provided $t_m \rightarrow \infty$ and $\varphi_{0,m}^{(\beta)} \in B(\tau-t_m, \theta_{-t_m}\omega)$.*

Proof. We first let $t_m \rightarrow \infty$, $B \in \mathcal{D}$, and $\varphi_{0,m}^{(\beta)} \in B(\tau-t_m, \theta_{-t_m}\omega)$. By Lemma 4.1, $\{\varphi^{(\beta)}(\tau, \tau-t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\}$ is bounded in $\mathcal{H}(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exists $M_1 = M_1(\tau, \omega, B) > 0$ such for all $m > M_1$,

$$\|\varphi^{(\beta)}(\tau, \tau-t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\|_{\mathcal{H}(\mathbb{R}^n)}^2 \leq \varrho^2(\tau, \omega). \tag{5.1}$$

In addition, it follows from Lemma 4.3 that there exist $r_1 = r_1(\tau, \varepsilon, \omega) > 0$ and $\widehat{M}_2 = \widehat{M}_2(\tau, B, \varepsilon, \omega) > 0$, such that for every $m \geq \widehat{M}_2$,

$$\|\varphi^{(\beta)}(\tau, \tau-t_m, \theta_{-\tau}\omega, \varphi_{0,m}^{(\beta)})\|_{\mathcal{H}(\mathbb{R}^n \setminus \mathbb{H}_{r_1})}^2 \leq \varepsilon. \tag{5.2}$$

Next, by using Lemma 4.4, there are $N = N(\tau, \varepsilon, \omega) > 0$, $r_2 = r_2(\tau, \varepsilon, \omega) \geq r_1$ and $\widehat{M}_3 = \widehat{M}_3(\tau, B, \varepsilon, \omega) > 0$, such that for every $m \geq \widehat{M}_3$,

$$\|(I - P_N)\widehat{\varphi}^{(\beta)}(\tau, \tau-t_m, \theta_{-\tau}\omega, \widehat{\varphi}_{0,m}^{(\beta)})\|_{\mathcal{H}(\mathbb{H}_{2r_2})}^2 \leq \varepsilon. \tag{5.3}$$

Using (4.44) and (5.1), we find that $\{P_N \widehat{\varphi}^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau} \omega, \widehat{\varphi}_{0,m}^{(\beta)})\}$ is bounded in the finite-dimensional space $P_N \mathcal{H}(\mathbb{H}_{2r_2})$, which together with (5.3) implies that $\{\widehat{\varphi}^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau} \omega, \widehat{\varphi}_{0,m}^{(\beta)})\}$ is precompact in $H^2(\mathbb{H}_{2r_2}) \times L^2(\mathbb{H}_{2r_2})$.

Note that $\widehat{\rho}(\frac{|x|^2}{r_2^2}) = 1$ for $|x| \leq r_2$. Recalling (4.44), we find that $\{\varphi^{(\beta)}(\tau, \tau - t_m, \theta_{-\tau} \omega, \varphi_{0,m}^{(\beta)})\}$ is precompact in $\mathcal{H}(\mathbb{H}_{r_2})$, which along with (5.2) shows that the precompactness of this sequence in $\mathcal{H}(\mathbb{R}^n)$. This completes the proof. \square

Theorem 5.1 Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then the continuous cocycle Φ_β associated with problem (3.9)-(3.10) has a unique \mathcal{D} -pullback attractor $\mathcal{A}_\beta = \{\mathcal{A}_\beta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{H}(\mathbb{R}^n)$.

Proof. Note that the cocycle Φ_β is pullback \mathcal{D} -asymptotically compact in $\mathcal{H}(\mathbb{R}^n)$ by Lemma 5.1. On the other hand, the cocycle Φ_β has a pullback \mathcal{D} -absorbing set by Lemma 4.1. Then the existence and uniqueness of a pullback \mathcal{D} -attractor of Φ_β follow from Proposition 2.1 immediately. \square

6 Upper semicontinuity of pullback attractors

First, we present a criteria concerning upper semicontinuity of non-autonomous random attractors with respect to a parameter in [23].

Theorem 6.1 Let $(X, \|\cdot\|_X)$ be a separable Banach space and Φ_0 be an autonomous dynamical system with the global attractor \mathcal{A}_0 in X . Given $\beta > 0$, suppose that Φ_β is the perturbed random dynamical system with a random attractor $\mathcal{A}_\beta \in \mathcal{D}$ and a random absorbing set $E_\beta \in \mathcal{D}$. Then for $P - a.e. \tau \in \mathbb{R}, \omega \in \Omega$,

$$d_H(\mathcal{A}_\beta(\tau, \omega), \mathcal{A}_0) \rightarrow 0, \text{ as } \beta \rightarrow 0,$$

if the following conditions are satisfied:

(i) there exists some deterministic constant c such that, for $P - a.e. \tau \in \mathbb{R}, \omega \in \Omega$

$$\limsup_{\beta \rightarrow 0} \|E_\beta(\tau, \omega)\|_X \leq c.$$

(ii) there exists a $\beta_0 > 0$, such that for $P - a.e. \tau \in \mathbb{R}, \omega \in \Omega$,

$$\bigcup_{0 < \beta \leq \beta_0} \mathcal{A}_\beta(\tau, \omega) \text{ is precompact in } X.$$

(iii) for $P - a.e. \tau \in \mathbb{R}, \omega \in \Omega, t \geq 0, \beta_n \rightarrow 0$, and $x_n, x \in X$ with $x_n \rightarrow x$, it holds that

$$\lim_{n \rightarrow \infty} \Phi_{\beta_n}(t, \tau, \omega)x_n = \Phi_0(t)x,$$

where $\|E_\beta(\tau, \omega)\|_X = \sup_{x \in E_\beta(\tau, \omega)} \|x\|_X$.

Next, we will use Theorem 6.1 to consider an upper semicontinuity of random attractors $\mathcal{A}_\beta(\omega)$ when $\beta \rightarrow 0$. To indicate the dependence of solutions on β , we respectively write the solutions of problem (3.9)-(3.10) as $u^{(\beta)}$ and $v^{(\beta)}$, that is, $(u^{(\beta)}, v^{(\beta)})$ satisfies

$$\begin{cases} \frac{du^{(\beta)}}{dt} = v^{(\beta)} - \delta u^{(\beta)} + \beta z(\theta_t \omega), \\ \frac{dv^{(\beta)}}{dt} = (\delta - \alpha - A)v^{(\beta)} + [\delta(-\delta + \alpha + A) - \lambda - A]u^{(\beta)} + \beta[1 - (\alpha + A - \delta)]z(\theta_t \omega) - f(x, u^{(\beta)}) + g(x, t), \\ u^{(\beta)}(\tau, \tau, x) = u_0^{(\beta)}(x), \quad v^{(\beta)}(\tau, \tau, x) = v_0^{(\beta)}(x). \end{cases} \quad (6.1)$$

When $\beta = 0$, the random problem (3.9)-(3.10) reduces to a deterministic one:

$$\begin{cases} \frac{du^{(0)}}{dt} = v^{(0)} - \delta u^{(0)}, \\ \frac{dv^{(0)}}{dt} = (\delta - \alpha - A)v^{(0)} + [\delta(-\delta + \alpha + A) - \lambda - A]u^{(0)} - f(x, u^{(0)}) + g(x, t), \\ u^{(0)}(\tau, \tau, x) = u_0^{(0)}(x), \quad v^{(0)}(\tau, \tau, x) = v_0^{(0)}(x). \end{cases} \quad (6.2)$$

Accordingly, by Theorem 5.1 the deterministic and autonomous system Φ_0 generated by (6.2) is readily verified to admit a global attractor \mathcal{A}_0 in $\mathcal{H}(\mathbb{R}^n)$.

Theorem 6.2 Assume that $h \in H^2(\mathbb{R}^n)$, (F1)-(F4) and (3.5) hold. Then the random dynamical system Φ_β generated by (3.9)-(3.10) has a unique \mathcal{D} -pullback attractor $\{\mathcal{A}_\beta(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ in $\mathcal{H}(\mathbb{R}^n)$. Moreover, the family $\{\mathcal{A}_\beta\}_{\beta > 0}$ of random attractors is upper semicontinuous.

Proof. By Lemma 4.2 and Theorem 5.1, Φ_β has a closed measurable random absorbing set $E_\beta(\tau, \omega)$ and a unique random attractor \mathcal{A}_β .

(i) since Lemma 4.2 has proved that system Φ_β possesses a closed random absorbing set $E_\beta = \{E_\beta(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ in \mathcal{D} , which is given by

$$E_\beta(\tau, \omega) = \{(u, v) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|u\|_{H^2(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2 \leq R_1(\tau, \omega)\}$$

with

$$R_1(\tau, \omega) = c + c\beta^2 \int_{-\infty}^0 e^{\sigma s} (1 + \|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds,$$

it is readily to obtain that

$$\limsup_{\beta \rightarrow 0} \|E_\beta(\tau, \omega)\|_{\mathcal{H}(\mathbb{R}^n)} \leq c, \tag{6.3}$$

which deduces condition (i) immediately.

(ii) Given $\beta \in (0, 1]$, let $E_1(\tau, \omega) = \{(u, v) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|u\|_{H^2(\mathbb{R}^n)}^2 + \|v\|_{L^2(\mathbb{R}^n)}^2 \leq R_*(\tau, \omega)\}$, where $R_*(\tau, \omega) = c + c \int_{-\infty}^0 e^{\sigma s} (1 + \|\Delta z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|^2 + \|z(\theta_s \omega)\|_{H^2}^{k+1}) ds$, then

$$\bigcup_{0 < \beta \leq 1} \mathcal{A}_\beta(\tau, \omega) \subseteq \bigcup_{0 < \beta \leq 1} E_\beta(\omega) \subseteq E_1(\tau, \omega). \tag{6.4}$$

For one thing, by (6.4), Lemma 4.3 and the invariance of $\mathcal{A}_\beta(\tau, \omega)$, we find that for every $\beta > 0$ and P -a.e. $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exists $r_0 = r_0(\omega, \varepsilon) \geq 1$ such that

$$\int_{|x| \geq r_0} (\|u(x)\|^2 + \|\Delta u(x)\|^2 + \|v(x)\|^2) dx \leq \varepsilon, \text{ for all } (u, v) \in \bigcup_{0 < \beta \leq 1} \mathcal{A}_\beta(\tau, \omega). \tag{6.5}$$

For another, by (6.4), the proof of Lemma 5.1, Lemma 4.4 and the invariance of $\mathcal{A}_\beta(\tau, \omega)$, we know that there exists $r_1 = r_1(\omega, \varepsilon) \geq r_0$ such that for all $r \geq r_1$, the set $\bigcup_{0 < \beta \leq 1} \mathcal{A}_\beta(\tau, \omega)$ is precompact in $\mathcal{H}(\mathbb{H}_r)$, which together with (6.5) implies that $\bigcup_{0 < \beta \leq 1} \mathcal{A}_\beta(\tau, \omega)$ is precompact in $\mathcal{H}(\mathbb{R}^n)$.

(iii) Let $\varphi^{(0)} = (u^{(0)}, v^{(0)})$ be a mild solution of (6.2) with initial data $\varphi_0^{(0)} = (u_0^{(0)}, v_0^{(0)})$, and $U = u^{(\beta)} - u^{(0)}$, $V = v^{(\beta)} - v^{(0)}$. It follows from (6.1) and (6.2) that

$$\begin{cases} \frac{dU}{dt} = V - \delta U + \beta z(\theta_t \omega), \\ \frac{dV}{dt} = (\delta - \alpha - A)V + [\delta(-\delta + \alpha + A) - \lambda - A]U - f(x, u^{(\beta)}) + f(x, u^{(0)}) + \beta[1 - (\alpha + A - \delta)]z(\theta_t \omega), \\ U(\tau, \tau, x) = U_0(x), \quad V(\tau, \tau, x) = V_0(x). \end{cases} \tag{6.6}$$

First taking the inner product of the second equation of (6.6) with V in $L^2(\mathbb{R}^n)$, and then using the first equation of (6.6) to simplify the resulting equality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|V\|^2 + (\delta^2 + \lambda - \delta\alpha)\|U\|^2 + (1 - \delta)\|\Delta U\|^2) \\ & \leq -\frac{3}{4}(\alpha - \delta)\|V\|^2 - \frac{3}{4}\delta(\delta^2 + \lambda - \delta\alpha)\|U\|^2 - \frac{3}{4}\delta(1 - \delta)\|\Delta U\|^2 + (-f(x, u^{(\beta)}) \\ & \quad + f(x, u^{(0)}), v) + c\beta^2(1 + \|\Delta z(\theta_t \omega)\|^2 + \|z(\theta_t \omega)\|^2). \end{aligned} \tag{6.7}$$

By (F4), the nonlinear term in (6.7) satisfies

$$|(f(x, u^{(0)}) - f(x, u^{(\beta)}), V)| \leq c\|U\|^2 + c\|V\|^2, \tag{6.8}$$

which along with (6.7) implies

$$\begin{aligned} \frac{d}{dt}(\|V\|^2 + (\delta^2 + \lambda - \delta\alpha)\|U\|^2 + (1 - \delta)\|\Delta U\|^2) &\leq c(\|V\|^2 + (\delta^2 + \lambda - \delta\alpha)\|U\|^2 + (1 - \delta)\|\Delta U\|^2) \\ &\quad + c\beta^2(1 + \|\Delta z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|^2). \end{aligned} \tag{6.9}$$

Applying Gronwall inequality to (6.9) over (τ, t) , we have

$$\begin{aligned} &\|u^{(\beta)}(t, \tau, \omega, u_0^{(\beta)}) - u^{(0)}(t, \tau, \omega, u_0^{(0)})\|_{H^2(\mathbb{R}^n)}^2 + \|v^{(\beta)}(t, \tau, \omega, v_0^{(\beta)}) - v^{(0)}(t, \tau, \omega, v_0^{(0)})\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq ce^{c(t-\tau)}(\|u_0^{(\beta)} - u_0^{(0)}\|_{H^2(\mathbb{R}^n)}^2 + \|v_0^{(\beta)} - v_0^{(0)}\|_{L^2(\mathbb{R}^n)}^2) + c\beta^2 \int_{\tau}^t e^{c(t-s)}(1 + \|\Delta z(\theta_s\omega)\|^2 \\ &\quad + \|z(\theta_s\omega)\|^2)ds, \end{aligned}$$

which along with (i),(ii) and Theorem 6.1 completes the proof. □

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References

- [1] Barbosaa A. R. A., Ma T. F., Long-time dynamics of an extensible plate equation with thermal memory, *J. Math. Anal. Appl.*, 2014, 416, 143–165.
- [2] Khanmamedov A. Kh., A global attractor for the plate equation with displacement-dependent damping, *Non. Anal.*, 2011, 74, 1607–1615.
- [3] Khanmamedov A. Kh., Existence of global attractor for the plate equation with the critical exponent in an unbounded domain, *Appl. Math. Lett.*, 2005, 18, 827–832.
- [4] Khanmamedov A. Kh., Global attractors for the plate equation with a localized damping and a critical exponent in an unbounded domain, *J. Differ. Equ.*, 2006, 225, 528–548.
- [5] Silva M. A. J., Ma T. F., Long-time dynamics for a class of Kirchhoff models with memory, *J. Math. Phys.*, 2013, 54, 021505.
- [6] Wu H., Long-time behavior for a nonlinear plate equation with thermal memory, *J. Math. Anal. Appl.*, 2008, 348, 650–670.
- [7] Xiao H. B., Asymptotic dynamics of plate equations with a critical exponent on unbounded domain, *Non. Anal.*, 2009, 70, 1288–1301.
- [8] Yang L., Zhong C. K., Global attractor for plate equation with nonlinear damping, *Non. Anal.*, 2008, 69, 3802–3810.
- [9] Yue G. C., Zhong C. K., Global attractors for plate equations with critical exponent in locally uniform spaces, *Non. Anal.*, 2009, 71, 4105–4114.
- [10] Zhou J., Global existence and blow-up of solutions for a Kirchhoff type plate equation with damping, *Appl. Math. Comput.*, 2015, 265, 807–818.
- [11] Yang L., Uniform attractor for non-autonomous plate equations with a localized damping and a critical nonlinearity, *J. Math. Anal. Appl.*, 2008, 338, 1243–1254.
- [12] Ma Q. Z., Ma W. J., Asymptotic behavior of solutions for stochastic plate equations with strongly damped and white noise, *J. Northwest Norm. Univ. Nat. Sci.*, 2014, 50, 6–17.
- [13] Ma W. J., Ma Q. Z., Attractors for the stochastic strongly damped plate equations with additive noise, *Electron. J. Differ. Equ.*, 2013, 111, 1–12.
- [14] Shen X. Y., Ma Q. Z., Existence of random attractors for weakly dissipative plate equations with memory and additive white noise, *Comput. Math. Appl.*, 2017, 73, 2258–2271.
- [15] Bates P. W., Lu K., Wang B. X., Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differ. Equ.*, 2009, 246, 845–869.
- [16] Wang Z., Zhou S., Random attractor for stochastic reaction-diffusion equation with multiplicative noise on unbounded domains, *J. Math. Anal. Appl.*, 2011, 384, 160–172.

- [17] Wang B., Random attractors for the stochastic FitzHugh-Nagumo system on unbounded domains, *Non. Anal.*, 2009, 71, 2811–2525.
- [18] Li Y., Yin J., A modified proof of pullback attractors in a Sobolev space for stochastic FitzHugh-Nagumo equations, *Discrete Contin. Dyn. Syst. Ser. B*, 2016, 21, 1203–1223.
- [19] Ju N., The H^1 -compact global attractor for the solutions to the Navier-Stokes equations in two-dimensional unbounded domains, *Nonlinearity*, 2000, 13, 1227–1238.
- [20] Wang B., Asymptotic behavior of stochastic wave equations with critical exponents on \mathbb{R}^3 , *Tran. Amer. Math. Soc.*, 2011, 363, 3639–3663.
- [21] Wang B., Gao X., Random attractors for wave equations on unbounded domains, *Discr. Contin. Dyn. Syst. Syst Special*, 2009, 800–809.
- [22] Wang Z., Zhou S., Random attractor for non-autonomous stochastic strongly damped wave equation on unbounded domains, *J. Math. Anal. Appl.*, 2015, 5, 363–387.
- [23] Wang B., Upper semicontinuity of random attractors for non-compact random dynamical systems, *Electron. J. Differential Equations*, 2009, 139, 1–18.
- [24] Wang B., Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, 2012, 253, 1544–1583.
- [25] Crauel H., Kloeden P.E., Yang M., Random attractors of stochastic reaction-diffusion equations on variable domains, *Stoch. Dyn.*, 2011, 11, 301–314.
- [26] Li Y., Gu A., Li J., Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations, *J. Differential Equations*, 2015, 258, 504–534.
- [27] Crauel H., *Random Probability Measure on Polish Spaces*, Taylor and Francis, London, 2002.
- [28] Crauel H., Debussche A., Flandoli F., Random attractors, *J. Dynam. Differential Equations*, 1997, 9, 307–341.
- [29] Crauel H., Flandoli F., Attractors for random dynamical systems, *Probab. Th. Re. Fields*, 1994, 100, 365–393.
- [30] Flandoli F., Schmalfuss B., Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stochastics*, 1996, 59, 21–45.
- [31] Wang Z., Zhou S., Gu A., Random attractor for a stochastic damped wave equation with multiplicative noise on unbounded domains, *Nonlinear Anal. Real World Appl.*, 2011, 12, 3468–3482.
- [32] Duan J., Lu K., Schmalfuss B., Invariant manifolds for stochastic partial differential equations, *Ann. Probab.*, 2003, 31, 2109–2135.
- [33] Shen Z., Zhou S., Shen W., One-dimensional random attractor and rotation number of the stochastic damped sine-Gordon equation, *J. Differential Equations*, 2010, 248, 1432–1457.
- [34] Pazy A., *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- [35] Temam R., *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1998.