ASYMPTOTIC BEHAVIOR OF A GENERALIZED TCP CONGESTION AVOIDANCE ALGORITHM

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The Transmission Control Protocol (TCP) is a Transport Protocol used in the Internet. In [8], a more general class of candidate Transport Protocols called "protocols in the TCP Paradigm" is introduced. The long run objective of studying this class is to find protocols with promising performance characteristics. This paper studies Markov chain models derived from protocols in the TCP Paradigm.

Protocols in the TCP Paradigm, as TCP, protect the network from congestion by reducing the "Congestion Window" (the amount of data allowed to be sent but not yet acknowledged) when there is packet loss or packet marking, and increasing it when there is no loss. When loss of different packets are assumed to be independent events and the probability p of loss is assumed to be constant, the protocol gives rise to a Markov chain $\{W_n\}$, where W_n is the size of the congestion window after the transmission of the n-th packet.

For a wide class of such Markov chains, we prove weak convergence results, after appropriate rescaling of time and space, as $p \to 0$. The limiting processes are defined by stochastic differential equations. Depending on certain parameter values, the stochastic differential equation can define an Ornstein-Uhlenbeck process or can be driven by a Poisson process.

1. Introduction. The Congestion Avoidance algorithm of TCP is designed to prevent network congestion during the transmission of data over

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a computer network. It does this by controlling the congestion window, i.e. the amount of data "transmitted but not yet acknowledged" by a sender. What follows is a simplified description of a more general class of Transport Protocols.

Under appropriate units, the congestion window W determines the maximum amount of data that a source can send without acknowledgement. The "TCP Paradigm" (see [8]) is a class of protocols that includes TCP (and other Transport Protocols). For each protocol in the TCP Paradigm there are two functions, $incr(\cdot)$ and $decr(\cdot)$. If, while the congestion window equals W, a packet is found to be lost (or marked, under ECN – see [2] and [12]), then the congestion window is reduced by $decr(W)$. However, the congestion window is never reduced below some fixed minimum value $\ell \geq 0$. If there are no lost packets, then the congestion window is increased by $incr(W)$. For protocols in the TCP Paradigm, $incr(W) = c_1 W^\alpha$ and $decr(W) = c_2 W^\beta$. In the special case of TCP, we have $c_1 = 1$, $\alpha = -1$, $c_2 = 1/2$, and $\beta = 1$. Another special case of interest is when $\alpha = 0$ and $\beta = 1$. This is the algorithm which Tom Kelly calls "Scalable TCP" in [3] and [4].

Let W_n denote the size of the congestion window after the transmission of the n-th packet, and let χ_n be the indicator function of the event that the n-th packet is lost. We shall assume that the χ_n's are independent and identically distributed. In particular, we are assuming that $p = P(\chi_n = 1)$ is a constant that does not change with time. Under these assumptions, we are led to the parameterized family of Markov processes

$$W_{p,n+1} = (W_{p,n} + c_1 W_{p,n}^\alpha (1 - \chi_{p,n+1}) - c_2 W_{p,n}^\beta \chi_{p,n+1}) \vee \ell. \tag{1.1}$$

The assumptions we place on the various parameters in the model are:

$$\{\chi_{p,n}\}_{n=1}^\infty \text{ is an iid sequence of } \{0,1\}\text{-valued random variables,} \tag{1.2}$$

$$p = P(\chi_{p,n} = 1), \tag{1.3}$$

$$c_1 > 0 \text{ and } c_2 > 0, \tag{1.4}$$

$$-\infty < \alpha < \beta \leq 1 \text{ and } \ell \geq 0, \tag{1.5}$$

$$\text{if } \beta = 1, \text{ then } c_2 < 1, \text{ and} \tag{1.6}$$

$$\text{if } \beta < 1, \text{ then } \ell > 0. \tag{1.7}$$

We will frequently drop the dependence on p from our notation and simply refer to the processes $\{\chi_n\}$ and $\{W_n\}$.

We are interested in studying the asymptotic behavior of $\{W_n\}$ as $p \to 0$. To this end, we define the continuous time process

$$Z_p(t) = p^\gamma W_{\lfloor tp^{-\nu} \rfloor}, \text{ where } \gamma = (\beta - \alpha)^{-1} \text{ and } \nu = (1 - \alpha)\gamma. \tag{1.8}$$

In the case that $\beta = 1$, we will show that Z_p converges weakly as $p \to 0$ to the process Z defined by

$$Z(t) = Z(0) + c_1 \int_0^t Z(s)^\alpha \, ds - c_2 \int_0^t Z(s-) \, dN(s), \tag{1.9}$$

where N is a unit rate Poisson process, independent of $Z(0) = \lim Z_p(0)$. (Note that this is the conjecture given on page 362 of [8].) We will also show that, when $\ell > 0$, the stationary distributions of the discrete time Markov chains $\{p^\gamma W_n\}$ converge weakly to the unique stationary distribution of Z. Questions about the convergence of the stationary distributions when $\beta = 1$, as well as the rate of convergence, are addressed in [9] using techniques that differ from those used in this paper.

In the case that $\beta < 1$, we will show that Z_p converges to the process ζ defined by

$$\zeta(t) = \zeta(0) + \int_0^t (c_1 \zeta(s)^\alpha - c_2 \zeta(s)^\beta) \, ds, \tag{1.10}$$

where $\zeta(0) = \lim Z_p(0)$. With the exception of the initial condition, the process ζ is entirely deterministic. The convergence of Z_p to ζ is therefore a law of large numbers type of result. Hence, in the case $\beta < 1$, we can extend our analysis and study the fluctuations of Z_p around this central tendency. Unfortunately, it will not suffice to center Z_p by ζ. We must rather define

$$\zeta_p(t) = \zeta_p(0) + \int_0^t (c_1(1-p)\zeta_p(s)^\alpha - c_2 \zeta_p(s)^\beta) \, ds, \tag{1.11}$$

where $\zeta_p(0) \to \zeta(0)$, and consider the processes

$$\xi_p(t) = p^{-\tau}(Z_p(t) - \zeta_p(t)), \text{ where } \tau = (\nu - 1)/2. \tag{1.12}$$

We will show that ξ_p converges weakly as $p \to 0$ to the process ξ defined by

$$\begin{aligned} \xi(t) = \xi(0) &+ \int_0^t (c_1 \alpha \zeta(s)^{\alpha-1} - c_2 \beta \zeta(s)^{\beta-1}) \xi(s) \, ds \\ &- c_2 \int_0^t \zeta(s)^\beta \, dB(s), \end{aligned} \tag{1.13}$$

where B is a Brownian motion and $\xi(0) = \lim \xi_p(0)$.

A special case of this last result is worth mentioning. For each $p \in [0, 1)$, define

$$c_p = (c_1(1-p)/c_2)^\gamma, \tag{1.14}$$

so that $\zeta_p(t) = c_p$ is an invariant solution to (1.11). Also, $\zeta(0) = \lim \zeta_p(0) = c_0$ is an invariant solution to (1.10). Hence, for an appropriate choice of $Z_p(0)$, ξ_p converges to the Ornstein-Uhlenbeck process defined by

$$d\xi = -\mu\xi dt + \sigma dW, \tag{1.15}$$

where $W = -B$,

$$\begin{aligned}\mu &= c_2\beta(c_1/c_2)^{\gamma(\beta-1)} - c_1\alpha(c_1/c_2)^{\gamma(\alpha-1)}\\ &= (\beta-\alpha)c_1^{-(1-\beta)/(\beta-\alpha)}c_2^{(1-\alpha)/(\beta-\alpha)},\end{aligned}$$

and

$$\sigma = c_2(c_1/c_2)^{\gamma\beta} = c_1^{\beta/(\beta-\alpha)}c_2^{-\alpha/(\beta-\alpha)}.$$

(Note that this is the conjecture given on page 364 of [8].) We will also show that the stationary distributions of the discrete time Markov chains $\{p^{-\tau}(p^\gamma W_n - c_p)\}$ converge weakly to the unique stationary distribution of the above Ornstein-Uhlenbeck process.

2. Main Results.

We first consider the case $\beta = 1$ and begin by cataloging some properties of the limit process Z.

LEMMA 2.1. *If $Z(0) > 0$ a.s., then the stochastic differential equation (1.9) has a unique solution Z. With probability one, $Z(t) > 0$ for all $t \geq 0$. Moreover, if $\tau = \inf\{t \geq 0 : Z(t) = c_0\}$, where c_0 is given by (1.14), then $\tau < \infty$ a.s.*

Proof. For each realization of the Poisson process, (1.9) can be solved deterministically and the solution is unique. Let

$$T = \inf\{t \geq 0 : Z(t) \notin (0,\infty)\}.$$

Since Z decreases only at the jump times of the Poisson process, and, with probability one, these jump times have no accumulation points, it follows that $T = \infty$ a.s.

To show that $\tau < \infty$ a.s., it will suffice to assume that $Z(0) = x > 0$ is deterministic. We first consider the case $x \leq c_0$. Suppose $\tau(\omega) = \infty$. Then $Z(t,\omega) < c_0$ for all $t \geq 0$. Find $u > r$ such that $u - r > \gamma c_2^{-1}$ and $N(u,\omega) = N(r,\omega)$. Then for all $t \in (r,u]$,

$$Z(t,\omega) = Z(r,\omega) + c_1\int_r^t Z(s,\omega)^\alpha\, ds.$$

Since the solution to this integral equation is unique,

$$Z(t,\omega) = (c_1(1-\alpha)(t-r) + Z(r,\omega)^{1-\alpha})^\gamma.$$

Therefore,

$$c_0 > Z(u,\omega) > (c_1(1-\alpha)(u-r))^\gamma > c_0,$$

a contradiction. Hence, $\tau < \infty$ a.s.

We next consider the case $x > c_0$. Define

$$\sigma_1 = \inf\{t \geq 0 : Z(t) < c_0\} \text{ and } \sigma_2 = \inf\{t \geq \sigma_1 : Z(t) = c_0\},$$

so that $\tau \leq \sigma_2$, and it will suffice to show that $\sigma_2 < \infty$ a.s. Fix $L > x$ and define $\rho = \inf\{t \geq 0 : Z(t) \notin [c_0, L]\}$. Suppose $\rho(\omega) = \infty$. Then $Z(t,\omega) \in [c_0, L]$ for all $t \geq 0$. Let

$$K = \inf\{u^\alpha : c_0 \leq u \leq L\} > 0.$$

Find $u > r$ such that $u - r > (L - c_0)/(c_1 K)$ and $N(u,\omega) = N(r,\omega)$. Then

$$L \geq Z(u,\omega) = Z(r,\omega) + c_1 \int_r^u Z(s,\omega)^\alpha \, ds \geq c_0 + c_1(u-r)K > L,$$

a contradiction. Hence, $\rho < \infty$ a.s.

Now, observe that

$$Z(t \wedge \rho) = x + \int_0^{t\wedge\rho} (c_1 Z(s)^\alpha - c_2 Z(s)) \, ds - c_2 \int_0^{t\wedge\rho} Z(s-) \, dM(s),$$

where $M(t) = N(t) - t$ is the compensated Poisson process. If $s < t \wedge \rho$, then $Z(s) \geq c_0 = (c_1/c_2)^\gamma$. This implies that $c_1 Z(s)^\alpha - c_2 Z(s) \leq 0$. Since M is a martingale, $E[Z(t \wedge \rho)] \leq x$. Letting $t \to \infty$ gives $E[Z(\rho)] \leq x$. Hence, $P(Z(\rho) = L) \leq x/L$. Note that either $Z(\rho) = L$ or $Z(\rho) < c_0$. Therefore,

$$P(\sigma_1 = \infty) \leq P(Z(\rho) = L) \leq x/L.$$

Letting $L \to \infty$ shows $\sigma_1 < \infty$ a.s.

As in Theorem V.6.35 in [11], Z is a strong Markov process. Therefore,

$$P(\sigma_2 = \infty) = E[P^{Z(\sigma_1)}(\tau = \infty)].$$

But $Z(\sigma_1) < c_0$, and we have already shown that $P^x(\tau = \infty) = 0$ for all $x \leq c_0$. Hence, $\sigma_2 < \infty$ a.s. □

We are now prepared to state our main results for the case $\beta = 1$. If μ_p and μ are Borel measures on a metric space S, then the notation $\mu_p \Rightarrow \mu$ will mean that μ_p converges weakly to μ as $p \to 0$, that is, $\int_S f\,d\mu_p \to \int_S f\,d\mu$ as $p \to 0$ for all bounded, continuous $f : S \to \mathbb{R}$. If X_p and X are S-valued random variables, then $X_p \Rightarrow X$ will mean that $PX_p^{-1} \Rightarrow PX^{-1}$. When X_p and X are processes, we will take our metric space to be $D_{\mathbb{R}^d}[0,\infty)$, the space of cadlag functions from $[0,\infty)$ to $\mathbb{R}^d$, with the Skorohod metric. See [1] for details.

THEOREM 2.2. *Suppose $\beta = 1$. Let the processes Z_p be given by (1.8) and suppose that $Z_p(0) \Rightarrow Z(0)$, where $Z(0) > 0$ a.s. Let Z be the unique solution to (1.9). Then $Z_p \Rightarrow Z$.*

THEOREM 2.3. *Suppose $\beta = 1$ and $\ell > 0$. Then the Markov chain $\{W_n\}$ has a unique stationary distribution. Moreover, the process Z given by (1.9) has a unique stationary distribution η on $(0,\infty)$. For each $p > 0$, let η_p be the stationary distribution for the Markov chain $\{p^\gamma W_n\}$. Then $\eta_p \Rightarrow \eta$.*

For some results on stationary distributions in the case $\beta = 1$ and $\ell = 0$, see [10].

For the case $\beta < 1$, we need some preliminary definitions. Assume that for all $p \in (0,1)$, the processes $\{W_{p,n}\}$ are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Define the σ-algebra

$$\mathcal{F}_0 = \sigma(W_{p,0} : 0 < p < 1) \vee \mathcal{N}, \tag{2.1}$$

where $\mathcal{N}$ denotes the collection of events $D \in \mathcal{F}$ with $P(D) = 0$.

THEOREM 2.4. *Suppose $\beta < 1$. Let the processes Z_p be given by (1.8). Suppose that $Z_p(0) \Rightarrow \zeta(0)$, where $\zeta(0) > 0$ a.s. Let ζ the unique solution to (1.10). Then $Z_p \Rightarrow \zeta$. Moreover, if $Z_p(0) \to \zeta(0)$ in probability, then $Z_p \to \zeta$ in probability.*

THEOREM 2.5. *Suppose $\beta < 1$. Let the processes Z_p be given by (1.8). For each $p \in (0,1)$, let $\zeta_p(0)$ be a strictly positive random variable defined on $(\Omega, \mathcal{F}, P)$. Assume that $\zeta_p(0)$ is $\mathcal{F}_0$-measurable and $Z_p(0) - \zeta_p(0) \to 0$ in probability. Define ζ_p and ξ_p by (1.11) and (1.12), respectively.*

Suppose that there exists a pair of random variables $(\xi(0), \zeta(0))$, defined on $(\Omega, \mathcal{F}, P)$, such that $\zeta(0) > 0$ a.s., $\zeta_p(0) \to \zeta(0)$ in probability, and $(\xi_p(0), \zeta_p(0)) \Rightarrow (\xi(0), \zeta(0))$. Let B be a standard Brownian motion independent of $(\xi(0), \zeta(0))$ and define the processes ζ and ξ by (1.10) and (1.13), respectively. Then $(\xi_p, \zeta_p) \Rightarrow (\xi, \zeta)$.

THEOREM 2.6. *Suppose $\beta < 1$. Then the Markov chain $\{W_n\}$ has a unique stationary distribution. For each $p > 0$, let η_p be the stationary distribution for the Markov chain $\{p^{-\tau}(p^\gamma W_n - c_p)\}$. Then $\eta_p \Rightarrow \eta$, where η is the stationary distribution of the Ornstein-Uhlenbeck process given by* (1.15).

3. General Definitions.

Define

$$\Lambda_n = (\ell - W_{n-1} - c_1 W_{n-1}^\alpha (1-\chi_n) + c_2 W_{n-1}^\beta \chi_n) \vee 0,$$

so that

$$W_{n+1} = W_n + c_1 W_n^\alpha - (c_1 W_n^\alpha + c_2 W_n^\beta)\chi_{n+1} + \Lambda_{n+1}.$$

If we let $W(t) = W_{\lfloor t \rfloor}$, then we can rewrite this recursive relation as the integral equation

$$\begin{aligned} W(t) = W(0) &+ c_1 \int_0^t W(s-)^\alpha \, dm(s) \\ &- \int_0^t (c_1 W(s-)^\alpha + c_2 W(s-)^\beta) \, dS(s) + L(t), \end{aligned}$$

where

$$m(t) = \lfloor t \rfloor, \; S(t) = \sum_{j=1}^{\lfloor t \rfloor} \chi_j, \; \text{and } L(t) = \sum_{j=1}^{\lfloor t \rfloor} \Lambda_j.$$

Using (1.8), it is then easy to see that

$$\begin{aligned} Z_p(t) = Z_p(0) &+ c_1 \int_0^t Z_p(s-)^\alpha \, dm_p(s) \\ &- c_1 p \int_0^t Z_p(s-)^\alpha \, dS_p(s) - c_2 \int_0^t Z_p(s-)^\beta \, dS_p(s) + L_p(t), \end{aligned} \tag{3.1}$$

where

$$m_p(t) = p^\nu m(tp^{-\nu}), \; S_p(t) = p^{\nu-1} S(tp^{-\nu}), \; \text{and } L_p(t) = p^\gamma L(tp^{-\nu}).$$

Note that if we define the filtration

$$\mathcal{F}_t^p = \mathcal{F}_0 \vee \sigma(\chi_{p,j} : j \le \lfloor tp^{-\nu} \rfloor),$$

then m_p, S_p, and L_p are all $\{\mathcal{F}_t^p\}$-adapted.

Define the $\mathbb{R}^2$-valued cadlag $\{\mathcal{F}_t^p\}$-semimartingale

$$Y_p = (m_p, S_p)^T$$

and define the function $G_p : \mathbb{R}^2 \to \mathbb{R}$ by

$$G_p(x) = (c_1 x^\alpha, -c_1 p x^\alpha - c_2 x^\beta) 1_{\{x>0\}}.$$

Then (3.1) becomes

$$Z_p(t) = Z_p(0) + \int_0^t G_p(Z_p(s-))\, dY_p(s) + L_p(t).$$

To show that Z_p converges as $p \to 0$, we will apply the theorems in [5]. This approach, however, comes with two technical difficulties. The first is the presence of the local time term L_p; the second is the fact that G_p may have a singularity at the origin. To deal with these issues, we introduce the process Z_p^ε, defined as the unique solution to

$$Z_p^\varepsilon(t) = Z_p(0) + \int_0^t G_p^\varepsilon(Z_p^\varepsilon(s-))\, dY_p(s), \tag{3.2}$$

where $G_p^\varepsilon = G_p(\varepsilon) 1_{(-\infty,\varepsilon)} + G_p 1_{[\varepsilon,\infty)}$. To quantify the sense in which Z_p and Z_p^ε are close, we define the functional $h_\varepsilon : D_{\mathbb{R}^d}[0,\infty) \to [0,\infty]$ by

$$h_\varepsilon(x) = \inf\{t \ge 0 : |x(t)| \wedge |x(t-)| \le \varepsilon\},$$

and the stopping times $\tau_p(\varepsilon) = h_\varepsilon(Z_p^\varepsilon)$, and we observe that

$$L_p = 0 \text{ and } Z_p = Z_p^\varepsilon \text{ on } [0, \tau_p(\varepsilon \vee p^\gamma \ell)). \tag{3.3}$$

By (3.5.2) in [1], if two cadlag functions x and y agree on the interval $[0,t)$, then $d(x,y) \le e^{-t}$, where d is the metric on $D_{\mathbb{R}^d}[0,\infty)$.

4. Convergence of $\boldsymbol{Z_p}$. In this section, we will prove Theorems 2.2 and 2.4 by applying the theorems in [5] to the processes Z_p^ε given by (3.2). We must therefore define the processes to which they converge in the cases $\beta = 1$ and $\beta < 1$.

Let $G(x) = (c_1 x^\alpha, -c_2 x^\beta) 1_{\{x>0\}}$ and $G^\varepsilon = G(\varepsilon) 1_{(-\infty,\varepsilon)} + G 1_{[\varepsilon,\infty)}$, and note that $G_p^\varepsilon \to G^\varepsilon$ uniformly on compacts as $p \to 0$. Let N be a unit rate Poisson process, define

$$Y(t) = (t, N(t))^T \text{ and } y(t) = (t,t)^T,$$

and let Z^ε and ζ^ε be the unique solutions to

$$Z^\varepsilon(t) = Z(0) + \int_0^t G^\varepsilon(Z^\varepsilon(s-))\, dY(s), \tag{4.1}$$

$$\zeta^\varepsilon(t) = \zeta(0) + \int_0^t G^\varepsilon(\zeta^\varepsilon(s-))\, dy(s), \tag{4.2}$$

where $Z(0)$ and N are independent. Note that if $\beta = 1$, then $Z^\varepsilon = Z$ on $[0, h_\varepsilon(Z^\varepsilon))$ and $h_\varepsilon(Z^\varepsilon) = h_\varepsilon(Z) \to \infty$ a.s. as $\varepsilon \to 0$. Hence, $d(Z^\varepsilon, Z) \leq \exp(-h_\varepsilon(Z)) \to 0$ a.s. That is, $Z^\varepsilon \to Z$ a.s. in $D_{\mathbb{R}}[0,\infty)$. Similarly, if $\beta < 1$, then $\zeta^\varepsilon = \zeta$ on $[0, h_\varepsilon(\zeta^\varepsilon))$, $h_\varepsilon(\zeta^\varepsilon) = h_\varepsilon(\zeta) \to \infty$ a.s., and $\zeta^\varepsilon \to \zeta$ a.s. in $D_{\mathbb{R}}[0,\infty)$.

We will show that $Z_p^\varepsilon \Rightarrow Z^\varepsilon$ and $\zeta_p^\varepsilon \Rightarrow \zeta^\varepsilon$. To pass from this to the conclusions of Theorems 2.2 and 2.4, we will need the following lemma, which is easily proved using the Prohorov metric. (See Section 3.1 in [1].)

Lemma 4.1. *Let (S,d) be a complete and separable metric space. Let $\{X_p\}_{p>0}$ be a family of S-valued random variables and suppose, for each ε, there exists a family $\{X_p^\varepsilon\}_{p>0}$ such that*

$$\limsup_{p\to 0} E[d(X_p, X_p^\varepsilon)] \leq \delta_\varepsilon,$$

where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Suppose also that for each ε, there exists Y^ε such that $X_p^\varepsilon \Rightarrow Y^\varepsilon$ as $p \to 0$. Then there exists X such that $X_p \Rightarrow X$ and $Y^\varepsilon \Rightarrow X$.

Proof of Theorem 2.2. Suppose $\beta = 1$, Z_p is given by (1.8), and $Z_p(0) \Rightarrow Z(0)$, where $Z(0) > 0$ a.s. Let Z be the solution to (1.9).

Let Z_p^ε and Z^ε be given by (3.2) and (4.1). We first show that $Z_p^\varepsilon \Rightarrow Z^\varepsilon$. Recall that $G_p^\varepsilon \to G^\varepsilon$ uniformly on compacts. Also observe that $S_p \Rightarrow N$ (see, for example, Problem 7.1 in [1]). Hence, since $Z_p(0)$ and Y_p are independent, $(Z_p(0), Y_p) \Rightarrow (Z(0), Y)$ in $D_{\mathbb{R}^3}[0,\infty)$. Hence, by Theorem 5.4 in [5], it will suffice to show that Y_p has a semimartingale decomposition $Y_p = M_p + A_p$ into a martingale part and a bounded variation part such that for each $t \geq 0$,

$$\sup_p E[[M_p]_t + T_t(A_p)] < \infty, \tag{4.3}$$

where $[M_p]_t$ is the quadratic variation process of M_p and $T_t(A_p)$ is the total variation of A_p on the interval $[0,t]$.

For this, define

$$\tilde{S}_p(t) = S_p(t) - m_p(t) = p^{\nu-1} \sum_{j=1}^{\lfloor tp^{-\nu}\rfloor} (\chi_j - p),$$

so that $\tilde{S}_p$ is an $\{\mathcal{F}_t^p\}$-martingale. Note that $T_t(m_p) = m_p(t)$ and

$$E[\tilde{S}_p]_t = p^{2\nu-2} \sum_{j=1}^{\lfloor tp^{-\nu}\rfloor} E|\chi_j - p|^2 = p^{2\nu-2}\lfloor tp^{-\nu}\rfloor p(1-p) \leq tp^{\nu-1}. \tag{4.4}$$

Since $\beta = 1$ implies $\nu = 1$, this verifies (4.3) and shows that $Z_p^\varepsilon \Rightarrow Z^\varepsilon$.

By passing to a subsequence, we can assume there exists a $[0, \infty]$-valued random variable $\sigma(\varepsilon)$ such that $(Z_p^\varepsilon, h_\varepsilon(Z_p^\varepsilon)) \Rightarrow (Z^\varepsilon, \sigma(\varepsilon))$. By (3.3),

$$\begin{aligned}\limsup_{p\to 0} E[d(Z_p, Z_p^\varepsilon)] &\le \limsup_{p\to 0} E[\exp(-\tau_p(\varepsilon \vee p^\gamma \ell))]\\ &= \limsup_{p\to 0} E[\exp(-h_\varepsilon(Z_p^\varepsilon))]\\ &= E[\exp(-\sigma(\varepsilon))].\end{aligned}$$

We claim that $E[\exp(-\sigma(\varepsilon))] \le E[\exp(-h_\varepsilon(Z^\varepsilon))]$. To see this, let us assume by the Skorohod Representation Theorem (see Theorem 3.1.8 in [1]) that $(Z_p^\varepsilon, h_\varepsilon(Z_p^\varepsilon)) \to (Z^\varepsilon, \sigma(\varepsilon))$ a.s. Then $h_\varepsilon(Z^\varepsilon) \le \sigma(\varepsilon)$ a.s., which proves the claim.

Since $h_\varepsilon(Z^\varepsilon) = h_\varepsilon(Z) \to \infty$ a.s. as $\varepsilon \to 0$, we can apply Lemma 4.1 to conclude that $Z_p \Rightarrow Z$. □

Proof of Theorem 2.4. Suppose $\beta < 1$, Z_p is given by (1.8), and $Z_p(0) \Rightarrow \zeta(0)$, where $\zeta(0) > 0$ a.s. Let ζ be the solution to (1.10).

Note that $\beta < 1$ implies $\nu > 1$. Hence, (4.4) implies that (4.3) is satisfied and $\tilde{S}_p \to 0$ in probability. Therefore, $(Z_p(0), Y_p) \Rightarrow (Z(0), y)$ in $D_{\mathbb{R}^3}[0, \infty)$. By Theorem 5.4 in [5], $Z_p^\varepsilon \Rightarrow \zeta^\varepsilon$. By Corollary 5.6 in [5], if $Z_p(0) \to \zeta(0)$ in probability, then $Z_p^\varepsilon \to \zeta^\varepsilon$ in probability. By the same argument as above, this implies that Z_p converges to ζ in distribution or in probability, respectively. □

5. Fluctuations of Z_p. In this section, we prove Theorem 2.5. Let us first recall the setting of that theorem. We have $\beta < 1$ and Z_p given by (1.8). Recall that the processes Z_p are all defined on the same probability space $(\Omega, \mathcal{F}, P)$. For each $p > 0$, $\zeta_p(0)$ is an $\mathcal{F}_0$-measurable random variable, where $\mathcal{F}_0$ is given by (2.1), such that $\zeta_p(0) > 0$ a.s. and $Z_p(0) - \zeta_p(0) \to 0$ in probability. The processes ζ_p and ξ_p are then given by (1.11) and (1.12).

To apply the theorems in [5], we wish to write ξ_p as the solution to a stochastic differential equation. By (1.11) and (3.1), we have

$$\begin{aligned}\xi_p(t) = \xi_p(0) &+ c_1(1-p)\int_0^t p^{-\tau}(Z_p(s-)^\alpha - \zeta_p(s)^\alpha)\,dm_p(s)\\ &- c_2\int_0^t p^{-\tau}(Z_p(s-)^\beta - \zeta_p(s)^\beta)\,dS_p(s)\\ &- c_2\int_0^t \zeta_p(s)^\beta\,dB_p(s) + R_p(t),\end{aligned}\tag{5.1}$$

where

$$B_p(t) = p^{-\tau}(S_p(t) - m_p(t)) = p^{(\nu-1)/2} \sum_{j=1}^{\lfloor tp^{-\nu} \rfloor} (\chi_j - p)$$

and

$$(5.2) \qquad \begin{aligned} R_p(t) = p^{-\tau} &\int_0^t (c_1(1-p)\zeta_p(s)^\alpha - c_2\zeta_p(s)^\beta)\, d(m_p(s) - s) \\ &- c_1 p \int_0^t Z_p(s-)^\alpha\, dB_p(s) + p^{-\tau} L_p(t). \end{aligned}$$

Given a real number r, let us define the continuous function $F_r : (0,\infty)^2 \to \mathbb{R}$ by

$$F_r(x,y) = \frac{x^r - y^r}{x - y} 1_{\{x \neq y\}} + r y^{r-1} 1_{\{x=y\}}.$$

Using this, (5.1) becomes

$$(5.3) \qquad \begin{aligned} \xi_p(t) = \xi_p(0) &+ c_1(1-p) \int_0^t \xi_p(s-)\mathcal{D}_p^\alpha(s-)\, dm_p(s) \\ &- c_2 \int_0^t \xi_p(s-)\mathcal{D}_p^\beta(s-)\, dS_p(s) - c_2 \int_0^t \zeta_p(s)^\beta\, dB_p(s) + R_p(t), \end{aligned}$$

where $\mathcal{D}_p^r = F_r(Z_p, \zeta_p)$.

Proof of Theorem 2.5. Suppose that there exists a pair of random variables $(\xi(0), \zeta(0))$, defined on $(\Omega, \mathcal{F}, P)$, such that $\zeta(0) > 0$ a.s., $\zeta_p(0) \to \zeta(0)$ in probability, and $(\xi_p(0), \zeta_p(0)) \Rightarrow (\xi(0), \zeta(0))$. Since the map that takes a point $x > 0$ to the unique solution to (1.11) with $\zeta_p(0) = x$ is continuous, $\zeta_p \to \zeta$ in probability and $(\xi_p(0), \zeta_p) \Rightarrow (\xi(0), \zeta)$. Also, since F_r is continuous, $\mathcal{D}_p^r \to r\zeta(\cdot)^{r-1}$ in probability.

Let

$$\begin{aligned} &\mathcal{U}_p(t) = \xi_p(0) - c_2 \int_0^t \zeta_p(s)^\beta\, dB_p(s) + R_p(t), \text{ and} \\ &\mathcal{Y}_p(t) = c_1(1-p) \int_0^t \mathcal{D}_p^\alpha(s-)\, dm_p(s) - c_2 \int_0^t \mathcal{D}_p^\beta(s-)\, dS_p(s), \end{aligned}$$

so that (5.3) becomes

$$(5.4) \qquad \xi_p(t) = \mathcal{U}_p(t) + \int_0^t \xi_p(s-)\, d\mathcal{Y}_p(s).$$

We will apply the theorems in [5] to this integral equation.

We first show that $R_p \to 0$ in probability. By the Martingale Central Limit Theorem (Theorem 7.1.4 in [1]), $B_p \Rightarrow B$, where B is a standard Brownian motion; by Theorem 2.4, $Z_p \to \zeta$ in probability; and by (4.4), $\{B_p\}$ satisfies (4.3). Hence, by Theorem 2.2 in [5],

$$c_1 p \int_0^t Z_p(s-)^\alpha \, dB_p(s) \to 0$$

in probability. By (3.3), $p^{-\tau} L_p = 0$ on $[0, h_{p^\gamma \ell}(Z_p))$. Since $h_{p^\gamma \ell}(Z_p) \to \infty$ in probability, $p^{-\tau} L_p \to 0$ in probability.

For the final term in (5.2), note that $p^{-\tau}|m_p(t) - t| \le p^{\nu-\tau}$ and $\nu - \tau = (\nu+1)/2 > 0$. Hence, $p^{-\tau}(m_p(t) - t) \to 0$ uniformly. Let $f_p(s) = c_1(1-p)\zeta_p(s)^\alpha - c_2\zeta_p(s)^\beta$. Since $\zeta_p \to \zeta$ in probability, we can pass to a subsequence and assume that $\zeta_p \to \zeta$ uniformly on $[0,t]$, a.s. By (1.11), this implies that $\zeta_p' \to \zeta'$ uniformly on $[0,t]$. Hence, f_p and f_p' converge uniformly. Integrating by parts, we have

$$\begin{aligned} p^{-\tau} \int_0^t f_p(s) \, d(m_p(s) - s) &= p^{-\tau} f_p(t)(m_p(t) - t) \\ &\quad - p^{-\tau} \int_0^t (m_p(s) - s) f_p'(s) \, ds, \end{aligned}$$

which goes to zero uniformly and completes the proof that $R_p \to 0$ in probability.

It now follows from Theorem 5.2 in [5] that $(\mathcal{U}_p, \mathcal{Y}_p, \zeta_p) \Rightarrow (\mathcal{U}, \mathcal{Y}, \zeta)$, where

$$\begin{aligned} \mathcal{U}(t) &= \xi(0) - c_2 \int_0^t \zeta(s)^\beta \, dB(s), \text{ and} \\ \mathcal{Y}(t) &= c_1 \int_0^t \alpha\zeta(s)^{\alpha-1} \, ds - c_2 \int_0^t \beta\zeta(s)^{\beta-1} \, ds, \end{aligned}$$

and B is a standard Brownian motion independent of $(\xi(0), \zeta(0))$. By Remark 2.5 in [5], we may apply Theorem 5.4 in [5] to (5.4) and conclude that $(\xi_p, \zeta_p) \Rightarrow (\xi, \zeta)$, where ξ is the unique solution to (1.13). □

6. Stationary Distributions. In this section, we prove Theorems 2.3 and 2.6. For this, we make time continuous in a slightly different manner than before. Let N be a unit rate Poisson process independent of $\{W_n\}$ and let $X(t) = W_{N(t)}$. Then X is a continuous time Markov chain on $E = [\ell, \infty)$ with generator

$$A\varphi(x) = p(\varphi(x - g(x)) - \varphi(x)) + (1-p)(\varphi(x + c_1 x^\alpha) - \varphi(x)),$$

where $g(x) = (c_2 x^\beta) \wedge (x - \ell)$. When $\beta = 1$, we will study the process

$$\hat{Z}_p(t) = p^\gamma X(tp^{-1}),$$

whereas when $\beta < 1$, we will consider

$$\hat{\xi}_p(t) = p^{-\tau}(p^\gamma X(tp^{-\nu}) - c_p),$$

where c_p is given by (1.14). It is easy to see that a probability measure is a stationary distribution for $\{p^\gamma W_n\}$ or $\{p^{-\tau}(p^\gamma W_n - c_p)\}$ if and only if it is a stationary distribution for $\hat{Z}_p$ or $\hat{\xi}_p$, respectively.

LEMMA 6.1. *If $\ell > 0$, then $\{W_n\}$ has a unique stationary distribution.*

Proof. It will suffice to show that X has a unique stationary distribution. Let $\varphi(x) = x$ so that

$$A\varphi(x) = -pg(x) + (1-p)c_1 x^\alpha.$$

Since $g(x) = c_2 x^\beta$ for x sufficiently large, $A\varphi$ is bounded above and $A\varphi(x) \to -\infty$ as $x \to \infty$. By Lemmas 4.9.5 and 4.9.7 in [1], the family of probability measures $\{\mu_t\}_{t \geq 1}$ defined by

$$\mu_t(\Gamma) = \frac{1}{t}\int_0^t P^x(X(s) \in \Gamma)\, ds$$

is relatively compact. By Theorem 4.9.3 in [1], any subsequential weak limit of $\{\mu_t\}$ is a stationary distribution for X.

To show that the stationary distribution is unique, it will suffice to show that for all $x \in E$,

$$\tau = \inf\{t \geq 0 : X(t) = \ell\} < \infty, \quad P^x\text{-a.s.}$$

(See, for example, Problem 4.36 in [1].) Let $x \in E$ be arbitrary and let $\varepsilon > 0$. Choose M such that $\mu_t([\ell, M]) \geq 1 - \varepsilon$ for all $t \geq 0$. Note that there exists $K > 0$ such that $P^y(\tau < \infty) \geq K$ for all $y \in [\ell, M]$.

Define the stopping times $\tau_0 = 0$ and

$$\tau_{j+1} = \inf\{t \geq \tau_j + 1 : X(t) \leq M\},$$

and note that $\tau_j \to \infty$ a.s. By the strong Markov property,

$$\begin{aligned} P(\tau = \infty, \tau_j < \infty) &= E[1_{\{\tau \geq \tau_j, \tau_j < \infty\}} P^{X(\tau_j)}(\tau = \infty)] \\ &\leq (1-K) P(\tau \geq \tau_j, \tau_j < \infty) \end{aligned}$$

Letting $j \to \infty$ shows that $P(\{\tau = \infty\} \cap D) = 0$, where D is the event that $\tau_j < \infty$ for all j. Note that

$$1_{D^c} \le \liminf_{t\to\infty} \frac{1}{t} \int_0^t 1_{\{X(s)>M\}}\, ds.$$

Hence, by Fatou's Lemma, $P(D^c) \le \liminf_{t\to\infty} \mu_t((M,\infty)) \le \varepsilon$. Therefore, $P(\tau = \infty) = P(\{\tau = \infty\} \cap D^c) \le \varepsilon$. Since ε was arbitrary, $\tau < \infty$ P^x-a.s. and the stationary distribution is unique. □

Proof of Theorem 2.3. In what follows, C and K will denote strictly positive, finite constants that do not depend on p and may change value from line to line.

Suppose $\beta = 1$, $\ell > 0$, and η_p is the stationary distribution for $\{p^\gamma W_n\}$. Then η_p is the stationary distribution for $\hat{Z}_p$, which is a continuous time Markov chain on $E_p = [p^\gamma \ell, \infty)$ with generator

$$\begin{aligned} A_p\varphi(x) &= \varphi(x - p^\gamma g(p^{-\gamma}x)) - \varphi(x) \\ &\qquad + p^{-1}(1-p)(\varphi(x + pc_1x^\alpha) - \varphi(x)). \end{aligned}$$

Let $\varphi(x) = x + x^{-1}$, so that

$$\begin{aligned} A_p\varphi(x) &= -p^\gamma g(p^{-\gamma}x) + (1-p)c_1x^\alpha \\ &\qquad + \frac{p^\gamma g(p^{-\gamma}x)}{x(x - p^\gamma g(p^{-\gamma}x))} - \frac{(1-p)c_1x^\alpha}{x(x + pc_1x^\alpha)}. \end{aligned}$$

Since $x \mapsto 1 + pc_1x^{\alpha-1}$ is decreasing,

$$1 + pc_1x^{\alpha-1} \le 1 + pc_1(p^\gamma\ell)^{\alpha-1} = 1 + c_1\ell^{\alpha-1}$$

for all $x \in E_p$. Hence,

$$A_p\varphi(x) \le -p^\gamma g(p^{-\gamma}x) + Cx^\alpha + \frac{p^\gamma g(p^{-\gamma}x)}{x(x - p^\gamma g(p^{-\gamma}x))} - Kx^{\alpha-2}$$

whenever $p < 1/2$.

If $x \ge p^\gamma\ell/(1-c_2)$, then $g(p^{-\gamma}x) = c_2p^{-\gamma}x$ and

$$A_p\varphi(x) \le -Kx + Cx^\alpha + Cx^{-1} - Kx^{\alpha-2}.$$

If $x < p^\gamma\ell/(1-c_2)$, then $g(p^{-\gamma}x) = p^{-\gamma}x - \ell$ and

$$A_p\varphi(x) \le Cx^\alpha + \frac{x - p^\gamma\ell}{xp^\gamma\ell} - Kx^{\alpha-2} \le Cx^\alpha + (p^\gamma\ell)^{-1} - Kx^{\alpha-2}.$$

But in this case, $(p^\gamma \ell)^{-1} < Cx^{-1}$. It therefore follows that

$$A_p\varphi(x) \le C - Kx - Kx^{\alpha-2}$$

for all $x \in E_p$.

Let $\varepsilon > 0$. Define

$$L = \sup_{p<1/2} \sup_{x\in E_p} A_p\varphi(x) < \infty$$

and let $m = L(1-\varepsilon)/\varepsilon$. Choose $M > 0$ such that $x \notin [M^{-1}, M]$ implies $A_p\varphi(x) < -m$ for all $p < 1/2$. By Corollary 4.9.8 in [1],

$$\eta_p([M^{-1}, M]) \ge \eta_p(\{x : A_p\varphi(x) \ge -m\}) \ge \frac{m}{L+m} = 1 - \varepsilon.$$

The family of measures $\{\eta_p\}$ is therefore relatively compact on $(0,\infty)$. By passing to a subsequence, we can assume that $\eta_p \Rightarrow \eta$ for some probability measure η on $(0,\infty)$.

Now let $p^\gamma W_0$ have distribution η_p and let Z_p be given by (1.8). By Theorem 2.2, $Z_p \Rightarrow Z$, where Z satisfies (1.9) with $PZ(0)^{-1} = \eta$. Fix $t_1 \le \cdots \le t_n$. Then

$$\begin{aligned}(Z_p(t_1), \dots, Z_p(t_n)) &= p^\gamma (W_{\lfloor t_1 p^{-1}\rfloor}, \dots, W_{\lfloor t_n p^{-1}\rfloor}) \\ &\overset{d}{=} p^\gamma (W_0, W_{\lfloor t_2 p^{-1}\rfloor - \lfloor t_1 p^{-1}\rfloor}, \dots, W_{\lfloor t_n p^{-1}\rfloor - \lfloor t_1 p^{-1}\rfloor}) \\ &= (Z_p(0), Z_p(t_2 - t_1), \dots, Z_p(t_n - t_1)) + \varepsilon,\end{aligned}$$

where $\varepsilon_j = Z_p(h_j) - Z_p(t_j - t_1)$ and $h_j = (\lfloor t_j p^{-1}\rfloor - \lfloor t_1 p^{-1}\rfloor)p$. Note that $h_j \to t_j - t_1$ as $p \to 0$ and, for fixed t, Z is almost surely continuous at t. Hence, $\varepsilon \to 0$ a.s., which gives

$$(Z_p(t_1), \dots, Z_p(t_n)) \Rightarrow (Z(0), Z(t_2 - t_1), \dots, Z(t_n - t_1)).$$

But

$$(Z_p(t_1), \dots, Z_p(t_n)) \Rightarrow (Z(t_1), \dots, Z(t_n)),$$

so Z is a stationary process, and η is a stationary distribution for Z. The uniqueness of η follows from Lemma 2.1. □

For the proof of Theorem 2.6, note that $\hat{\xi}_p$ is a continuous time Markov chain on $E_p = [p^{-\tau}(p^\gamma \ell - c_p), \infty)$ with generator

$$\begin{aligned}A_p\varphi(x) = p^{-\nu+1}&(\varphi(x - p^{\gamma-\tau} g(p^{\tau-\gamma}x + p^{-\gamma}c_p)) - \varphi(x)) \\ &+ p^{-\nu}(1-p)(\varphi(x + p^{\gamma-\tau} c_1 (p^{\tau-\gamma}x + p^{-\gamma}c_p)^\alpha) - \varphi(x)).\end{aligned} \tag{6.1}$$

We will use the same argument as in the proof of Theorem 2.3, this time using the Lyapunov function $\varphi(x) = |x|^r$, where r is sufficiently large. Our key estimate on $A_p\varphi(x)$ is given in the following lemma and is valid as long as $|x|$ is not too large.

LEMMA 6.2. *Suppose $\beta < 1$. Let $\varphi(x) = |x|^r$, where $r \geq 2$, and let A_p be given by (6.1). Let $0 < \delta < M < \infty$ be arbitrary. Then there exists $p_0 > 0$ and strictly positive, finite constants C and K such that*

$$A_p\varphi(x) \leq C - K|x|^r$$

for all $p \leq p_0$ and all $x \in E_p$ satisfying $\delta \leq p^\tau x + c_p \leq M$.

Proof. For notational simplicity, let us define $y_p(x) = p^\tau x + c_p$ so that

$$\begin{aligned} A_p\varphi(x) = {} & p^{-\nu+1}(\varphi(x - p^{\gamma-\tau}g(p^{-\gamma}y_p)) - \varphi(x)) \\ & + p^{-\nu}(1-p)(\varphi(x + p^{\gamma-\tau}c_1(p^{-\gamma}y_p)^\alpha) - \varphi(x)). \end{aligned}$$

Either $g(x) = c_2x^\beta$ or $g(x) < c_2x^\beta$. Note that there exists $x_0 > \ell$ such that $g(x) = c_2x^\beta$ if and only if $x \geq x_0$. Hence, if $g(p^{-\gamma}y_p) < c_2(p^{-\gamma}y_p)^\beta$, then $p^{-\gamma}y_p < x_0$, which implies $x < p^{-\tau}(p^\gamma x_0 - c_p)$. If p is sufficiently small, this implies $x < 0$. Since φ is decreasing on $(-\infty, 0]$, it follows that

$$\begin{aligned} A_p\varphi(x) \leq {} & p^{-\nu+1}(\varphi(x - p^{\gamma-\tau-\gamma\beta}c_2y_p^\beta) - \varphi(x)) \\ & + p^{-\nu}(1-p)(\varphi(x + p^{\gamma-\tau-\gamma\alpha}c_1y_p^\alpha) - \varphi(x)) \end{aligned}$$

for all $x \in E_p$.

Observe that

$$\begin{aligned} |\varphi(z) - \varphi(x) - \varphi'(x)(z-x)| &= \left|\int_x^z (z-u)\varphi''(u)\,du\right| \\ &\leq C|z-x|^2(|x|^{r-2} + |z|^{r-2}) \\ &\leq C|x|^{r-2}|z-x|^2 + C|z-x|^r. \end{aligned}$$

Hence,

$$\begin{aligned} A_p\varphi(x) \leq {} & -\varphi'(x)p^{-\tau}(p^{-\nu+1+\gamma-\gamma\beta}c_2y_p^\beta - p^{-\nu+\gamma-\gamma\alpha}c_1(1-p)y_p^\alpha) \\ & + C|x|^{r-2}(p^{-\nu+1+2\gamma-2\tau-2\gamma\beta}c_2^2y_p^{2\beta} + p^{-\nu+2\gamma-2\tau-2\gamma\alpha}c_1^2y_p^{2\alpha}) \\ & + C(p^{-\nu+1+r\gamma-r\tau-r\gamma\beta}c_2^ry_p^{r\beta} + p^{-\nu+r\gamma-r\tau-r\gamma\alpha}c_1^ry_p^{r\alpha}). \end{aligned}$$

We can simplify these exponents by observing that

$$\begin{aligned}
-\nu + \gamma - \gamma\alpha &= 0 \\
-\nu + 1 + \gamma - \gamma\beta &= 0 \\
-\nu + 2\gamma - 2\tau - 2\gamma\alpha &= 1 \\
-\nu + 1 + 2\gamma - 2\tau - 2\gamma\beta &= 0 \\
-\nu + 1 + r\gamma - r\tau - r\gamma\beta &= \tau(r-2) \\
-\nu + r\gamma - r\tau - r\gamma\alpha &= r - 1 + \tau(r-2).
\end{aligned}$$

Thus,

$$\begin{aligned}
A_p\varphi(x) \le &-\varphi'(x)p^{-\tau}(c_2 y_p^\beta - c_1(1-p)y_p^\alpha) + C|x|^{r-2}(y_p^{2\beta} + p y_p^{2\alpha}) \\
&+ C(p^{\tau(r-2)}y_p^{r\beta} + p^{r-1+\tau(r-2)}y_p^{r\alpha}).
\end{aligned}$$

Since $\varphi'(x)$ and $c_2 y_p^\beta - c_1(1-p)y_p^\alpha$ have the same sign, this gives

$$\begin{aligned}
A_p\varphi(x) \le &-r|x|^{r-1}p^{-\tau}|c_2 y_p^\beta - c_1(1-p)y_p^\alpha| + C|x|^{r-2}(y_p^{2\beta} + p y_p^{2\alpha}) \\
&+ C(p^{\tau(r-2)}y_p^{r\beta} + p^{r-1+\tau(r-2)}y_p^{r\alpha})
\end{aligned} \tag{6.2}$$

for all $x \in E_p$.

If $r \ge 2$ and $\delta \le y_p \le M$, then

$$A_p\varphi(x) \le -r|x|^{r-1}p^{-\tau}c_2 y_p^\alpha |y_p^{\beta-\alpha} - c_p^{\beta-\alpha}| + C|x|^{r-2} + C.$$

By the Mean Value Theorem,

$$\begin{aligned}
\psi_p(x) &\le -K|x|^{r-1}p^{-\tau}|y_p - c_p| + C|x|^{r-2} + C \\
&= -K|x|^r + C|x|^{r-2} + C,
\end{aligned}$$

which completes the proof. □

The following two lemmas provide the needed estimates on $A_p\varphi$ in the extreme regimes.

LEMMA 6.3. *Suppose $\beta < 1$. Let $\varphi(x) = |x|^r$, where $r \ge 2$, and let A_p be given by (6.1). Then there exists $p_0 > 0$, $M < \infty$ and $K > 0$ such that*

$$A_p\varphi(x) \le -K|x|^{(r-1)\wedge(r-1+\beta)}$$

for all $p \le p_0$ and all $x \in E_p$ satisfying $p^\tau x + c_p > M$.

Proof. Let $p \leq p_0$ and $y_p = p^\tau x + c_p > M$. If p_0 is sufficiently small and M is sufficiently large, then $x \geq Kp^{-\tau}$ and $y_p \leq x$. By (6.2),

$$\begin{aligned} A_p\varphi(x) &\leq -K|x|^{r-1}y_p^\beta + C|x|^{r-2}y_p^{2\beta} + Cy_p^{r\beta} \\ &= -|x|^{r-1}y_p^\beta(K - C|x|^{-1}y_p^\beta - C|x|^{-r+1}y_p^{\beta(r-1)}). \end{aligned}$$

If $\beta \leq 0$, then for p sufficiently small,

$$A_p\varphi(x) \leq -|x|^{r-1}y_p^\beta(K - C|x|^{-1} - C|x|^{-r+1}) \leq -K|x|^{r-1+\beta}.$$

If $\beta > 0$, then

$$A_p\varphi(x) \leq -|x|^{r-1}y_p^\beta(K - C|x|^{\beta-1} - C|x|^{(\beta-1)(r-1)}),$$

so for p sufficiently small, $A_p\varphi(x) \leq -K|x|^{r-1}y_p^\beta \leq -K|x|^{r-1}$. □

Lemma 6.4. *Suppose $\beta < 1$. Let $\varphi(x) = |x|^r$, where $r \geq 2$, and let A_p be given by (6.1). Then there exists $p_0 > 0$, $\delta > 0$ and $K > 0$ such that*

$$A_p\varphi(x) \leq -K|x|^{r\wedge(r-2\alpha/(1-\beta))}$$

for all $p \leq p_0$ and all $x \in E_p$ satisfying $p^\tau x + c_p < \delta$.

Proof. Let $p \leq p_0$ and $y_p = p^\tau x + c_p < \delta$. Note that since $x \in E_p$, $y_p \geq p^\gamma \ell$. If p_0 and δ are sufficiently small, then $x < 0$ and $Kp^{-\tau} \leq |x| \leq Cp^{-\tau}$. By (6.2), for δ sufficiently small,

$$\begin{aligned} A_p\varphi(x) &\leq -|x|^r y_p^\alpha(K|y_p^{\beta-\alpha} - c_p^{\beta-\alpha}| - C(p^{2\tau}y_p^{2\beta-\alpha} + p^{2\tau+1}y_p^\alpha) \\ &\qquad - C(p^{\tau r+\tau(r-2)}y_p^{r\beta-\alpha} + p^{\tau r+r-1+\tau(r-2)}y_p^{r\alpha-\alpha})) \\ &\leq -|x|^r y_p^\alpha(K - C(p^{2\tau}y_p^{2\beta-\alpha} + p^{2\tau(r-1)}y_p^{r\beta-\alpha}) \\ &\qquad - C(p^{2\tau+1}y_p^\alpha + p^{(2\tau+1)(r-1)}y_p^{\alpha(r-1)})). \end{aligned}$$

Let us first estimate the term $p^{2\tau}y_p^{2\beta-\alpha}$. If $2\beta-\alpha \geq 0$, then $p^{2\tau}y_p^{2\beta-\alpha} \leq Cp^{2\tau}$. If $2\beta - \alpha < 0$, then $p^{2\tau}y_p^{2\beta-\alpha} \leq Cp^{2\tau+\gamma(2\beta-\alpha)}$. Note that $2\tau + \gamma(2\beta - \alpha) = \gamma + 1$. Hence, for all values of α and β, there exists some $s > 0$ such that $p^{2\tau}y_p^{2\beta-\alpha} \leq p^s$.

Similarly, for the remaining terms in the above inequality, we observe that

$$\begin{aligned} 2\tau(r-1) + \gamma(r\beta - \alpha) &= (2\tau + \gamma\beta)(r-1) + 1 = \gamma(r-1) + 1 \\ 2\tau + 1 + \gamma\alpha &= \gamma \\ (2\tau + 1)(r-1) + \gamma\alpha(r-1) &= \gamma(r-1). \end{aligned}$$

Therefore, if p_0 is sufficiently small, then $A_p\varphi(x) \le -K|x|^r y_p^\alpha$. If $\alpha < 0$, then $A_p\varphi(x) \le -K|x|^r$. If $\alpha \ge 0$, then

$$A_p\varphi(x) \le -K|x|^r p^{\gamma\alpha} \le -K|x|^{r-\gamma\alpha/\tau}.$$

Since $\gamma\alpha/\tau = 2\alpha/(1-\beta)$, this completes the proof. □

Proof of Theorem 2.6. Suppose $\beta < 1$ and η_p is the stationary distribution for $\{p^{-\tau}(p^\gamma W_n - c_p)\}$. Then η_p is the stationary distribution for $\hat{\xi}_p$. Let $\varphi(x) = |x|^r$, where $r \ge 2$. By Lemmas 6.2, 6.3, and 6.4, if r is sufficiently large, there exists $p_0 > 0$ and strictly positive, finite constants C and K such that

$$A_p\varphi(x) \le C - K|x|^s$$

for some $s > 0$ and all $p \le p_0$ and $x \in E_p$. As in the proof of Theorem 2.3, this implies that the family of measures $\{\eta_p\}$ is relatively compact on $\mathbb{R}$. By passing to a subsequence, we can assume that $\eta_p \Rightarrow \eta$ for some probability measure η on $\mathbb{R}$.

Let $p^{-\tau}(p^\gamma W_0 - c_p)$ have distribution η_p, let Z_p be given by (1.8), and let ξ_p be given by (1.12) with $\zeta_p \equiv c_p$. Note that $\xi_p(0)$ converges in distribution, so $p^\tau \xi_p(0) = Z_p(0) - \zeta_p(0) \to 0$ in probability. Hence, by Theorem 2.5, $\xi_p \Rightarrow \xi$, where ξ satisfies (1.15) with $P\xi(0)^{-1} = \eta$. As in the proof of Theorem 2.3, ξ is a stationary process, so η is the stationary distribution for ξ. □

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