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ASYMPTOTIC BEHAVIOR OF BAYES PROCEDURES FOR TESTING SIMPLE HYPOTHESES IN MULTIPARAMETER EXPONENTIAL FAMILIES

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The asymptotic form of the Bayes acceptance region is derived for testing simple null-hypotheses in multiparameter exponential families. This result suggests a reasonable definition for tests which might be called "almost-Bayes". The rate at which the risk of the Bayes test converges to zero is obtained, showing the nature of its dependence on the prior distribution and providing a basis for comparison of almost-Bayes procedures. Concluding remarks contain a brief discussion of some asymptotic consequences of poor prior guessing.

1. Introduction and summary. Consider the problem of testing $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$ in the following model:

(i) We observe $X_1, X_2, \cdots, X_n$, which are i.i.d. with k-dimensional density (under θ)

$$f(x, \theta) = e^{\theta' x - \psi(\theta)}$$

with respect to some nondegenerate measure μ (i.e., its support is not contained in a $k-1$ dimensional subspace).

(ii) The k-dimensional parameter θ ranges over the natural parameter space Ω which includes θ_0 as an interior point.

(iii) For convenience zero-one loss functions are used. Modifications which accommodate more general loss functions are discussed in Section 3.

(iv) The prior distribution ν assigns probability γ to $\theta = \theta_0$ and density (w.r.t. Lebesgue measure) ρ to $\theta \neq \theta_0$. This density satisfies

$$\rho(\theta) = |\theta - \theta_0|^p h\left(\frac{\theta - \theta_0}{|\theta - \theta_0|}\right) + o(|\theta - \theta_0|^p) \qquad \text{as} \quad |\theta - \theta_0| \to 0\,, \tag{1}$$

where h is a continuous, positive function defined on the unit sphere.

In this setting the acceptance region for the Bayes test is

$$\begin{aligned} \bar{X} \in C_n &= \{x \in R^k : \textstyle\int e^{n(\theta' x - \psi(\theta))} \rho(\theta)\, d\theta \leqq \gamma e^{n(\theta_0' x - \psi(\theta_0))}\} \\ &= \{x : \textstyle\int e^{n[(\theta - \theta_0)'(x - \nabla\psi(\theta_0)) - I(\theta_0, \theta)]} \rho(\theta)\, d\theta \leqq \gamma\}\,, \end{aligned} \tag{2}$$

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where $I(\theta_0, \theta) = \nabla\psi(\theta_0)'(\theta_0 - \theta) - (\psi(\theta_0) - \psi(\theta))$ is the Kullback–Leibler information. Asymptotically the region C_n shrinks to the mean vector $\nabla\psi(\theta_0)$ under $\theta = \theta_0$; more precisely, C_n behaves like

$$(n^{-1} \log n)^{\frac{1}{2}} D + \nabla\psi(\theta_0) ,$$

where D is the ellipsoid

$$D = \{x : x'\Sigma_0^{-1}x \leqq k + p\} \tag{3}$$

and

$$\Sigma_0 = \left\|\frac{\partial^2\psi(\theta_0)}{\partial\theta_i\partial\theta_j}\right\| \qquad \text{(covariance matrix under} \quad \theta = \theta_0) . \tag{4}$$

The risk of the Bayes test is

$$B_n = \gamma P_{\theta_0}(\bar{X} \notin C_n) + \int P_\theta(\bar{X} \in C_n)\rho(\theta)\, d\theta . \tag{5}$$

An investigation of the rates at which each of these two terms converges to zero shows the type II risk to be the dominate term, behaving like

$$C_{k,p}(n^{-1} \log n)^{(k+p)/2} .$$

The type I risk goes to zero more rapidly by the factor $(\log n)^{-1}$.

This work generalizes some of the results obtained by Johnson and Truax (1974), where an extensive study was made for one-dimensional parameter exponential families. Working in a somewhat different framework, Rubin and Sethuraman (1965) obtained similar convergence rates for the type I and type II risks.

The form of the Bayes acceptance region suggests consideration of tests which accept whenever

$$n^{\frac{1}{2}}(\bar{X} - \nabla\psi(\theta_0)) \in (\log n)^{\frac{1}{2}}E ,$$

where E is a bounded convex set. Tests of this form, which might be called "almost-Bayes", are studies in Section 4. Some interesting and surprising results are obtained when the convergence rates are compared for various E sets. In conclusion, some asymptotic consequences of poor prior guessing are discussed.

2. Main results. This paper is centered around two principal theorems which are presented in this section. The first describes the rate at which the Bayes acceptance region C_n shrinks to the mean vector under $\theta = \theta_0$; the second is concerned with the asymptotic behavior of the Bayes risk and shows the nature of its dependence on choice of prior distribution. Several technical lemmas, which are required for the proofs of these two theorems, have been relegated to the final section.

We begin by showing that C_n has the limiting behavior described in Section 1. More precisely,

$$d((n/\log n)^{\frac{1}{2}}(C_n - \nabla\psi(\theta_0)), D) \to 0 \qquad \text{as} \quad n \to \infty ,$$

where d is the Hausdorff metric. Define

$$D_n = (n/\log n)^{\frac{1}{2}}(C_n - \nabla\psi(\theta_0)) .$$

We need to prove that given $\varepsilon > 0$, D is contained in an ε-neighborhood of D_n for all n sufficiently large; and, conversely, D_n is contained in an ε-neighborhood of D. Clearly, it will be enough to prove that for all sufficiently large n

$$\{x : x'\Sigma_0^{-1}x \leqq k + p - \varepsilon\} \subset D_n \subset \{x : x'\Sigma_0^{-1}x \leqq k + p + \varepsilon\} .$$

Actually, we prove a somewhat stronger result.

THEOREM 1. *Given $\varepsilon > 0$ we have for all sufficiently large n*

$$E_n^- \subset D_n \subset E_n^+ ,$$

where

$$E_n^\pm = \left\{x : x'\Sigma_0^{-1}x \leqq k + p - p\frac{\log\log n}{\log n} + \frac{c^\pm \pm \varepsilon}{\log n}\right\} ,$$

and

$$c^\pm = 2\log\left\{\frac{\gamma(\det\Sigma_0)^{\frac{1}{2}}}{(\lambda^\mp)^{p/2}h^\mp(2\pi)^{k/2}(k + p)^{p/2}}\right\} , \qquad h^\mp = \frac{\inf}{\sup}\{h(t) : |t| = 1\} ,$$

$$\lambda^\mp = \frac{\min}{\max} \textit{ eigenvalue of } \Sigma_0^{-1} .$$

PROOF. To simplify notation without losing generality, we assume that θ_0 and the mean vector $\nabla\phi(\theta_0)$ under H_0 are both the zero vector. Then $\psi(\theta)$ is the Kullback–Leibler information number $I(\theta_0, \theta)$. We must show that, given $\varepsilon > 0$, we have, for all sufficiently large n, $E_n^- \subset D_n$. That is, for all sufficiently large n, $x \in E_n^-$ implies

$$\int e^{(n\log n)^{\frac{1}{2}}\theta'x - n\psi(\theta)}\rho(\theta)\, d\theta \leqq \gamma . \tag{6}$$

Write the left hand side of (6) as the sum of two integrals (7) and (8).

$$\int_{|\theta| > n^{-\frac{1}{2}}\log n} e^{(n\log n)^{\frac{1}{2}}\theta'x - n\psi(\theta)}\rho(\theta)\, d\theta , \tag{7}$$

$$\int_{|\theta| \leqq n^{-\frac{1}{2}}\log n} e^{(n\log n)^{\frac{1}{2}}\theta'x - n\psi(\theta)}\rho(\theta)\, d\theta . \tag{8}$$

Since $\bigcup_n E_n^-$ is a bounded set we have by Lemma 4 (see Section 6) the existence of an integer N_1 so that if $n \geqq N_1$

$$\int_{|\theta| > n^{-\frac{1}{2}}\log n} e^{(n\log n)^{\frac{1}{2}}\theta'x - n\psi(\theta)}\rho(\theta)\, d\theta < \gamma(1 - e^{-\varepsilon/4}) \tag{9}$$

for all $x \in E_n^-$.

At this stage we remark that we may as well assume that $\Sigma_0 = I$. If it is not, write $\Sigma_0^{-1} = B'B$ where B is nonsingular. $Y = BX$ then has an exponential distribution with parameter $\eta = B'^{-1}\theta$ and when $\eta = 0$, $\operatorname{Cov}(Y) = I$. If θ has prior density $\rho(\theta)$ then η has density

$$\tilde\rho(\eta) = (\det\Sigma_0)^{-\frac{1}{2}}\rho(B'\eta) ,$$

and if $\rho(\theta) = |\theta|^p h(\theta/|\theta|) + o(|\theta|^p)$, then

$$\tilde\rho(\eta) = |\eta|^p \tilde h\left(\frac{\eta}{|\eta|}\right) + o(|\eta|^p)$$

where

$$\tilde{h}\left(\frac{\eta}{|\eta|}\right) = \frac{|B'\eta|^p}{|\eta|^p}(\det \Sigma_0)^{-\frac{1}{2}} h\left(\frac{B'\eta}{|B'\eta|}\right).$$

If h is bounded by h^+, h^- on the unit sphere, then $\tilde{h}$ is bounded by $(\det \Sigma_0)^{-\frac{1}{2}}(\lambda^+)^{p/2}h^+$, $(\det \Sigma_0)^{-\frac{1}{2}}(\lambda^-)^{p/2}h^-$ where $\lambda^\pm$ are the max and min eigenvalues of BB' (or $B'B = \Sigma_0^{-1}$).

Now, making the simplifying assumption that $\Sigma_0 = I$, (8) can be bounded by

$$e^{\gamma_n} \int_{|\theta| \leq n^{-\frac{1}{2}}\log n} e^{(n\log n)^{\frac{1}{2}}\theta' x - n|\theta|^2/2} \rho(\theta)\, d\theta \tag{10}$$

where

$$\gamma_n = n \sup_{|\theta| \leq n^{-\frac{1}{2}}\log n} (|\theta|^2/2 - \psi(\theta)).$$

Make a change of variables to express (10) as

$$n^{-k/2} e^{\gamma_n} \int_{|\theta| \leq \log n} e^{(\log n)^{\frac{1}{2}}\theta' x - \frac{1}{2}|\theta|^2} \rho(\theta/n^{\frac{1}{2}})\, d\theta .$$

Notice that for any $\delta > 0$

$$\rho(\theta/n^{\frac{1}{2}}) \leq (1+\delta)|\theta|^p h\left(\frac{\theta}{|\theta|}\right) n^{-p/2} \qquad \text{if} \quad |\theta| \leq \log n$$

for all n sufficiently large, so that another upper bound for (8) is

$$e^{\gamma_n}(1+\delta) n^{\frac{1}{2}(|x|^2 - k - p)} \int_{|\theta| \leq \log n} |\theta|^p h\left(\frac{\theta}{|\theta|}\right) e^{-\frac{1}{2}|\theta - (\log n)^{\frac{1}{2}} x|^2}\, d\theta .$$

But, we only increase this by integrating over the whole space, and

$$\int |\theta + (\log n)^{\frac{1}{2}} x|^p h\left(\frac{\theta + (\log n)^{\frac{1}{2}} x}{|\theta + (\log n)^{\frac{1}{2}} x|}\right) e^{-\frac{1}{2}|\theta|^2}\, d\theta \sim (\log n)^{p/2} |x|^p h\left(\frac{x}{|x|}\right) (2\pi)^{k/2} .$$

Thus, for n sufficiently large (say $n \geq N_2$), (8) is bounded above by

$$\begin{aligned}
& e^{\gamma_n}(1+\delta)^2 h\left(\frac{x}{|x|}\right) n^{\frac{1}{2}(|x|^2-k-p)} (2\pi)^{k/2} (\log n)^{p/2} |x|^p \\
& \quad \leq (1+\delta)^3 h^+ (k+p)^{p/2} (2\pi)^{k/2} n^{\frac{1}{2}(|x|^2-k-p)} (\log n)^{p/2} \\
& \quad \leq (1+\delta)^3 h^+ (k+p)^{p/2} (2\pi)^{k/2} n^{-(p/2)(\log\log n/\log n) + (c^- - \varepsilon)/(2\log n)} (\log n)^{p/2} \\
& \quad = (1+\delta)^3 \gamma e^{-\varepsilon/2} < \gamma e^{-\varepsilon/4} ,
\end{aligned}$$

if we choose δ sufficiently small.

For the second part of the theorem we want to show that for any $\varepsilon > 0$ there is an N such that if $n \geq N$, (11) implies (12)

$$\int e^{(n\log n)^{\frac{1}{2}}\theta' x - n\psi(\theta)} \rho(\theta)\, d\theta \leq \gamma , \tag{11}$$

$$|x|^2 \leq k + p - p\,\frac{\log\log n}{\log n} + \frac{c^+ + \varepsilon}{\log n} . \tag{12}$$

The proof is by contradiction. Suppose there is some $\varepsilon > 0$ such that for infinitely many n, there is an x_n satisfying (11) and not (12). From (11)

$$\begin{aligned}
\gamma & \geq \int_{|\theta| \leq n^{-\frac{1}{2}}\log n} e^{(n\log n)^{\frac{1}{2}}\theta' x_n - n\psi(\theta)} \rho(\theta)\, d\theta \\
& \geq n^{-k/2} e^{\gamma_n^*} \int_{|\theta| \leq \log n} e^{(\log n)^{\frac{1}{2}}\theta' x_n - \frac{1}{2}|\theta|^2} \rho(\theta/n^{\frac{1}{2}})\, d\theta
\end{aligned}$$

where $\gamma_n^* = n \inf_{|\theta| \leq n^{-\frac{1}{2}}\log n} (\frac{1}{2}|\theta|^2 - \psi(\theta))$.

By choosing n sufficiently large we have, for a fixed δ to be chosen later,

$$\rho(\theta/n^{\frac{1}{2}}) \geqq (1-\delta)|\theta|^p h\left(\frac{\theta}{|\theta|}\right) n^{-p/2} \qquad \text{if} \quad |\theta| \leqq \log n\ .$$

Thus, for all sufficiently large n

$$\gamma \geqq n^{-(k+p)/2} e^{\gamma_n^*}(1-\delta)\int_{|\theta|\leqq \log n} e^{(\log n)^{\frac{1}{2}}\theta' x_n - \frac{1}{2}|\theta|^2}|\theta|^p h\left(\frac{\theta}{|\theta|}\right) d\theta$$

$$= e^{\gamma_n^*}(1-\delta) n^{\frac{1}{2}(|x_n|^2-k-p)} \int_{|\theta|\leqq \log n} e^{-\frac{1}{2}|\theta-(\log n)^{\frac{1}{2}} x_n|^2}|\theta|^p h\left(\frac{\theta}{|\theta|}\right) d\theta\ .$$

Since $\{x_n\}$ is bounded (Lemma 5) the above integral is asymptotically equivalent to the integral over the whole space. Also, since $\{x_n\}$ is bounded away from the origin (because (12) fails) the integral is asymptotically equivalent to

$$(2\pi)^{k/2}(\log n)^{p/2}|x_n|^p h\left(\frac{x_n}{|x_n|}\right)\ . \tag{13}$$

Further, because (12) fails, we have for all sufficiently large n that (13) is greater than

$$(2\pi)^{k/2}(1-\delta)^{p/2}(k+p)^{p/2} h^-(\log n)^{p/2}\ .$$

Finally, we have for all n sufficiently large

$$\gamma \geqq e^{\gamma_n^*}(1-\delta)^2 n^{\frac{1}{2}(|x_n|^2-k-p)}(2\pi)^{k/2}(1-\delta)^{p/2}(k+p)^{p/2} h^-(\log n)^{p/2}\ .$$

Also, $e^{\gamma_n^*} \geqq 1-\delta$ for n large enough, and using the fact that (12) fails for sufficiently large n

$$\begin{aligned}\gamma &\geqq h^-(1-\delta)^{\frac{1}{2}(p+6)}(\log n)^{-p/2} e^{c^+/2} e^{\varepsilon/2}(2\pi)^{k/2}(k+p)^{p/2}(\log n)^{p/2}\\ &= (1-\delta)^{\frac{1}{2}(p+6)} e^{\varepsilon/2}\gamma\ .\end{aligned}$$

We get a contradiction by choosing δ so that

$$(1-\delta)^{\frac{1}{2}(p+6)} e^{\varepsilon/2} > 1\ .$$

Thus, when $\Sigma_0 = I$, we have proved the result.

In the general case, make the transformation $Y = BX$, where $\Sigma_0^{-1} = B'B$, and let h be replaced by

$$\tilde{h}\left(\frac{\eta}{|\eta|}\right) = \frac{|B'\eta|^p}{|\eta|^p}(\det \Sigma_0)^{-\frac{1}{2}} h\left(\frac{B'\eta}{|B'\eta|}\right)\ .$$

$$\begin{aligned}\tilde{h}^+ = \sup_\eta \tilde{h}\left(\frac{\eta}{|\eta|}\right) &\leqq \sup_\eta \frac{(\eta' BB'\eta)^{p/2}}{|\eta|^p}(\det \Sigma_0)^{-\frac{1}{2}} \sup_\eta h\left(\frac{B'\eta}{|B'\eta|}\right)\\ &\leqq (\text{max. eigenvalue of } BB')^{p/2}(\det \Sigma_0)^{-\frac{1}{2}} h^+\\ &= (\text{max. eigenvalue of } \Sigma_0^{-1})^{p/2}(\det \Sigma_0)^{-\frac{1}{2}} h^+\ .\end{aligned}$$

Similarly,

$$\tilde{h}^- \geqq (\lambda^-)^{p/2}(\det \Sigma_0)^{-\frac{1}{2}} h^-\ .$$

□

Next, we examine the asymptotic behavior of the Bayes risk

$$B_n = \gamma P_{\theta_0}(\bar{X} \notin C_n) + \int P_\theta(\bar{X} \in C_n)\rho(\theta)\, d\theta\ . \tag{14}$$

THEOREM 2. *The risk of the Bayes test satisfies*

$$B_n = C_{k,p}(n^{-1}\log n)^{(k+p)/2}(1 + o_n(1)) ,$$

where

$$C_{k,p} = \int_{u'\Sigma_0 u \leqq k+p} |u|^p h\left(\frac{u}{|u|}\right) du .$$

PROOF. As in the proof of Theorem 1, we assume $\theta_0 = 0$ and the mean vector $\nabla\psi(\theta_0) = 0$ (so $\psi(\theta) = I(\theta_0, \theta)$). We will estimate the rate at which each of the terms in (14) converges to zero. Write

$$\int P_\theta(\bar{X} \in C_n)\rho)\theta)\, d\theta = \int_{|\theta| \leqq n^{-\frac{1}{2}}\log n} + \int_{|\theta| > n^{-\frac{1}{2}}\log n} .$$

First, look at the integral

$$(15)\qquad \int_{|\theta| \leqq n^{-\frac{1}{2}}\log n} P_\theta(\bar{X} \in C_n)\rho(\theta)\, d\theta = \int_{|\theta| \leqq n^{-\frac{1}{2}}\log n} \int_{n^{\frac{1}{2}}C_n} e^{-n\psi(\theta) + n^{\frac{1}{2}}\theta' v}\, dP_{0n}(v)\rho(\theta)\, d\theta ,$$

where P_{0n} is the distribution of $n^{\frac{1}{2}}\bar{X}$ when $\theta = 0$. To make our calculations simpler, let us assume that $\Sigma_0 = I$. We can then recapture the general case as in the proof of Theorem 1. After a change in the order of integration, (15) can be expressed as

$$\int_{n^{\frac{1}{2}}C_n} \int_{|\theta| \leqq n^{-\frac{1}{2}}\log n} e^{-(n/2)|\theta|^2 + n^{\frac{1}{2}}\theta' v}(e^{n(\frac{1}{2}|\theta|^2 - \psi(\theta))})\rho(\theta)\, d\theta\, dP_{0n}(v) .$$

The quantity in parentheses converges to one uniformly for $|\theta| \leqq n^{-\frac{1}{2}} \log n$, and using

$$\rho(\theta) \sim |\theta|^p h\left(\frac{\theta}{|\theta|}\right) \qquad \text{as} \quad |\theta| \to 0$$

we see that the above integral is asymptotically equivalent to

$$(16)\qquad n^{-(k+p)/2} \int_{n^{\frac{1}{2}}C_n} e^{\frac{1}{2}|v|^2} \int_{|\theta| \leqq \log n} |\theta|^p h\left(\frac{\theta}{|\theta|}\right) e^{-\frac{1}{2}|\theta - v|^2}\, d\theta\, dP_{0n}(v) .$$

For $v \in n^{\frac{1}{2}}C_n$ we have $v = O((\log n)^{\frac{1}{2}})$ so that

$$\int_{|\theta| \leqq \log n} |\theta|^p h\left(\frac{\theta}{|\theta|}\right) e^{-\frac{1}{2}|\theta - v|^2}\, d\theta$$

is asymptotically equivalent to

$$\int |\theta|^p h\left(\frac{\theta}{|\theta|}\right) e^{-\frac{1}{2}|\theta - v|^2}\, d\theta$$

uniformly for $v \in n^{\frac{1}{2}}C_n$. Thus (16) is asymptotically

$$(17)\qquad n^{-(k+p)/2} \int_{(\log n)^{\frac{1}{2}}D_n} e^{\frac{1}{2}|v|^2} \int |\theta + v|^p h\left(\frac{\theta + v}{|\theta + v|}\right) e^{-\frac{1}{2}|\theta|^2}\, d\theta\, dP_{0n}(v) .$$

We want to show that (17) is asymptotically equivalent to the integral obtained when Φ is substituted for P_{0n}; Φ is $N(0, I)$. Consider Theorem 15.4, page 154 of Bhattacharya and Rao (1976). Taking $r = 0$ and

$$f(v) = e^{\frac{1}{2}|v|^2} M(v) I_{\{v \in (\log n)^{\frac{1}{2}}D_n\}}(v)$$

in this theorem, where

$$M(v) = \int |\theta + v|^p h\left(\frac{\theta + v}{|\theta + v|}\right) e^{-\frac{1}{2}|\theta|^2}\, d\theta ,$$

one obtains

$$\begin{aligned}(18)\qquad &\int_{(\log n)^{\frac12}D_n} e^{\frac12|v|^2}M(v)\,dP_{0n}(v)\\ &\quad= \int f(v)\,dP_{0n}(v)\\ &\quad= \int f(v)\,d\Phi(v) + \textstyle\sum_{j=1}^{s-2} n^{-j/2}\int f(v)\,dP_j(-\Phi)(v) + R_n\,.\end{aligned}$$

Here

$$(19)\qquad |R_n| \leqq \omega_f(R^k)\,o(n^{-(s-2)/2}) + \bar{\omega}_f(\eta:\Phi) + o(1)\,,$$

and

$$\begin{aligned}&\omega_f(R^k) = \sup\{|f(x)-f(y)| : x, y \in R^k\}\,,\\ (20)\qquad &\eta = O(n^{-\frac12}\log n)\,,\\ &\omega_f(x:\eta) = \sup\{|f(y)-f(x)| : |y-x| < \eta\}\,,\\ &\bar{\omega}_f(\eta:\Phi) = \int \omega_f(v:\eta)\,d\Phi(v)\,.\end{aligned}$$

If $v \in (\log n)^{\frac12}D_n$, then for some constant a

$$|v|^2 \leqq (k+p)\log n - p\log\log n + a\,.$$

Since $M(v) \leqq h^+ \int |\theta + v|^p\, e^{-\frac12|\theta|^2}\,d\theta$ and the right hand side is an increasing function of $|v|^2$, we have $M(v) \leqq b(\log n)^{p/2}$ for some constant b. It is easy to show that

$$\begin{aligned}&\omega_f(R^k) \leqq e^{a/2}bn^{(k+p)/2}\,,\\ &\omega_f(v:\eta) \leqq d_1 e^{\frac12|v|^2}\{(\log n)^{p/2}(\eta|v| + \eta^2/2) + \omega_M(v:\eta)\}\\ (21)\qquad &\qquad\times I_{\{|v|^2\leqq(k+p)\log n - p\log\log n + a + \eta\}}(v)\,,\\ &\omega_M(v:\eta) \leqq d_2 \int |\theta|^p(\tfrac12\eta^2 + |\theta - v|\eta)\exp\{-\tfrac12|\theta - v|^2 + \tfrac12\eta^2 + |\theta - v|\eta\}\,d\theta\\ &\qquad\leqq d_3(1 + |v|^{p+1})\eta\,,\end{aligned}$$

where d_1, d_2, and d_3 are positive numbers (independent of n). Hence,

$$\begin{aligned}&\bar{\omega}_f(\eta:\Phi) \leqq (2\pi)^{-k/2}\int_{\{|v|^2\leqq(k+p)\log n - p\log\log n + a + \eta\}}((\log n)^{p/2}(\eta|v| + \eta^2/2)\\ (22)\qquad &\qquad + d_3(1 + |v|^{p+1})\eta)\,dv\\ &\quad = o(1)\,.\end{aligned}$$

Taking $s - 2 \geqq k + 2$, the relations (18)–(22) yield

$$\begin{aligned}&\int_{(\log n)^{\frac12}D_n} e^{\frac12|v|^2}M(v)\,dP_{0n}(v)\\ &\quad= \int_{(\log n)^{\frac12}D_n} e^{\frac12|v|^2}M(v)\,d[\Phi + \textstyle\sum_{j=1}^{s-2} n^{-j/2}P_j(-\Phi)](v) + o(1)\,.\end{aligned}$$

Now, $dP_j(-\Phi)(v) = Q_j(v)e^{-\frac12|v|^2}\,dv$ where Q_j is a polynomial.

$$\begin{aligned}&|n^{-j/2}\int_{(\log n)^{\frac12}D_n} e^{\frac12|v|^2}M(v)Q_j(v)e^{-\frac12|v|^2}\,dv|\\ &\quad= |n^{-j/2}\int_{(\log n)^{\frac12}D_n} M(v)Q_j(v)\,dv|\\ &\quad\leqq n^{-j/2}\max_{v\in(\log n)^{\frac12}D_n}|Q_j(v)|M(v)\int_{(\log n)^{\frac12}D_n} dv = n^{-j/2}O((\log n)^{\alpha})\end{aligned}$$

for some α (and this is also $o(1)$). Hence,

$$\int_{(\log n)^{\frac12}D_n} e^{\frac12|v|^2}M(v)\,dP_{0n}(v) = \int_{(\log n)^{\frac12}D_n} e^{\frac12|v|^2}M(v)\,d\Phi(v) + o(1)\,.$$

Finally, we need to calculate

$$\int_{(\log n)^{\frac{1}{2}}D_n} e^{\frac{1}{2}|v|^2} M(v)\, d\Phi(v) = (2\pi)^{-k/2} \int_{(\log n)^{\frac{1}{2}}D_n} M(v)\, dv\ .$$

Take $\{\alpha_n\}$ to be a sequence such that $\alpha_n \to \infty$, $(\log n)^{\frac{1}{2}}\alpha_n \to 0$. Then

$$\begin{aligned}(2\pi)^{-k/2} &\int_{(\log n)^{\frac{1}{2}}D_n - S(0,\alpha_n)} M(v)\, dv \\ &= (2\pi)^{-k/2} \int\int_{(\log n)^{\frac{1}{2}}D_n - S(0,\alpha_n)} |\theta + v|^p h\left(\frac{\theta + v}{|\theta + v|}\right) e^{-\frac{1}{2}|\theta|^2}\, d\theta\, dv\ ,\end{aligned}$$

where $S(0, \alpha_n) = \{v\colon |v| \leqq \alpha_n\}$. If $|v| \geqq \alpha_n$ then

$$\frac{(2\pi)^{-k/2} \int |\theta + v|^p e^{-\frac{1}{2}|\theta|^2} h\left(\frac{\theta + v}{|\theta + v|}\right) d\theta}{|v|^p h\left(\frac{v}{|v|}\right)} \to 1$$

uniformly in $|v| \geqq \alpha_n$. Thus, the above integral is asymptotically equivalent to

$$\int_{(\log n)^{\frac{1}{2}}D_n - S(0,\alpha_n)} |v|^p h\left(\frac{v}{|v|}\right) dv = (\log n)^{(k+p)/2} \int_{D_n - S(0,\beta_n)} |u|^p h\left(\frac{u}{|u|}\right) du\ ,$$

where $\beta_n = (\log n)^{-\frac{1}{2}}\alpha_n \to 0$, and in turn, this is asymptotically equivalent to

$$(\log n)^{(k+p)/2} \int_{|u|^2 \leqq k+p} |u|^p h\left(\frac{u}{|u|}\right) du\ .$$

Also,

$$\begin{aligned}(2\pi)^{-k/2} \int_{S(0,\alpha_n)} M(v)\, dv &= \int_{S(0,\alpha_n)} \int |\theta + v|^p h\left(\frac{\theta + v}{|\theta + v|}\right) \phi(\theta)\, d\theta\, dv \\ &= \alpha_n{}^k \int_{S(0,1)} \int |\theta + \alpha_n v|^p h\left(\frac{\theta + \alpha_n v}{|\theta + \alpha_n v|}\right) \phi(\theta)\, d\theta\, dv \\ &= O(\alpha_n{}^{k+p}) = o((\log n)^{(k+p)/2})\ .\end{aligned}$$

Finally,

$$\int_{|\theta| \leqq n^{-\frac{1}{2}}\log n} P_\theta(\bar{X} \in C_n)\rho(\theta)\, d\theta \sim C_{k,p}(n^{-1}\log n)^{(k+p)/2}\ ,$$

where

$$C_{k,p} = \int_{|u|^2 \leqq k+p} |u|^p h\left(\frac{u}{|u|}\right) du\ .$$

In the general case we get the same result with h replaced by $\tilde{h}$. That is,

$$\begin{aligned}C_{k,p} &= (\det \Sigma_0)^{-\frac{1}{2}} \int_{|u|^2 \leqq k+p} |B'u|^p h\left(\frac{B'u}{|B'u|}\right) du \\ &= \int_{u'\Sigma_0 u \leqq k+p} |u|^p h\left(\frac{u}{|u|}\right) du\ .\end{aligned}$$

Finally, we have to examine

$$\begin{aligned}\int_{|\theta| > n^{-\frac{1}{2}}\log n} &P_\theta(\bar{X} \in C_n)\rho(\theta)\, d\theta \\ &= \int_{|\theta| > n^{-\frac{1}{2}}\log n} \int_{(\log n)^{\frac{1}{2}}D_n} e^{-n[\psi(\theta) - n^{-\frac{1}{2}}\theta' v]}\, dP_{0n}(v)\rho(\theta)\, d\theta\ .\end{aligned}$$

If $v \in (\log n)^{\frac{1}{2}} D_n$, then $|n^{-\frac{1}{2}} v| \to 0$ so that $\psi(\theta) - n^{-\frac{1}{2}}\theta' v$ has its minimum in the region $|\theta| \geqq n^{-\frac{1}{2}} \log n$ attained at θ_n on the boundary

$$\begin{aligned}\psi(\theta_n) - n^{-\frac{1}{2}}\theta_n' v &\geqq c_1|\theta_n|^2 + o(|\theta_n|^2) - c_2|\theta_n|(n^{-1}\log n)^{\frac{1}{2}} \\ &= c_1 n^{-1}(\log n)^2 - c_2 n^{-1}(\log n)^{\frac{3}{2}} + o(n^{-1}(\log n)^2)\,,\end{aligned}$$

so there exists $c_3 > 0$ such that

$$\psi(\theta_n) - n^{-\frac{1}{2}}\theta_n' v \geqq c_3 n^{-1}(\log n)^2\,, \qquad \text{for all sufficiently large} \quad n\,.$$

Hence,

$$\int_{|\theta| > n^{-\frac{1}{2}}\log n} \int_{(\log n)^{\frac{1}{2}} D_n} e^{-n[\psi(\theta) - n^{-\frac{1}{2}}\theta' v]}\, dP_{0n}(v)\rho(\theta)\, d\theta \leqq e^{-c_3(\log n)^2} \leqq n^{-(k+p)/2}\,.$$

Putting all these results together gives us

$$\int P_\theta(\bar{X} \in C_n)\rho(\theta)\, d\theta \sim C_{k,p}(n^{-1}\log n)^{(k+p)/2}\,.$$

To complete the calculation of the Bayes risk one must examine

$$\gamma P_{\theta_0}(\bar{X} \notin C_n) = \gamma(1 - P_{0n}((\log n)^{\frac{1}{2}} D_n))\,.$$

Again using Theorem 15.4, page 154 of Bhattacharya and Rao (1976), this time with $r = 0$ and $f(v) = I_{\{v \notin (\log n)^{\frac{1}{2}} D_n\}}(v)$, one obtains

$$\gamma P_{\theta_0}(\bar{X} \notin C_n) = \gamma(1 - \Phi((\log n)^{\frac{1}{2}} D_n)) - \gamma \textstyle\sum_{j=1}^{s-2} n^{-j/2} P_j(-\Phi)((\log n)^{\frac{1}{2}} D_n) + R_n\,,$$

where $|R_n| \leqq o(n^{-(s-2)/2}) + \gamma\bar{\omega}_f(\eta : \psi^+)$, and $d\psi(v) = d(\Phi + \sum_{j=1}^{s-2} n^{-j/2} P_j(-\Phi))(v)$. First, we estimate the terms

$$P_j(-\Phi)((\log n)^{\frac{1}{2}} D_n) = (2\pi)^{-k/2} \int_{(\log n)^{\frac{1}{2}} D_n} e^{-\frac{1}{2}|x|^2} Q_j(x)\, dx\,.$$

Noting that $P_j(-\Phi)(C) = -P_j(-\Phi)(C^c)$ we have (assuming $\Sigma_0 = I$) for given $\varepsilon > 0$

$$(2\pi)^{-k/2} \int_{(\log n)^{\frac{1}{2}} D_n^c} e^{-\frac{1}{2}|x|^2} |Q_j(x)|\, dx \leqq (2\pi)^{-k/2} e^{-\frac{1}{2}(1-\varepsilon)|x_n|^2} \int e^{-(\varepsilon/2)|x|^2} |Q_j(x)|\, dx\,,$$

where $|x_n|^2 \geqq (\log n)(k + p - p(\log\log n/\log n) + ((c^- - \varepsilon)/\log n))$, ($n$ sufficiently large). Thus,

$$|P_j(-\Phi)((\log n)^{\frac{1}{2}} D_n)| \leqq (\text{constant}) n^{-\frac{1}{2}(k+p)(1-\varepsilon)} (\log n)^{p(1-\varepsilon)/2}\,,$$

and by choosing $\varepsilon < (k + p)^{-1}$ we have

$$n^{-j/2} P_j(-\Phi)((\log n)^{\frac{1}{2}} D_n) = o(n^{-(k+p)/2})\,.$$

To estimate $1 - \Phi((\log n)^{\frac{1}{2}} D_n)$ we need to make use of Theorem 1.

$$1 - \Phi((\log n)^{\frac{1}{2}} E_n^+) \leqq 1 - \Phi((\log n)^{\frac{1}{2}} D_n) \leqq 1 - \Phi((\log n)^{\frac{1}{2}} E_n^-)\,.$$

To calculate the left and right hand side we make use of the following lemma.

LEMMA.

$$(2\pi)^{-k/2} \int_{|x| \geqq \alpha_n} e^{-\frac{1}{2}|x|^2}\, dx \sim \frac{2\alpha_n^{k-2} e^{-\alpha_n^2/2}}{2^{k/2}\Gamma(k/2)}\,, \qquad \textit{as} \quad \alpha_n \to \infty\,.$$

Thus,

$$1 - \Phi((\log n)E_n{}^{\pm}) \sim \frac{2(k+p)^{(k-2)/2}}{2^{k/2}\Gamma(k/2)} (\log n)^{(k-2)/2} e^{-\frac{1}{2}[(k+p)(\log n) - p(\log\log n) + c^{\pm} \pm \varepsilon]}$$

$$= \frac{2(k+p)^{(k-2)/2}}{2^{k/2}\Gamma(k/2)} \frac{(\log n)^{(k+p-2)/2}}{n^{(k+p)/2}} e^{-c^{\pm}/2} e^{\pm\varepsilon/2} .$$

Recall,

$$C_{k,p} = \int_{|u|^2 \leqq k+p} |u|^p h\left(\frac{u}{|u|}\right) du = h^* \int_{|u|^2 \leqq k+p} |u|^p \, du$$

$$= h^* \frac{2\pi^{k/2}}{\Gamma(k/2)} (k+p)^{(k+p-2)/2} ,$$

where h^* lies between $h^{\pm}$. Thus,

$$\gamma(1 - \Phi((\log n)^{\frac{1}{2}} E_n{}^{\pm})) \sim \frac{h^{\mp}}{h^*} C_{k,p} n^{-(k+p)/2} (\log n)^{(k+p-2)/2} e^{\pm\varepsilon/2} ,$$

giving us

$$\gamma(1 - \Phi((\log n)^{\frac{1}{2}} E_n{}^{\pm})) = C_{k,p} n^{-(k+p)/2} (\log n)^{(k+p-2)/2} e^{O_n(1)} ,$$

where $h^-/h^+ \leqq \overline{\lim}\, e^{O_n(1)} \leqq h^+/h^-$. These results also show

$$\bar{\omega}_f(\eta : \psi^+) = \psi^+(\partial\{v \notin (\log n)^{\frac{1}{2}} D_n\}^{\eta})$$

$$\leqq |\psi| \left\{v : |v| > (\log n)^{\frac{1}{2}} \left(k + p - p\frac{\log\log n}{\log n} + \frac{c^- - \varepsilon}{\log n}\right)^{\frac{1}{2}} - \eta\right\} , \quad \eta = O(n^{-\frac{1}{2}} \log n)$$

$$= O(n^{-(k+p)/2} (\log \mathrm{n})^{(k+p-2)/2}) .$$

Taking $s - 2 \geqq k + p$ we see that the dominant term in the risk of the Bayes test is the type II risk

$$C_{k,p}(n^{-1} \log n)^{(k+p)/2} . \qquad \square$$

3. Generalized loss functions. Suppose that, instead of zero-one loss functions, we had assumed the more general structure:

$$L_0(\theta) = |\theta - \theta_0|^{\alpha} g\left(\frac{\theta - \theta_0}{|\theta - \theta_0|}\right) + o(|\theta - \theta_0|^{\alpha}) \qquad \text{as} \quad |\theta - \theta_0| \to 0 ,$$

$$L_1(\theta) = \delta \qquad \text{if} \quad \theta = \theta_0$$

$$= 0 \qquad \text{if} \quad \theta \neq \theta_0 .$$

Then the acceptance region for the Bayes test (2) would have been

$$(2') \qquad \bar{X} \in C_n{}' = \{x : \int e^{n(\theta' x - \psi(\theta))} L_0(\theta)\rho(\theta)\, d\theta \leqq \delta\gamma e^{n(\theta_0' x - \psi(\theta_0))}\} ,$$

and the Bayes risk (3) would have been

$$(3') \qquad B_n{}' = \delta\gamma P_{\theta_0}(\bar{X} \notin C_n{}') + \int P(\bar{X} \in C_n{}') L_0(\theta)\rho(\theta)\, d\theta .$$

Comparing (2) with (2′) and (3) with (3′), it is obvious that the results of Section 2 can be adapted to this generalized setting by formally replacing γ by $\delta\gamma$, p by $p + \alpha$, and h by gh.

4. Almost-Bayes tests. Now, let us look at the risk of tests that might be called "almost-Bayes." Recall that the Bayes test had an acceptance region like

$$n^{\frac{1}{2}}(\bar{X} - \nabla\psi(\theta_0)) \in (\log n)^{\frac{1}{2}}\{x : x'\Sigma_0^{-1}x \leqq k + p\}.$$

Let E be an arbitrary bounded convex set and consider tests which accept whenever

$$n^{\frac{1}{2}}(\bar{X} - \nabla\psi(\theta_0)) \in (\log n)^{\frac{1}{2}}E.$$

For simplicity we will suppose $\Sigma_0 = I$. This can always be accomplished by looking at BX instead of X, where $\Sigma_0^{-1} = B'B$. The set BE is also convex if E is.

For the type I risk one needs to evaluate

$$\gamma(1 - P_{0n}((\log n)^{\frac{1}{2}}E)),$$

where P_{0n} is the distribution of $n^{\frac{1}{2}}(\bar{X} - \nabla\psi(\theta_0))$ when $\theta = \theta_0$. By the same argument as in the proof of Theorem 2, we have

$$\gamma[1 - P_{0n}((\log n)^{\frac{1}{2}}E)] \sim \gamma[1 - \Phi((\log n)^{\frac{1}{2}}E)]$$

and the latter is

$$\gamma(2\pi)^{-k/2} \int_{(\log n)^{\frac{1}{2}}E^c} e^{-\frac{1}{2}|x|^2}\,dx = \frac{\gamma(\log n)^{k/2}}{(2\pi)^{k/2}} \int_{E^c} e^{-\frac{1}{2}(\log n)|x|^2}\,dx.$$

Let x_0 be a point such that $|x_0|^2 = \inf_{x\in E^c}|x|^2$. Then

$$\frac{\gamma(\log n)^{k/2}}{(2\pi)^{k/2}} \int_{E^c} e^{-\frac{1}{2}(\log n)|x|^2}\,dx = \frac{\gamma(\log n)^{k/2}}{(2\pi)^{k/2}n^{\frac{1}{2}|x_0|^2}} \int_{E^c} e^{-\frac{1}{2}(\log n)(|x|^2-|x_0|^2)}\,dx.$$

Write

$$\begin{aligned}\int_{E^c} e^{-\frac{1}{2}(\log n)(|x|^2-|x_0|^2)}\,dx &= \int_0^\infty \lambda\left\{x\in E^c : |x|^2 - |x_0|^2 < \frac{-2\log t}{\log n}\right\}dt \\ &= \int_0^1 \lambda\left\{x\in E^c : |x|^2 - |x_0|^2 < \frac{-2\log t}{\log n}\right\}dt \\ &= \tfrac{1}{2}(\log n)\int_0^\infty e^{-\frac{1}{2}(\log n)y}\lambda\{x\in E^c : |x|^2 - |x_0|^2 < y\}\,dy.\end{aligned}$$

Next, we have to calculate $\lambda\{x\in E^c : |x|^2 - |x_0|^2 > y\}$. The two most extreme cases are when E is a sphere and when E is a half-space. In the second case we need

$$\lambda\{x : |x_0| < x_k, \textstyle\sum_{i=1}^k x_i^2 \leqq |x_0|^2 + y\} \sim \frac{\pi^{(k-1)/2}}{2\Gamma((k+3)/2)|x_0|}y^{(k+1)/2}, \qquad \text{as } y\to 0.$$

At the other extreme

$$\lambda\{x : |x_0|^2 \leqq \textstyle\sum_{i=1}^k x_i^2 \leqq |x_0|^2 + y\} \sim \frac{k\pi^{k/2}}{2\Gamma(k/2+1)}|x_0|^{k/2}y, \qquad \text{as } y\to 0.$$

Recall that

$$\tfrac{1}{2}(\log n)\int_0^\infty y^{\alpha-1}e^{-\frac{1}{2}(\log n)y}\,dy = 2^{\alpha-1}\Gamma(\alpha)(\log n)^{\alpha-1}.$$

Thus, the type I risk for a half-space is asymptotically

$$\begin{aligned}&\frac{\gamma(\log n)^{k/2}}{(2\pi)^{k/2}n^{\frac{1}{2}|x_0|^2}}\cdot\frac{\pi^{(k-1)/2}}{2\Gamma((k+3)/2)|x_0|}\cdot\frac{2^{(k+1)/2}\Gamma((k+3)/2)}{(\log n)^{(k+1)/2}} \\ &\qquad = \gamma(2\pi)^{-\frac{1}{2}}|x_0|^{-1}(\log n)^{-\frac{1}{2}}n^{-\frac{1}{2}|x_0|^2}.\end{aligned}$$

For the sphere the type I risk is asymptotically

$$\frac{\gamma(\log n)^{k/2}}{(2\pi)^{k/2}n^{\frac{1}{2}|x_0|^2}}\frac{k\pi^{k/2}}{2\Gamma(k/2+1)}|x_0|^{k-2}\frac{2\Gamma(2)}{(\log n)}=\frac{\gamma(\log n)^{(k-2)/2}k|x_0|^{k-2}}{2^{k/2}n^{\frac{1}{2}|x_0|^2}\Gamma(k/2+1)}.$$

Thus, in general one can say that the type I risk is asymptotically

$$\gamma n^{-\frac{1}{2}|x_0|^2}\rho_n,$$

where $c_1(\log n)^{-\frac{1}{2}} \leqq \rho_n \leqq c_2(\log n)^{(k-2)/2}$ for some positive constants c_1, c_2.

If we had not assumed $\Sigma_0 = I$, the result would be the same except that

$$|x_0|^2 = \inf\{y'\Sigma_0^{-1}y : y \notin E\}.$$

The next problem is to compute the type II risk for almost-Bayes tests. Just as in the Bayes case one shows that the type II risk is asymptotically (again with $\Sigma_0 = I$)

$$\begin{aligned}&n^{-(k+p)/2}\int_{(\log n)^{\frac{1}{2}}E} e^{\frac{1}{2}|v|^2}\int|\theta+v|^p h\left(\frac{\theta+v}{|\theta+v|}\right)e^{-\frac{1}{2}|\theta|^2}\,d\theta\,d\Phi(v)\\&\quad=(2\pi)^{-k/2}n^{-(k+p)/2}\int_{(\log n)^{\frac{1}{2}}E}\int|\theta+v|^p h\left(\frac{\theta+v}{|\theta+v|}\right)e^{-\frac{1}{2}|\theta|^2}\,d\theta\,dv\end{aligned}$$

which, in turn, is asymptotically equivalent to

$$\begin{aligned}&\frac{(\log n)^{(k+p)/2}}{(2\pi)^{k/2}n^{(k+p)/2}}\int_E\int\left|\frac{\theta}{(\log n)^{\frac{1}{2}}}+v\right|^p h\left(\frac{\theta/(\log n)^{\frac{1}{2}}+v}{|\theta/(\log n)^{\frac{1}{2}}+v|}\right)e^{-\frac{1}{2}|\theta|^2}\,d\theta\,dv\\&\quad\sim(n^{-1}\log n)^{(k+p)/2}\int_E|v|^p h\left(\frac{v}{|v|}\right)dv.\end{aligned}$$

For the general case replace E by $\Sigma_0^{-1}E$ in this expression.

It is interesting to note that if in place of the Bayes set

$$D_n = \left\{x : x'\Sigma_0^{-1}x \leqq k+p+p\frac{\log\log n}{\log n}+\cdots\right\}$$

one uses $D = \{x : x'\Sigma_0^{-1}x \leqq k+p\}$ the type II risk stays the same but the type I risk is reduced by a power of $\log n$.

If a test with acceptance set $(\log n)^{\frac{1}{2}}E$ has $|x_0|^2 < k+p$, then the risk is asymptotically the type I risk and is larger than the Bayes risk by a power of n. However, if E has $|x_0|^2 \geqq k+p$ then the risk is larger only by the factor

$$\frac{\int_{\Sigma_0^{-1}E}|v|^p h\left(\frac{v}{|v|}\right)dv}{\int_{\Sigma_0^{-1}D}|v|^p h\left(\frac{v}{|v|}\right)dv}.$$

One can define (similar to Rubin and Sethuraman (1965)) an index of efficiency of a test based on the acceptance region $(\log n)^{\frac{1}{2}}E$ by letting

$$e(E)=\left(\frac{\int_{\Sigma_0^{-1}D}|v|^p h\left(\frac{v}{|v|}\right)dv}{\int_{\Sigma_0^{-1}E}|v|^p h\left(\frac{v}{|v|}\right)dv}\right)^{2/(k+p)},$$

because if

$$(n_1^{-1}\log n_1)^{(k+p)/2}\int_{\Sigma_0^{-1}D}|v|^p h\left(\frac{v}{|v|}\right)dv = (n_2^{-1}\log n_2)^{(k+p)/2}\int_{\Sigma_0^{-1}E}|v|^p h\left(\frac{v}{|v|}\right)dv$$

as $n_1 \to \infty$, $n_2 \to \infty$, we must have

$$n_1/n_2 \to e(E)\ .$$

Note that this is only for E containing D. Otherwise, if n_1 and n_2 are chosen to make the risks equal, n_1/n_2 goes to zero faster than a power of $1/n_2$.

5. Consequences of poor prior guessing. An almost-Bayes test results from wrongly guessing the exponential family. Let ζ^0 denote the mean vector under the null-hypothesis, and suppose that the assumed exponential family yields the covariance matrix A under H_0 instead of the actual $\Sigma_0 = I$. Then the (almost-Bayes) acceptance region

$$n^{\frac{1}{2}}(\bar{X} - \zeta^0) \in (\log n)^{\frac{1}{2}}\{x : x'A^{-1}x \leqq k + p\}$$

would be used; so the type I risk is

$$\gamma n^{-\frac{1}{2}|x_0|^2}\rho_n\ , \qquad \text{where} \quad |x_0|^2 = \inf\{|x|^2 : x'A^{-1}x > k + p\}$$
$$= (\text{min e.v. of } A)(k + p)\ ,$$

and the type II risk is

$$(n^{-1}\log n)^{(k+p)/2}(2\pi)^{-k/2}\int_{\{x:x'A^{-1}x\leqq k+p\}}|v|^p h\left(\frac{v}{|v|}\right)dv\ .$$

Therefore, unless all the eigenvalues of A are $\geqq 1$, the risk of the Bayes test is much smaller (approximately by the factor $1/n^{(1-\lambda_{\min})(k+p)/2}$).

Suppose next that one guesses the wrong prior

$$\tilde{\rho}(\theta) = |\theta - \theta_0|^{\tilde{p}}\tilde{h}\left(\frac{\theta - \theta_0}{|\theta - \theta_0|}\right) + o(|\theta - \theta_0|^{\tilde{p}})\ .$$

The calculated acceptance region would then be

$$n^{\frac{1}{2}}(\bar{X} - \zeta^0) \in (\log n)^{\frac{1}{2}}\{x : x'x \leqq k + \tilde{p}\}\ ,$$

and $|x_0|^2 = k + \tilde{p}$. Again, unless $\tilde{p} \geqq p$, the actual Bayes risk is much smaller, this time roughly by a factor $1/n^{(p-\tilde{p})/2}$.

6. Lemmas. This section contains the technical lemmas needed to prove the main results (see Section 2). Here we again assume without loss of generality that $\theta_0 = 0$ and the mean vector $\nabla\psi(\theta_0) = 0$ (so $\psi(\theta) = I(\theta_0, \theta)$ and is strictly convex with absolute minimum zero at $\theta_0 = 0$).

LEMMA 1. *Let $r > 0$ and $\xi_n \to 0$. Then there exists $\delta > 0$ such that*

$$1 \geqq \int_{|\theta|\leqq r;\theta' t\geqq\xi_n}\rho(\theta)\,d\theta \geqq \delta > 0 \qquad \textit{for all} \quad |t| = 1\ .$$

PROOF. $1 \geqq \int_{|\theta| \leqq r;\theta' t \geqq \xi_n} \rho(\theta)\, d\theta \geqq \int_{|\theta| \leqq r_0:\theta' t \geqq \xi_n} \rho(\theta)\, d\theta$, where we have chosen $r_0 < r$ so small that

$$\frac{\rho(\theta)}{|\theta|^p} > \tfrac{1}{2} \inf_{|u|=1} h(u) > 0 \qquad \text{if} \quad |\theta| < r_0 .$$

Thus,

$$\begin{aligned} 1 \geqq \int_{|\theta| \leqq r;\theta' t \geqq \xi_n} \rho(\theta)\, d\theta &\geqq \tfrac{1}{2} \inf_{|u|=1} h(u) \int_{|\theta| \leqq r_0;\theta' t \geqq \xi_n} |\theta|^p\, d\theta \\ &= \tfrac{1}{2} \inf_{|u|=1} h(u) \int_{|\theta| \leqq r_0;\theta' t_0 \geqq \xi_n} |\theta|^p\, d\theta , \end{aligned}$$

where $t_0 = (1, 0, 0, \cdots, 0)$. For n sufficiently large

$$\int_{|\theta| \leqq r_0;\theta' t_0 \geqq \xi_n} |\theta|^p\, d\theta \geqq \tfrac{1}{4} \int_{|\theta| \leqq r_0} |\theta|^p\, d\theta > 0 . \qquad \square$$

LEMMA 2. *If $|y_n| \to \infty$, there exists $c > 1$ such that*

$$\int e^{n[\theta' y_n - \psi(\theta)]} \rho(\theta)\, d\theta \geqq c^n ,$$

for all n sufficiently large.

PROOF. Let $\varepsilon > 0$ be given and choose r so small that $M = \sup_{|\theta| \leqq r} \psi(\theta) < \varepsilon$. Then

$$\begin{aligned} \int e^{n[\theta' y_n - \psi(\theta)]} \rho(\theta)\, d\theta &\geqq \int_{|\theta| \leqq r} e^{n[\theta' y_n - \psi(\theta)]} \rho(\theta)\, d\theta \\ &\geqq e^{-nM} \int_{|\theta| \leqq r;\theta' y_n \geqq \varepsilon} e^{n\theta' y_n} \rho(\theta)\, d\theta \\ &\geqq e^{n(\varepsilon - M)} \int_{|\theta| \leqq r;\theta' y_n/|y_n| \geqq \varepsilon/|y_n|} \rho(\theta)\, d\theta \\ &\geqq \delta e^{n(\varepsilon - M)} , \qquad \text{by Lemma 1} \\ &\geqq c^n \quad \text{if} \quad n \quad \text{is sufficiently large,} \end{aligned}$$

where $1 < c < e^{\varepsilon - M}$. $\square$

LEMMA 3. *Suppose $x_n \in C_n = \{x : \int e^{n(\theta' x - \psi(\theta))} \rho(\theta)\, d\theta \leqq \gamma\}$ for all n. Then* $\lim_n x_n = 0$.

PROOF. First, note that $\{x_n\}$ is a bounded sequence. Otherwise, there would be a subsequence $\{x_{n_k}\}$ such that $|x_{n_k}| \to \infty$; and along this subsequence we would have

$$\gamma \geqq \int e^{n_k[\theta' x_{n_k} - \psi(\theta)]} \rho(\theta)\, d\theta ,$$

which would contradict Lemma 2.

Since the sequence is bounded, let x_0 be a finite limit point. It entails no loss to suppose $x_n \to x_0$. Then, a standard argument shows

$$n^{-1} \log [\int e^{n[\theta' x_n - \psi(\theta)]} \rho(\theta)\, d\theta] \to \sup_{\{\theta : \rho(\theta) > 0\}} [\theta' x_0 - \psi(\theta)] .$$

Because $\theta' x_0 - \psi(\theta)$ is strictly concave, its maximum must occur at some $\tilde{\theta}$ such that $x_0 = \nabla\psi(\tilde{\theta})$. Also, $\theta' x_0 - \psi(\theta)$ is zero at $\theta = 0$, so $\sup_{\{\theta : \rho(\theta) > 0\}} [\theta' x_0 - \psi(\theta)] > 0$ unless $\tilde{\theta} = 0$. Since

$$0 \geqq n^{-1} \log \gamma \geqq n^{-1} \log \int e^{n(\theta' x_n - \psi(\theta))} \rho(\theta)\, d\theta ,$$

we must have $\tilde{\theta} = 0$ (hence $x_0 = \nabla\psi(0) = 0$). $\square$

LEMMA 4. *Let* $\alpha_n = (n^{-1}\log n)^{\frac{1}{2}}$. *Then*

$$\int_{|\theta|\geq n^{-\frac{1}{2}}\log n} e^{n[\alpha_n\theta' x-\psi(\theta)]}\rho(\theta)\,d\theta \to 0 \qquad as \quad n\to\infty\,,$$

uniformly for x in a bounded set.

PROOF. $\alpha_n\theta' x - \psi(\theta)$ is a strictly concave function whose maximum value occurs when $\nabla\psi(\theta_n) = \alpha_n x$. Since Σ_0 is nonsingular, it follows that $\theta_n = O(\alpha_n)$, hence, that $|\theta_n| \leq n^{-\frac{1}{2}}\log n$ for all x in a bounded set, provided n is sufficiently large . That is, the maximum value of $\alpha_n\theta' x - \psi(\theta)$ for $|\theta| \geq n^{-\frac{1}{2}}\log n$ occurs when $|\bar\theta_n| = n^{-\frac{1}{2}}\log n$; so that for all n sufficiently large

$$\int_{|\theta|\geq n^{-\frac{1}{2}}\log n} e^{n[\alpha_n\theta' x-\psi(\theta)]}\rho(\theta)\,d\theta \leq e^{n[\alpha_n\bar\theta_n' x-\psi(\bar\theta_n)]}$$

where $\bar\theta_n$ is the point on the sphere $\{\theta\colon |\theta| = n^{-\frac{1}{2}}\log n\}$ where the maximum occurs. From Taylor's theorem

$$n[\alpha_n\bar\theta_n' x - \psi(\bar\theta_n)] = (\log n)^{\frac{3}{2}}\bar\omega_n' x - \tfrac{1}{2}(\log n)^2\bar\omega_n'\Sigma_0\bar\omega_n + O\left(\frac{(\log n)^3}{n^{\frac{1}{2}}}\right),$$

where $|\bar\omega_n| = 1$. This converges to $-\infty$ uniformly for x in a bounded set. □

LEMMA 5. $\bigcup_{n=1}^{\infty} D_n$ *is bounded* (D_n *is defined in Section* 2).

PROOF. Suppose there is a sequence $\{y_n\}$ contained in $\bigcup_{n=1}^{\infty} D_n$ with $|y_n| \to \infty$. It is easy to show that each D_n is a bounded set, so that without loss we can suppose $y_n \in D_n$ for all n. By Lemma 3 we have $(n^{-1}\log n)^{\frac{1}{2}}y_n \to 0$. If $y_n \in D_n$ we have

$$\int e^{n[\theta' y_n (n^{-1}\log n)^{\frac{1}{2}}-\psi(\theta)]}\rho(\theta)\,d\theta \leq \gamma\,.$$

We will obtain a contradiction by showing that the left-hand side converges to ∞. Set $\beta_n = (n^{-1}\log n)^{\frac{1}{2}}y_n$. Choose $d > 0$ so small that $\psi(\theta) \leq |\theta|^2/2\sigma^2$ for all $|\theta| \leq d$, where σ^2 is some suitably chosen positive constant. This is possible because

$$\psi(\theta) = \tfrac{1}{2}\theta'\Sigma_0\theta + O(|\theta|^3)\,.$$

Then

$$\begin{aligned}\int e^{n[\theta'\beta_n-\psi(\theta)]}\rho(\theta)\,d\theta &\geq \int_{|\theta|\leq d} e^{n[\theta'\beta_n-|\theta|^2/2\sigma^2]}\rho(\theta)\,d\theta\\ &= \int_{|\theta|\leq d} e^{(n/\sigma^2)(\theta'\sigma^2\beta_n-\frac{1}{2}|\theta|^2)}\rho(\theta)\,d\theta\\ &\geq e^{\frac{1}{2}n\sigma^2|\beta_n|^2}\tfrac{1}{2}\inf_\theta h\left(\frac{\theta}{|\theta|}\right)\int_{|\theta|\leq d} e^{-(n/2\sigma^2)|\theta-\sigma^2\beta_n|^2}|\theta|^p\,d\theta\end{aligned}$$

(We also choose d so small that $\rho(\theta)/|\theta|^p \geq \frac{1}{2}\inf_{|\omega|=1} h(\omega)$ for $|\theta| \leq d$.)

$$\begin{aligned}&= e^{\frac{1}{2}n\sigma^2|\beta_n|^2}(\tfrac{1}{2}\inf h)\int_{|\theta+\sigma^2\beta_n|\leq d} e^{-n|\theta|^2/2\sigma^2}|\theta+\sigma^2\beta_n|^p\,d\theta\\ &\sim e^{\frac{1}{2}n\sigma^2|\beta_n|^2}(\tfrac{1}{2}\inf h)\int e^{-n|\theta|^2/2\sigma^2}|\theta+\sigma^2\beta_n|^p\,d\theta\\ &\sim e^{\frac{1}{2}n\sigma^2|\beta_n|^2}(\tfrac{1}{2}\inf h)|\sigma^2\beta_n|^p(2\pi/n)^{k/2}\sigma^k\,.\end{aligned}$$

But

$$\begin{aligned}&\tfrac{1}{2}n\sigma^2|\beta_n|^2 + p\log|\beta_n| - \tfrac{1}{2}k\log n\\ &\quad= \tfrac{1}{2}\sigma^2|y_n|^2\log n + p\log|y_n| + \tfrac{1}{2}p(\log\log n - \log n) - \tfrac{1}{2}k\log n\\ &\quad= \tfrac{1}{2}(\log n)[\sigma^2|y_n|^2 - (p+k)] + p\log|y_n| + \tfrac{1}{2}p\log\log n\\ &\quad\to \infty\,.\end{aligned}$$

□

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