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ASYMPTOTIC BEHAVIOR OF ELEMENTARY SOLUTIONS OF ONE-DIMENSIONAL GENERALIZED DIFFUSION EQUATIONS

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We give the asymptotic estimate for large t of elementary solutions of one-dimensional generalized diffusion equations with regularly varying Green functions. As a corollary we obtain the precise asymptotic behavior of the semigroup $T_t f(x)$ for all $f \in L_1(dm)$ if the speed measure function $m(x)$ is regularly varying as $x \to \pm\infty$.

1. Introduction. Let $p(t, x, y)$ be an elementary solution of a generalized diffusion equation

(1.1) $$\partial u(t, x)/\partial t = \mathfrak{G} u(t, x), \quad t > 0$$

on an interval S in the sense of McKean [9]. Its asymptotic behavior for small $t > 0$ is obtained in general form by Watanabe in [8]. In this paper we study that for large t under the assumption that the Green function $G(\alpha, x, y)$ is regularly varying as α tends to 0 from the right, i.e.,

(1.2) $$G(\alpha, 0, 0) \sim \alpha^{-\rho} L(1/\alpha) \quad \text{as} \quad \alpha \downarrow 0,$$

for some slowly varying function $L(t)$ and $0 < \rho \le 1$. Here the description $a(\alpha) \sim b(\alpha)$ as $\alpha \downarrow 0$ [$\alpha \to \infty$] stands for $\lim_{\alpha \downarrow 0} a(\alpha)/b(\alpha) = 1$ [resp. $\lim_{\alpha \to \infty} a(\alpha)/b(\alpha) = 1$]. More precisely, after noting the pointwise asymptotic formula

(1.3) $$p(t, x, y) \sim \Gamma(\rho)^{-1} t^{\rho - 1} L(t) \quad \text{as} \quad t \to \infty, \quad x, y \in S,$$

we will show that the following global asymptotic estimate

(1.4) $$\limsup_{t \to \infty} t^{1-\rho} L(t)^{-1} \sup_{y \in S} p(t, x, y) < \infty, \; x \in S$$

holds provided $0 < \rho < 1$ and an extra condition is assumed (see (2.5) and (2.6) below). It should be noted that the formula (1.3) together with estimate (1.4) implies

(1.5) $$\int_S p(t, x, y) f(y)\, dm(y) \sim \Gamma(\rho)^{-1} t^{\rho-1} L(t) \int_S f(y)\, dm(y) \quad \text{as} \quad t \to \infty,$$

for all $f \in L_1(dm)$, where $dm(x)$ is the speed measure. As a by-product, we will also note that (1.4) and (1.5) remain valid for $\rho = 1$ if $L(t)$ converges to a positive constant as $t \to \infty$.

Actually our first interest in this subject was inspired by a physicist, Masuo Suzuki, who gave (1.5) for long-time tail phenomena in statistical physics ([12] and [13], also cf. [2] and [7]). Our results here verify his results rigorously as far

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as the external fluctuation force is Gaussian white noise. We will illustrate it in Section 6. Finally, we note that some sufficient conditions for our assumption (1.2) are given in [5] and [6]. We will mention that in Corollary 2 and Remark 2 in the next section. As is mentioned in [5], the class of one-dimensional diffusion processes which satisfy (1.2) coincides with that introduced by Darling and Kac [1] for the limit theorem of normalized occupation time law (see also Stone [11]).

2. Statement of results. Let $S = (\ell_1, \ell_2)$ be an open interval with $-\infty \leq \ell_1 < 0 < \ell_2 \leq \infty$ and $m(x)$ a real valued nontrivial right continuous nondecreasing function on it with $m(0) = 0$. The support of the measure $dm(x)$ on S induced by $m(x)$ is denoted as S_m. Let $C(S)$ and $C(S_m)$ be the spaces of all complex valued bounded continuous functions on S and S_m, respectively, and $D(\mathfrak{G})$ the space of all those functions $u(x)$ in $C(S)$ satisfying the following two conditions. a) There are two complex constants a, b and a function $g(x)$ in $C(S_m)$ such that

$$u(x) = a + bx + \int_{0+}^{x+} (x - y)g(y)\, dm(y), \quad x \in S, \tag{2.1}$$

where the integral is read as

$$\int_{0+}^{x+} f(y)\, dm(y) = \begin{cases} \displaystyle\int_{(0,x]} f(y)\, dm(y) & \text{if} \quad x \in [0, \ell_2), \\ \displaystyle -\int_{(x,0]} f(y)\, dm(y) & \text{if} \quad x \in (\ell_1, 0). \end{cases}$$

b) If ℓ_i is finite, then

$$\lim_{x \to \ell_i, x \in S} u(x) = 0, \quad i = 1, 2. \tag{2.2}$$

The linear operator $\mathfrak{G}$ from $D(\mathfrak{G})$ into $C(S_m)$ is defined by

$$D(\mathfrak{G}) \ni u \mapsto \mathfrak{G}u = g \in C(S_m).$$

We note that the above setting includes all cases of sticky elastic boundary conditions for regular boundaries as well. Indeed, if ℓ_2 is a regular boundary and if we want to set the reflecting boundary condition at ℓ_2 for instance, then we have only to reset $S = (\ell_1, \infty)$ and $m(x) = m(\ell_2-)$ for all $x \in [\ell_2, \infty)$ (for details see [8] and [16]). Now we can define the elementary solution $p(t, x, y)$ of the diffusion equation (1.1) following McKean [9] (see Section 3 below for precise definition). The corresponding Green function $G(\alpha, x, y)$ is given by

$$G(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y)\, dt + \Phi(x, y), \quad \alpha > 0, \quad x, y \in S, \tag{2.3}$$

where $\Phi(x, y)$ is a nonnegative function defined in (3.9) below. We note that the correction function $\Phi(x, y)$ is equal to zero if $[x \wedge y, x \vee y] \cap S_m \neq \varnothing$, where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Hence the formula (2.3) is reduced to the usual relation between the Green function and the elementary solution for diffusion processes, if $[x \wedge y, x \vee y] \cap S_m \neq \varnothing$, especially if the generalized diffusion equation (1.1) is a diffusion equation (i.e., $S_m = S$).

PROPOSITION. *The Green function satisfies* (1.2) *if and only if the elementary solution satisfies* (1.3). *Further in this case it holds that*

$$\limsup_{t\to\infty} t^{1-\rho}L(t)^{-1}\sup_{x,y\in K}p(t, x, y) < \infty \tag{2.4}$$

for every compact set K in S.

REMARK 1. Assume that the formula (1.2) (and hence (1.3)) is satisfied. Then $\ell_1 = -\infty$ and $\ell_2 = \infty$. Further if $0 < \rho < 1$, then either $J_1 = \infty$ or $J_2 = \infty$ holds, where

$$J_i = \lim_{x\to\ell_i, x\in S} \int_{0+}^{x+} y\, dm(y), \quad i = 1, 2.$$

THEOREM. *Let $0 < \rho < 1$ and the Green function satisfy* (1.2). *In the case of $J_1 = J_2 = \infty$, let also*

$$\liminf_{c\to\infty}\{cm(c)\}^{-1} \int_0^c (m(c) - m(x))\, dx > 0, \tag{2.5}$$

$$\liminf_{c\to-\infty}\{cm(c)\}^{-1} \int_c^0 (m(x) - m(c))\, dx > 0. \tag{2.6}$$

Then the global asymptotic estimate (1.4) *holds. Further the formula* (1.5) *is valid for all $f \in L_1(dm)$.*

COROLLARY 1. *The Green function satisfies* (1.2) *with $\rho = 1$ and the slowly varying function $L(t)$ converging to a positive finite limit $L(\infty)$ as $t \to \infty$ if and only if $\ell_1 = -\infty$, $\ell_2 = \infty$, $0 < m(\infty) - m(-\infty) = 1/L(\infty) < \infty$. Further in this case the formulae* (1.3)–(1.5) *hold.*

Note that the assumption (2.5) [(2.6)] is satisfied if $m(\infty) = \infty$ [resp. $m(-\infty) = -\infty$] and

$$\liminf_{c\to\infty} m(cx_0)/m(c) > 1 \quad [\text{resp. } \liminf_{c\to-\infty} m(cx_0)/m(c) > 1]$$

for some positive x_0. In particular, it is satisfied if $m(x)$ [resp. $-m(-x)$] is regularly varying with positive exponent as $x \to \infty$. Hence, due to [5] and [6], we have the following

COROLLARY 2. *Assume that $\ell_1 = -\infty$, $\ell_2 = \infty$, $J_1 < \infty$, $J_2 = \infty$ and*

$$m(x) \sim x^{1/\rho-1}K(x) \quad as \quad x \to \infty, \tag{2.7}$$

for some $0 < \rho < 1$ and slowly varying function $K(x)$. Then the assertions of the Theorem hold with a slowly varying function $L(t)$ satisfying

$$K(t^\rho L(t)) \sim \{\rho(1 - \rho)\}^{-1}\{\Gamma(1 + \rho)/\Gamma(1 - \rho)\}^{1/\rho}L(t)^{-1/\rho} \quad as \quad t \to \infty. \tag{2.8}$$

REMARK 2. Suppose that

$$\lim_{t\to\infty} k_2(t)/k_1(t) = \hat{\theta} \tag{2.9}$$

for some $0 \le \hat{\theta} < \infty$, where $k_1(t)$ and $k_2(t)$ are the inverse functions of the mapping $[0, -\ell_1) \ni x \mapsto -xm(-x)$ and $[0, \ell_2) \ni x \mapsto xm(x)$, respectively. Then the Green function satisfies (1.2) if and only if the speed measure satisfies (2.7) up to a multiplicative positive constant ([6]). Thus conditions (2.5) and (2.6) are not necessary for the Green function formula (1.2). On the other hand, (1.2) together with (2.5) and (2.6) does not imply (2.9) either.

3. Preliminaries. In this section we define the elementary solution $p(t, x, y)$ of the generalized diffusion equation (1.1) and list some of its properties. Most of the arguments in the following, except for Lemma 1, are the analogues of those for diffusion equations in McKean [9] and Yoshida [17; Chapter 5], and can be obtained by tracing their proofs (see also [4] and [8] for the reason why the generalized diffusion equation should be studied).

Let S and $m(x)$ be those given in Section 2. For each $\alpha \in \mathbb{C}$, let $\varphi_1(x, \alpha)$ and $\varphi_2(x, \alpha)$ be the solutions of the integral equations

$$\begin{aligned} \varphi_1(x, \alpha) &= 1 + \alpha \int_{0+}^{x+} (x - y)\varphi_1(y, \alpha)\, dm(y), \\ \varphi_2(x, \alpha) &= x + \alpha \int_{0+}^{x+} (x - y)\varphi_2(y, \alpha)\, dm(y), \end{aligned} \qquad x \in S, \tag{3.1}$$

respectively. Then for each $\alpha > 0$, there exist the limits

$$\begin{aligned} h_1(\alpha) &= -\lim_{x\downarrow\ell_1} \varphi_2(x, \alpha)/\varphi_1(x, \alpha), \\ h_2(\alpha) &= \lim_{x\uparrow\ell_2} \varphi_2(x, \alpha)/\varphi_1(x, \alpha). \end{aligned} \tag{3.2}$$

Here and hereafter, we use the usual convention $1/\infty = 0$, $(\pm a)/0 = \pm\infty$, $\infty \pm a = \infty$ and $-\infty \pm a = -\infty$ for positive a. Define the function $h(\alpha)$ by the equality

$$1/h(\alpha) = 1/h_1(\alpha) + 1/h_2(\alpha) \tag{3.3}$$

and $u_i(x, \alpha)$, $i = 1, 2$, $\alpha > 0$, $x \in S$, by

$$u_i(x, \alpha) = \varphi_1(x, \alpha) + (-1)^{i+1}\varphi_2(x, \alpha)/h_i(\alpha). \tag{3.4}$$

Then it is well known that $u_1(x, \alpha)$ $[u_2(x, \alpha)]$ is positive and nondecreasing [resp. nonincreasing] in $x \in S$ with $u_1(0, \alpha) = u_2(0, \alpha) = 1$ (see [6; page 178]). Let

$$\begin{aligned} &h_{11}(\alpha) = h(\alpha), \quad h_{22}(\alpha) = -(h_1(\alpha) + h_2(\alpha))^{-1}; \\ &h_{12}(\alpha) = h_{21}(\alpha) = -h(\alpha)/h_2(\alpha). \end{aligned} \tag{3.5}$$

Then it is seen that all these functions $h_{ij}(\alpha)$, $i, j = 1, 2$ can be analytically continued to the exterior of the half line $(-\infty, 0]$ in the complex plane. The spectral measures $\sigma_{ij}(d\lambda)$, $i, j = 1, 2$ are given by

$$(3.6)\qquad \sigma_{ij}([\lambda_1, \lambda_2]) = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} Imh_{ij}(-\lambda - \sqrt{-1}\varepsilon)\, d\lambda$$

for all continuity points $\lambda_1 < \lambda_2$. Then the matrix valued measure $[\sigma_{ij}(d\lambda)]_{i,j=1,2}$ is symmetric and nonnegative definite. Now we define the Green function and the elementary solution of the generalized diffusion equation by

$$(3.7)\qquad G(\alpha, x, y) = G(\alpha, y, x) = h(\alpha)u_1(x, \alpha)u_2(y, \alpha), \quad \alpha > 0, \quad x \le y, \quad x, y \in S,$$

$$(3.8)\qquad p(t, x, y) = \sum_{i,j=1}^{2} \int_{0-}^{\infty} e^{-\lambda t}\varphi_i(x, -\lambda)\varphi_j(y, -\lambda)\sigma_{ij}(d\lambda), \quad t > 0, \quad x, y \in S.$$

As will be given in (3.11) below, there is another candidate $\tilde{G}(\alpha, x, y)$ for the Green function of the generalized diffusion equation (1.1). The two functions make sense substantially only for $x, y \in S_m$. Further, they coincide with each other for those points, and we find no drastic difference between the two definitions. We adopted our definition (3.7) because it is simpler than (3.11). Instead, we need a correction function $\Phi(x, y)$ to combine the Green function $G(\alpha, x, y)$ and $p(t, x, y)$. Denote $S\backslash S_m = \bigcup_{k=1}^{\infty} I_k$, where $I_1, I_2, \cdots$ are disjoint open intervals (some or all of them may be null) with the end points (if they exist) belonging to $S_m \cup \{\ell_1, \ell_2\}$. For each $x, y \in S$ with $x \le y$, we set

$$(3.9)\qquad \Phi(x, y) = \Phi(y, x) = \begin{cases} (x - x_1)(x_2 - y)/(x_2 - x_1), & -\infty < x_1 < x_2 < \infty, \\ x - x_1, & -\infty < x_1 < x_2 = \infty, \\ x_2 - y, & -\infty = x_1 < x_2 < \infty, \end{cases}$$

if $x, y \in \bar{I}_k \equiv [x_1, x_2]$ for some I_k, and $= 0$ otherwise.

LEMMA 1. *The equality* (2.3) *holds. Hence it follows that*

$$(3.10)\qquad G(\alpha, x, y) = \tilde{G}(\alpha, x, y) + \Phi(x, y), \quad \alpha > 0, \quad x, y \in S,$$

where

$$(3.11)\qquad \tilde{G}(\alpha, x, y) = \sum_{i,j=1}^{2} \int_{0-}^{\infty} (\alpha + \lambda)^{-1}\varphi_i(x, -\lambda)\varphi_j(y, -\lambda)\sigma_{ij}(d\lambda).$$

PROOF. Tracing the arguments in [17; Chapter 5], we see that the function $p(t, x, y)$ in (3.8) is no other than the elementary solution in [9], if $x, y \in S_m$. Hence (3.10) is valid for $x, y \in S_m$, and we have only to show it for $x \in S\backslash S_m$ or $y \in S\backslash S_m$.

Assume first that $-\infty < x_1 < x \le y < x_2 < \infty$ for some $I_k = (x_1, x_2)$. Noting that the functions $\varphi_i(x, -\lambda)$, $i = 1, 2$ are linear in $x \in I_k$ by the integral equations (3.1), we have

$$p(t, x, y) = [(x_2 - x)(x_2 - y)p(t, x_1, x_1) + (x - x_1)(y - x_1)p(t, x_2, x_2) + \{(x - x_1)(x_2 - y) + (x_2 - x)(y - x_1)\}p(t, x_1, x_2)]/(x_2 - x_1)^2.$$

Similarly,

$$G(\alpha, x, y) = \{(x_2 - x)(x_2 - y)G(\alpha, x_1, x_1) + (x - x_1)(y - x_1)G(\alpha, x_2, x_2) + (x - x_1)(x_2 - y)h(\alpha)u_1(x_2, \alpha)u_2(x_1, \alpha) + (x_2 - x)(y - x_1)G(\alpha, x_1, x_2)\}/(x_2 - x_1)^2.$$

Hence

$$G(\alpha, x, y) - \int_0^\infty e^{-\alpha t}p(t, x, y)\,dt = \Phi(x, y)h(\alpha)\,\frac{u_1(x_2, \alpha)u_2(x_1, \alpha) - u_1(x_1, \alpha)u_2(x_2, \alpha)}{x_2 - x_1}.$$

But $u_i(x_2, \alpha) = u_i(x_1, \alpha) + (x_2 - x_1)u_i^+(x_1, \alpha)$, $i = 1, 2$. Further, the Wronskian of $u_1(x, \alpha)$ and $u_2(x, \alpha)$ is constant:

$$u_1^+(x, \alpha)u_2(x, \alpha) - u_1(x, \alpha)u_2^+(x, \alpha) = h(\alpha)^{-1}, \quad \alpha > 0, \quad x \in S, \tag{3.12}$$

where in general $u^+(x) = \lim_{\varepsilon\downarrow 0}\{u(x + \varepsilon) - u(x)\}/\varepsilon$. Thus we obtain (2.3) for this case. Assume next that $-\infty < x_1 < x \le y < x_2 = \ell_2 = \infty$. Then we have $h_2(\alpha) = \varphi_2^+(x_1, \alpha)/\varphi_1^+(x_1, \alpha)$, and the expression (3.8) is reduced to

$$p(t, x, y) = \int_{0-}^\infty e^{-\lambda t}\varphi(x, -\lambda)\varphi(y, -\lambda)\sigma_{11}(d\lambda)$$

where $\varphi(x, -\lambda) = \varphi_1(x, -\lambda) - \varphi_2(x, -\lambda)\varphi_1^+(x_1, -\lambda)/\varphi_2^+(x_1, -\lambda)$. This implies $p(t, x, y) = p(t, x_1, x_1)$. Further we have $u_1(x, \alpha) = u_1(x_1, \alpha) + (x - x_1)u_1^+(x, \alpha)$ and $u_2(y, \alpha) = u_2(x_1, \alpha)$. Hence we have (3.10) by (3.12). The proof for the other cases is similar and will be omitted.

We note that Schwarz's and Jensen's inequalities together with (3.8) and (3.11) imply

$$p(t, x, y) \le p(t, x, x)^{1/2}p(t, y, y)^{1/2}, \quad t > 0, \quad x, y \in S, \tag{3.13}$$

$$p(t, x, x) \le t^{-1}\tilde{G}(t^{-1}, x, x), \quad t > 0, \quad x \in S. \tag{3.14}$$

For each $a \in S$, let $p_a(t, x, y)$ and $G_a(\alpha, x, y)$ be the elementary solution and the Green function of the generalized diffusion equation (1.1) restricted on the interval $S_a = (a, \ell_2)$ with the boundary condition (2.2) at a. Then, also tracing the arguments in [17; Chapter 5], we have the following spectral representations. Let

$$\varphi_{1,a}(x, \alpha) = \varphi_2^+(a, \alpha)\varphi_1(x, \alpha) - \varphi_1^+(a, \alpha)\varphi_2(x, \alpha),$$

$$\varphi_{2,a}(x, \alpha) = -\varphi_2(a, \alpha)\varphi_1(x, \alpha) + \varphi_1(a, \alpha)\varphi_2(x, \alpha),$$

$$h_{2,a}(\alpha) = \lim_{x\uparrow\ell_2}\varphi_{2,a}(x, \alpha)/\varphi_{1,a}(x, \alpha) = \{\varphi_1(a, \alpha)h_2(\alpha) - \varphi_2(a, \alpha)\}/\{-\varphi_1^+(a, \alpha)h_2(\alpha) + \varphi_2^+(a, \alpha)\}.$$

Also define the spectral measure $\sigma_a(d\lambda)$ by

$$\sigma_a([\lambda_1, \lambda_2]) = -\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} Im\left(\frac{1}{h_{2,a}(-\lambda - \sqrt{-1}\varepsilon)}\right) d\lambda \tag{3.15}$$

for all continuity points $\lambda_1 < \lambda_2$. We note that $\sigma_a(\{0\}) = 0$ since the limit $1/h_{2,a}(0+)$ is finite. Now we can check that in this case the equalities (3.7), (3.8) and (3.10) are reduced to

$$G_a(\alpha, x, y) = G_a(\alpha, y, x) = \varphi_{2,a}(x, \alpha)u_2(y, \alpha)/u_2(a, \alpha), \qquad \alpha > 0, \quad a < x \le y < \ell_2, \tag{3.16}$$

$$p_a(t, x, y) = \int_{0+}^{\infty} e^{-\lambda t}\varphi_{2,a}(x, -\lambda)\varphi_{2,a}(y, -\lambda)\sigma_a(d\lambda), \qquad t > 0, \quad x, y \in (a, \ell_2), \tag{3.17}$$

$$G_a(\alpha, x, y) = \int_{0+}^{\infty} (\alpha + \lambda)^{-1}\varphi_{2,a}(x, -\lambda)\varphi_{2,a}(y, -\lambda)\sigma_a(d\lambda) + \Phi_a(x, y), \qquad \alpha > 0, \quad x, y \in (a, \ell_2), \tag{3.18}$$

where $\Phi_a(x, y)$ is defined on S_a in a similar way as $\Phi(x, y)$ is defined on S.

Following [9], we define a nonnegative function $q_a(t, y)$ by

$$q_a(t, y) = \lim_{x\downarrow a} \partial p_a(t, x, y)/\partial x.$$

The correction function $\Psi_a(y)$ is defined as $= (x_2 - y)/(x_2 - a)$ if $a, y \in \bar{I}_k = [x_1, x_2]$, $a \neq x_2$, for some I_k with $x_2 < \infty$; $= 1$ if $a, y \in \bar{I}_k = [x_1, \infty)$; and $= 0$ for all the rest. Then we have by the same arguments as those in the proof of [9; Theorem 4.2] and those in the proof of Lemma 1,

$$\frac{\partial^n}{\partial t^n} q_a(t, y) = \int_{0+}^{\infty} (-\lambda)^n e^{-\lambda t}\varphi_{2,a}(y, -\lambda)\sigma_a(d\lambda), \qquad t > 0, \quad y \in (a, \ell_2), \quad n = 0, 1, 2, \cdots, \tag{3.19}$$

$$\frac{u_2(y, \alpha)}{u_2(a, \alpha)} = \int_0^{\infty} e^{-\alpha t} q_a(t, y)\, dt + \Psi_a(y), \quad \alpha > 0, \quad y \in (a, \ell_2). \tag{3.20}$$

Further from (2.3), (3.7) and (3.20) it follows that

$$\begin{aligned} &p(t, x, y) \\ &= \int_0^t p(t - s, x, a)q_a(s, y)\, ds + \Phi(x, a)q_a(t, y) + p(t, x, a)\Psi_a(y), \\ &\qquad t > 0, \quad \ell_1 < x \le a < y < \ell_2. \end{aligned} \tag{3.21}$$

Finally, let $m_a(x) = m(x + a) - m(a)$ and $m^{(c,a)}(x) = m_a(cx)/m_a(c)$, $\ell_1 - a < x < \ell_2 - a$, for each $c > 0$ with $m_a(c) > 0$. We denote the corresponding items for $m^{(c,a)}(x)$ in place of $m(x)$ by $\varphi_i^{(c,a)}(x, \alpha)$, $h_i^{(c,a)}(\alpha)$, $i = 1, 2$, $\sigma_0^{(c,a)}(d\lambda)$, $q_0^{(c,a)}(t, y)$, etc. Then it is well known that $\varphi_{1,a}(x, \alpha) = \varphi_1^{(c,a)}((x - a)/c, cm_a(c)\alpha)$, $\varphi_{2,a}(x, \alpha) = c\varphi_2^{(c,a)}((x - a)/c, cm_a(c)\alpha)$ and $h_{2,a}(\alpha) = ch_2^{(c,a)}(cm_a(c)\alpha)$ for $\alpha > 0$ ([5]), which with

(3.15) and (3.19) imply

$$\sigma_a([0, \lambda]) = \sigma_0^{(c,a)}([0, cm_a(c)\lambda])/c^2 m_a(c), \quad \lambda \geq 0, \tag{3.22}$$

$$q_a(t, y) = \frac{q_0^{(c,a)}(t/cm_a(c), (y-a)/c)}{cm_a(c)}, \quad t > 0, \quad y \in (a, \ell_2 - a). \tag{3.23}$$

4. Proof of the Proposition. The following simple proof of the next lemma is due to T. Shiga.

LEMMA 2. *The Green function for the generalized diffusion equation* (1.1) *satisfies*

$$G(\alpha, y, y) \leq G(\alpha, x, x) + |y - x|, \quad \alpha > 0, \quad x, y \in S. \tag{4.1}$$

PROOF. We give the proof only for $\ell_1 < x \leq y < \ell_2$. By (3.7) and (3.12)

$$G(\alpha, y, y)$$
$$= h(\alpha)u_2(y, \alpha)^2 \frac{u_1(y, \alpha)}{u_2(y, \alpha)} = h(\alpha)u_2(y, \alpha)^2 \left\{\frac{u_1(x, \alpha)}{u_2(x, \alpha)} + \int_x^y \frac{1}{h(\alpha)u_2(z, \alpha)^2}\, dz\right\}.$$

Since $u_2(x, \alpha)$ is nonincreasing in $x \in S$, this implies (4.1).

PROOF OF THE PROPOSITION. Note first that the formula (1.2) obviously follows from (1.3) by the well-known Abelian theorem.

Assume (1.2). Since $G(\alpha, 0, 0) = h(\alpha)$ for all $\alpha > 0$, it follows that $h(0+) = \infty$. Hence in view of (3.3) and (3.4) we have $h_1(0+) = h_2(0+) = \infty$ and $u_1(x, 0+) = u_2(x, 0+) = 1$ for $x \in S$. This with (3.7) implies

$$G(\alpha, x, y) \sim \alpha^{-\rho} L(1/\alpha) \quad \text{as} \quad \alpha \downarrow 0, \tag{4.2}$$

for every fixed $x, y \in S$. Due to the Hardy-Littlewood-Karamata theorem ([10; Theorem 2.3], e.g.), we have from (2.3) and (3.7) that

$$\int_0^t p(s, x, y)\, ds \sim \Gamma(\rho + 1)^{-1} t^{\rho} L(t) \quad \text{as} \quad t \to \infty. \tag{4.3}$$

In the case of $x = y$, the density function $p(t, x, x)$ is nonincreasing in t by means of the representation (3.8). Hence it follows that

$$p(t, x, x) \sim \Gamma(\rho)^{-1} t^{\rho - 1} L(t) \quad \text{as} \quad t \to \infty \tag{4.4}$$

([10; Theorem 2.4], e.g.). In the case of $x \neq y$, the density function $p(t, x, y)$ is not monotone in general, and we have to study it more closely. From the estimate (3.13) and the asymptotic formula (4.4), it follows that

$$\limsup_{t \to \infty} \Gamma(\rho) t^{1-\rho} L(t)^{-1} p(t, x, y) \leq 1. \tag{4.5}$$

In order to prove the reverse inequality

$$\liminf_{t \to \infty} \Gamma(\rho) t^{1-\rho} L(t)^{-1} p(t, x, y) \geq 1, \tag{4.6}$$

we note that (3.21) implies

$$p(t, x, y) \geq p(t, x, x)\left\{\int_0^t q_x(s, y)\, ds + \Psi_x(y)\right\}, \quad x < y, \quad x, y \in S. \tag{4.7}$$

Further, letting $\alpha \downarrow 0$ in (3.20), we have

$$\int_0^\infty q_x(s, y)\, ds + \Psi_x(y) = 1, \quad x < y, \quad x, y \in S. \tag{4.8}$$

Now (4.6) follows from (4.4), (4.7) and (4.8). The first assertion is proved.

For the proof of (2.4), note that Lemma 2 together with (3.13) and (3.14) implies

$$p(t, x, y) \leq t^{-1}h(t^{-1})(1 + |x|/h(t^{-1}))^{1/2}(1 + |y|/h(t^{-1}))^{1/2}, \quad t > 0, \quad x, y \in S.$$

This and (4.2) prove (2.4).

5. Proof of the Theorem. First we prepare some properties of the function $q_a(t, y)$.

LEMMA 3. 1) *The spectral measure* $\sigma_0(d\lambda)$ *defined by* (3.15) *satisfies*

$$\frac{1}{\alpha h_2(\alpha)} = \frac{1}{\alpha \ell_2} + \int_{0+}^\infty \frac{\sigma_0(d\lambda)}{\lambda(\alpha + \lambda)} = \int_{0+}^{\ell_2} \frac{dm(z)}{\varphi_2^+(x, \alpha)^2}, \quad \alpha > 0. \tag{5.1}$$

2) *For each* $a \in S$,

$$\int_0^t q_a(s, y)\, ds \leq \frac{te}{t + \int_a^y (m(y) - m(z))\, dz}, \quad t > 0, \quad y \in (a, \ell_2). \tag{5.2}$$

3) *If* $\ell_2 = \infty$ *and* $m(\infty) < \infty$, *then*

$$\int_0^\infty s q_a(s, y)\, ds = (y - a)(m(\infty) - m(y)) + \int_{a+}^{y+} (z - a)\, dm(z), \tag{5.3}$$
$$\ell_1 < a < y.$$

PROOF. 1) The assertions are implied from those in [4; Section 12]. We shall explain it. Let $m^{-1}(x)$ be the inverse function of $m(x)$ on $[0, \ell_2)$, i.e., $m^{-1}(x) = \sup\{y\colon m(y) \leq x\}$, $x \in [0, m(\ell_2-))$. This is called dual string to $m(x)$ and the corresponding characteristic function $h_*(\alpha)$ is given by $1/\alpha h_2(\alpha)$ (see [4; (12.5)] and [8; (1.10)]). By virtue of [4; (12.6)] and [4; (12.7)], the spectral measure $\sigma_*(d\lambda)$ corresponding to $h_*(\alpha)$ coincides with $\sigma_0(d\lambda)/\lambda$ for $\lambda > 0$ and $\sigma_*(\{0\}) = 1/\ell_2$. Further, the left end point of the support of $dm^{-1}(x)$ is equal to 0 and the corresponding α-harmonic function is given by $\varphi_2^+(m^{-1}(x), \alpha)$. Hence (5.1) follows from Krein's correspondence theory ([4; (12.4)]).

2) Let $y \in (a, \ell_2)$. Then

$$u_2(a, \alpha) = u_2(y, \alpha) + (a - y)u_2^+(y, \alpha) + \alpha \int_a^y dz \int_{(z,y]} u_2(\xi, \alpha)\, dm(\xi).$$

Since $u_2(x, \alpha)$ is nonincreasing in x, this implies

$$u_2(a, \alpha) \geq u_2(y, \alpha)\left\{1 + \alpha \int_a^y (m(y) - m(z))\, dz\right\}.$$

Hence we have by (3.20) that

$$\int_0^t q_a(s, y)\, ds \leq e \int_0^\infty e^{-s/t} q_a(s, y)\, ds \leq \frac{et}{t + \int_a^y (m(y) - m(z))\, dz},$$

proving (5.2).

3) We will give the proof only for $a = 0$ for simplicity. Note first that the condition $\ell_2 = \infty$ together with (3.4) and (3.20) guarantees (4.7). In view of

$$u_2(y, \alpha) = 1 + y u_2^+(0, \alpha) + \alpha \int_{0+}^{y+} (y - z) u_2(z, \alpha)\, dm(z),$$

we then have from (3.4) and (5.1) that

$$\begin{aligned} \alpha^{-1}(1 - u_2(y, \alpha)) &= \frac{y}{\alpha h_2(\alpha)} - \int_{0+}^{y+} (y - z) u_2(z, \alpha)\, dm(z) \\ &= y \int_{y+}^\infty \frac{dm(z)}{\varphi_2^+(z, \alpha)^2} + y \int_{0+}^{y+} \left\{\frac{1}{\varphi_2^+(z, \alpha)^2} - u_2(z, \alpha)\right\} dm(z) \\ &\quad + \int_{0+}^{y+} z u_2(z, \alpha)\, dm(z), \quad \alpha > 0, \quad y > 0. \end{aligned}$$

Letting $\alpha \downarrow 0$, we obtain (5.3) from (3.20).

For the proof of our key Lemma 5 in the following, we need one more preparation from calculus:

LEMMA 4. *Let $f \in C^2((0, 1))$ and*

$$|f(t)| \leq M_1 t^p, \quad |f''(t)| \leq M_2 t^q, \quad t \in (0, 1)$$

for some real constants p, q, M_1 and M_2, and set $r = \min\{p - 1, (p + q)/2\}$. Then

$$|f'(t)| \leq \{(2^{|p|} + 1)M_1 + 2^{|q|} M_2\} t^r, \quad t \in (0, 1/2).$$

LEMMA 5. *Let $\ell_1 < a < b < \ell_2 = \infty$, $m(b) - m(a) > 0$ and*

$$\inf_{c \geq b}\{c(m(c) - m(a))\}^{-1} \int_a^c (m(c) - m(x))\, dx > 0. \tag{5.4}$$

Then it follows that

$$\sup_{y \geq b, t > 0} t q_a(t, y) < \infty. \tag{5.5}$$

PROOF. Step 1. Without loss of generality, we may assume $a = 0$ and $m(b) > 0$. Then the function $[b, \infty) \ni y \mapsto y m(y)$ is strictly increasing and we denote its inverse function by $k_2(t)$. Further we denote by $m^{(c)}(x)$, $\varphi_2^{(c)}(x, \alpha)$, $\sigma_0^{(c)}(d\lambda)$ and $q_0^{(c)}(t, y)$ the corresponding items for this case. In view of (3.23), we have

$$\sup_{y\geq b, t>0} tq_0(t, y) \leq \sup_{y\geq b, 0<t<ym(y)/2}(t/ym(y))q_0^{(y)}(t/ym(y), 1)$$
$$+ \sup_{y\geq b, k_2(2t)\geq y} \tfrac{1}{2} q_0^{(k_2(2t))}(\tfrac{1}{2}, y/k_2(2t)).$$

Hence it is enough for (5.5) to show that

$$\sup_{c\geq b, 0<t<1/2} tq_0^{(c)}(t, 1) < \infty, \tag{5.6}$$

and

$$\sup_{c\geq b, 0<y\leq 1} q_0^{(c)}(\tfrac{1}{2}, y) < \infty. \tag{5.7}$$

Step 2. On this step we will show (5.6). For each $c \geq b$, set

$$f_c(t) = \int_0^t q_0^{(c)}(s, 1)\, ds, \quad 0 < t < 1.$$

Then by (5.2)

$$f_c(t) \leq te \Big/ \int_0^1 (m^{(c)}(1) - m^{(c)}(z))\, dz$$
$$= te \Big/ \{cm(c)\}^{-1} \int_0^c (m(c) - m(z))\, dz.$$

Hence (5.4) assures

$$0 \leq f_c(t) \leq M_1 t, \quad 0 < t < 1, \quad c \geq b \tag{5.8}$$

for some constant M_1. On the other hand, due to the representation (3.19) for $q_0^{(c)}(t, y)$, we have

$$f_c''(t) = \frac{\partial q_0^{(c)}(t, 1)}{\partial t} = -\int_{0+}^\infty \lambda e^{-\lambda t} \varphi_2^{(c)}(1, -\lambda)\sigma_0^{(c)}(d\lambda), \quad 0 < t < 1.$$

But there is an absolute constant M_3 such that

$$(\lambda t)^{3/2} t(\lambda + 1/t)e^{-\lambda t} \leq M_3, \quad \lambda \geq 0, \quad t > 0.$$

Hence it follows that

$$|f_c''(t)| \leq M_3 t^{-5/2} \int_{0+}^\infty \lambda^{-1/2}\left(\lambda + \frac{1}{t}\right)^{-1} |\varphi_2^{(c)}(1, -\lambda)|\, \sigma_0^{(c)}(d\lambda)$$

$$\leq M_3 t^{-5/2}\left(\int_{0+}^\infty \frac{(\varphi_2^{(c)}(1, -\lambda))^2}{\lambda + 1/t}\, \sigma_0^{(c)}(d\lambda)\right)^{1/2}\left(\int_{0+}^\infty \frac{\sigma_0^{(c)}(d\lambda)}{\lambda(\lambda + 1/t)}\right)^{1/2} \tag{5.9}$$

$$\leq M_3 t^{-2} G_0^{(c)}(1/t, 1, 1)^{1/2} h_2^{(c)}(1/t)^{-1/2}$$

by (3.18) and (5.1). By Lemma 2 applied to $G_0^{(c)}(t, x, y)$,

$$G_0^{(c)}(1/t, x, x) \leq G_0^{(c)}(1/t, 0+, 0+) + x = x, \quad x > 0. \tag{5.10}$$

From [15; (3.7)],

$$h_2^{(c)}(1/t)^{-1} \leq 3k_2^{(c)}(t)^{-1}, \quad t > 0,$$

where $k_2^{(c)}(t)$ is the inverse function of $xm^{(c)}(x)$. Noting that $xm^{(c)}(x) = xm(cx)/m(c) \leq x$ for $x \in (0, 1)$ and $= 1$ for $x = 1$, we have $k_2^{(c)}(t) \geq t$ for $t \in (0, 1)$. Hence it follows that

$$h_2^{(c)}(1/t)^{-1} \leq 3t^{-1}, \quad 0 < t \leq 1. \tag{5.11}$$

This with (5.9) and (5.10) yields

$$|f_c''(t)| \leq M_2 t^{-5/2}, \quad 0 < t < 1, \tag{5.12}$$

where $M_2 = 3^{1/2}M_3$. Further it is obvious that $f_c \in C^2((0, 1))$. Due to Lemma 4, the inequalities (5.8) and (5.12) prove (5.6).

Step 3. The proof of (5.7) is now easy. Indeed, as in (5.9), we have

$$0 \leq q_0^{(c)}(1/2, y) \leq M_4 G_0^{(c)}(1, y, y)^{1/2} h_2^{(c)}(1)^{-1/2}, \quad c \geq b$$

for some constant M_4. This with (5.10) and (5.11) verifies (5.7). The lemma is proved.

PROOF OF THE THEOREM. Suppose that the assumptions of the Theorem are fulfilled. Then $\ell_1 = -\infty$ and $\ell_2 = \infty$ as is noted in Remark 1. Without loss of generality, we may assume that $t^{\rho-1}L(t)$ is nonincreasing. By the exchange of the roles of $-\infty$ and ∞ if necessary, it is enough for (1.4) to show that

$$\sup_{t\geq 1, y\geq x} t^{1-\rho}L(t)^{-1}p(t, x, y) < \infty, \quad x \in S. \tag{5.13}$$

Let r_2 be the supremum of the support S_m of $dm(x)$. If $r_2 < \infty$, then as in the proof of Lemma 1 we have

$$p(t, x, y) = p(t, x \wedge r_2, r_2), \quad t > 0, \quad r_2 \leq y. \tag{5.14}$$

Hence (5.13) follows from (2.4) and (5.14). Now we assume $r_2 = \infty$. Then, for each fixed $x \in S$, we can choose two points a and b such that $x < a < b$, $m(a) - m(x) > 0$ and $m(b) - m(a) > 0$. Fixing such a and b, we have

$$M_5 = \sup_{t>0} p(t, x, a) < \infty, \quad \sup_{t>0} t^{-1}p(t, x, a) < \infty \tag{5.15}$$

(see [9]). Further, by (2.4) again, it is enough for (5.13) to verify

$$\sup_{t\geq 1, y\geq b} t^{1-\rho}L(t)^{-1}p(t, x, y) < \infty. \tag{5.16}$$

For the proof of (5.16), we note that $\Phi(x, a) = \Psi_a(y) = 0$ for $y \geq b$ and decompose the expression (3.21) as

$$\begin{aligned} p(t, x, y) &= \int_0^{t/2} p(t - s, x, a)q_a(s, y)\, ds + \int_{t/2}^{t} p(t - s, x, a)q_a(s, y)\, ds \\ &\equiv I_1(t, y) + I_2(t, y), \quad t \geq 1, \quad y \geq b. \end{aligned} \tag{5.17}$$

Due to (1.3)

$$I_1(t, y) \le M_6 \int_0^{t/2} (t - s)^{\rho-1} L(t - s) q_a(s, y)\, ds$$
$$\le 2M_6 t^{\rho-1} L(t) \int_0^{t/2} L(t)^{-1} L(t(1 - s/t)) q_a(s, y)\, ds$$

for some constant M_6. But $L(t\lambda)/L(t) \to 1$ as $t \to \infty$ uniformly in $\lambda \in [½, 1]$ ([10; Theorem 1.1]). Hence we have from(4.7) that

$$\sup_{t\ge 1, y\ge b} t^{1-\rho} L(t)^{-1} I_1(t, y) < \infty. \tag{5.18}$$

At this point we divide our arguments into two cases.

Case 1. Suppose that $J_2 < \infty$. Then we have $m(\infty) < \infty$ and also, owing to (5.3)

$$t \int_{t/2}^{t} q_a(s, y)\, ds \le 2 \int_0^{\infty} s q_a(s, y)\, ds$$
$$= 2\left\{(y - a)(m(\infty) - m(y)) + \int_{a+}^{y+} (z - a)\, dm(z)\right\} \le M_7, \quad t \ge 1, \quad y \ge b,$$

for some constant M_7. Hence it follows from (5.15) that

$$\sup_{t\ge 1, y\ge b} t^{1-\rho} L(t)^{-1} I_2(t, y) \le M_5 M_7 \sup_{t\ge 1} t^{-\rho} L(t)^{-1} < \infty. \tag{5.19}$$

Case 2. Suppose that $J_2 = \infty$. In case of $J_1 = \infty$, we may assume (5.4) by (2.5). When $J_1 < \infty$, it is well known that (1.2) is equivalent to (2.7) ([5] and [6]). Hence we may assume (5.4) again. Due to Lemma 5, we thus have (5.5) in both cases. Now fix an $\varepsilon \in (0, \rho)$. Then we can find a slowly varying function $L_\varepsilon(t)$ such that $t^{\rho-\varepsilon} L_\varepsilon(t)$ is nondecreasing, $L(t) \le L_\varepsilon(t)$ and $L(t) \sim L_\varepsilon(t)$ as $t \to \infty$ ([10; page 20]). It follows from (1.3), (5.5) and (5.15) that

$$I_2(t, y) \le M_8 \int_{t/2}^{t} (t - s)^{\rho-1} L(t - s) s^{-1}\, ds$$
$$\le M_8 \int_{t/2}^{t} (t - s)^{\rho-\varepsilon} L_\varepsilon(t - s)(t - s)^{\varepsilon-1} s^{-1}\, ds \tag{5.20}$$
$$\le M_8 t^{\rho-\varepsilon} L_\varepsilon(t) t^{\varepsilon-1} \int_{1/2}^{1} (1 - s)^{\varepsilon-1} s^{-1}\, ds$$
$$\le M_9 t^{\rho-1} L(t), \quad t \ge 1, \quad y \ge b,$$

for some constants M_8 and M_9.

The formulae (5.17)–(5.20) verify (5.16), completing the proof of (1.4). The second assertion is a direct consequence of (1.3), (1.4) and Lebesgue's dominated convergence theorem.

PROOF OF COROLLARY 1. Suppose first that the formula (1.2) holds with $\rho = 1$ and the slowly varying function $L(t)$ satisfying $0 < L(\infty) < \infty$. Then as in the proof of the Proposition we have $h_1(0+) = h_2(0+) = \infty$. But

$$\lim_{\alpha\to 0}\alpha \int_{0+}^{\infty} \frac{\sigma_0(d\lambda)}{\lambda(\alpha + \lambda)} = 0,$$

since

$$\int_{0+}^{\infty} \frac{\sigma_0(d\lambda)}{\lambda(1 + \lambda)} \leq \frac{1}{h_2(1)} < \infty.$$

Hence, due to (5.1), we have $\ell_2 = \infty$. Similarly, it holds that $\ell_1 = -\infty$. Further, by (5.1) and the similar relations for the characteristic function $h_1(\alpha)$, we have

$$\frac{1}{\alpha h(\alpha)} = \int_{(\ell_1,\ell_2)} \frac{dm(z)}{\varphi_2^+(z, \alpha)^2}. \tag{5.21}$$

Hence we obtain the formula $1/L(\infty) = m(\infty) - m(-\infty)$. The converse assertion also follows from (5.21).

We will show the second assertion. The implication of (1.3) from (1.2) is included in the Proposition. On the other hand, with the same choice of $x < a < b$ as in the proof of the Theorem, we have from (3.21) and (5.15) that

$$p(t, x, y) \leq M_5 \int_0^t q_a(s, y)\, ds, \quad t > 0, \quad y \geq b.$$

The rest of the arguments are the same as those in the proof of the Theorem.

6. Examples and application to long-tail phenomena.

EXAMPLE 1 (Brownian motion). Let $S = (-\infty, \infty)$ and $m(x) = 2x$. Then the equation (1.1) is reduced to the standard heat equation

$$\partial u(t, x)/\partial t = \partial^2 u(t, x)/2\partial x^2, \quad t > 0, \quad x \in \mathbb{R}. \tag{6.1}$$

In this case, it is known that

$$\begin{aligned}
\varphi_1(x, \alpha) &= \{\exp(\sqrt{2\alpha}x) + \exp(-\sqrt{2\alpha}x)\}/2\\
\varphi_2(x, \alpha) &= \{\exp(\sqrt{2\alpha}x) - \exp(-\sqrt{2\alpha}x)\}/2\sqrt{2\alpha},\\
h_1(\alpha) &= h_2(\alpha) = 1/\sqrt{2\alpha},\\
\sigma_{11}(d\lambda)/d\lambda &= \begin{cases} 1/2\pi\sqrt{2\lambda}, & \lambda > 0,\\ 0, & \lambda \leq 0,\end{cases}\\
\sigma_{22}(d\lambda)/d\lambda &= \begin{cases} \sqrt{2\lambda}/2\pi, & \lambda > 0,\\ 0, & \lambda \leq 0,\end{cases}\\
\sigma_{12}(d\lambda) &= \sigma_{21}(d\lambda) = 0,
\end{aligned}$$

(see [17; Chapter 5]). Hence, by (3.8), we have as usual

$$p(t, x, y) = (1/2\sqrt{2\pi t})e^{-(x-y)^2/2t} \sim 1/2\sqrt{2\pi t} \quad \text{as} \quad t \to \infty,$$

and

$$\sup_{x,y\in S} 2\sqrt{2\pi t}\, p(t, x, y) \leq 1.$$

Thus we have (1.3) and (1.4) with $\rho = ½$ and $L(t) \equiv 1/2\sqrt{2}$.

EXAMPLE 2 (Brownian motion with reflecting boundary). Let $S = (-\infty, \infty)$ and

$$m(x) = \begin{cases} 2x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then we have (6.1) for $x > 0$ also, and due to (2.2), the domain $D(\mathfrak{G})$ of the operator $\mathfrak{G}$ consists of the functions $u(x)$ such that

$$u(x) = \begin{cases} a + 2\displaystyle\int_0^x (x - y)g(y)\, dy, & x > 0, \\ a, & x \leq 0, \end{cases}$$

for some constant a and $g \in C((0, \infty))$. In this case

$$\varphi_1(x, \alpha) = \begin{cases} \{\exp(\sqrt{2\alpha}x) + \exp(-\sqrt{2\alpha}x)\}/2, & x \geq 0, \\ 1, & x < 0, \end{cases}$$

$$\varphi_2(x, \alpha) = \begin{cases} \{\exp(\sqrt{2\alpha}x) - \exp(-\sqrt{2\alpha}x)\}/2\sqrt{2\alpha}, & x \geq 0, \\ x, & x > 0, \end{cases}$$

$$h_1(\alpha) = \infty, \quad h_2(\alpha) = 1/\sqrt{2\alpha},$$

$$\frac{\sigma_{11}(d\lambda)}{d\lambda} = \begin{cases} 1/\pi\sqrt{2\lambda}, & \lambda > 0, \\ 0, & \lambda \leq 0, \end{cases}$$

$$\sigma_{22}(d\lambda) = \sigma_{12}(d\lambda) = \sigma_{21}(d\lambda) = 0.$$

Hence we have

$$\begin{aligned} p(t, x, y) &= p(t, y, x) \\ &= \begin{cases} (1/2\sqrt{2\pi t})\{e^{-(x-y)^2/2t} + e^{-(x+y)^2/2t}\}, & x, y > 0, \\ (1/\sqrt{2\pi t})e^{-y^2/2t}, & x \leq 0 < y, \\ 1/\sqrt{2\pi t}, & x, y \leq 0. \end{cases} \end{aligned}$$

Thus we have

$$p(t, x, y) \sim 1/\sqrt{2\pi t} \quad \text{as} \quad t \to \infty,$$

and

$$\sup_{x,y\in S} \sqrt{2\pi t}\, p(t, x, y) \leq 1$$

again.

Now we illustrate an application to long-time phenomena, which is one of our main objects.

Let $a \in C^1((0, \infty))$, $b \in C((0, \infty))$ with $a(x) > 0$ and $(W(t), P)$ be the standard

Brownian motion. Let $x(t)$ be the solution of the stochastic differential equation

$$dx(t) = a(x(t)) \circ dW(t) + b(x(t))dt, \quad t > 0$$

with the initial condition $x(0) = x \in (0, \infty)$, where the stochastic integral is taken in Stratonovich's sense. Then the stochastic process $(x(t), P)$ $(x(0) = x \in (0, \infty))$ is a diffusion process with the generator

$$L = a(x)^2 d^2/2dx^2 + (a(x)a'(x)/2 + b(x))d/dx.$$

Hence the scale function $s(x)$ and the speed measure $dm(y)$ with the natural scale $y = s(x)$ can be taken as

$$y = s(x) = \int_1^x a(z)^{-1} e^{-B(z)}\, dz, \quad x \in (0, \infty),$$

$$m(y) = 2 \int_1^{s^{-1}(y)} a(z)^{-1} e^{B(z)}\, dz, \quad y \in (s(0), s(\infty)),$$

where $B(z) = \int_1^z \{2b(\xi)/a(\xi)^2\}\, d\xi$. Thus if we assume

$$\int_0^1 \left| \frac{b(x)}{a(x)^2} \right| dx < \infty, \quad \int_0^1 a(x)^{-1}\, dx = \infty, \tag{6.2}$$

$$\int_1^\infty a(x)^{-1} e^{B(x)} \int_1^x a(z)^{-1} e^{-B(z)}\, dz\, dx < \infty, \tag{6.3}$$

and impose the reflecting boundary condition at r_2 in case of

$$r_2 \equiv \int_1^\infty a(x)^{-1} e^{-B(x)}\, dx < \infty, \tag{6.4}$$

then we have $\ell_1 = s(0+) = -\infty$, $\ell_2 = \infty$, $\int_0^\infty y\, dm(y) < \infty$ and

$$m(y) \sim 2e^{2B(0)} y \quad \text{as} \quad y \to -\infty.$$

In this case, the feature of the elementary solution is basically the same as that in Example 2 (the orient of S is reversed). Indeed, it follows from Corollary 2 that

$$G(\alpha, 0, 0) \sim 2^{-1/2} e^{-B(0)} \alpha^{-1/2} \quad \text{as} \quad \alpha \downarrow 0, \tag{6.5}$$

$$p(t, x, y) \sim (2\pi t)^{-1/2} e^{-B(0)} \quad \text{as} \quad t \to \infty. \tag{6.6}$$

But

$$E[f(x(t))] = \int_0^\infty p(t, s(x), s(z)) f(z)\, dm(s(z)),$$

where $E[f(x(t))]$ stands for the expectation of $f(x(t))$ with respect to the probability measure P. Hence we have

$$E[f(x(t))] \sim (2\pi t)^{-1/2} 2 \int_0^\infty a(x)^{-1} f(x) e^{B(x) - B(0)}\, dx \quad \text{as} \quad t \to \infty \tag{6.7}$$

for all f such that the integral in the right-hand side converges absolutely.

If we assume instead of (6.2) and (6.3) that

$$M \equiv 2 \int_0^\infty a(x)^{-1} e^{B(x)}\, dx < \infty \tag{6.8}$$

and impose the reflecting boundary condition at r_1 $[r_2]$ in case of

$$r_1 \equiv -\int_0^1 a(x)^{-1} e^{-B(x)}\, dx > -\infty \tag{6.9}$$

[resp. (6.4)], then we have $\ell_1 = -\infty$, $\ell_2 = \infty$ and

$$m(\infty) - m(-\infty) = M.$$

Hence it follows from Corollary 1 that

$$G(\alpha, 0, 0) \sim 1/\alpha M \quad \text{as} \quad \alpha \downarrow 0, \tag{6.10}$$

$$p(t, x, y) \sim 1/M \quad \text{as} \quad t \to \infty, \tag{6.11}$$

and

$$E[f(x(t))] \sim M^{-1} 2 \int_0^\infty a(x)^{-1} f(x) e^{B(x)}\, dx \quad \text{as} \quad t \to \infty \tag{6.12}$$

for all f such that the integral in the right-hand side converges absolutely.

Following Suzuki, Kaneko and Takesue [14], let

$$a(x) \sim x^n, \quad b(x) \sim \gamma x - c x^m \quad \text{as} \quad x \downarrow 0 \quad \text{and} \quad x \to \infty, \tag{6.13}$$

where n, m, γ and c are real constants such that $n \geq 1$, $m > 1$, $\gamma \geq 0$ and $c > 0$. Suppose first that $\gamma = 0$ and $(m + 1)/2 > n \geq 1$. Then it follows that (6.2), (6.3) and $r_2 = \infty$ hold. Hence we have the formulae (6.5)–(6.7), which coincide with those in [12], [13] and [14]. Suppose next that $\gamma > 0$. Then (6.8) follows, and we have the formulae (6.10)–(6.12).

Finally we note that if $\gamma = 0$, $(m + 1)/2 = n > 2c + 1$ and

$$\int_x^1 \frac{b(y)}{a(y)^2}\, dy = -c \log 1/x + o(1) \quad \text{as} \quad x \downarrow 0, \tag{6.14}$$

then the exponent ρ in (1.5) differs from ½. Indeed, in this case, it follows that $r_2 = s(\infty) < \infty$, $m(\infty) < \infty$, $s(0) = -\infty$ and

$$m(y) \sim -2(n + 2c - 1)^{-1}\{(n - 2c - 1)(-y)\}^{1/\rho - 1} \quad \text{as} \quad y \to -\infty, \tag{6.15}$$

where $\rho = (n - 2c - 1)/2(n - 1)$. Hence if we impose the reflecting boundary condition at r_2, then we have (1.5) with

$$L(t) \equiv 2^\rho (n - 1)^{2\rho} \Gamma(1 + \rho)/(n - 2c - 1)\Gamma(1 - \rho)$$

for all $f \in L_1(dm)$. Unfortunately this says nothing about the formula [14; (3.24)] because no functions of the form $f(x) = x^p$ belong to $L_1(dm(s(x)))$ in this case.

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