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Asymptotic behavior of large solutions of elliptic equations

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Abstract. In this paper we give a survey of the result concerning the asymptotic behavior of the solutions of $\triangle u = f(u)$ in D which blow up at the boundary. We concentrate ourselves to the case where the blowup occurs on the whole boundary. The main tools to derive sharp estimates are the comparison principle and the method of upper and lower solutions. A list of references is given which are closely related to the specific aspects discussed in this survey. This list is by no means complete.

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1. Introduction

Let $D \subset \mathbb{R}^N$ be an arbitrary domain the boundary of which satisfies an inner and outer sphere condition, and let f be a positive, increasing function. In this paper we recount some results on the asymptotic behavior of the solutions of

$$\triangle u = f(u) \quad \text{in} \quad D, \quad u(x) \to \infty \quad \text{as} \quad x \to \partial D. \tag{1}$$

From the maximum principle it follows that $u(x) \geq v(x)$ for any other solution of $\triangle v = f(v)$ with bounded boundary values. Therefore $u(x)$ is called a *large solution*. Problem (1) is best understood by first looking at the one-dimensional case

$$\phi'' = f(\phi), \quad x > 0, \quad \phi(x) \to \infty \quad \text{as} \quad x \to 0. \tag{2}$$

The solutions are given implicitely by

$$x = \int_\phi^\infty \frac{ds}{\sqrt{2F(s)}} =: \psi(\phi), \quad \text{where} \quad F' = f.$$

Clearly blow-up occurs if and only if

$$\int^\infty \frac{ds}{\sqrt{2F(s)}} < \infty. \tag{3}$$

Observe that this condition is independent of the particular choice of the primitive F.

It turns out [14] that also in general bounded domains (3) is necessary and sufficient for the existence of a solution of (1).

The following observation [7] will be crucial for some of our next results.

Remark 1.1. *Let ϕ and ϕ_c solutions of (2) corresponding to the primitives F and $F + c$, respectively. Then $\phi(x) - \phi_c(x) \to 0$ as $x \to 0$.*

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Surprisingly the asymptotic behavior of the solutions of (1) does not depend, in the first order approximation, on the geometry of D. In fact, if $\delta(x)$ stands for the distance of a point $x \in D$ to the boundary, then [6], [1]

$$\lim_{x \to \partial D} \psi[u(x)]/\delta(x) = 1. \tag{4}$$

In order to conclude that

$$\lim_{x \to \partial D} \frac{u(x)}{\phi(\delta(x))} = 1 \tag{5}$$

an additional assumption on ψ, e.g.

$$\liminf_{t \to \infty} \psi(\beta t)(\psi(t) > 1, \quad \forall \beta \in (0,1) \tag{6}$$

is needed. The latter condition is not a serious restriction. It is clearly satisfied for the classical model cases such as $f(t) = t^p$ and e^t.

Problem (1) has a long history which can be traced back to Bieberbach [8] who studied the existence and asymptotic behavior for $f(t) = e^t$ in planar domains. He showed that in this case $|u(x) - \phi(\delta(x))| < c$ near the boundary. Because of Remark 1 this is true for any solution ϕ. In simply connected domains the large solutions are expressed by means of the conformal map $f : D \to \{|z| < 1\}$. Liouville cf. also [8] showed that

$$u(z) = \log \frac{8|f'(z)|^2}{(1 - |f(z)|^2)^2}, \quad z \in D.$$

Lazer and McKenna [18], motivated by Bieberbach's result asked the following question: *Under what conditions on f do the solutions of (1) satisfy*

$$u(x) - \phi(\delta(x)) \to 0, \quad \text{as} \quad x \to \partial D?$$

They were able to establish this property for a class of functions with fast growth at infinity such as $f = e^t$ and $f = t^p$, for $p > 3$. Some generalizations and further discussion are found in [7].

The question we want to address here is: *when does the geometry of D come into play in the asymptotic development?*

A first answer comes from the following result for large radial solutions in an annulus $\{R_0 < |x| < R_1\}$, [7], namely:

$$u(\delta) = \phi\left(\delta - \frac{N-1}{2R_1}(1 + o(1)) \int_0^\delta \Gamma(\phi(s))ds\right) \quad \text{as} \quad r \to R_1,$$

$$u(\delta) = \phi\left(\delta + \frac{N-1}{2R_0}(1 + o(1)) \int_0^\delta \Gamma(\phi(s))ds\right) \quad \text{as} \quad r \to R_0,$$

$$\text{where} \quad \Gamma(t) := \frac{\int_0^t \sqrt{2F(s)}ds}{F(t)}.$$

Condition (3) implies that $\lim_{t \to \infty} \Gamma(t) = 0$ (cf. first remark at the end of the section). Therefore the term $\int_0^\delta \Gamma(\phi)ds$ represents a secondary effect in the blowup behavior of the solutions. The expressions $-1/R_0$ and $1/R_1$ can be interpreted as the mean curvature of the inner, resp. outer boundary.

The result for the annulus leads to the conjecture that the second order effect involves the mean curvature H_0 of the projection σ of x on the boundary. The first result in this direction is due to del Pino and Letelier [9] who proved that for $f(t) = t^p$ with $p < 3$

$$u(x) = \Phi(\delta(x)) \left(1 + \frac{N-1}{p+3} H_0(\sigma(x))\delta(x) + o(\delta(x))\right) \text{ as } x \to \partial D,$$
$$\text{where} \quad \Phi(\delta) = c_p \delta^{-\frac{2}{p-1}}, \quad c_p := ((p-1)/\sqrt{2(p+1)})^{-2/(p-1)}. \tag{7}$$

It follows from the results in [4] that (7) holds for all $p > 1$. A different approach via Fuchsian reduction was used by Kichenassamy [15] for the Loewner-Nirenberg problem where $f(t) = N(N-2)t^{\frac{N+2}{N-2}}$. He was able to establish (7) for this particular case. For $f(t) = e^t$ it turns out [4], cf. also [16] that

$$u(x) = \log \frac{2}{\delta^2(x)} + (N-1)H_0(\sigma(x))\delta(x) + o(\delta(x)) \text{ as } x \to \partial D. \tag{8}$$

At present, the most general result on the influence of the mean curvature is given in [4]. In order to state it we have to introduce the following expressions .

$$B(t) = \frac{f(t)}{\sqrt{2F(t)}} = \frac{d}{dt}\sqrt{2F(t)},$$
$$J(t) = \frac{N-1}{2}\int_0^t \Gamma(\phi(s))ds,$$

where ϕ is any solution of (2).

Suppose that

$$(i) \qquad \lim_{\delta \to 0} \frac{B(\phi(\delta(1+o(1)))}{B(\phi(\delta))} = 1,$$
$$(ii) \qquad \limsup_{t \to \infty} B(t)\Gamma(t) < \infty. \tag{9}$$

Then

Assume (3), (6), (9) and $\partial D \in C^4$. Then the large solutions satisfy

$$|u(x) - \phi(\delta - H_0 J(\delta))| \le \phi(\delta) o(\delta). \tag{10}$$

As a consequence we get (7) and (8) for domains with smooth boundaries.

By means of a scaling argument [1] and [5] it is possible to determine the asymptotic behavior of the gradient of a solution of (1) when $f(t)$ behaves for large t like a power t^p, $p > 1$ or like the exponential e^t. In fact we have

$$\frac{|\nabla u|^2}{\phi'^2(\delta(x))} \to 1 \quad \text{as} \quad x \to \partial D, \tag{11}$$

where ϕ is a solution of (2) corresponding to $f(t) = t^p$ or e^t.

1.1. Remarks and open problems.

(1) For monotone functions condition (3) implies that

$$B(t) \to \infty, \quad \text{as} \quad t \to \infty, \quad \left(B(t) = -\frac{\phi''}{\phi'}\right).$$

From here it follows that $\frac{F(t)}{t^2} \to \infty$ as $t \to \infty$. This observation is well-known. The simplest proof is found in [12]. Assumption (9) are satisfied for the standard nonlinearities. It would be interesting to know whether or not this is a serious restriction and for what nonlinearities it does not hold.

(2) It is clear that in non-smooth domains even (5) can not be true. The first result in non-smooth domains seems to go back to [25]. If the domain is Lipschitz and $f(t) = t^p$, (5) remains valid if x tends to a regular point on ∂D, [25]. However if the limiting point is irregular, e.g. a corner, the rate of blow-up depends on the direction of approach. This behavior is illustrated by the large solutions for $f(t) = t^p$, $1 < p < \frac{N}{N-2}$ in open cones $\mathcal{C} = \{x = r\theta : r > 0, \theta \in \Omega \subset \mathbb{S}^{N-1}\}$, $\mathbb{S}^{N-1} = \{|x| = 1\}$. Let $\triangle_\theta$ be the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$ and let $\alpha(\theta)$ be the large solution of

$$\triangle_\theta \alpha = \alpha^p - \frac{2}{p-1}(2 - N + 2/(p-1))\alpha \quad \text{in} \quad \Omega.$$

The large solutions in the cone can then be written as

$$u(x) = r^{-\frac{2}{p-1}}\alpha(\theta),$$

where in accordance with the observation in Sec. 3.1

$$\Big((p-1)/\sqrt{2(p+1)}\Big)^{2/(p-1)} \tilde{\delta}(\theta)^{2/(p-1)}\alpha(\theta) \to 1, \quad \text{as} \quad \theta \to \partial\Omega,$$
$$\tilde{\delta}(\theta) = \text{distance}\{\theta, \partial\Omega\} \quad \text{on} \quad \mathbb{S}^{N-1}.$$

Similarly for $f(t) = e^t$, [22] there exist large solutions in $\mathcal{C}$ of the form

$$u(x) = \alpha(\theta) - 2\log r,$$

where α solves

$$\triangle_\theta \alpha = 2(N-2) + e^\alpha \quad \text{in} \quad \Omega, \quad \alpha \to \infty \quad \text{if} \quad \theta \to \partial\Omega.$$

In [22] J. Matero studied large solutions in fractal domains and derived estimates near the boundary. In his thesis [24] he gave a fairly complete list of references for the blow-up problem up to 1997.

(3) It has already been observed in the seminal paper by Loewner and Nirenberg [20] that (5) leads to the uniqueness of large solutions if $f(t)/t$ is increasing. This applies to all power nonlinearities $f(t) = t^p$ with $p > 1$. The uniqueness for the exponential function was discussed in [19]. The case of non-smooth domains where (5) does not hold precisely, was studied for $f(t) = t^p$ in [25], cf. also [11] for a problem with variable coeffficients. A further uniqueness result follows from (10). Namely if

$$\delta\phi(\delta(1 + o(1))) < \infty \quad \text{for} \quad \delta < \delta_0$$
$$\text{or equivalently} \quad \frac{F(t)}{t^4} < \infty \quad \text{as} \quad t \to \infty,$$

then there exists a unique large solution. Indeed since the difference of two large solutions tends to zero as x approaches the boundary, by the maximum principle they have to coincide.

If there exists more than one large solution there is a maximal one, $U(x)$ obtained by the Perron process

$$U(x) = \sup\{v(x) : v(x) \quad \text{solution of} \quad \triangle v = f(v) \quad \text{in} \quad D\}.$$

The minimal large solution, $\omega(x)$ can be constructed by means of the iteration process

$$\triangle\omega_n = f(\omega_{n-1}) \quad \text{in} \quad D, \quad \omega_n = n \quad \text{on} \quad \partial D,$$

if we set $\omega = \lim_{n\to\infty}\omega_n$. All other large solutions satisfy $\omega(x) \le u(x) \le U(x)$ in D.

Except in strips (cf. Remark 1) it is not yet clear whether the large solution is unique only under the assumption that f is monotone. No example with two large solutions is known up to now. It is possible that the uniqueness property is not only related to the nonlinearity but also to the geometry of the domain.

(4) The global geometry of D can already appear in the third term of the asymptotic behavior. For instance if $f = e^t$ and D is a strip, then the general solutions which blow up at the origin are

$$\phi(x) = \begin{cases} \log \frac{a^2}{\sinh^2(ax/\sqrt{2})} \\ \text{or} \\ \log \frac{a^2}{\sin^2(ax/\sqrt{2})} \end{cases}$$

The constant a is determined by the width of the strip.

Near the origin ϕ assumes the form

$$\phi(x) = \log \frac{2}{x^2} \pm \frac{(ax)^2}{6} + o((ax)^2).$$

2. Tools

The main tools for deriving estimates are the comparison principle and upper and lower solutions.

Comparison principle *Let $D_1 \subset D_2$ and let u_1 and u_2 be corresponding large solutions. Then $u_1 \geq u_2$ in D_1.*

Definition 2.1. *A function $\overline{u}$, $(\underline{u})$ is called an upper (lower) solution of (1) in Ω if*

$$\triangle \overline{u} \leq f(\overline{u}) \quad in \quad \Omega.$$

For the lower solution the inequality sign is reversed.

From the maximum principle it follows that, if $u \leq \overline{u}$ on $\partial\Omega$ then $u \leq \overline{u}$ in Ω and similarly if $u \geq \underline{u}$ on $\partial\Omega$ then $u \geq \underline{u}$ in Ω .

The asymptotic estimate (10) follows from suitable choice of upper and lower solutions in the parallel strip near the boundary $D_\rho = \{x \in D : \text{dist}(x, \partial D) < \rho\}$.

Candidates are cf. [4] and [9], $\overline{u} := \nu^+$ and $\underline{u} := \nu^-$, where

$$v^{\pm}(\delta, \sigma) := \Phi\left(\delta - \frac{N-1}{2}(H_0(\sigma) \pm \nu) \int_0^\delta \Gamma(\phi(s)) ds\right). \tag{12}$$

Here ϕ is a solution of the one-dimensional problem (2). Then under the assumptions of Theorem 1 , there exists $\rho_0 > 0$ and a decreasing function $\nu : (0, \rho_0) \mapsto (0, \infty)$ such that $\nu(\rho) \to 0$ as $\rho \to 0$ and, for every $\rho \in (0, \rho_0)$, the function v^+ (resp. v^-) with $\nu = \nu(\rho)$, is an upper (resp. lower) solution in D_ρ. By a shift $\delta \pm \epsilon$, it can be achieved that $u \leq \nu^+$ and $u \geq \nu^-$ on the boundary of D_ρ which implies that these inequalities hold in the whole parallel strip D_ρ.

2.1. Remark and open problem. In order to show that $\nu^\pm$ are upper and lower solutions in the classical sense the mean curvature H_0 has to be in C^2. This implies that $\partial D \in C^4$. The approach by Kichenassamy [15] and [16] for $f(t) = e^t$ and $f(t) = t^{(N+2)/(N-2)}$ indicates that $\partial D \in C^{2+\alpha}$ should suffice. What is the third order term in non-smooth domains?

3. Generalizations

3.1. Inhomogeneous problems. The results of Section 1 have been generalized in many directions. In [5] the Laplace operator was replaced by a second order uniformly elliptic operator

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i}, \quad b_i, \quad a_{ij} = a_{ji} \in C^\alpha(\overline{D}).$$

and by a inhomogeneous nonlinearity of the following type

$$g(x,t)/f(t) \to h(x) > 0 \quad \text{in} \quad \overline{D}, \text{as} \quad t \to \infty.$$

It turns out that the asymptotic behavior of

$$Lu = g(x,u) \quad \text{in} \quad D, \quad u(x) \to \infty, \quad as \quad x \to \partial D$$

is similar as for the homogeneous problem (1). If $\tilde{\delta}$ stands for the distance with respect to the Riemannian metric given by

$$ds^2 = \sum_{i,j=1}^{N} A_{ij} dx_i dx_j, \quad A_{ij} \quad \text{inverse matrix of} \quad a_{ij},$$

then

$$\frac{u(x)}{\phi(\sqrt{h(x)}\tilde{\delta}(x))} \to 1 \quad \text{as} \quad x \to \partial D. \tag{13}$$

Notice that this expression depends only on the principal part of L.

From this result we obtain immediately the asymptotic behavior of large solutions of (1) on manifolds. Indeed if $\triangle_{\mathcal{M}}$ is the Laplace Beltrami operator on the manifold $\mathcal{M}$ then the large solutions of

$$\triangle_{\mathcal{M}} u = h(x)f(u) \quad \text{in} \quad D \subset \mathcal{M}, \quad u(x) \to \infty \quad \text{on} \quad \partial D$$

satisfy $u(x)\phi(\sqrt{h(x)}\tilde{\delta}(x)) \to 1$ as x tends to the boundary. Here $\tilde{\delta}$ denotes the distance related to metric of $\mathcal{M}$.

3.2. Nonlinear differential operators. A lot of attention has been given to large solutions in the case of more general operators. J. Matero extended the results of [1] concerning the asymptotic behaviour of the large solutions and theirs gradients to the pseudo-Laplacian

$$\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

The results are very similar to the ones for the ordinary Laplacian. For instance the blow-up condition (4) reads as

$$\int^\infty \frac{ds}{F^{1/p}(s)} < \infty.$$

and (5) holds if ϕ is replaced by the inverse ϕ_p of

$$\psi_p(t) = \int_t^\infty \left[\frac{p}{p-1}F(s)\right]^{-1/p} ds.$$

An interesting extension is the case with nonlinear gradient terms [2],

$$\triangle u \pm |\nabla u|^q = f(u) \quad , q > 0.$$

The presence of the nonlinear gradient term has a significant influence on the existence and asymptotic behavior of large solutions. Different mechanisms are in competition, diffusion, convection and absorption. For instance, if $f(t) = e^t$ and $q < 2$ then the diffusion prevails and we have $u(x)/\log \delta^{-1}(x) \to 2$ as $x \to \partial D$ whereas if $q \geq 2$ in the case of positive convection $(+|\nabla u|^q)$, we find $\log u(x)/\log \delta^{-1}(x) \to q$ as $x \to \partial D$. Hence the behavior of the large solutions at the boundary is governed by the convection.

Problems of the type

$$\mathrm{div} g(|\nabla u|\nabla u) = f(u)k(|\nabla u|)$$

were discussed in [3] by means of the methods introduced in [18].

3.3. Problems and Remarks.

(1) An unexpected observation was made in [17] where it was shown that under condition (3) the problem

$$\triangle u = h(x)f(u) \quad \text{in} \quad D, \quad u(x) \to \infty \quad \text{as} x \to \partial D,$$

has a large solution even if h vanishes on ∂D provided that it is positive in a neighborhood of ∂D. It would be interesting to see if this is still true if h vanishes on a connected set inside D, reaching the boundary, e.g. if $D = \{|x| < 1, x \in \mathbb{R}^2\}$, $h = 0$ on $x_2 = 0$, and if it is positive elsewhere in $\overline{D}$. It is clear that (13) makes no sense if h is zero on the boundary.

(2) It would be interesting to know if the mean of curvature plays a role in the case of nonlinear operators and where does it appear.

(3) The gradient estimate (11) extends to problems with variable coefficients considered in Section 3.1 [5].

(4) A challenging problem posed by Dynkin and Kuznetzov [10] in connection with the study of superdiffusion is to characterize the trace of $Lu = |u|^{p-1}u$ in D. For recent progress on this question cf. [21].

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