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Asymptotic behavior of flows in networks

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Abstract. Using functional analytical and graph theoretical methods, we extend the results of [12] to more general transport processes in networks allowing space dependent velocities and absorption. We characterize asymptotic periodicity and convergence to an equilibrium by conditions on the underlying directed graph and the (average) velocities.

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1 Introduction

The study of networks is becoming an important and popular field of mathematical research with many recent applications ranging from classical sciences to the internet. However, dynamical processes in such networks have not been studied widely (see the remark on p. 224 of [19]). Mostly second order problems are handled in the literature. We refer to the monographs of Lagnese, Leugering et al. [13] and [14] that treat second order equations also on more complicated multi-link-structures. Dáger and Zuazua in [8] and [9] investigate controllability and stabilizability of wave equations on graphs. F. Ali Mehmeti, J. von Below et al. ([1], [2], [3], [4], [5]) used functional analytic methods to treat diffusion and wave propagation in networks (and on more general structures). Recently, M. Kramar and E. Sikolya [12] have developed a semigroup approach to certain transport processes in networks. They described the asymptotic behavior of these processes under conditions on the flow velocities on the edges of the underlying directed graph.

Our aim is—based on the paper [12]—to handle more general transport processes allowing space dependent velocities and absorption in networks. We model the problem by a system of partial differential equations on a directed graph where the vertices serve as linking points between the edges. We show that the flow is (up to

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rescaling) asymptotically periodic if the average flow velocities are linear dependent over $\mathbb{Q}$ (see condition $(\mathrm{LD})_{\mathbb{Q}}$ in Definition 4.8), while it converges strongly to an equilibrium in case of linear independence.

Furthermore, we extend the result to the situation when—instead of functions—we consider measures on the edges.

Generalizing the setting in [12], we describe a flow in a finite network by the equations

$$
\text{(F)} \quad \begin{cases} \frac{\partial}{\partial t} u_j(s,t) = c_j(s) \frac{\partial}{\partial s} u_j(s,t) + q_j(s) \cdot u(s,t), & s \in (0,1),\ t \geq 0, \\ u_j(s,0) = f_j(s), \quad s \in (0,1), & \text{(IC)} \\ \phi_{ij}^- u_j(1,t) = \omega_{ij} \sum_{k=1}^{m} \phi_{ik}^+ u_k(0,t), \quad t \geq 0, & \text{(BC)} \end{cases}
$$

for $i = 1, \ldots, n$, and $j = 1, \ldots, m$.

For the model of the network we use the terminology as in [12, Section 1] and associate it to a directed, topological graph $G = (V, E)$ having vertices $V = \{v_1, \ldots, v_n\}$ and directed edges (or *arcs*) $E = \{e_1, \ldots, e_m\}$, normalized as $e_j = [0,1]$. The arcs are parameterized contrary to the direction of the flow. For graph theoretical notions see also [6].

The distribution of material along an edge e_j at time $t \geq 0$ is described by the functions $u_j(s,t)$ for $s \in [0,1]$ (see [12, Section 1]). The functions $c_j(\cdot)$ are the space dependent velocities of the flow on each arc e_j, while the functions $q_j(\cdot)$ describe the absorption along the edges. We arrange them into the diagonal matrices

$$
(1) \qquad C(s) := \begin{pmatrix} c_1(s) & & 0 \\ & \ddots & \\ 0 & & c_m(s) \end{pmatrix}, \quad Q(s) := \begin{pmatrix} q_1(s) & & 0 \\ & \ddots & \\ 0 & & q_m(s) \end{pmatrix}.
$$

We assume that the absorption functions q_j and velocities c_j are bounded, that is belong to $L^\infty[0,1]$, and in addition that $c_j(s) \geq \varepsilon > 0$ for a.e. $s \in [0,1]$ and for every $j = 1, \ldots, m$.

The boundary conditions (BC) are described by the following matrices, again as used in [12, Section 1]. First, we define the *outgoing incidence matrix* $\Phi^- = (\phi_{ij}^-)_{n \times m}$ with

$$
\phi_{ij}^- := \begin{cases} 1, & v_i = e_j(1), \\ 0, & \text{otherwise.} \end{cases}
$$

Respectively, we define the *incoming incidence matrix* $\Phi^+ = (\phi_{ij}^+)_{n \times m}$ with

$$\phi_{ij}^{+} := \begin{cases} 1, & v_i = e_j(0), \\ 0, & \text{otherwise.} \end{cases}$$

We also define the *weighted outgoing incidence matrix* $\Phi_w^- = (\omega_{ij})_{n\times m}$ with entries $0 \le \omega_{ij} \le 1$ expressing the proportion of the mass leaving the vertex v_i into the edge e_j. We require that

$$\text{(2)} \qquad \omega_{ij} = \phi_{ij}^{-}\omega_{ij} \quad \text{and} \quad \sum_{j=1}^{m} \omega_{ij} = 1 \quad \text{for all } i = 1, \ldots, n,\ j = 1, \ldots, m,$$

and we assume that if e_j is an outgoing edge of v_i, then $\omega_{ij} \neq 0$. Again, as in [12], the *Kirchhoff law* is satisfied in every vertex. Indeed, the boundary conditions (BC) together with (2) implies

$$\text{(3)} \qquad \sum_{j=1}^{m} \phi_{ij}^{-} u_j(1,t) = \sum_{j=1}^{m} \phi_{ij}^{+} u_j(0,t), \quad i = 1, \ldots, n.$$

Hence in each vertex the total outgoing flow equals to the total incoming flow.

To treat our problem (F) we use methods as in [12]. First, we rewrite (F) in the form of an abstract Cauchy problem and prove its well-posedness using semigroup methods (see also [10]). We then investigate the spectral properties of the generator of the solution semigroup. Also, we extend this treatment to transport involving measures. Finally, in Section 4, we give an accurate characterization for the asymptotic behavior of the solutions.

2 Well-posedness of the problem

Our aim is to study the asymptotic behavior of the solutions in the *state spaces* $\mathfrak{L} := ((L^1[0,1])^m, \|.\|_{\mathfrak{L}})$ and $\mathfrak{M} := ((\mathscr{M}[0,1])^m, \|.\|_{\mathfrak{M}})$, where $\|.\|_{\mathfrak{L}}$ is the m-fold product of the usual L^1 norm defined with respect to Lebesgue measure, while $\|.\|_{\mathfrak{M}}$ is the m-fold product of the total variation norm on the Banach space of (finite) Borel measures $\mathscr{M}[0,1]$. Note that $\mathfrak{L}$ can be regarded as a subspace of $\mathfrak{M}$ by identifying the absolutely continuous measures with their Radon-Nikodym derivative. In the sequel we use this convention without further mention.

For both state spaces we obtain the appropriate semigroups by abstract methods similarly to [12, Section 2]. The analysis of the resulting operators will then be carried out in the next section.

Consider first $\mathfrak{L}$. Denoting by M_{q_j} the multiplication operator with the function q_j, we define the operator

$$(4)\qquad A_w := \begin{pmatrix} c_1(s)\dfrac{d}{ds} + M_{q_1} & & 0 \\ & \ddots & \\ 0 & & c_m(s)\dfrac{d}{ds} + M_{q_m} \end{pmatrix}$$

with (dense) domain

$$(5)\qquad D(A_w) := \{g = (g_1, \dots, g_m) \in (W^{1,1}[0,1])^m \mid g(1) \in \operatorname{ran}(\Phi_w^-)^\top\}.$$

We call

$$\partial\mathfrak{X} := \mathbb{C}^n$$

the *boundary space* and introduce the *outgoing boundary operator* $L : \mathfrak{X} \to \partial\mathfrak{X}$,

$$L := \Phi^- \otimes \delta_1, \quad D(L) := (W^{1,1}[0,1])^m,$$

where δ_1 is the point evaluation at 1, and the *incoming boundary operator* $M : \mathfrak{X} \to \partial\mathfrak{X}$,

$$(6)\qquad M := \Phi^+ \otimes \delta_0, \quad D(M) := (W^{1,1}[0,1])^m,$$

where δ_0 is the point evaluation at 0. Similarly as in [12, Definition 2.2], the operator corresponding to the problem (F) is the following.

Definition 2.1. On the Banach space $\mathfrak{X}$ we define the operator

$$(7)\qquad D(A) := \{g \in D(A_w) \mid Lg = Mg\},$$

$$Ag := A_w g.$$

A simple calculation shows that the conditions in the domain of A are equivalent to (BC), hence the Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = u_0 \end{cases}$$

with $u_0 = (f_j)_{j=1,\dots,m}$ is an abstract version of our original problem. In the following, if we write $u(t)$, we understand it as $u(t) = (u_1(t), \dots, u_m(t)) = (u_1(\cdot\,, t), \dots, u_m(\cdot\,, t)) \in L^1([0,1], \mathbb{R}^m) \cong (L^1[0,1])^m$. By standard semigroup theory (see [10, Theorem II.6.7]) this problem is well-posed if and only if A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathfrak{X}$. In this case, the solutions of (2) have the form $u(t) = T(t)u_0$ yielding solutions for (F) too. To show the generator property we will use the

Phillips theorem as in [12, Lemma 2.4]. We recall from [17, Section C-II.1] the definition of dispersive operators on Banach lattices.

Definition 2.2. An operator A on a Banach lattice X is called *dispersive* if for every $g \in D(A)$ one has $\operatorname{Re}\langle Ag, \phi\rangle \leq 0$ for some $\phi \in X'_+$ such that $\|\phi\| \leq 1$ and $\langle g, \phi\rangle = \|g^+\|$.

Based on this property, we can show that the operator A generates a semigroup of *positive* operators on the Banach lattice $\mathfrak{L}$.

Proposition 2.3. *The operator* $(A, D(A))$ *generates a positive strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on* $\mathfrak{L}$.

Proof. Our operator A can be written as the sum

$$A = \begin{pmatrix} c_1(s)\dfrac{\partial}{\partial s} & & 0 \\ & \ddots & \\ 0 & & c_m(s)\dfrac{\partial}{\partial s} \end{pmatrix} + \begin{pmatrix} M_{q_1} & & 0 \\ & \ddots & \\ 0 & & M_{q_m} \end{pmatrix} = A_c + A_q.$$

First we show that A_c generates a positive C_0-semigroup $(T(t))_{t\geq 0}$. To this purpose we introduce a new, but equivalent lattice norm on $\mathfrak{L}$ defined as

$$\|g\|_c := \sum_{j=1}^{m} \int_0^1 \frac{|g_j(s)|}{c_j(s)}\, ds. \tag{8}$$

Using the same computation as in the proof of [12, Lemma 2.4] we can show that the operator $(A_c, D(A))$ is dispersive on the Banach lattice $((L^1[0,1])^m, \|\cdot\|_c)$. Since $(A_c, D(A))$ is also closed, densely defined and as we will see in Corollary 3.6, its resolvent set is not empty, we can use the Phillips Theorem from [17, Theorem C-II.1.2]. From this we obtain that $(A_c, D(A))$ generates a positive semigroup $(U(t))_{t\geq 0}$ on $(\mathfrak{L}, \|\cdot\|_c)$, hence on $(\mathfrak{L}, \|\cdot\|_{\mathfrak{L}})$.

By the assumptions on q_j, A_q is a bounded real multiplication operator on $\mathfrak{L}$, hence it generates a positive multiplication semigroup $(S(t))_{t\geq 0}$ with $\|S(t)\| \leq e^{\omega t}$ for some $\omega > 0$. To the positive semigroups $(U(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ we can apply the *Trotter product formula* (see [10, Corollary III.5.8] obtaining

$$T(t)x = \lim_{n\to\infty} [U(t/n)S(t/n)]^n x, \quad x \in X.$$

This formula clearly defines again a positive semigroup $(T(t))_{t\geq 0}$. □

This yields to the following result.

Corollary 2.4. *The problem* (F) *is well-posed.*

Consider now our second state space $\mathfrak{M}$. We show that it is possible to extend the semigroup $(T(t))_{t\geq 0}$ on $\mathfrak{M}$ using the density of $\mathfrak{L}$ with respect to the weak*-topology on $(\mathscr{M}[0,1])^m$, that is in the space

$$\mathfrak{M}_{w^\star} := ((\mathscr{M}[0,1])^m, w^\star) = (C[0,1]^m, \|.\|_\infty)'.$$

We denote the $w^\star$ convergence by $\rightarrow_{w^\star}$.

Proposition 2.5. *There is an extension of $(T(t))_{t\geq 0}$ from $\mathfrak{L}$ to a $w^\star$-continuous semigroup onto the state space $\mathfrak{M}_{w^\star}$, that is for every $t > 0$, $T(t)f_k \rightarrow_{w^\star} T(t)\mu$ if $f_k \rightarrow_{w^\star} \mu$ for a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathfrak{L}$ and $\mu \in \mathfrak{M}$.*

Proof. It is enough to show that $T(t) : \mathfrak{L} \rightarrow \mathfrak{L}$ is continuous with respect to the weak*-topology on $\mathfrak{L}$ since then $T(t)\mu$ can be defined as the weak*-limit of $T(t)f_k$ where $f_k \rightarrow_{w^\star} \mu \in \mathfrak{M}$, $\|f_k\|_{\mathfrak{L}} \leq 2\|\mu\|_{\mathfrak{M}}$, as follows. Since $T(t)$ is norm bounded on $\mathfrak{L}$, the set $\{T(t)f_k : k \in \mathbb{N}\}$ is bounded, hence it has a $w^\star$-accumulation point (see [23, Theorem IV.11.2]). By weak*-continuity on $\mathfrak{L}$, this accumulation point is unique; let $T(t)\mu$ be defined as this point. Then this extension will clearly form a semigroup.

Observe that because of the metrizability of norm bounded sets in $(\mathfrak{L}, w^\star)$ it is enough to show that $(T(t))_{t\geq 0}$ is sequentially continuous. So take f, $(f_k)_{k\in\mathbb{N}} \subset \mathfrak{L}$ such that $f_k \rightarrow_{w^\star} f$. Consider the adjoint semigroup $(T'(t))_{t\geq 0}$ on $\mathfrak{L}' = L^\infty([0,1], \mathbb{C}^m)$ (see e.g. [10, Section I.5.14]). By definition, for every $\varphi \in C[0,1]^m$ we have that

$$\langle T(t)f_k - T(t)f, \varphi\rangle = \langle f_k - f, T'(t)\varphi\rangle \rightarrow 0$$

if $k \rightarrow \infty$. This shows that $T(t)f_k \rightarrow_{w^\star} T(t)f$ and proves the $w^\star$-continuity of $T(t)$ on $\mathfrak{L}$. □

3 Spectral properties

To analyse the spectrum of A, we again use methods as in [12, Section 3] and introduce the operator

$$A_0 := A_w|_{\ker L}, \quad D(A_0) = \{g \in D(A_w) : Lg = 0\}.$$

The domain of A_0 can be rewritten as

$$D(A_0) = \{g \in (W^{1,1}[0,1])^m : g(1) = 0\}.$$

The corresponding Cauchy problem

$$\text{(9)} \qquad \begin{cases} \dot{u}(t) = A_0 u(t), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

is well-posed since $(A_0, D(A_0))$ generates a nilpotent semigroup $(T_0(t))_{t\geq 0}$ on $\mathfrak{L}$. To write explicitly this semigroup, we introduce two notations.

Definition 3.1. Take $j = 1, \ldots, m$ and $s_1, s_2 \in [0,1]$. If the edge e_j points from v_i towards v_p, we set

$$\text{(10)} \qquad \tau_j(s_1, s_2) := \tau_{i,p}(s_1, s_2) := \int_{s_1}^{s_2} \frac{ds}{c_j(s)}$$

and

$$\text{(11)} \qquad \xi_j(s_1, s_2) := \xi_{i,p}(s_1, s_2) := \int_{s_1}^{s_2} \frac{q_j(s)}{c_j(s)} ds.$$

We denote $\tau_+ := \max_{1\leq j\leq m} \tau_j(0,1)$ and $\tau_- := \min_{1\leq j\leq m} \tau_j(0,1)$.*

The value $\tau_j(s_1, s_2)$ is exactly the time needed to pass on the edge e_j from s_1 to s_2 moving with speed $c_j(s)$ at every point $s \in [s_1, s_2]$, while $\xi_j(s_1, s_2)$ is the rate of the mass gain or lost on this journey resulting from the factor $q_j(s)$. Note that our assumptions on the flow speed and the absorption functions imply that the integrals in (10) and (11) are finite. With these notations the resolvent of A_0, which exists for every $\lambda \in \mathbb{C}$, can be computed explicitly.

Lemma 3.2. *For every $\lambda \in \mathbb{C}$ and with the matrices $C(s)$ and $Q(s)$ defined in* (1), *we have*

$$\text{(12)} \qquad (R(\lambda, A_0)f)(s) = \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1} C(\tau)^{-1} f(\tau)\, d\tau, \quad s \in [0,1],\ f \in \mathfrak{L},$$

where

$$\text{(13)} \qquad \epsilon_\lambda(s) := \operatorname{diag}(e^{-\xi_i(0,s)+\lambda\tau_i(0,s)})_{i=1,\ldots,m}, \quad s \in [0,1].$$

Proof. An easy calculation shows that

$$\text{(14)} \qquad \epsilon'_\lambda(s) = \epsilon_\lambda(s)(-Q(s) + \lambda)C(s)^{-1}.$$

* Throughout this paper, an edge parameter with one index number corresponds to the edge with the same index, while pairs of index numbers refer to the edge pointing from the vertex with the first index towards the vertex with the second index.

Clearly, the function

$$g(s) := \int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau, \quad s \in [0,1]$$

is contained in $D(A_0)$. By applying $\lambda - A_0$ to it and using (14) we obtain

$$\begin{aligned}
((\lambda - A_0)g)(s) &= \lambda g(s) - C(s)g'(s) - Q(s)g(s) \\
&= \lambda g(s) - C(s)\epsilon_\lambda'(s)\int_s^1 \epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&\quad + C(s)\epsilon_\lambda(s)\epsilon_\lambda(s)^{-1}C(s)^{-1}f(s) - Q(s)g(s) \\
&= \lambda\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&\quad - C(s)\epsilon_\lambda(s)(-Q(s)+\lambda)C(s)^{-1}\int_s^1 \epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&\quad + f(s) - Q(s)\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&= \lambda\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&\quad + Q(s)\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&\quad - \lambda\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau \\
&\quad + f(s) - Q(s)\int_s^1 \epsilon_\lambda(s)\epsilon_\lambda(\tau)^{-1}C(\tau)^{-1}f(\tau)\,d\tau = f(s),
\end{aligned}$$

using the fact that the diagonal matrices commute. A similar argument, using the above form for the derivative of ϵ_λ, yields that formula (12) gives also the left inverse of $\lambda - A_0$. □

We also obtain a formula for the semigroup generated by A_0.

Lemma 3.3. *Let the edge e_j be fixed, and, with the notations of Definition 3.1, let $\tilde{s}(t) \in [0,1]$ be the location where the flow moves to on the edge e_j from the point s* *during time* $t \le \tau_j(s,1)$. *Hence the function* $\tilde{s} \in C[0, \tau_j(s,1)]$ *is defined by* $\tau_j(s, \tilde{s}(t)) = t$. *Then the jth coordinate of the semigroup* $(T_0(t))_{t\ge 0}$ *generated by* $(A_0, D(A_0))$ *is given by*

$$(15) \qquad (T_0(t)f)_j(s) = \begin{cases} e^{\xi_j(s,\tilde{s}(t))} f_j(\tilde{s}(t)), & \textit{if } 0 \le t \le \tau_j(s,1), \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. If we write

$$(16) \qquad (S_0(t)f)_j(s) = \begin{cases} e^{\xi_j(s,\tilde{s}(t))} f_j(\tilde{s}(t)), & \text{if } 0 \le t \le \tau_j(s,1), \\ 0, & \text{otherwise}, \end{cases}$$

we have to prove that $S_0(t) = T_0(t)$ for every $t \ge 0$. Observe that ξ_j is continuous with $\xi_j(s,s) = 0$ for every $s \in [0,1]$. Furthermore, because of the continuity of τ_j,

$$(17) \qquad \tilde{s}(0) = s \quad \text{and} \quad \lim_{t\to 0} \tilde{s}(t) = s.$$

From these properties follows that $S_0(0)f = f$ and $(S_0(t))_{t\ge 0}$ is strongly continuous in $t = 0$.

Since

$$\tau_j(s, \widetilde{\tilde{s}(u)}(t)) = \tau_j(s, \tilde{s}(u)) + \tau_j(\tilde{s}(u), \widetilde{\tilde{s}(u)}(t)) = u + t = \tau_j(s, \tilde{s}(u+t)),$$

we have

$$\widetilde{\tilde{s}(u)}(t) = \tilde{s}(u+t),$$

and so

$$(S_0(u+t)f)_j(s) = e^{\xi_j(s,\tilde{s}(u+t))} f_j(\tilde{s}(u+t))$$

$$= e^{\xi_j(s,\tilde{s}(u))} e^{\xi_j(\tilde{s}(u), \widetilde{\tilde{s}(u)}(t))} f_j(\widetilde{\tilde{s}(u)}(t)) = (S_0(u)(S_0(t)f))_j(s),$$

hence $(S_0(t))_{t\ge 0}$ is a C_0-semigroup. Let B_0 denote its generator; we have to show that $B_0 = A_0$.

We know that if $f \in D(B_0)$, then

$$(18) \qquad \lim_{t\to 0} \frac{S_0(t)f - f}{t} = B_0 f$$

in $\mathfrak{L}$. Therefore there exists a sequence $t_n \to 0$ such that

$$\lim_{n\to\infty} \frac{(S_0(t_n)f)(s) - f(s)}{t_n} = (B_0 f)(s) \quad \text{for almost all } s \in [0,1].$$

For all $s \in [0,1]$, we can calculate the pointwise limit as follows. Let j be fixed. Then

$$\frac{1}{t}[(S_0(t)f)_j(s) - f_j(s)] = \frac{1}{t}[e^{\xi_j(s,\tilde{s}(t))}f_j(\tilde{s}(t)) - f_j(s)]$$

$$= \frac{1}{t}[e^{\int_s^{\tilde{s}(t)}(q_j(s)/c_j(s))\,dx}f_j(\tilde{s}(t)) - e^{\int_s^{\tilde{s}(t)}(q_j(s)/c_j(s))\,dx}f_j(s) + e^{\int_s^{\tilde{s}(t)}(q_j(s)/c_j(s))\,dx}f_j(s) - f_j(s)]$$

$$= e^{\int_s^{\tilde{s}(t)}(q_j(s)/c_j(s))\,dx}\frac{1}{t}[f_j(\tilde{s}(t)) - f_j(s)] + \frac{1}{t}[e^{\int_s^{\tilde{s}(t)}(q_j(s)/c_j(s))\,dx} - 1]f_j(s).$$

Since $\tilde{s} \in W^{1,1}[0,1]$, we have that $\tilde{s}'(0)$ exists and $\tilde{s}'(0) = c_j(s)$ for almost every $s \in [0,1]$. From this and using (17) we obtain that

$$\lim_{t\to 0}\frac{((S_0(t)f)_j(s) - f_j(s))}{t} = 1 \cdot f_j'(s) \cdot c_j(s) + 1 \cdot \frac{q_j(s)}{c_j(s)} \cdot c_j(s) \cdot f_j(s)$$

$$= c_j(s)f_j'(s) + q_j(s)f_j(s).$$

Because of the uniqueness of the limit, we have

$$(B_0f)_j = c_j \cdot f_j' + q_j \cdot f_j \quad \text{for all } f \in D(B_0).$$

Since the L^1-limit in (18) exists for all $f \in (C^\infty[0,1])^m$ (the pointwise convergence implies L^1-convergence because of Lebesgue's dominated convergence theorem), $D := (C^\infty[0,1])^m \subset D(B_0)$ and $B_0 = A_0$ on a dense subspace $D \subset X$. Clearly the subspace D is invariant for $S_0(t)$, $t \geq 0$, hence it is a core for $D(B_0)$, see [10, Proposition II.1.7]. That means

$$\bar{D}^{\|\cdot\|_{B_0}} = \bar{D}^{\|\cdot\|_{A_0}} = D(B_0).$$

Since $D \subset D(A_0)$ and A_0 is closed, we have $\bar{D}^{\|\cdot\|_{A_0}} = D(B_0) \subset D(A_0)$. Hence we obtained that

$$B_0 = A_0|_{D(B_0)}, \quad \text{that is } B_0 \subset A_0.$$

As we have seen in Lemma 3.2, $\rho(A_0) = \mathbb{C}$, hence $\rho(A_0) \cap \rho(B_0) \neq \emptyset$ and so by [10, IV.1.21(5)], $B_0 = A_0$ and hence $S_0 = T_0$. □

In order to compute the spectrum of the generator A we use the operator matrix techniques as in [12, Section 3], developed by A. Rhandi (see [21]) and R. Nagel (see [18]).

Using the results in [11] by Greiner, observe that $L|_{\ker(\lambda - A_w)}$ is invertible for every $\lambda \in \rho(A_0) = \mathbb{C}$. Its inverse will play an important role in the characterization of the spectrum of A and we denote it by

$$D_\lambda := (L|_{\ker(\lambda - A_w)})^{-1} : \partial\mathfrak{L} \to \ker(\lambda - A_w).$$

In order to determine D_λ explicitly we use (13) and the notation

$$E_\lambda := \epsilon_\lambda(1) = \operatorname{diag}(e^{-\xi_j(0,1)+\lambda\tau_j(0,1)})_{j=1,\dots,m} \tag{19}$$

$$= \operatorname{diag}\left(\exp\left(\int_0^1 \frac{-q_j(s)+\lambda}{c_j(s)}\,ds\right)\right)_{j=1,\dots,m}. \tag{20}$$

Lemma 3.4. *The operator D_λ has the form*

$$D_\lambda = \epsilon_\lambda E_\lambda^{-1}(\Phi_w^-)^\top, \tag{21}$$

that is

$$(D_\lambda d)(s) = \epsilon_\lambda(s) \cdot [E_\lambda^{-1}(\Phi_w^-)^\top]d \quad \textit{for any } d \in \partial\mathfrak{L},\ s \in [0,1].$$

The proof is an easy computation, and we refer to [12, Lemma 3.1] for more details.

By the same operator matrix and perturbation methods as in [12, Section 3], we obtain the following characterization for the spectrum of A.

Proposition 3.5. *For every $\lambda \in \mathbb{C}$ we have*

$$\lambda \in \sigma(A) \Leftrightarrow 1 \in \sigma(MD_\lambda).$$

Furthermore, the resolvent of A has the form

$$R(\lambda, A) = (I_\mathfrak{L} + D_\lambda(\mathbf{1} - MD_\lambda)^{-1}M)R(\lambda, A_0) \tag{22}$$

Proof. See [12, Proposition 3.3]. □

The operator MD_λ appearing in the characteristic equation is actually an $n \times n$ matrix:

$$MD_\lambda = (\Phi^+ \otimes \delta_0)(\epsilon_\lambda E_\lambda^{-1}(\Phi_w^-)^\top) = \Phi^+ E_\lambda^{-1}(\Phi_w^-)^\top =: \mathbb{A}_\lambda$$

having entries

$$(\mathbb{A}_\lambda)_{ip} = \begin{cases} \omega_{pj} e^{\xi_j(0,1)-\lambda\tau_j(0,1)}, & \text{if } v_i = e_j(0) \text{ and } v_p = e_j(1), \\ 0, & \text{else.} \end{cases} \tag{23}$$

It is a *weighted (transposed) adjacency matrix* of G. This means that its entry a_{ip} is different from zero if and only if there is an arc from the vertex v_p to the vertex v_i. Let us investigate the matrix $\mathbb{A}_0$ using $\sum_{j=1}^m \omega_{ij} = 1$. If $q_l \le 0$ for all l, then the column sums of $\mathbb{A}_0$ are all less than or equal to 1. Therefore in this case $\|\mathbb{A}_\lambda\|_1 < \|\mathbb{A}_0\|_1 \le 1$ for $\operatorname{Re}\lambda > 0$, and we obtain the following

Corollary 3.6. *For every $\lambda \in \mathbb{C}$ we have*

$$(24) \qquad \lambda \in \sigma(A) \Leftrightarrow \det(\mathbf{1} - \mathbb{A}_\lambda) = 0.$$

In particular, if $q_l \le 0$ for all l, this implies

$$(25) \qquad \lambda \in \rho(A) \quad \textit{for } \operatorname{Re}\lambda > 0.$$

In the following we use the *spectral bound*

$$(26) \qquad \tilde{q} := s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$$

of A and recall an important property of positive semigroups on L^p-spaces (see e.g. [10, Section VI.1.b]).

Remark 3.7. For the *growth bound*

$$\omega_0 = \inf\{w \in \mathbb{R} : \exists M_w \ge 1 \text{ such that } \|T(t)\| \le M_w e^{wt} \text{ for all } t \ge 0\}$$

of the semigroup we have

$$(27) \qquad \omega_0 = \tilde{q} \in \sigma(A),$$

if $\tilde{q} > -\infty$.

Finally we extend to $\mathfrak{M}$ the results obtained in $\mathfrak{L}$. First, we introduce a notation and prove a technical result.

Definition 3.8. With the notations of Definition 3.1, for $j \in \{1, \dots, m\}$ and $0 \le t \le \tau_j(0,1)$ let $s_j(t) \in [0,1]$ be the location where the flow moves to on the edge e_j from the point 0 during time t. Hence $s_j \in C[0, \tau_j(0,1)]$ is defined by $\tau_j(0, s_j(t)) = t$.

Lemma 3.9. *With the above notation,*

1. $u_l(\sigma, t) = e^{\xi_l(\sigma,1)} u_l(1, t - \tau_l(\sigma, 1))$ *for* $\tau_l(\sigma, 1) \le t$ *and*
2. $u_j(0, t - \tau_l(\sigma, 1)) = e^{\xi_j(0, s_j(\tau_l(0,\sigma)))} u_j(s_j(\tau_l(0,\sigma)), t - \tau_l(0,1))$ *for* $\tau_l(0,1) \le t$, $\tau_l(0,\sigma) \le \tau_j(0,1)$

hold for every $1 \le j, l \le m$, *and a.e.* σ.

Proof. For both of the statements, we use Lemma 3.3. For any fixed l and σ satisfying the conditions assumed in 1. and 2., the action of $T(t)$ and $T_0(t)$ coincide on the subintervals of e_l we consider. That is

$$u_l(\sigma, t) = (T(\tau_l(\sigma, 1))u(\cdot, t - \tau_l(\sigma, 1)))_l(\sigma)$$

$$= (T_0(\tau_l(\sigma, 1))u(\cdot, t - \tau_l(\sigma, 1)))_l(\sigma) = e^{\xi_l(\sigma, 1)} u_l(1, t - \tau_l(\sigma, 1)),$$

and

$$u_j(0, t - \tau_l(\sigma, 1)) = (T(\tau_l(0, \sigma))u(\cdot, t - \tau_l(0, 1)))_j(0)$$

$$= (T_0(\tau_l(0, \sigma))u(\cdot, t - \tau_l(0, 1)))_j(0)$$

$$= e^{\xi_j(0, s_j(\tau_l(0, \sigma)))} u_j(s_j(\tau_l(0, \sigma)), t - \tau_l(0, 1)). \qquad \square$$

The w^*-convergence makes possible to extend all the formulae obtained in $\mathfrak{L}$ to their corresponding weak versions. Since for measures point evaluation can not be defined in general, we have to prove first a generalized version of the Kirchhoff law in (3).

Proposition 3.10. *The measures $(u_j(\cdot, t))_{1 \le j \le m}$ obtained as the orbits of the extended semigroup $(T(t))_{t \ge 0}$ satisfy the generalized Kirchhoff law, that is, using the notation of Definition 3.1 and 3.8, for every fixed $\tau_+ < t$, $1 \le i \le n$ and l such that $v_i = e_l(1)$, we have*

$$(28) \qquad \int_0^{s_l(\tau_-)} \varphi(\sigma)\, du_l(\sigma, t)$$

$$= \omega_{il} \sum_{j=1}^{m} \phi_{ij}^{+} \int_0^{s_l(\tau_-)} \varphi(\sigma) e^{\xi_l(\sigma, 1) + \xi_j(0, s_j(\tau_l(0, \sigma)))}\, du_j(s_j(\tau_l(0, \sigma)), t - \tau_l(0, 1)),$$

where φ is any continuous function on $[0, 1]$.

Proof. It is enough to prove the statement for absolutely continuous measures, then (28) holds by Proposition 2.5.

So let $u(\cdot, t)$ be in $\mathfrak{L}$. According to Lemma 3.9.1, we have

$$u_l(\sigma, t) = e^{\xi_l(\sigma, 1)} u(1, t - \tau_l(\sigma, 1)),$$

by (7),

$$u_l(1, t - \tau_l(\sigma, 1)) = \omega_{il} \sum_{j=1}^{m} \phi_{ij}^{+} u_j(0, t - \tau_l(\sigma, 1)),$$

while by Lemma 3.9.2,

$$u_j(0, t - \tau_l(\sigma, 1)) = e^{\xi_j(0, s_j(\tau_l(0,\sigma)))} u_j(s_j(\tau_l(0, \sigma)), t - \tau_l(0, 1)) \quad \text{a.e.}$$

Putting these together we have

$$u_l(\sigma, t) = \omega_{il} \sum_{j=1}^{m} \phi_{ij}^{+} e^{\xi_l(\sigma, 1) + \xi_j(0, s_j(\tau_l(0,\sigma)))} u_j(s_j(\tau_l(0, \sigma)), t - \tau_l(0, 1))$$

a.e., as required. □

The preceding spectral analysis of the operator A is based on the fact that it generates a strongly continuous semigroup on the state space $\mathfrak{L}$. Since this is not true for the state space $\mathfrak{M}$, we have to study directly the semigroup $(T(t))_{t \geq 0}$. Here, instead of the spectrum of A we are looking for periodic orbits of the semigroup.

Instead of working with the measure functions $u_j(\cdot\,, t)$ describing the flow on the *edges* of our network, we consider the flow through the vertices of the graph. In $\mathfrak{L}$ is easy to make precise this intuitive notion, namely for every $v_i \in V$ one has to take

$$f[u]_{v_i}(t) := f_i(t) := \sum_{j=1}^{m} \phi_{ij}^{+} u_j(0, t), \quad t \geq 0.$$

For measures the analogous expression is more complicated. Since point evaluation is not possible in general, for every $1 \leq i \leq n$ we "put together" the one-parameter family of measures $\{u_j(\cdot\,, t) : \phi_{ij}^{+} = 1\}$ into one measure $\mu[u]_{v_i}$ in the following way. Let $\mu[u]_{v_i} := \mu_i \in \mathscr{M}(\mathbb{R}^+)$ be the complete history of the flow through the vertex v_i, that is the unique measure which for any $t \in \mathbb{R}^+$ and $\varphi \in C[t, t + \tau_-]$ satisfies

$$\text{(29)} \qquad \int_0^{\tau_-} \varphi(t + s)\, \mathrm{d}\mu_i(t + s) = \sum_{j=1}^{m} \phi_{ij}^{+} \int_0^{\tau_-} \varphi(t + s) e^{\xi_j(0, s_j(s))}\, \mathrm{d}u_j(s_j(s), t).$$

We now introduce a notation turning out to be very useful in the subsequent computations.

Definition 3.11. Fix a $p \in \{1, \dots, n\}$. Let $l(i)$, $i = 1, \dots, n$ denote the index of the edge pointing from v_i to v_p if it exists. Otherwise let $l(i)$ be any edge index. We define

$$\omega_{i,p} := \begin{cases} \omega_{il(i)}, & \text{if } e_{l(i)} \text{ points from } v_i \text{ to } v_p, \\ 0, & \text{else.} \end{cases}$$

The measures μ_i, by the Kirchhoff law in (28), are related in the following way.

Lemma 3.12. *Let δ_r denote the left shift on $\mathbb{R}^+$ with r. Using the notations of Definition 3.1 and 3.11,*

$$\mu_p = \sum_{i=1}^{n} \omega_{i,p} e^{\xi_{i,p}(0,1)} \mu_i \delta_{\tau_{i,p}(0,1)}$$

holds on $[\tau_+, \infty)$ *for every* $p = 1, \dots, n$.

Proof. Fix a vertex v_p and let $l(i)$ be defined as in Definition 3.11. Clearly, it is enough to show that

$$\int_0^{\tau_-} \varphi(t+s)\, \mathrm{d}\mu_p(t+s) = \sum_{i=1}^{n} \omega_{i,p} e^{\xi_{l(i)}(0,1)} \int_0^{\tau_-} \varphi(t+s)\, \mathrm{d}\mu_i(t - \tau_{l(i)}(0,1) + s) \tag{30}$$

holds for every $t > \tau_+$ and $\varphi \in C[t, t+\tau_-]$.

By (29), for the two sides of (30) we have

$$\int_0^{\tau_-} \varphi(t+s)\, \mathrm{d}\mu_p(t+s) = \sum_{l=1}^{m} \phi_{pl}^{+} \int_0^{\tau_-} \varphi(t+s) e^{\xi_l(0, s_l(s))}\, \mathrm{d}u_l(s_l(s), t) \tag{31}$$

and

$$\begin{aligned} &\sum_{i=1}^{n} \omega_{i,p} e^{\xi_{l(i)}(0,1)} \int_0^{\tau_-} \varphi(t+s)\, \mathrm{d}\mu_i(t - \tau_{l(i)}(0,1) + s) \\ &= \sum_{i=1}^{n} \omega_{i,p} e^{\xi_{l(i)}(0,1)} \sum_{j=1}^{m} \phi_{ij}^{+} \int_0^{\tau_-} \varphi(t+s) e^{\xi_j(0, s_j(s))}\, \mathrm{d}u_j(s_j(s)), t - \tau_{l(i)}(0,1)). \end{aligned} \tag{32}$$

Applying the Kirchhoff law (28) for the measures $u_{l(i)}(\cdot, t)$ with variable $\sigma = s_{l(i)}(s)$, (31) equals

$$\begin{aligned} &\sum_{i=1}^{n} \omega_{i,p} \sum_{j=1}^{m} \phi_{ij}^{+} \int_0^{\tau_-} \varphi(t+s) e^{\xi_{l(i)}(0, s_{l(i)}(s))} e^{\xi_{l(i)}(s_{l(i)}(s), 1) + \xi_j(0, s_j(\tau_{l(i)}(0, s_{l(i)}(s))))} \\ &\qquad \mathrm{d}u_j(s_j(\tau_{l(i)}(0, s_{l(i)}(s)))), t - \tau_{l(i)}(0,1)). \end{aligned} \tag{33}$$

Since by Definition 3.1

$$\xi_{l(i)}(0, s_{l(i)}(s)) + \xi_{l(i)}(s_{l(i)}(s), 1) = \xi_{l(i)}(0,1)$$

and $\tau_{l(i)}(0, s_{l(i)}(s)) = s$, (33) equals

$$\sum_{i=1}^{n} \omega_{i,p} \sum_{j=1}^{m} \phi_{ij}^{+} \int_0^{\tau_-} \varphi(t+s) e^{\xi_{l(i)}(0,1) + \xi_j(0, s_j(s))}\, \mathrm{d}u_j(s_j(s), t - \tau_{l(i)}(0,1)).$$

This is exactly the right hand side of (32), so the proof is complete. □

Since we are interested in (asymptotically) periodic solutions, it is natural to discretize our problem by Fourier transformation. For every $\tau > 0$, $q \in \mathbb{R}$, $p \in \{1, \dots, n\}$ and $k \in \mathbb{Z}$, let

$$\mathscr{F}[u]_p^{\tau,q}(k) = \int_0^\tau e^{(-q+2\pi \mathrm{i}k/\tau)t}\,\mathrm{d}\mu_p(t). \tag{34}$$

If, rescaling with the spectral bound $\tilde{q}$ in (26), $\{e^{-\tilde{q}t}u_j(\cdot\,, t)\colon j = 1, \dots, m\}$ corresponds to a periodic solution in the state space $\mathfrak{M}$ with period τ, then from Lemma 3.12 we have

$$\begin{aligned}
\mathscr{F}[u]_p^{\tau,\tilde{q}}(k) &= \int_0^\tau e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)t}\,\mathrm{d}\mu_p(t) \\
&= \sum_{i=1}^n \omega_{i,p} e^{\xi_{i,p}(0,1)} \int_0^\tau e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)t}\,\mathrm{d}\mu_i(t - \tau_{i,p}(0,1)) \\
&= \sum_{i=1}^n \omega_{i,p} e^{\xi_{i,p}(0,1)} \int_{-\tau_{i,p}(0,1)}^{\tau-\tau_{i,p}(0,1)} e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)(t+\tau_{i,p}(0,1))}\,\mathrm{d}\mu_i(t) \\
&= \sum_{i=1}^n \omega_{i,p} e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)\tau_{i,p}(0,1)+\xi_{i,p}(0,1)} \int_{-\tau_{i,p}(0,1)}^{\tau-\tau_{i,p}(0,1)} e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)t}\,\mathrm{d}\mu_i(t) \\
&= \sum_{i=1}^n \omega_{i,p} e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)\tau_{i,p}(0,1)+\xi_{i,p}(0,1)} \mathscr{F}[u]_i^{\tau,\tilde{q}}(k) \quad \text{for every } k \in \mathbb{Z}.
\end{aligned} \tag{35}$$

For the sake of simplicity, let us denote the weighted adjacency matrix by

$$A_{\tau,\tilde{q}}(k) := \mathbb{A}_{\tilde{q}-2\pi \mathrm{i}k/\tau} = \left(\omega_{i,p} e^{(-\tilde{q}+2\pi \mathrm{i}k/\tau)\tau_{i,p}(0,1)+\xi_{i,p}(0,1)}\right)_{i,p=1}^n. \tag{36}$$

Then the vectors of the Fourier coefficients

$$\mathscr{F}[u]_{\tau,\tilde{q}}(k) := \begin{pmatrix} \mathscr{F}[u]_1^{\tau,\tilde{q}}(k) \\ \vdots \\ \mathscr{F}[u]_n^{\tau,\tilde{q}}(k) \end{pmatrix} \tag{37}$$

satisfy

$$\mathscr{F}[u]_{\tau,\tilde{q}}(k) = A_{\tau,\tilde{q}}(k)\mathscr{F}[u]_{\tau,\tilde{q}}(k).$$

We summarize what we have obtained.

Proposition 3.13. *The rescaled semigroup* $(e^{-\tilde{q}t}T(t))_{t\geq 0}$, *where* $\tilde{q}$ *is defined in* (26), *has a periodic orbit on the state space* $\mathfrak{M}$ *with period* τ *if and only if for every* $k \in \mathbb{Z}$,

$\mathscr{F}[u]_{\tau,\tilde{q}}(k)$ from (34) *is an eigenvector of the weighted adjacency matrix $A_{\tau,\tilde{q}}(k)$ with eigenvalue* 1*, that is*

$$\mathscr{F}[u]_{\tau,\tilde{q}}(k) = A_{\tau,\tilde{q}}(k)\mathscr{F}[u]_{\tau,\tilde{q}}(k), \quad k \in \mathbb{Z}. \tag{38}$$

In the following section we will use the consequences of the spectral properties of A and of $(T(t))_{t\geq 0}$ to describe the asymptotic behavior of the system.

4 Asymptotic behavior on strongly connected graphs

To obtain results on the asymptotic behavior of the semigroup on $\mathfrak{L}$, we will proceed as in [12, Section 4] and restrict ourselves to special networks yielding a semigroup with useful additional properties. The next property (see [17, Definition C-III.3.1]) is essential to this purpose.

Definition 4.1. A positive semigroup on $L^1(\Omega,\mu)$, μ a σ-finite measure, with generator A is *irreducible* if for all $\lambda > s(A)$—the spectral bound defined in (26)—and $f > 0$, the resolvent satisfies $(R(\lambda,A)f)(s) > 0$ for almost all $s \in \Omega$.

Throughout this section, we will consider networks having the following type of underlying graph.

Definition 4.2. A directed graph is called *strongly connected* if for every two vertices in the graph there are paths connecting them in both directions.

As stated in [15, Theorem IV.3.2], a directed graph is strongly connected if and only if its adjacency matrix is irreducible. Using this fact we can relate the irreducibility of our semigroup on $\mathfrak{L}$ to the strong connectedness of the underlying graph.

Lemma 4.3. *Let the graph G be strongly connected. Then the semigroup $(T(t))_{t\geq 0}$ is irreducible.*

Proof. The proof is based on the above Definition 4.1, on the form of the resolvent of A in (22) and works as the proof of [12, Lemma 4.4]. Observe that $R(\lambda,A_0)$ in (12) is again positive. We still need the strict positivity of $(\mathbf{1}-\mathbb{A}_\lambda)^{-1}$ for $\lambda > \tilde{q}$. Since $\mathbb{A}_\lambda$ is positive irreducible, from the form (23) of its entries follows by [22, Corollary I.6.4] that its spectral radius $r(\mathbb{A}_\lambda)$ is a (continuous) strictly monotone decreasing function of λ. Because of the positivity, $r(\mathbb{A}_\lambda) \in \sigma(\mathbb{A}_\lambda)$, see [22, Proposition I.2.3]. These facts imply that $r(\mathbb{A}_{\tilde{q}}) = 1 > r(\mathbb{A}_\lambda)$ for every $\lambda > \tilde{q}$. Now from [22, Proposition I.6.2] follows that $(\mathbf{1}-\mathbb{A}_\lambda)^{-1}$ is strictly positive for $\lambda > \tilde{q}$. The rest of the proof is analogous to that one in [12]. □

Using the irreducibility of the adjacency matrix, we can state the following result on the asymptotic behavior of $(T(t))_{t\geq 0}$.

Proposition 4.4. *Assume that G is strongly connected. If $q_l \leq 0$ for all l and there exists at least one index j such that $q_j \neq 0$ (that is $q_j < 0$ on a set of positive measure), then $\tilde{q} < 0$, hence the semigroup $(T(t))_{t \geq 0}$ is uniformly exponentially stable.*

Proof. From Corollary 3.6 follows that $\tilde{q} \leq 0$ holds, hence we only have to prove that $\tilde{q} \neq 0$. Let $\mathbb{A}_{0,1} := \mathbb{A}_0$ and call $\mathbb{A}_{0,2}$ the weighted adjacency matrix for $\lambda = 0$ in the case when we replace q_j by the 0. Observe that both matrices are irreducible. With the notation of [22], from the form (11) of ξ_j and from (23) follows that $|\mathbb{A}_{0,1}| \leq |\mathbb{A}_{0,2}|$, and there is at least one entry in the first matrix that is strictly less than the same entry in the second one. Using [22, Corollary I.6.4] we obtain that $r(\mathbb{A}_{0,1}) < r(|\mathbb{A}_{0,2}|)$. Since $r(|\mathbb{A}_{0,2}|) \leq \|\mathbb{A}_{0,2}\|_1 \leq 1$, we have that $\det(\mathbf{1} - \mathbb{A}_{0,1}) \neq 0$ and so $0 \notin \sigma(A)$. Using (27) we obtain that $\tilde{q} \neq 0$, hence $\tilde{q} < 0$. □

In the following we always assume the graph G to be strongly connected, hence our semigroup to be irreducible on $\mathfrak{L}$. The behavior of this semigroup $(T(t))_{t \geq 0}$ is governed by the growth bound $\tilde{q}$: for $\tilde{q} > 0$ the flow blows up, while for $\tilde{q} < 0$ it vanishes. To obtain a finer description, we work with the rescaled semigroup $\tilde{T}(t) := e^{-\tilde{q}t} T(t)$. In the following lemma we summarize the basic properties of $(\tilde{T}(t))_{t \geq 0}$.

Lemma 4.5. *The rescaled semigroup $(\tilde{T}(t))_{t \geq 0}$ is positive and strongly continuous on $\mathfrak{L}$ and its generator $\tilde{A} := A - \tilde{q}I$ satisfies $s(\tilde{A}) = 0 \in \sigma(\tilde{A})$. Furthermore, if the graph is strongly connected, $(\tilde{T}(t))_{t \geq 0}$ is irreducible.*

We also obtain that our semigroup is bounded.

Theorem 4.6. *If the graph is strongly connected, the semigroup $(\tilde{T}(t))_{t \geq 0}$ is bounded on $\mathfrak{L}$.*

Proof.[†] From the Banach-Steinhaus theorem follows that it is enough to prove that for all $g \in \mathfrak{L}$, $f \in \mathfrak{L}'$ there exists $K_{g,f} > 0$ such that

$$(39) \qquad |\langle g, \tilde{T}(t)' f \rangle| \leq K_{g,f}, \quad t \geq 0,$$

where $\mathfrak{L}' = L^\infty([0,1], \mathbb{C}^m)$ is the dual space of $\mathfrak{L}$. Using the positivity and irreducibility of the semigroup $(\tilde{T}(t))_{t \geq 0}$ and the compactness of $R(\lambda, \tilde{A})$, we obtain that $s(\tilde{A}) = 0$ is a (first order) pole of the resolvent, and by [17, Proposition C-III.3.5] admits a (strictly) positive eigenvector also for $\tilde{A}'$. Let us fix such an eigenvector $h \in \mathfrak{L}'$. From the form of $\tilde{A}$ and $\tilde{A}'$ follows that h is an exponential function, hence we can assume that $h \geq 1$. Since $\tilde{A}' h = 0$, we have that $\tilde{T}'(t) h = h$ for all $t \geq 0$, see [10, Proposition IV.2.18] and [10, Theorem IV.3.7]. To prove (39), take an arbitrary $f \in \mathfrak{L}'$. Then $|f| \leq \|f\|_\infty \cdot h$. From the positivity of the adjoint semigroup follows that

[†] The proof is due to Bálint Farkas.

$$|\tilde{T}(t)'f| \leq \tilde{T}(t)'|f| \leq \|f\|_\infty \cdot (\tilde{T}(t)'h),$$

hence

$$\|\tilde{T}(t)'f\|_\infty \leq \|f\|_\infty \cdot \|\tilde{T}(t)'h\|_\infty = \|f\|_\infty \cdot \|h\|_\infty.$$

From this we obtain

$$|\langle g, \tilde{T}(t)'f\rangle| \leq \|g\|_1 \cdot \|f\|_\infty \cdot \|h\|_\infty, \quad t \geq 0, \text{ for all } g \in \mathfrak{L},\ f \in \mathfrak{L}',$$

where $\|h\|_\infty$ is fixed. □

Having a bounded irreducible semigroup $(\tilde{T}(t))_{t\geq 0}$ with $R(\lambda, \tilde{A})$ compact, we immediately obtain the following decomposition of the state space $\mathfrak{L}$.

Proposition 4.7. *Using the above notation, the following properties hold.*

1. $\mathfrak{L}_0 := \ker \tilde{A}$ *is one dimensional and is spanned by a positive eigenvector.*

2. *There is a projection* $Q : \mathfrak{L} \to \mathfrak{L}$, *hence a decomposition* $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$, *such that* $\tilde{T}(t)x \to 0$ *if and only if* $x \in \mathfrak{L}_2 := \ker Q$ *while* $\mathfrak{L}_1 := \operatorname{ran} Q = \overline{\operatorname{lin}}\{x \in D(\tilde{A}) \mid \exists \alpha \in \mathbb{R} : \tilde{A}x = \mathbf{i}\alpha x\}$.

Proof. Using the fact that $\tilde{A}$ has compact resolvent, we obtain by [10, Corollary IV.1.19] that all the elements of the (point) spectrum of $\tilde{A}$ are poles of the resolvent $R(\lambda, \tilde{A})$ with finite algebraic multiplicity. Being $s(\tilde{A}) = 0$ an element of the spectrum (see (27)), and using that the semigroup $(\tilde{T}(t))_{t\geq 0}$ is irreducible, [17, Proposition C-III.3.5] implies that 0 is an algebraically simple pole and admits a positive eigenvector for $\tilde{A}$, hence 1. holds.

Since $\tilde{A}$ has compact resolvent, the second statement is Corollary V.2.15 of [10]. □

It turns out that the following properties of the velocities strongly influence the spectral properties of $\tilde{A}$ and the asymptotic behavior of $(\tilde{T}(t))_{t\geq 0}$.

Definition 4.8. We say that $(\mathrm{LD}_{\mathbb{Q}})$, [$(\mathrm{LI}_{\mathbb{Q}})$ resp.] holds if the numbers

$$\{\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1) : e_{j_1}, \ldots, e_{j_k} \text{ form a cycle in } G\}$$

are linearly dependent [independent, resp.] over $\mathbb{Q}$.

In the sequel we treat these two alternatives separately.

4.1 $(\mathrm{LD}_{\mathbb{Q}})$ case

As we will see, rationally dependent average speeds produce nontrivial periodic orbits for $(\tilde{T}(t))_{t\geq 0}$. Asymptotically, these periodic solutions govern the dynamics of the flow, and even in the state space $\mathfrak{M}$ the orbits converge uniformly in norm to a rotation.

Let us investigate the characteristic equation (24) in the case when condition $(\mathrm{LD}_{\mathbb{Q}})$ holds. Using the definition of the determinant and (23), we obtain that

$$\text{(40)} \qquad \begin{aligned} h(\lambda) &:= \det(\mathbf{1} - \mathbb{A}_\lambda) \\ &= 1 + a_1(\lambda) + \cdots + a_n(\lambda) \end{aligned}$$

with

$$a_r(\lambda) = \sum_{p=1}^{r} (-1)^p \sum_{\substack{k_1+\cdots+k_p=r \\ Z_1,\ldots,Z_p}} \prod_{j=1}^{p} \left(w_j e^{-\lambda \sum_{e_i \in Z_j} \tau_i(0,1)}\right).$$

Here the second sum runs over all positive integers $k_1, \ldots, k_p$ having sum r such that there exist vertex disjoint cycles $Z_1, \ldots, Z_p$ in the graph G having $k_1, \ldots, k_p$ vertices, respectively. The numbers w_j are defined as

$$w_j := \prod_{e_k \in Z_j} \omega_{ik} \cdot e^{\xi_k(0,1)},$$

where $\omega_{ik} \neq 0$ is uniquely determined by e_k in the cycle Z_j. For more details on this representation of $\det(\mathbf{1} - \mathbb{A}_\lambda)$ see a generalization of the Sachs theorem in [7, Theorem 3.1].

The condition $(\mathrm{LD}_{\mathbb{Q}})$ implies that there exists a real number c such that

$$c(\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1)) \in \mathbb{N}$$

for all $e_{j_1}, \ldots, e_{j_k}$ that form a cycle in G. Take the greatest common divisor of these numbers

$$l(c) := \gcd\{c(\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1)); e_{j_1}, \ldots, e_{j_k} \text{ form a cycle in } G\}$$

and observe that the fraction $\frac{l(c)}{c}$ does not depend on the special choice of c. Therefore the number

$$\text{(41)} \qquad \gamma := \frac{l(c)}{c}$$

is well-defined. This leads to the following expression for the terms $a_r(\lambda)$ in (23)

$$\text{(42)} \qquad a_r(\lambda) = \sum_{p=1}^{r} (-1)^p \sum_{\substack{k_1+\cdots+k_p=r \\ Z_1,\ldots,Z_p}} \prod_{j=1}^{p} w_j (e^{-\lambda\gamma})^{l_j}$$

with

$$l_j := \frac{1}{\gamma} \sum_{e_i \in Z_j} \tau_i(0,1) \in \mathbb{N}.$$

The form (42) implies that $h(\lambda)$ can be written as

$$h(\lambda) = q(e^{-\lambda\gamma}) \tag{43}$$

with a polynomial q. This immediately leads to the following result on the spectrum of $\tilde{A}$.

Lemma 4.9. *Suppose that the condition* ($\mathrm{LD}_{\mathbb{Q}}$) *is fulfilled. Then the eigenvalues of A, hence that of $\tilde{A}$ lie on finitely many vertical lines.*

Proof. By (24) and (43) the zeros of $q(e^{-\lambda\gamma})$ are exactly the eigenvalues of A, hence the statement follows. □

We are now able to relate the spectral properties of the generator to those of the semigroup as already shown in [12, Proposition 3.8].

Proposition 4.10 (Circular Spectral Mapping Theorem). *Suppose that the condition* ($\mathrm{LD}_{\mathbb{Q}}$) *holds. Then the semigroup $(\tilde{T}(t))_{t\geq 0}$ satisfies the so called* circular spectral mapping theorem, *that is*

$$\Gamma \cdot e^{t\sigma(\tilde{A})} = \Gamma \cdot \sigma(\tilde{T}(t)) \backslash \{0\} \quad \textit{for every } t \geq 0,$$

where Γ denotes the unit circle.

Proof. The proof uses the form of the resolvent of $\tilde{A}$ and is analogous to the proof of [12, Proposition 3.8]. □

The Circular Spectral Mapping Theorem and the above Lemma 4.9 imply that the spectrum $\sigma(\tilde{T}(t))$ lies on finitely many circles, where the largest one is the unit circle Γ (see Lemma 4.5). This immediately allows the following decomposition of the semigroup.

Proposition 4.11. *Suppose that condition* ($\mathrm{LD}_{\mathbb{Q}}$) *holds. Then for the decomposition in Proposition 4.7.2 the following assertions are true.*

1. *The operators $S(t) := \tilde{T}(t)|_{\mathfrak{L}_1}$, $t \geq 0$, yield a bounded C_0-group on $\mathfrak{L}_1$.*
2. *The semigroup $(\tilde{T}(t)|_{\mathfrak{L}_2})_{t\geq 0}$ is uniformly exponentially stable, hence*

$$\|\tilde{T}(t) - S(t)\|_{\mathfrak{L}} \leq Me^{-\varepsilon t}$$

for some constants $M \geq 1$, $\varepsilon > 0$.

Proof. Use Theorem 4.10 and [10, Theorem V.1.17] in a rescaled form. □

The Perron-Frobenius theory for positive irreducible semigroups (and the compactness of $R(\lambda, \tilde{A})$) implies that

$$\sigma(\tilde{A}) \cap \mathbf{i}\mathbb{R} = \mathbf{i}\alpha\mathbb{Z} \quad \text{for some } \alpha \geq 0,$$

where each $\mathbf{i}\alpha k$ is a simple pole of the resolvent (see [10, Theorem VI.1.12] or [17, Section C-III]). In the following we want to identify α. The statement (27) and the form (43) of the characteristic equation (24) imply that $\lambda = \tilde{q}$ is a zero of $q(e^{-\lambda\gamma})$, therefore all the numbers $\lambda = \tilde{q} + \mathbf{i}2\pi\frac{1}{\gamma}k$, $k \in \mathbb{Z}$, are also zeros of $h(\lambda)$—hence eigenvalues of A. So we obtain

$$\mathbf{i}2\pi\frac{1}{\gamma}\mathbb{Z} \subseteq \sigma_b(\tilde{A}), \tag{44}$$

where $\sigma_b(\tilde{A}) = \sigma(\tilde{A}) \cap \mathbf{i}\mathbb{R}$ denotes the boundary spectrum of $\tilde{A}$. To obtain equality in (44), we need the following lemma.

Lemma 4.12. *Let* $\mathbb{B}_0 = (a_{i,p})_{i,p=1}^n$ *be a real irreducible matrix, so that* $a_{i,p} \geq 0$, $i,p = 1,\dots,n$, *with a positive vector* $\underline{b} = (b_1,\dots,b_n)$ *satisfying*

$$\mathbb{B}_0\underline{b} = \underline{b}. \tag{45}$$

Let $\mathbb{B}$ *denote any matrix obtained form* $\mathbb{B}_0$ *by multiplying each of its entries by a complex number having absolute value* 1, *that is*

$$(\mathbb{B})_{i,p} = e^{\mathbf{i}\vartheta_{i,p}}a_{i,p}.$$

Then $\det(\mathbb{B} - \mathbf{1}) = 0$ *if and only if*

$$\prod_{l=1}^{s} e^{\mathbf{i}\vartheta_{i_l,p_l}} = 1$$

for every sequence $(i_1,p_1),\dots,(i_s,p_s)$ *with* $i_{s+1} = i_1$, $p_l = i_{l+1}$, $l = 1,\dots,s$.

Proof. The "if" part being trivial, we only prove the other implication. Suppose that $\det(\mathbb{B} - \mathbf{1}) = 0$. Then the columns of the matrix $\mathbb{B} - \mathbf{1}$ are not linearly independent over $\mathbb{C}$, so there exist coefficients $c_1,\dots,c_n$ satisfying

$$\sum_{i=1}^{n} c_i z_i = 0, \tag{46}$$

where z_i denotes the i^{th} column of $\mathbb{B}$.

Choose c_{i_0} to be one coefficient such that

$$\frac{|c_i|}{b_i} \le \frac{|c_{i_0}|}{b_{i_0}} \tag{47}$$

for every $i = 1, \dots, n$. From (46) in the i_0^{th} coordinate we obtain

$$\begin{aligned} c_1 e^{\mathbf{i}\vartheta_{1,i_0}} a_{1,i_0} + \cdots + c_{i_0-1} e^{\mathbf{i}\vartheta_{i_0-1,i_0}} a_{i_0-1,i_0} + c_{i_0}(e^{\mathbf{i}\vartheta_{i_0,i_0}} a_{i_0,i_0} - 1) \\ + c_{i_0+1} e^{\mathbf{i}\vartheta_{i_0+1,i_0}} a_{i_0+1,i_0} + \cdots + c_n e^{\mathbf{i}\vartheta_{n,i_0}} a_{n,i_0} = 0, \end{aligned} \tag{48}$$

hence

$$\begin{aligned} c_1 e^{\mathbf{i}\vartheta_{1,i_0}} a_{1,i_0} + \cdots + c_{i_0-1} e^{\mathbf{i}\vartheta_{i_0-1,i_0}} a_{i_0-1,i_0} + c_{i_0} e^{\mathbf{i}\vartheta_{i_0,i_0}} a_{i_0,i_0} \\ + c_{i_0+1} e^{\mathbf{i}\vartheta_{i_0+1,i_0}} a_{i_0+1,i_0} + \cdots + c_n e^{\mathbf{i}\vartheta_{n,i_0}} a_{n,i_0} = c_{i_0}. \end{aligned} \tag{49}$$

We also know from (45) that

$$b_1 a_{1,i_0} + \cdots + b_{i_0-1} a_{i_0-1,i_0} + b_{i_0} a_{i_0,i_0} + b_{i_0+1} a_{i_0+1,i_0} + \cdots + b_n a_{n,i_0} = b_{i_0}. \tag{50}$$

Since, by (47), c_{i_0} is the relatively greatest coordinate, we expect by (50) that (49) is possible only if

$$\frac{c_i}{b_i} e^{\mathbf{i}\vartheta_{i,i_0}} = \frac{c_{i_0}}{b_{i_0}} \tag{51}$$

for every $i = 1, \dots, n$ with $a_{i,i_0} \neq 0$. This holds since

$$\begin{aligned} c_{i_0} &= c_1 e^{\mathbf{i}\vartheta_{1,i_0}} a_{1,i_0} + \cdots + c_n e^{\mathbf{i}\vartheta_{n,i_0}} a_{n,i_0} \\ &= \frac{c_1}{b_1} e^{\mathbf{i}\vartheta_{1,i_0}} b_1 a_{1,i_0} + \cdots + \frac{c_n}{b_n} e^{\mathbf{i}\vartheta_{n,i_0}} b_n a_{n,i_0}. \end{aligned} \tag{52}$$

Hence, by the triangle inequality and (50),

$$\begin{aligned} |c_{i_0}| &\le \frac{|c_1|}{b_1} b_1 a_{1,i_0} + \cdots + \frac{|c_n|}{b_n} b_n a_{n,i_0} \\ &\le \frac{|c_{i_0}|}{b_{i_0}} b_1 a_{1,i_0} + \cdots + \frac{|c_{i_0}|}{b_{i_0}} b_n a_{n,i_0} = |c_{i_0}|. \end{aligned} \tag{53}$$

Equality is possible only if all the complex numbers point into the same direction and if (47) holds with equality for every i with $a_{i,i_0} \neq 0$. This is exactly (51).

But then (48), (49) and hence (51) hold for every k instead of i_0 with $a_{k,i_0} \neq 0$, all those c_k's satisfying (47). Since these k's are the indices of the nonzero entries in the i_0^{th} row, by the irreducibility of the matrix and by repeating the argument, we obtain that

$$\frac{c_i}{b_i} e^{\mathbf{i}\vartheta_{i,k}} = \frac{c_k}{b_k} \tag{54}$$

for every k and i with $a_{k,i} \neq 0$.

Take now indices $i_1, \ldots, i_s$ satisfying, with the notational convention $i_{s+1} = i_1$, $a_{i_l, i_{l+1}} \neq 0$ for every $l = 1, \ldots, s$. By (54),

$$\prod_{l=1}^{s} e^{\mathbf{i}\vartheta_{i_l, i_{l+1}}} = \prod_{l=1}^{s} \frac{c_{l+1}/b_{l+1}}{c_l/b_l} = 1,$$

since $a_{i_l, i_{l+1}} \neq 0$ for every $l = 1, \ldots, s$. This proves the statement. □

Corollary 4.13. *Suppose that* $(\mathrm{LD}_{\mathbb{Q}})$ *is satisfied.*

1. *The boundary spectrum* $\sigma_b(\tilde{A})$ *in the state space* $\mathfrak{L}$ *satisfies*

$$\mathbf{i}2\pi \frac{1}{\gamma} \mathbb{Z} = \sigma_b(\tilde{A}). \tag{55}$$

2. *In* $\mathfrak{M}$*, the periods of periodic orbits of* $(\tilde{T}(t))_{t \geq 0}$ *are exactly* $\gamma/\mathbb{Z}$.

Proof. Since a spectrum point $\mathbf{i}\beta \in \sigma_b(\tilde{A})$ corresponds to a periodic orbit with period $\frac{2\pi}{\beta}$, by (44) we only need to prove that every period in $\mathfrak{M}$ is of the form $\gamma/\mathbb{Z}$.

Suppose that μ is a periodic orbit with period τ and consider the Fourier coefficients of (37) and the characteristic equation in (38). For every $k \in \mathbb{Z}$, we have that either $\mathscr{F}_{\tau,\tilde{q}}(k)$ is null or $\mathscr{F}_{\tau,\tilde{q}}(k)$ is an eigenvector of the matrix $A_{\tau,\tilde{q}}(k)$, defined in (36), with eigenvalue 1, that is $\det(A_{\tau,\tilde{q}}(k) - \mathbf{1}) = 0$.

Using (36) we apply Lemma 4.12 with

$$\mathbb{B}_0 = A_{\tau,\tilde{q}}(0) = \left(\omega_{i,p} e^{-\tilde{q}\tau_{i,p}(0,1) + \xi_{i,p}(0,1)}\right)_{i,p=1}^{n}$$

and

$$\mathbb{B} = A_{\tau,\tilde{q}}(k) = \left(\omega_{i,p} e^{(-\tilde{q} + 2\pi\mathbf{i}k/\tau)\tau_{i,p}(0,1) + \xi_{i,p}(0,1)}\right)_{i,p=1}^{n},$$

that is with $\vartheta_{i,p} = 2\pi k \frac{\tau_{i,p}(0,1)}{\tau}$ for every $k \in \mathbb{Z}$. According to Proposition 4.7 and Definition 4.20, $\underline{b} = \mathscr{F}_{\tau,\tilde{q}}[u_0](0)$ is a positive vector satisfying (45), so we have that either $\mathscr{F}_{\tau,\tilde{q}}(k)$ is null for every $k \neq 0$ or

$$\prod_{l=1}^{s} e^{\mathbf{i}\vartheta_{i_l,p_l}} = \prod_{l=1}^{s} e^{2\pi\mathbf{i}k(\tau_{i,p}(0,1)/\tau)} = \exp\left(\frac{2k\pi\mathbf{i}}{\tau}\sum_{l=1}^{s}\tau_{i_l,p_l}(0,1)\right) = 1$$

whenever the edges $e_{v_{i_1},v_{p_1}},\dots,e_{v_{i_s},v_{p_s}}$ form a cycle in G. Hence τ is a possible period only if

$$\frac{k}{\tau}\sum_{l=1}^{s}\tau_{i_l,p_l}(0,1) \in \mathbb{Z}$$

for every cycle and k is such that $\mathscr{F}_{\tau,\tilde{q}}(k) \neq 0$. Since then the actual period of the orbit is $\tau = \tau/\gcd\{k : \mathscr{F}_{\tau,\tilde{q}}(k) \neq 0\}$ (cf. (35)), we have that $\gcd\{k : \mathscr{F}_{\tau,\tilde{q}}(k) \neq 0\} = 1$. This implies that

$$\frac{1}{\tau}\sum_{l=1}^{s}\tau_{i_l,p_l}(0,1) \in \mathbb{Z}$$

for every cycle. In particular, in the definition of γ, c can be taken as $1/\tau$. So $\gamma = \frac{l(1/\tau)}{1/\tau} = \tau l(1/\tau)$ where $l(1/\tau) \in \mathbb{Z}$, so indeed τ is of the form $\gamma/\mathbb{Z}$. This finishes the proof. □

Applying now the result of Nagel [16, Theorem 4.3] generalized in [12, Theorem 4.5] we obtain that under ($\mathrm{LD}_{\mathbb{Q}}$) and strong connectivity of the graph, the rescaled semigroup $(\tilde{T}(t))_{t\geq 0}$ behaves asymptotically as a periodic group on a function space.

Theorem 4.14. *Suppose that the condition* ($\mathrm{LD}_{\mathbb{Q}}$) *holds and that the graph G is strongly connected. Then the decomposition* $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ *from Proposition 4.11 has the following additional properties.*

1. $\mathfrak{L}_1$ *is a closed sublattice of* $\mathfrak{L}$ *isomorphic to* $L^1(\Gamma)$, *where* Γ *is the unit circle.*
2. *The group* $(S(t))_{t\geq 0}$ *is isomorphic to the rotation group on* $L^1(\Gamma)$ *with period*

$$\tau = \gamma = \frac{1}{c}\gcd\{c(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)); e_{j_1},\dots,e_{j_k} \text{ form a cycle in } G\}, \tag{56}$$

where γ *is defined in* (41) *and c is any number such that* $c(\tau_{j_1}(0,1)+\cdots+\tau_{j_k}(0,1)) \in \mathbb{N}$ *for all* $e_{j_1},\dots,e_{j_k}$ *which form a cycle in G.*

Proof. By Lemma 4.5, the semigroup $(\tilde{T}(t))_{t\geq 0}$ is irreducible, positive and bounded. Since $s(\tilde{A}) = 0$ and because of the compactness of the resolvent, 0 is a pole of $R(\lambda,\tilde{A})$. By the above corollary, we also know that there are nonzero spectral points on the imaginary axis. So, all the conditions of [17, C-IV, Lemma 2.12] and [17, C-IV, Theorem 2.14] are fulfilled, and we obtain the statements 1. and the first half of 2.

By [17, C-IV, Lemma 2.12 (c)] the period τ equals $\frac{2\pi}{\alpha}$, where $\alpha \in \mathbb{R}$ is determined by

$$\sigma(\tilde{A}) \cap \mathbf{i}\mathbb{R} = i\alpha\mathbb{Z}.$$

Due to (55), formula (56) holds. □

In less technical terms the above result can be expressed as follows.

Corollary 4.15. *Under the assmptions of Theorem 4.14 the rescaled semigroup* $(\tilde{T}(t))_{t\geq 0}$ *is* asymptotically periodic *with period*

$$\tau = \frac{1}{c}\,\mathrm{gcd}\{c(\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1)); e_{j_1}, \ldots, e_{j_k} \textit{ form a cycle in } G\},$$

where c is any number such that $c(\tau_{j_1}(0,1) + \cdots + \tau_{j_k}(0,1)) \in \mathbb{N}$ for all $e_{j_1}, \ldots, e_{j_k}$ that form a cycle in G.

Remark 4.16. Observe that the period does not depend on the weights on the edges.

We now state the corresponding result on the state space $\mathfrak{M}$.

Corollary 4.17. *If* $(\mathrm{LD}_{\mathbb{Q}})$ *holds,* $\tilde{T}(t)\mu$ *converges in* $\|.\|_{\mathfrak{M}}$*-norm to a periodic solution for every* $\mu \in \mathfrak{M}$*, as well.*

Proof. With the notation of Proposition 4.11, we show first that for any positive C,

$$\|\tilde{T}(t)\mu - \tilde{T}(s)\mu\|_{\mathfrak{M}} \leq 3Me^{-\varepsilon C}$$

whenever $s, t \geq C$ and $t - s$ is a multiple of τ, the period of $(S(t))$. Suppose that this is not true, i.e., for a suitable functional $\varphi \in C[0,1]^m$ with $\|\varphi\|_\infty = 1$ we have

$$|\langle \tilde{T}(t)\mu - \tilde{T}(s)\mu, \varphi\rangle| > 3Me^{-\varepsilon C}. \tag{57}$$

Since $(\tilde{T}(t))_{t\geq 0}$ on $\mathfrak{M}$ is the $w^\star$-extension of $(\tilde{T}(t))_{t\geq 0}$ on $\mathfrak{L}$, there is an $f \in \mathfrak{L}$ such that

$$|\langle \tilde{T}(t)\mu - \tilde{T}(t)f, \varphi\rangle| + |\langle \tilde{T}(s)\mu - \tilde{T}(s)f, \varphi\rangle| \leq Me^{-\varepsilon C}.$$

Then by Proposition 4.11.2,

$$\begin{aligned}|\langle \tilde{T}(t)\mu - T(s)\mu, \varphi\rangle| \leq{}& |\langle \tilde{T}(t)\mu - \tilde{T}(t)f, \varphi\rangle| \\ &+ |\langle (\tilde{T}(t) - S(t))f - (\tilde{T}(s) - S(s))f, \varphi\rangle| \\ &+ |\langle \tilde{T}(s)f - \tilde{T}(s)\mu, \varphi\rangle| \leq 3Me^{-\varepsilon C}.\end{aligned}$$

This contradicts (57).

So for every fixed $t \geq 0$, the sequence $(\tilde{T}(t + n\tau)\mu)_{n \in N}$ is Cauchy; thus since $\mathfrak{M}$ is complete $(\tilde{T}(t + n\tau)\mu)_{n \in \mathbb{N}}$ converges. By the semigroup property, it defines a periodic orbit, which finishes the proof. □

In [24, Chapter 4] one can find some examples for this phenomena on concrete oriented graphs. In the following we present one of these. For the sake of simplicity, in the system (F) we set constant velocities and no absorption—that is, $c_j \equiv 1$ and $q_j \equiv 0$ on all edges. By Theorem 4.14 the system is periodic with period equal to gcd{cycle lengths}, hence in our case equal to 1. As we also have noticed in Remark 4.16, the period does not depend on the weights of the edges in (BC). Investigating the velocity of the convergence—that is the (optimal) value of ε in Proposition 4.11 for which

$$\|\tilde{T}(t) - S(t)\|_{\mathfrak{L}} \leq Me^{-\varepsilon t}$$

—, it turns out that the weights can have an influence to it. From the characteristic equation (24) for equal velocities (see also [12, Corollary 3.6]) and the Circular Spectral Mapping Theorem (Theorem 4.10) follows that

$$(58) \qquad \varepsilon = -\log r,$$

where r is the second largest absolute value—that is, the largest absolute value different from 1—of the spectral points of $\mathbb{A}_0$ in (23) (in the case $c_j \equiv 1$ and $q_j \equiv 0$).

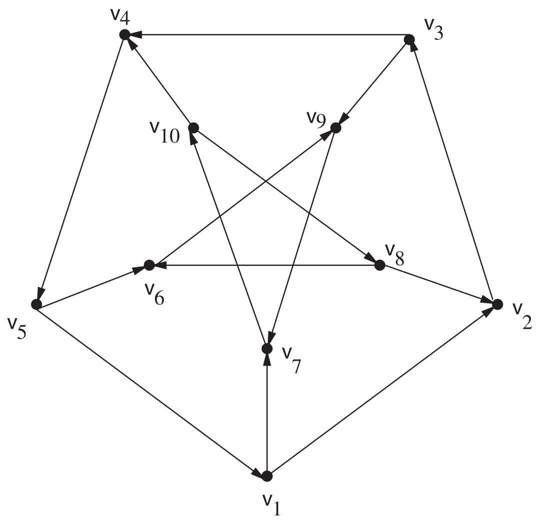

Fig. 1. An oriented Petersen graph

Example 4.18. We consider an orientation of the well-known *Petersen graph*.[‡]

1. We first investigate in general how the weight in v_1 as parameter effects the convergence speed to the periodic semigroup. Let

$$\mathbb{A}_0 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 1-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

for $0 < a < 1$. The characteristic polynomial becomes $p(z) = z^{10} - 0.5z^5 - 0.375z^4 - 0.0625(1+a) \cdot z^2 - 0.0625z + 0.0625a$. Clearly, $z_1 = 1$ is a root of $p(z)$. Dividing $p(z)$ by $z - 1$ yields a polynomial $p_1(z)$ whose root with the greatest absolute value—depending on a—is the value r occuring in (58). Actually, r is the greatest absolute value of the roots of the polynomial $\tilde{p}_1(z) = 16z^9 + 16z^8 + 16z^7 + 16z^6 + 16z^5 + 8z^4 + 2z^3 + 2z^2 + (1-a)z - a$.

One can compute that the value of r has a minimum at approximately $a \approx 0.6085 \pm 0.0003$, which means that in this case the convergence speed is maximal.

2. We now take equal outgoing flow proportions in v_1 and investigate the effect of different proportions in the "inner" vertex v_8.

$$\mathbb{A}_0 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 1-a & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The maximal convergence speed now is attained at $a \approx 0.56295 \pm 0.00006$.

‡ The next computations are made with Maple.

4.2 ($LI_{\mathbb{Q}}$) case

We have seen that linearly dependent average speeds result in periodic orbits. As we will show now, ($LD_{\mathbb{Q}}$) is also necessary for the existence of nontrivial limit flows. First we prove that under the condition ($LI_{\mathbb{Q}}$) the eigenvalue structure of $\tilde{A}$ is completely different from the ($LD_{\mathbb{Q}}$) case. Before giving the description of the spectrum, we rewrite ($LI_{\mathbb{Q}}$) in an equivalent form.

Lemma 4.19. *Under condition* ($LI_{\mathbb{Q}}$), *for every fixed* $\delta > 0$ *and* $K \in \mathbb{R}$ *there is a* $\rho > K$ *such that if the edges* $e_{i_1,p_1}, e_{i_2,p_2}, \dots, e_{i_k,p_k}$ *form a cycle in* G *one has*

$$|\rho(\tau_{i_1,p_1}(0,1) + \cdots + \tau_{i_k,p_k}(0,1)) - 2\pi l| < \delta \tag{59}$$

for an appropriate $l \in \mathbb{Z}$ *depending on the cycle.*

Proof. It follows from Diophantine approximation, see e.g. [20, Proposition V.58]. □

We now proceed to the description of the spectrum of $\tilde{A}$. Proposition 4.7.1 in the first part of this section allows the following definition.

Definition 4.20. We denote by $u_0 \in \mathfrak{L}$ the unique positive element of $\mathfrak{L}_0 := \ker \tilde{A}$ with $\mathscr{F}[u_0]_1^{1,\tilde{q}}(0) + \cdots + \mathscr{F}[u_0]_n^{1,\tilde{q}}(0) = 1$, while $Pr_{\mathfrak{L}_0} : \mathfrak{M} \to \mathfrak{L}_0$ is the projection mapping

$$u \mapsto \tilde{u} := (\mathscr{F}[u]_1^{1,\tilde{q}}(0) + \cdots + \mathscr{F}[u]_n^{1,\tilde{q}}(0))u_0.$$

Theorem 4.21. *Suppose that* ($LI_{\mathbb{Q}}$) *holds.*

1. *Then there is no nontrivial periodic orbit of* $(\tilde{T}(t))_{t\geq 0}$ *in the state space* $\mathfrak{M}$ *(so neither in* $\mathfrak{L}$*), or equivalently, with the notations of Proposition 4.7,* $\mathfrak{L}_1$ *contains only the trivial periodic solutions, that is* $\mathfrak{L}_0 = \mathfrak{L}_1$. *In particular,* $\sigma(\tilde{A}) \cap \mathbf{i}\mathbb{R} = \{0\}$.

2. *On the other hand, for every* $\eta \in \sigma(\tilde{A})$ *and every rectangle*

$$R_{\varepsilon,K} = \{z \in \mathbb{C} \mid \operatorname{Re}\eta - \varepsilon \leq \operatorname{Re} z \leq \operatorname{Re}\eta + \varepsilon, \operatorname{Im}\eta + K \leq \operatorname{Im} z\}$$

we have $\sigma(\tilde{A}) \cap R_{\varepsilon,K} \neq \emptyset$.

Proof. For 1, suppose that $\tilde{T}(t)\mu$ is a periodic orbit with period τ. We show that, with the notation of Definition 4.20, $\mu = \tilde{\mu}$. Consider the Fourier coefficients of (37) and the characteristic equation in (38). For every $k \in \mathbb{Z}$, we have that either $\mathscr{F}_{\tau,\tilde{q}}(k)$ is an eigenvector of the matrix $A_{\tau,\tilde{q}}(k)$ with eigenvalue 1, that is $\det(A_{\tau,\tilde{q}}(k) - \mathbf{1}) = 0$, or $\mathscr{F}_{\tau,\tilde{q}}(k)$ is null.

Similarly to the ($LD_{\mathbb{Q}}$) case we apply Lemma 4.12 with $\mathbb{B}_0 = A_{\tau,\tilde{q}}(0)$, $\mathbb{B} = A_{\tau,\tilde{q}}(k)$ for every $k \in \mathbb{Z}$. We have that either $\mathscr{F}_{\tau,\tilde{q}}(k)$ is null for every $k \neq 0$ or

$$\prod_{l=1}^{s} e^{\vartheta_{i_l,p_l}} = \prod_{l=1}^{s} e^{2\pi \mathbf{i} k(\tau_{i,p}(0,1)/\tau)} = \exp\left(\frac{2k\pi\mathbf{i}}{\tau}\sum_{l=1}^{s}\tau_{i_l,p_l}(0,1)\right) = 1$$

whenever the edges $e_{v_{i_1},v_{p_1}},\dots,e_{v_{i_s},v_{p_s}}$ form a cycle in G. This contradicts $(\mathrm{LI}_{\mathbb{Q}})$, so $\mathscr{F}_{\tau,\tilde{q}}(k)$ is null for every $k \neq 0$, that is $\mu = \tilde{\mu}$.

We now prove the second statement. We consider the holomorphic function $h(\lambda) = \det(\mathbf{1} - \mathbb{A}_\lambda)$. By (24), $\lambda \in \sigma(\tilde{A})$ if and only if $h(\lambda) = 0$.

For every $\eta = \alpha + \mathbf{i}\beta \in \sigma(\tilde{A})$ and rectangle $R := R_{\varepsilon,K}$ we construct another rectangle $R' \subset R$ such that the curve $h(\partial R')$ goes around zero with the multiplicity of the root η. This clearly proves the existence of a root of h in $R_{\varepsilon,K}$.

Fix a $\delta > 0$ and take $\rho > K$ according to Lemma 4.19. Being a determinant, $h(z)$ is a sum of terms of the form

$$ce^{z(\tau_{i_1,p_1}(0,1)+\cdots+\tau_{i_k,p_k}(0,1))}.$$

For $\alpha - \varepsilon \leq \Re z \leq \alpha + \varepsilon$ we have

$$(60)\qquad |ce^{(z+\mathbf{i}\rho)(\tau_{i_1,p_1}(0,1)+\cdots+\tau_{i_k,p_k}(0,1))} - ce^{z(\tau_{i_1,p_1}(0,1)+\cdots+\tau_{i_k,p_k}(0,1))}|$$

$$(61)\qquad \leq |c|e^{(\alpha+\varepsilon)(\tau_{i_1,p_1}(0,1)+\cdots+\tau_{i_k,p_k}(0,1))}|e^{\mathbf{i}\rho(\tau_{i_1,p_1}(0,1)+\cdots+\tau_{i_k,p_k}(0,1))} - 1|,$$

so by (59), there is a constant $C = C(A,\eta)$ for which

$$(62)\qquad |h(z+\mathbf{i}\rho) - h(z)| < \delta C$$

holds whenever $\alpha - \varepsilon \leq \operatorname{Re} z \leq \alpha + \varepsilon$.

Since $\sigma(\tilde{A})$ is discrete, one can find a rectangle

$$R'' \subset \{\alpha - \varepsilon \leq \operatorname{Re} z \leq \alpha + \varepsilon\}$$

such that $R'' \cap \sigma(\tilde{A}) = \{\eta\}$. Since η is a root of h, the curve $h(\partial R'')$ goes around zero with the multiplicity of η. If δ is small enough, by (62) for $R' := R'' + \mathbf{i}\rho$ the curve $h(\partial R')$ also goes around zero with the multiplicity of η, which completes the proof. □

In case $(\mathrm{LI}_{\mathbb{Q}})$, not only the periodic orbits but the uniform convergence is lost. The following theorem is the counterpart to the previous one.

Theorem 4.22. *Under condition* $(\mathrm{LI}_{\mathbb{Q}})$ *and using the notation of Definition 4.20 the following holds.*

1. $\tilde{T}(t)$ *converges strongly to* $Pr_{\mathfrak{L}_0}$ *on* $\mathfrak{L}$.

2. $\tilde{T}(t)$ *converges weakly uniformly to* $Pr_{\mathfrak{L}_0}$ *on* $\mathfrak{M}$.

Proof. The strong convergence on $\mathfrak{L}$ means that

$$(63) \qquad \|(\tilde{T}(t)f - \tilde{f})\|_{\mathfrak{L}} \to 0$$

for every $f \in \mathfrak{L}$. By Theorem 4.21.1, for the projection Q of Proposition 4.7.2 we have ran $Q = \mathfrak{L}_0$, so $Q = Pr_{\mathfrak{L}_0}$ which gives (63).

From this weak uniform convergence on $\mathfrak{M}$ immediately follows. For every function $\varphi \in C[0,1]^m$, we show that $\langle \tilde{T}(t)\mu - \tilde{\mu}, \varphi \rangle \to 0$ uniformly in μ on the unit ball of $\mathfrak{M}$ denoted by $B(\mathfrak{M})$. Fix an $\varepsilon > 0$. Since $\mathfrak{L}$ is dense in the weakly compact set $B(\mathfrak{M})$, we can find an ε-net $\{f_1, \dots, f_N\} \subset \mathfrak{L}$ to φ, that is for every $\mu \in B(\mathfrak{M})$ there exists an index $1 \le i \le N$ such that $|\langle \mu - f_i, \varphi \rangle| \le \varepsilon$. By the strong convergence on $\mathfrak{L}$, there is a $t_0 > 0$ such that $\|\tilde{T}(t)f_i - \tilde{f}_i\|_{\mathfrak{L}} \le \varepsilon$ for every $t > t_0$ and $1 \le i \le N$. Let now $\mu \in B(\mathfrak{M})$ be arbitrary and assume that $|\langle \mu - f_i, \varphi \rangle| \le \varepsilon$ for a proper $1 \le i \le N$. Then for $t > t_0$

$$(64) \qquad |\langle (\tilde{T}(t)\mu - \tilde{\mu}), \varphi \rangle|$$

$$= |\langle (\tilde{T}(t)(\mu - f_i) - (\tilde{\mu} - \tilde{f}_i), \varphi \rangle| + |\langle \tilde{T}(t)f_i - \tilde{f}_i, \varphi \rangle|$$

$$\le (M + 1 + \|\varphi\|_\infty)\varepsilon,$$

where M denotes a common norm bound of the projection $Pr_{\mathfrak{L}_0}$ and $(\tilde{T}(t))_{t \ge 0}$. This gives weak uniform convergence and finishes the proof. □

Remark 4.23. The weak uniform and the strong but not uniform convergence of $\tilde{T}(t)$ on $\mathfrak{M}$ and $\mathfrak{L}$ respectively proved in Theorem 4.22 are the best one can guarantee, in general. To illustrate this, consider the orbit of a Dirac mass δ under $(\tilde{T}(t))_{t \ge 0}$ associated to a network satisfying ($LI_{\mathbb{Q}}$). By Theorem 4.22.2, $\tilde{T}(t)\delta \to \tilde{\delta}$ weakly.

The measure $\tilde{\delta} = Pr_{\mathfrak{L}_0}\delta$ is nonzero, so $\|\tilde{\delta}\|_{\mathfrak{M}} \ne 0$. On the other hand, it is easy to see that $\tilde{T}(t)\delta$ is a finite sum of positive Dirac measures for every $t > 0$. So if t is fixed, for every τ sufficiently small one has

$$(65) \qquad \operatorname{supp} \tilde{T}(t+\tau)\delta \cap \operatorname{supp} \tilde{T}(t)\delta = \emptyset,$$

hence

$$(66) \qquad \|\tilde{T}(t+\tau)\delta - \tilde{T}(t)\delta\|_{\mathfrak{M}} = \|\tilde{T}(t+\tau)\delta\|_{\mathfrak{M}} + \|\tilde{T}(t)\delta\|_{\mathfrak{M}} \to 2\|\tilde{\delta}\|_{\mathfrak{M}}$$

as $t \to \infty$. So strong convergence on $\mathfrak{M}$ is not possible under condition ($LI_{\mathbb{Q}}$).

By approximating δ with absolutely continuous measures we immediately have that the convergence on $\mathfrak{L}$ cannot be uniform. If for some $t_0, t_1 > 0$, $\operatorname{supp} \tilde{T}(t_0)\delta \cap \operatorname{supp} \tilde{T}(t_1)\delta = \emptyset$ holds, one can find a function $\varphi \in C[0,1]^m$, $\|\varphi\|_\infty \le 1$, for which $\langle \tilde{T}(t_i), \varphi \rangle = i\|\tilde{T}(t_1)\delta\|_{\mathfrak{M}}$, $i = 0, 1$. Since $\tilde{T}(t)$ is weakly continuous and $\mathfrak{L}$ is weakly dense in $\mathfrak{M}$, we can find an $f_{t_0,t_1} \in \mathfrak{L}$, $\|f_{t_0,t_1}\|_{\mathfrak{L}} = 1$, satisfying

$$|\langle \tilde{T}(t_i)f_{t_0,t_1} - \tilde{T}(t_i)\delta, \varphi\rangle| \leq 1/3\|\tilde{\delta}\|, \quad i = 0, 1.$$

Applying this for any $t_0 = t$ and $t_1 = t + \tau$ satisfying (65) we have a function $f_t := f_{t,t+\tau}$ so that according to (66)

$$\begin{aligned}(67)\qquad \|\tilde{T}(t+\tau)f_t - \tilde{T}(t)f_t\|_{\mathfrak{L}} &\geq |\langle \tilde{T}(t+\tau)f_t - \tilde{T}(t)f_t, \varphi\rangle| \\ &\geq |\langle \tilde{T}(t+\tau)\delta - \tilde{T}(t)\delta, \varphi\rangle| \\ &\quad - |\langle \tilde{T}(t+\tau)\delta - \tilde{T}(t+\tau)f_t, \varphi\rangle| - |\langle \tilde{T}(t)f_t - \tilde{T}(t)\delta, \varphi\rangle| \\ &\geq \|\tilde{\delta}\|_{\mathfrak{M}} - 1/3\|\tilde{\delta}\|_{\mathfrak{M}} - 1/3\|\tilde{\delta}\|_{\mathfrak{M}} = 1/3\|\tilde{\delta}\|_{\mathfrak{M}}\end{aligned}$$

for every t sufficiently large. By Theorem 4.22.1, $\tilde{T}(t)f_t \to \tilde{f}_t$ in norm so by (67) this convergence cannot be uniform.

Note also that the converse of Theorem 4.22.2 is *not* true. To illustrate this, one might take a purely atomic measure μ so that in a suitable translated copy of μ the different atoms in μ cancel each other, that is $\tilde{T}(t)\mu = 0$ for every t sufficiently large.

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