The Annals of Statistics
1985, Vol. 13, No. 4, 1403–1417

ASYMPTOTIC BEHAVIOR OF M ESTIMATORS OF p REGRESSION PARAMETERS WHEN p^2/n IS LARGE; II. NORMAL APPROXIMATION

BY STEPHEN PORTNOY[1]

University of Illinois

In a general linear model, $Y = X\beta + R$ with Y and R n-dimensional, X a $n \times p$ matrix, and β p-dimensional, let $\hat{\beta}$ be an M estimator of β satisfying $0 = \Sigma x_i \psi(y_i - x_i'\beta)$. Let $p \to \infty$ such that $(p \log n)^{3/2}/n \to 0$. Then $\max_i |x_i'(\hat{\beta} - \beta)| \to_P 0$, and it is possible to find a uniform normal approximation for the distribution of $\hat{\beta}$ under which arbitrary linear combinations $a_n'(\hat{\beta} - \beta)$ are asymptotically normal (when appropriately normalized) and $(\hat{\beta} - \beta)'(X'X)(\hat{\beta} - \beta)$ is approximately χ_p^2.

1. Introduction. Consider the general linear model

$$Y = X\beta + R, \tag{1.1}$$

where Y and R are n-dimensional random vectors, X is a $n \times p$ matrix, β is p-dimensional, and the coordinates of R are independent and identically distributed. Let x_i denote the (column) vector in R^p whose coordinates form the ith row of X. Let ψ: $R \to R$ be given and consider the M estimator, $\hat{\beta}$, satisfying the vector equation

$$0 = \sum_{i=1}^{n} x_i \psi(Y_i - x_i'\beta). \tag{1.2}$$

In a companion paper (Portnoy, 1984a), hereafter called Part I, the author considered some recent history and some new results on the asymptotic behavior of $\hat{\beta}$ when p tends to infinity with n. Part I considered the analysis of variance and regression cases of (1.1) separately, providing consistency and asymptotic normality results in the ANOVA case and norm consistency ($\|\hat{\beta} - \beta\|^2 = \mathcal{O}_p(p/n)$) in the regression case essentially under the condition $p(\log p)/n \to 0$. In Section 3 of the present paper, the following results are presented:

(1) if $p^{3/2}(\log n)/n \to 0$ then $\max|x_i'(\hat{\beta} - \beta)| \to_P 0$;

(2) if $(p \log n)^{3/2}/n \to 0$ then for any sequence $\{a_n\}$ with $a_n \in R^p$, $a_n'(\hat{\beta} - \beta) \to_D \mathcal{N}(0, \sigma^2)$ [where $\sigma^2 = a_n'(X'X)^{-1}a_n E\psi^2(R)/(E\psi'(R))^2$], and (under stronger conditions) a uniform normal approximation for the distribution of $\hat{\beta}$ will hold which yields a χ_p^2 approximation for $(\hat{\beta} - \beta)'(X'X)(\hat{\beta} - \beta)$.

As noted in Part I, asymptotic normality for $a_n'(\hat{\beta} - \beta)$ should hold (under reasonable conditions) as long as $p^{1+\varepsilon}/n \to 0$ for some $\varepsilon > 0$. In fact, Portnoy

Received October 1982; revised November 1984.

[1]Research supported in part by National Science Foundation grants MCS-80-02340 and MCS-83-01834.

AMS 1980 *subject classifications.* Primary 62G35; secondary 62E20, 62J05.

Key words and phrases. M estimators, general linear model, asymptotic normality, consistency, robustness, regression.

(1984c) obtains higher order terms in the expansion of β which permits a weakening of the convergence rate to $(p^{11/8}\log^2 n)/n \to 0$. However, obtaining this rate requires four new terms (in addition to those of Lemma 3.4), two of which are fourth-order sums. The excessive complications in the expansions required to achieve so minor an improvement in rate clearly indicate the difficulty of typing to obtain rate $p^{1+\varepsilon}/n \to 0$. Huber (1981) conjectured that to obtain this rate may require that ψ be odd. However, the computations in Section 3 indicate that antisymmetry is not needed (antisymmetry appears only in condition P5 which is required only for a uniform normal approximation in R^p). Note that Section 3 also shows that error terms in the chi-square approximation for $(\hat{\beta} - \beta)'(X'X)(\hat{\beta} - \beta)$ are of order $p^{3/2}/n$, so that improved rates are not possible in complete generality.

The uniform normal approximation requires a Central Limit Theorem for $\Sigma x_i \psi(R_i)$ in R^p. When $p^2/n \to \infty$, known Central Limit Theorems do not apply. The best present results are discussed in Portnoy (1984b), which gives a counterexample showing that a general Central Limit Theorem can not hold if $p^2/n \to +\infty$. However, if $\{x_i\}$ form a sample from a mixed multivariate normal distribution [see (4.1)], then an appropriate normal approximation holds conditionally on X for $X \in B_n$ where $P(B_n) \to 1$ (and, hence, also unconditionally). The results are proved in Portnoy (1984d) and summarized in Section 4. It can also be shown that the conditions on X needed here hold in probability under (4.1) (see Portnoy, 1984a and 1984d).

2. The conditions. The results here require conditions of three types: conditions (denoted "N") on the rate at which p can increase, conditions (denoted "P") on the ψ function and jointly on the distribution of R, and conditions (denoted "X") on the design matrix.

For most results the following condition will be necessary:

$$N1\colon\ p^{3/2}(\log n)/n \to 0 \quad \text{as } n \to \infty.$$

A few results require a slightly stronger condition:

$$N1'\colon\ p^{3/2}(\log n)^{3/2}/n \to 0 \quad \text{as } n \to \infty.$$

Several results can be obtained under a weaker condition which will be listed separately:

$$N2\colon\ p(\log n)/n \to 0 \quad \text{as } n \to \infty.$$

REMARKS. (1) As noted in Part I, conditions "N" can actually be stated with a factor "$\log p$" instead of "$\log n$" since the conditions with "$\log p$" imply the conditions listed here. Also note that if N1 holds, then $p(\log n)^2/n \to 0$ as $n \to \infty$.

(2) In order to show that the "X" conditions hold in probability under (4.1), the condition $p/\log n \to \infty$ is needed. However, if $p/\log n \to 0$, it is not hard to use classical proofs to obtain the desired normality and consistency results.

The following conditions on ψ will be used:

P1: $E\psi(R) = 0$, $E\psi^2(R) = \sigma^2 < +\infty$;

P2: ψ is absolutely continuous with $d \equiv E\psi'(R) > 0$;

P3: For u and v real, define

$$Q(u, v) = \frac{\psi(u) - \psi(u - v)}{u} - d. \tag{2.1}$$

Then $Q(u, v)$ is uniformly bounded. Furthermore, $\psi'(u)$ is Lebesgue integrable and the distribution of R has a density, g, with a uniformly bounded first derivative.

P4: ψ has three bounded continuous derivatives and $\psi(R)$ has a finite moment generating function in a neighborhood of zero.

P5: $\psi(R)$ has a finite moment generating function in a neighborhood of zero and for some constant, $d > 0$, and for each n there exists a ψ function, ψ_n, satisfying the above part of P5 and condition P5′, such that

$$|\psi(u) - \psi_n(u)| \leq \frac{1}{n^{d+1}}. \tag{2.2}$$

P5′: ψ is an odd twice differentiable function, and g is an even differentiable density satisfying

$$e^{a\psi(r)} \cdot g(r)/\psi'(r) \to 0 \quad \text{as } r \to \pm\infty$$

for any sufficiently small constant, a. Also,

$$\left|\frac{g(r)}{\psi'(r)}\right| \leq Bn^{d'} \quad \text{for some constants } B \text{ and } d', \tag{2.3}$$

and $(d/dr)(g(r)/\psi'(r))$ has at most M sign changes (where M is a constant not depending on n).

REMARKS. (1) The results of Section 3 also require the consistency results of Part I which require that ψ be nondecreasing; and thus the monotonicity of ψ will be a tacit assumption in this paper.

(2) The differentiability conditions in P4 can be replaced by differentiability conditions on the density, g; and, hence, the results of Section 3 can be shown to hold for commonly suggested Huber- and Hampel-psi functions. In particular, if the differentiability conditions in P4 are replaced by the condition that $g^{(4)}$ be continuous, absolutely bounded and integrable, then it is possible to derive an expression for the joint density of $\hat{\beta}$ which is bounded by a normal density in R^p so that the results of Section 3 hold.

(3) Given ψ and g, it is generally trivial to construct the function ψ_n satisfying P5′ (as required in P5). In particular, since $E\exp\{t\psi(R)\}$ is finite (for t small), it is clear that as long as neither $\psi(r)$ or $g(r)$ have "spikes," $g(r)e^{t\psi(r)} \to 0$; and it is easy to smooth ψ so that P5′ holds.

Lastly, we impose the following conditions on the design matrix, X with rows $\{x_i'\}$. As noted earlier, it can be shown that these conditions hold in probability under (4.1).

X1: The maximum and minimum eigenvalues of $X'X$ satisfy (for constants B and b)

$$\lambda_{\max}(X'X) \leq Bn, \qquad \lambda_{\min}(X'X) \geq bn. \tag{2.4}$$

Note that $\Sigma(x_i'u)^2 \le \|u\|^2 \lambda_{\max}(X'X)$ so that if X1 holds and $\|\hat\beta\|^2 = \mathcal{O}_p(p/n)$ (as shown in Part I) then $\Sigma(x_i'\hat\beta)^2 = \mathcal{O}_p(p)$.

X2: $\{x_i\}$ are such that for any B

$$\sup_{\|w\|=1} \sup_{\|\beta\| \le pB/n} \sum_{i=1}^{n} (x_i'w)^2 Q(R_i, x_i'\beta) = \mathcal{O}_p(pn \log n)^{1/2}, \tag{2.5}$$

where Q is defined in (2.1).

X3: Define

$$y_i = (X'X)^{-1/2} x_i \qquad (i = 1, 2, \dots, n). \tag{2.6}$$

Then uniformly in $i = 1, \dots, n$ and $\ell = 1, \dots, n$,

$$y_i'y_\ell = \mathcal{O}\left(\sqrt{p \log n}\,/n\right), \quad \text{for } i \ne \ell. \tag{2.7}$$

Furthermore, there are values $\{s_i\colon\ i = 1, \dots, n\}$ uniformly bounded such that (uniformly in i)

$$\|y_i\|^2 = \frac{p}{n} s_i + \mathcal{O}\left(\sqrt{p}\log n/n\right). \tag{2.8}$$

X4: With y_i defined by (2.6),

$$\sum\sum_{i \ne \ell} (y_i'y_\ell)\|y_i\|^2 \|y_\ell\|^2 = \mathcal{O}\left(\frac{p^2(\log n)}{n}^{3/2}\right). \tag{2.9}$$

X5: With y_i defined by (2.6) and $\{a_n\}$ a sequence of fixed vectors in R^p with $\|a_n\|$ bounded, $\max\{|x_i'a_n|\colon\ i = 1, \dots, n\} = \mathcal{O}(\sqrt{\log n})$ and

$$\sum_{i=1}^{n} (x_i'a_n)\|y_i\|^2 = \mathcal{O}(p \log^3 n)^{1/2}. \tag{2.10}$$

3. The basic results. Some preliminary lemmas will be listed first. A basic consistency result is given in Theorem 3.1. Normality results for linear combinations $a'\hat\beta$ are given in Theorem 3.2, while Theorem 3.3 gives a result providing an asymptotic χ^2 approximation for the distribution of $\hat\beta'(X'X)\hat\beta$.

LEMMA 3.1. *Assume conditions* N2, X1, P1, *and* P2, *and suppose that Lemma* 3.2 *holds. Let Y be defined by* (2.6) *and let $\hat\beta$ be the M estimator satisfying* (1.2). *Define $P \in R^n$ to have coordinates $P_i = \psi(R_i)$ and let Q be a diagonal $n \times n$ matrix with diagonal elements*

$$Q_i = \frac{\psi(R_i) - \psi(R_i - x_i'\hat\beta)}{x_i'\hat\beta} - d, \tag{3.1}$$

where $d = E\psi'(R)$ and Q_i is defined to be zero if $x_i'\hat\beta = 0$. Then for any $k = 1, 2, \dots,$

$$\hat\beta = \frac{1}{d}(X'X)^{-1/2}\left(\sum_{\ell=0}^{k-1}\left(-\frac{1}{d}Y'QY\right)^{\ell}\right)Y'P + e_k, \tag{3.2}$$

where

$$\|e_k\|^2 = \mathcal{O}_p\left(\left(\frac{p}{n}\right)^{k+1}(\log n)^k\right). \tag{3.3}$$

PROOF. Rewrite (1.2) [using (3.1)]:

$$0 = \sum_{i=1}^{n} x_i\psi(R_i - x_i'\hat{\beta}) = \sum_{i=1}^{n} x_i\psi(R_i) - \sum_{i=1}^{n} x_i(x_i'\hat{\beta})Q_i - d\sum_{i=1}^{n} x_i x_i'\hat{\beta}. \tag{3.4}$$

Writing (3.4) in matrix form, multiplying by $(X'X)^{-1/2}$, and using the definition of Y and P yields

$$\begin{aligned} 0 &= Y'P - (X'X)^{-1/2}(X'QX + dX'X)\hat{\beta} \\ &= Y'P - d\left(\frac{1}{d}Y'QY + I\right)(X'X)^{1/2}\hat{\beta}, \\ \hat{\beta} &= \frac{1}{d}(X'X)^{-1/2}\left(I + \frac{1}{d}Y'QY\right)^{-1} Y'P. \end{aligned} \tag{3.5}$$

By Lemma 3.2, we have the expansion

$$\left(I + \frac{1}{d}Y'QY\right)^{-1} = \sum_{\ell=0}^{\infty}\left(-\frac{1}{d}Y'QY\right)^{\ell}$$

and, hence, (3.2) holds with

$$e_k = \frac{1}{d}(X'X)^{-1/2}\left(\sum_{\ell=k}^{\infty}\left(-\frac{1}{d}Y'QY\right)^{\ell}\right)Y'P.$$

Therefore (with $\lambda_{\max}$ denoting the maximum eigenvalue),

$$\|e_k\|^2 \le \frac{1}{d^2}\lambda_{\max}(X'X^{-1})\left(\sum_{\ell=k}^{\infty}\lambda_{\max}^2\left(\frac{1}{d}Y'QY\right)^{\ell}\right)\|Y'P\|^2. \tag{3.6}$$

Now

$$\begin{aligned} E\|Y'P\|^2 &= \sum_{j=1}^{p}\sum_{i=1}^{n}\sum_{\ell=1}^{n} y_{ij}y_{i\ell}E\psi(R_i)\psi(R_\ell) \\ &= \sum_{j=1}\sum_{i=1} y_{ij}^2 E\psi^2(R) = pE\psi^2(R), \end{aligned}$$

and, hence, $\|Y'P\|^2 = \mathcal{O}_p(p)$. By condition X1, $\lambda_{\max}(X'X)^{-1} = (\lambda_{\min}(X'X))^{-1} = \mathcal{O}(1/n)$. Lastly, by Lemma 3.2, there is a constant B such that, in probability,

$$\begin{aligned} \sum_{\ell=k}^{\infty}\lambda_{\max}^2(Y'QY)^{\ell} \le B\sum_{\ell=k}^{\infty}\left(p\frac{\log n}{n}\right)^{\ell} &= B\left(\frac{p\log n}{n}\right)^{k}\left(1 - \frac{p\log n}{n}\right)^{-1} \\ &= \mathcal{O}_p\left(\frac{p\log n}{n}\right)^{k}. \end{aligned}$$

Therefore, from (3.6),

$$\|e_k\|^2 = \mathcal{O}_p\left(\frac{1}{n}\right) \cdot \mathcal{O}_p\left(\frac{p \log n}{n}\right)^k \cdot \mathcal{O}_p(p) = \mathcal{O}_p\left(\left(\frac{p}{n}\right)^{k+1} (\log n)^k\right). \qquad \square$$

LEMMA 3.2. *Let Y be defined by* (2.6) *and let Q be defined by* (3.1). *Assume conditions* X1, X2, *and* P2 [*it can be shown that* X2 *will hold in probability under conditions* P1, P2, P3, N2, *and* (4.1)]. *Suppose that* $\|\hat{\beta}\|^2 = \mathcal{O}_p(p/n)$ (*see Theorem* 3.2 *of Part I*). *Then the maximum eigenvalue of* $Y'QY$ *satisfies*

$$\lambda_{\max}(Y'QY) = \mathcal{O}_p\left(\frac{p \log n}{n}\right)^{1/2}.$$

PROOF. By the condition on $\|\hat{\beta}\|^2$ and definition, for some $B > 0$,

$$\lambda_{\max}(Y'QY) \le \sup_{\|\beta\|^2 \le Bp/n} \sup_{\|u\|=1} u'(Y'QY)u$$
$$= \sup_{\|\beta\|^2 \le Bp/n} \sup_{\|u\|=1} \sum_{i=1}^{n} (y_i'u)^2 Q_i.$$

Let $w = (X'X)^{-1/2}u/\lambda_{\max}(X'X)^{-1/2}$. Then $\|w\|^2 \le \|u\|^2 = 1$, and (by condition X1) for some $B_0 > 0$,

$$(y_i'u)^2 = (x_i'w)^2 \cdot \lambda_{\max}(X'X)^{-1} \le \frac{B_0}{n}(x_i'w)^2.$$

Therefore, Lemma 3.2 follows directly from condition X2. $\square$

LEMMA 3.3. *Assume conditions* P1, X3, *and* N2, *and suppose that* $\psi(R)$ *has a finite moment generating function in a neighborhood of the origin. Then*

$$\sum_{\ell=1}^{n} (y_i'y_\ell)\psi(R_\ell) = \mathcal{O}_p\left(\frac{p \log n}{n}\right)^{1/2} \quad \textit{uniformly in } i = 1, 2, \ldots, n.$$

PROOF. First note that by P1 and the hypothesis on ψ, there are constants $b > 0$ and $\varepsilon > 0$ such that

$$Ee^{t\psi(R)} \le e^{bt^2} \quad \text{for } |t| \le \varepsilon. \tag{3.7}$$

thus, if $t > 0$ (depending on n) is chosen so that $t \max_\ell |y_i'y_\ell| \le \varepsilon$, the Markov inequality yields

$$P\left\{\sum_{\ell=1}^{n} (y_i'y_\ell)\psi(R_\ell) \ge c_n\right\} \le \exp\{-tc_n\} \cdot E\exp\left\{t\sum_{\ell=1}^{n} (y_i'y_\ell)\psi(R_\ell)\right\}$$

$$\le \exp\left\{-tc_n + bt^2 \sum_{\ell=1}^{n} (y_i'y_\ell)^2\right\} \tag{3.8}$$

$$= \exp\left\{-tc_n + bt^2\|y_i\|^2\right\}$$

$$\le \exp\left\{-tc_n + b^*t^2\frac{p}{n}\right\},$$

where b^* is a constant and the fact that $Y'Y = I$ and condition X3 are used.

Now, let $c_n = B(p \log n/n)^{1/2}$ and let $t = \delta(n \log n/p)^{1/2}$. Then by conditions X3 and N2 (uniformly in i)

$$t \max_{\ell} |y_i' y_\ell| = \mathcal{O}\left(\frac{n \log n}{p}\right)^{1/2} \cdot \mathcal{O}\left(\frac{p}{n}\right) = \mathcal{O}\left(\frac{p \log n}{n}\right)^{1/2} \to 0.$$

Therefore, δ may be chosen so that $t \max_\ell |y_i' y_\ell| \le \varepsilon$ and (3.8) holds. Also choose δ (perhaps smaller) so that $\delta b^* < 1$. Then from (3.8),

$$\begin{aligned} P\left\{ \sum_{\ell=1}^{n} (y_i' y_\ell) \psi(R_\ell) \ge B\left(\frac{p \log n}{n}\right)^{1/2} \right\} &\le \exp\{-\delta B \log n + b^* \delta^2 \log n\} \\ &= \exp\{-\delta B \log n (1 - b^* \delta/B)\} \\ &\le \exp\{-\delta B \log n(1 - 1/B)\}. \end{aligned}$$

Thus, if B is chosen greater than 3 and $\delta B \ge 3$, the above probability is bounded above by $\exp\{-3 \log n(2/3)\} = 1/n^2$ (uniformly in i). Thus,

$$P\left\{ \sum_{\ell=1}^{n} (y_i' y_\ell) \psi(R_\ell) \ge B\left(\frac{p \log n}{n}\right)^{1/2} \text{for some } i \right\} \le n \cdot \frac{1}{n^2} = \frac{1}{n} \to 0.$$

The proposition follows since the same argument works for the reverse inequality in (3.8). □

Although Part I provided norm consistency for $\{\hat{\beta}\}$, the following alternative form of consistency will also be needed here:

THEOREM 3.1. *Assume the conditions for Lemmas* 3.1, 3.2, *and* 3.3 *and suppose also that* N1 *holds and that*

$$\max\{\|x_i\|^2 : i = 1, 2, \dots, n\} = \mathcal{O}(p). \tag{3.9}$$

[*Note that* (3.9) *follows directly from conditions* X1 *and* X3.] *Then*

$$\max\{|x_i' \hat{\beta}| : i = 1, 2, \dots, n\} \to_P 0.$$

PROOF. By Lemma 3.1 with $k = 1$,

$$x_i' \beta = \frac{1}{d} x_i (X'X)^{-1/2} Y'P + x_i' e.$$

Now, using (3.3) and (3.9),

$$|x_i' e_1| \le \|x_i\| \, \|e_1\| = \mathcal{O}_p\left(\sqrt{p} \cdot \frac{p}{n} \sqrt{\log n}\right) = \mathcal{O}_p\left(\frac{p^3 \log n}{n^2}\right)^{1/2} \to_P 0,$$

uniformly in $i = 1, 2, \dots, n$. Furthermore,

$$x_i' (X'X)^{-1/2} Y'P = y_i' Y'P = \sum_{\ell=1}^{n} (y_i' y_\ell) \psi(R_\ell).$$

This term converges to zero uniformly in $i = 1, 2, \dots, n$ in probability by Lemma 3.3, and the theorem follows. □

To obtain asymptotic normality it will be necessary to use the error term e_2 and to consider the second term in (3.2). This requires a Taylor series expansion and, hence, somewhat stronger conditions.

LEMMA 3.4. *Assume conditions* P1, P2, P4, X1, X2, X3, X4, *and* N1′. *Let* $\sigma^2 = E\psi^2(R)$, $d = E\psi'(R)$ *and define*

$$\hat{\theta} = \frac{d}{\sigma}(X'X)^{1/2}\hat{\beta}.$$

Then

$$\hat{\theta} = W + \frac{1}{\sigma}A + \frac{\sigma}{6d^2}\tilde{A} - \left(\frac{1}{2d}\right)A^*W + e, \tag{3.10}$$

where $\|e\|^2 = \mathcal{O}_p(p^3(\log n)^{5/2}/n^2)$; W, A, *and* $\tilde{A}$ *in* R^p *and* A^* *a* $p \times p$ *matrix are defined by*

$$\begin{aligned} W_k &= \frac{1}{\sigma}\sum_{i=1}^{n} y_{ik}\psi(R_i) \\ A_j &= \sum_{i=1}^{n}\sum_{\ell=1}^{n} y_{ij}(y_i'y_\ell)\psi(R_\ell)(\psi'(R_i) - d) \\ A_{jk}^* &= \sum_{i=1}^{n}\sum_{\ell=1}^{n} y_{ij}y_{ik}(y_i'y_\ell)\psi(R_\ell)\psi''(R_i) \\ \tilde{A}_j &= \sum_{i=1}^{n}\sum_{\ell=1}^{n} y_{ij}(y_i'y_\ell)(y_i'\hat{\theta})^2\psi(R_\ell)\psi'''(S_i) \end{aligned} \tag{3.11}$$

for some S_i *between* R_i *and* $R_i - (x_i'\beta)$. *Furthermore, we have*

$$\|A\|^2 = \mathcal{O}_p\big(p^2(\log n)^{3/2}/n\big), \qquad \|\tilde{A}\|^2 = \mathcal{O}_p\big(p^{7/2}(\log n)^{3/2}/n^2\big), \tag{3.12}$$

and

$$|A^*| = \mathcal{O}_p(p\log n/n)^{1/2},$$

where $|A^*| = \sup\{u'A^*u\colon \|u\| = 1\}$.

PROOF. From Lemma 3.1,

$$\hat{\theta} = W + \frac{1}{\sigma}(Y'QY)Y'P + e_1 \quad \text{where } \|e_1\|^2 = \mathcal{O}_p(p^3\log^2 n/n^2). \tag{3.13}$$

Expanding (3.1) in a Taylor series and noting that $x_i'\hat{\beta} = (\sigma/d)y_i'\hat{\theta}$ yields

$$Q_i = (\psi'(R_i) - d) - \frac{1}{2}\frac{\sigma}{d}(y_i'\hat{\theta})\psi''(R_i) + \frac{1}{6}\left(\frac{\sigma}{d}\right)^2(y_i'\hat{\theta})^2\psi'''(S_i),$$

where S_i is between R_i and $R_i - (\sigma/d)(y_i'\hat{\theta})$. Substituting in (3.13) gives

$$\hat{\theta} = W + \frac{1}{\sigma}A - \frac{1}{2d}A^*\hat{\theta} + \frac{\sigma}{6d^2}\tilde{A} + e_1 \tag{3.14}$$

or

$$\left(I + \frac{1}{2d}A^*\right)\hat{\theta} = W + \frac{1}{\sigma}A + \frac{\sigma}{6d^2}\tilde{A} + e_1.$$

Now, using condition P4, Lemma 3.3, the fact that $\Sigma(y_i'u)^2 = \|u\|^2$, and condition N1,

$$\begin{aligned} |A^*| &= \sup_{\|u\|^2=1}\left|\sum_i\sum_\ell (y_i'u)^2(y_i'y_\ell)\psi(R_\ell)\psi''(R_i)\right| \\ &\le B \sup_{\|u\|^2=1}\sum_i (y_i'u)^2\left|\sum_\ell (y_i'y_\ell)\psi(R_\ell)\right| \qquad (3.15)\\ &= \mathcal{O}_p\left(\frac{p\log n}{n}\right)^{1/2} \to_p 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\theta} &= \left(I + \frac{1}{2d}A^*\right)^{-1}\left(W + \frac{1}{\sigma}A + \frac{\sigma}{6d^2}\tilde{A} + e_1\right) \\ &= W + \frac{1}{\sigma}A + \frac{\sigma}{6d^2}\tilde{A} - \frac{1}{2d}A^*W - \frac{1}{2\sigma d}A^*A - \frac{\sigma}{12d^3}A^*\tilde{A} \qquad (3.16)\\ &\quad - \frac{1}{2d}A^*e_1 + \sum_{k=2}^{\infty} A^{*k}\left(W + \frac{1}{\sigma}A + \frac{\sigma}{6d^2}\tilde{A} + e_1\right). \end{aligned}$$

Now bound the norms of the vectors W, A, and $\tilde{A}$:

$$E\|W\|^2 = E\sum_i\sum_\ell (y_i'y_\ell)\psi(R_i)\psi(R_\ell) = \sigma^2\sum_i\|y_i\|^2 = \mathcal{O}(p); \qquad \|W\| = \mathcal{O}_p(\sqrt{p}).$$

$$\begin{aligned} E\|A\|^2 &= \sum_{i_1}\sum_{i_2}\sum_{\ell_1}\sum_{\ell_2}(y_{i_1}'y_{i_2})(y_{i_1}'y_{\ell_1})(y_{i_2}'y_{\ell_2})E\psi(R_{\ell_1})\psi(R_{\ell_2}) \\ &\quad \cdot(\psi'(R_{i_1}) - d)(\psi'(R_{i_2}) - d) \\ &= (E\psi(R_1)(\psi'(R_2) - d))^2\left\{\sum\sum_{i\ne\ell}(y_i'y_\ell)\|y_i\|^2\|y_\ell\|^2 + \sum\sum_{i\ne\ell}(y_i'y_\ell)^3\right\} \qquad (3.17)\\ &\quad + E\psi^2(R)E(\psi'(R) - d)^2\sum\sum_{i\ne\ell}\|y_i\|^2(y_i'y_\ell)^2 \\ &\quad + E\psi^2(R)(\psi'(R) - d)^2\sum_i\|y_i\|^6. \end{aligned}$$

Using condition X3 and X4 (for the first term), it is straightforward to show that $E\|A\|^2 = \mathcal{O}(p^2(\log n)^{3/2}/n)$. Thus $\|A\|^2$ has this same order in probability.

Lastly,

$$\|\tilde{A}\|^2 = \sum_{i_1}\sum_{i_2}\sum_{\ell_1}\sum_{\ell_2}(y_{i_1}'y_{i_2})(y_{i_1}'y_{\ell_1})(y_{i_2}'y_{\ell_2})\psi(R_{\ell_1})\psi(R_{\ell_2})$$
$$\cdot(y_{i_1}'\hat{\theta})^2\psi'''(S_{i_1})(y_{i_2}'\hat{\theta})^2\psi'''(S_{i_2})$$
$$\le B\sum_{i_1}\sum_{i_2}|y_{i_1}'y_{i_2}|(y_{i_1}'\hat{\theta})^2(y_{i_2}'\hat{\theta})^2\left|\sum_{\ell_1}(y_{i_1}'y_{\ell_1})\psi(R_{\ell_1})\right|\left|\sum_{\ell_2}(y_{i_2}'y_{\ell_2})\psi(R_{\ell_2})\right|.$$

Thus, using Lemma 3.3, (2.7), and (from Theorem 3.2 of Part I and X1) the fact that

$$\frac{\sigma}{d}\sum_i(y_i'\hat{\theta})^2 = \hat{\beta}'(X'X)\hat{\beta} \le \lambda_{\max}(X'X)\|\hat{\beta}\|^2 = \mathcal{O}_p(p),$$

it follows that the sum over $i_1 \ne i_2$ contributes a term of order

$$\mathcal{O}_p\left(\frac{\sqrt{p\log n}}{n}\cdot p^2\cdot\frac{p\log n}{n}\right) = \mathcal{O}_p\left(p^{7/2}(\log n)^{3/2}/n^2\right).$$

Similarly, [using (2.8)], the sum over $i_1 = i_2$ is bounded by

$$\sum_{i=1}^{n}(y_i\theta)^4\mathcal{O}_p\left(\frac{p^2\log n}{n^2}\right) \le \mathcal{O}_p\left(\frac{p^3\log n}{n^2}\right)\max{}_i|y_i'\theta|^2,$$

which has even smaller order (by Theorem 3.1). Hence, $\|\tilde{A}\|^2$ has the required order.

Thus, (3.12) holds, and it remains to bound the last four terms in (3.16). The above computations show that the second factor in each summand (in the last term) is $\mathcal{O}_p(\sqrt{p})$. Thus, using (3.15) and the argument in Lemma 3.1, this infinite sum is $\mathcal{O}_p((p\log n/n)\sqrt{p})$ (in norm) and can be included in the error. Similarly,

$$\|A^*A\|^2 \le |A^*|^2\|A\|^2 = \mathcal{O}_p\left(\frac{p\log n}{n}\cdot\left(\frac{p^2(\log n)}{n}\right)^{3/2}\right)$$
$$= \mathcal{O}_p\left(\left(\frac{p^3(\log n)}{n^2}\right)^{5/2}\right)$$
$$\|A^*\tilde{A}\|^2 = \mathcal{O}_p\left(\frac{p\log n}{n}\left(\frac{p^{7/2}(\log n)}{n^2}\right)^{3/2}\right) = \mathcal{O}_p\left(\left(\frac{p^3(\log n)}{n^2}\right)^{5/3}\right)^{3/2}$$
$$\|A^*e_1\|^2 = o_p(1)\mathcal{O}_p\left(\frac{p^3\log^2 n}{n^2}\right).$$

Thus, the last four terms in (3.16) are of the required order for error, and the lemma is proven. □

THEOREM 3.2. *Suppose the hypotheses of Lemma* 3.4 *hold and assume condition* X5. *Then for any sequence of vectors* $\{a_n\}$ *with* $a_n \in R^p$ *and* $\|a_n\|$

bounded,

$$a_n'\hat{\beta}\left(\frac{d}{v\sigma}\right) \to_D \mathcal{N}(0,1)$$

where

$$v^2 = a_n'(X'X)^{-1}a_n. \tag{3.18}$$

PROOF. Let $b_n = \sqrt{n}(X'X)^{-1/2}a_n$. Then

$$v^2 = \|b_n\|^2/n \quad \text{and} \quad a_n'\hat{\beta}\left(\frac{d}{v\sigma}\right) = \frac{b_n'\hat{\theta}}{\|b_n\|}.$$

Thus, it suffices to show that if $\|b_n\| = 1$, $b_n'\hat{\theta} \to_D \mathcal{N}(0,1)$. So assume $\|b_n\| = 1$ and apply Lemma 3.4:

$$b_n'\hat{\theta} = b_n'W + \frac{1}{\sigma}b_n'A + \frac{\sigma}{bd^2}b_n'\tilde{A} - \frac{1}{2d}b_n'A^*W + b_n'e. \tag{3.19}$$

Now using (3.11), $b_n'W = (1/\sigma)\Sigma_{i=1}^n(y_i'b_n)\psi(R_i)$.

Since $\max_i|y_i'b_n| \to 0$ (by conditions X3 or X5) and $\Sigma(y_i'b_n)^2 = \|b_n\|^2 = 1$, the Central Limit Theorem implies that $b_n'W \to_D \mathcal{N}(0,1)$. So it remains to consider the remainder terms in (3.19).

First, by (3.11),

$$b_n'A = \sum_i\sum_\ell(y_i'b_n)(y_i'y_\ell)\psi(R_\ell)(\psi'(R_i) - d).$$

As in (3.17), for appropriate constants c_1, c_2, and c_3

$$\begin{aligned} E(b_n'A)^2 = {} & c_1\left\{\sum_{i\neq\ell}\sum(y_i'b_n)(y_\ell'b_n)\|y_i\|^2\|y_\ell\|^2 + \sum_{i\neq\ell}\sum(y_i'b_n)(y_\ell'b_n)(y_i'y_\ell)^2\right\} \\ & + c_2\sum_{i\neq\ell}\sum(y_i'b_n)^2(y_i'y_\ell)^2 + c_3\sum_i(y_i'b_n)^2\|y_i\|^4. \end{aligned} \tag{3.20}$$

By definition of b_n, $(y_i'b_n) = B(x_i'a_n)/\sqrt{n}$ (for some constant B). Thus, the first term in (3.20) is $(\Sigma(x_i'a_n)\|y_i\|^2/\sqrt{n})^2 - \Sigma(y_i'b_n)^2\|y_i\|^4$ which is $\mathcal{O}(\sqrt{(p/n)}\log^3 n)$ by condition X5. Similarly X5 and X3 show that the second and third terms are

$$\mathcal{O}\left(n^2 \cdot \frac{\log n}{n} \cdot \frac{p\log n}{n^2}\right) = \mathcal{O}\left(\frac{p\log^2 n}{n}\right) \to 0;$$

and the last term is $\mathcal{O}(\|b_n\|^2 \cdot p^2/n^2)$. Therefore, $E(b_n'A)^2 \to 0$ and $b_n'A \to_P 0$.

Next, by (3.11) again

$$b_n'\tilde{A} = \sum_i\sum_\ell(y_i'b_n)(y_i'y_\ell)\psi(R_\ell)(y_i'\hat{\theta})^2\psi'''(S_i).$$

Thus, using Lemma 3.3, condition X5, the boundedness of ψ''', and the fact that $\Sigma(y_i'\hat{\theta})^2 = \mathcal{O}_p(p)$,

$$|b_n'\tilde{A}| = \mathcal{O}_p\left(\sqrt{\frac{\log n}{n}} \cdot \sqrt{\frac{p\log n}{n}} \cdot p\right) = \mathcal{O}_p\left(\frac{p^{3/2}\log n}{n}\right) \to_P 0$$

by condition N1.

For the third remainder term, (3.11) yields

$$b_n'A^*W = \sum_i \sum_\ell \sum_\nu (y_i'b_n)(y_i'y_\ell)\psi(R_\ell)(y_i'y_\nu)\psi(R_\nu)\psi''(R_i).$$

As in (3.20), $E(b_n'A^*W)^2$ can be calculated as a six-fold sum over i_1, i_2, ℓ_1, ℓ_2, ν_1, and ν_2 in which these indices must be equal at least in pairs. Straightforward though tedious computations using condition X3 show that all terms are $\mathscr{O}(p/n)$ except for the following two terms where $\{i_1 = \ell_1 \equiv i,\ i_2 = \ell_2 \equiv \ell$, and $\nu_1 = \nu_2 \equiv \nu\}$ (or related terms) and $\{i_1 = \ell_1 = \nu_1 \equiv i$ and $i_2 = \ell_2 = \nu_2 \equiv \ell\}$:

$$(3.21)\quad \begin{aligned}&\sum\sum\sum_{i\neq\ell\neq\nu} \frac{(x_i'a_n)}{\sqrt{n}}\frac{(x_\ell'a_n)}{\sqrt{n}}\|y_i\|^2\|y_\ell\|^2(y_i'y_\nu)(y_\ell'y_\nu)\\ &= \frac{1}{n}\sum\sum_{i\neq\ell}(x_i'a_n)(x_\ell'a_n)\|y_i\|^2\|y_\ell\|^2\{(y_i'y_\ell) - (\|y_i\|^2 + \|y_\ell\|^2)(y_i'y_\ell)\}\end{aligned}$$

and $(\Sigma_i((x_i'a_n)/\sqrt{n})\|y_i\|^4)^2$. Using X3 and X5, the first term in (3.21) is

$$\mathscr{O}\left(\frac{1}{n}\cdot n^2\cdot\frac{\log n}{n}\cdot\frac{p^2}{n^2}\cdot\frac{\sqrt{p\log n}}{n}\right) = \mathscr{O}\left(\frac{p^{5/2}(\log n)^{3/2}}{n^2}\right),$$

and the second term is clearly $\mathscr{O}(p^4\log n/n^3)$. Thus, $E(b_n'A^*W)^2 \to 0$ (by N1) and $b_n'A^*W \to_P 0$. Lastly, $|b_n'e| \le \|b_n\|\,\|e\| \to_P 0$, and the proof is complete. □

REMARK. In the classical situation it would be possible to use Theorem 3.2 to obtain asymptotic normality of $b_n'\hat{\theta}$ for a sequence of constants $\{b_n\}$ by choosing $a_n = \sqrt{n}(X'X)^{-1/2}b_n$. This will not work here because condition X5 requires that a_n depend on neither X nor R. To obtain normality for $b_n'\hat{\theta}$ would require a condition of the form

$$\text{X5}'\text{: } \max_i (y_i'a)^2 = \mathscr{O}\left(\frac{\log n}{n}\right), \qquad \sum_i (y_i'a)\|y_i\|^2 \to 0$$

for a constant vector, a. It is possible but much more difficult to show that X5′ holds in probability under (4.1).

Lastly, a wide variety of asymptotic distribution results can be obtained by combining Lemma 3.4 and Theorem 4.1, which provides a uniform normal approximation for the distribution of W in R^p. For example, the following result provides a chi-square approximation for the distribution of $\|\hat{\beta}\|^2$ (for large p) by showing that $(\|\hat{\theta}\|^2 - p)/\sqrt{2p} \to_D \mathscr{N}(0,1)$ (if $p \to \infty$).

THEOREM 3.3. *Assume the hypotheses of Lemma* 3.4, *suppose* $p \to \infty$, *and suppose that Theorem* 4.1 *holds. Then*

$$\frac{\|\hat{\theta}\|^2 - p}{\sqrt{2p}} \to_D \mathscr{N}(0,1).$$

As a consequence, for any sequence of reals, $\{x_n\}$,

$$\left| P\left\{ \frac{d^2}{\sigma^2} \hat{\beta}'(X'X)\hat{\beta} \le x_n \right\} - P\left\{ \chi_p^2 \le x_n \right\} \right| \to 0 \quad \text{as } n \to \infty.$$

PROOF. From (3.10) (see Lemma 3.4),

$$\frac{\|\hat{\theta}\|^2 - p}{\sqrt{2p}} = \frac{\|W\|^2 - p}{\sqrt{2p}} + \frac{1}{\sqrt{p}} e^*,$$

where e^* consists of the sum of squared norms of the last four (error) terms in (3.10) plus the sum of all pairwise inner products of the five terms in (3.10). By Theorem 4.1, $(\|W\|^2 - p)/\sqrt{2p} \to_D \mathcal{N}(0,1)$; and, hence, it remains to show that each term in e^* is $o_p(\sqrt{p})$. From (3.12), using N1′ and the fact that $\|W\| = \mathcal{O}_p(\sqrt{p})$,

$$\|A\|^2 = \mathcal{O}_p\left(\sqrt{p} \cdot \left(\frac{p^{3/2}(\log n)^{3/2}}{n} \right) \right) = o_p(\sqrt{p}),$$

$$\|\tilde{A}\|^2 = \mathcal{O}_p\left(\sqrt{p} \left(\frac{p^3(\log n)^{3/2}}{n^2} \right) \right) = o_p(\sqrt{p}),$$

$$\|A^*W\|^2 \le |A^*|^2 \|W\|^2 = \mathcal{O}_p\left(\frac{p \log n}{n} \cdot p \right) = \mathcal{O}_p\left(\sqrt{p} \cdot \frac{p^{3/2} \log n}{n} \right) = o_p(\sqrt{p}).$$

Also $\|e\|^2 = o_p(1) = o_p(\sqrt{p})$; so it remains to consider the 10 inner product terms. Again using (3.12) as above,

$$|A'\tilde{A}| \le \|A\| \, \|\tilde{A}\| = \mathcal{O}_p\left(\sqrt{p} \cdot \frac{p^{9/4}(\log n)^{3/2}}{n^{3/2}} \right) = o_p(\sqrt{p}),$$

$$|A'A^*W| \le \|A\| \, |A^*| \, \|W\| = \mathcal{O}_p\left(\frac{p^{3/2}(\log n)^{5/4}}{n} \cdot \sqrt{p} \right) = o_p(\sqrt{p}),$$

$$|\tilde{A}'A^*W| \le \|\tilde{A}\| \, |A^*| \, \|W\| = \mathcal{O}_p\left(\frac{p^{9/4}(\log n)^{5/4}}{n^{3/2}} \cdot \sqrt{p} \right) = o_p(\sqrt{p}).$$

Furthermore, for $W'e$, $A'e$, $A'e$, and $(A^*W)'e$, the first factor has norm $\mathcal{O}_p(\sqrt{p})$ and $\|e\| = o_p(1)$; so these terms can be discarded. Thus, it remains to consider $W'A$, $W'\tilde{A}$, and $W'A^*W$. Now from (3.11),

$$W'A = \sum_i \sum_\ell \sum_\nu (y_i'y_\nu)\psi(R_\nu)(y_i'y_\ell)\psi(R_\ell)(\psi'(R_i) - d),$$

and, as before, it is not difficult to show that the dominating term in $E(W'A)^2$ is [since $\Sigma_\nu(a'y_\nu)(b'y_\nu) = a'b$]

$$\sum_i \sum_\ell \sum_\nu \|y_i\|^2 \|y_\ell\|^2 (y_i'y_\nu)(y_\ell'y_\nu) = \sum_i \sum_\ell \|y_i\|^2 \|y_\ell\|^2 (y_i y_\ell).$$

By condition X4, this term is $\mathcal{O}(p^2(\log n)^{3/2}/n)$. Thus,

$$|W'A| = \mathcal{O}_p\big(p(\log n)^{3/4}/\sqrt{n}\big) = \mathcal{O}_p\left(\sqrt{p}\cdot\sqrt{\frac{p(\log n)^{3/2}}{n}}\right) = o_p(\sqrt{p}).$$

Next by (3.11) again,

$$W'\tilde{A} = \sum_i\sum_\ell\sum_\nu (y_i'y_\nu)\psi(R_\nu)(y_i'y_\ell)\psi(R_\ell)(y_i'\hat{\theta})^2\psi'''(S_i).$$

Thus, using Lemma 3.3 and the fact that $\Sigma(y_i'\hat{\theta})^2 = \mathcal{O}_p(p)$,

$$|W'\tilde{A}| \le B\sum_i (y_i'\hat{\theta})^2\Big(\sum_\ell (y_i'y_\ell)\psi(R_\ell)\Big)^2 = \mathcal{O}_p\left(p\cdot\frac{p\log n}{n}\right) = \mathcal{O}_p\left(\sqrt{p}\cdot\frac{p^{3/2}\log n}{n}\right)$$
$$= o_p(\sqrt{p}).$$

Lastly, from (3.11)

$$W'A^*W = \sum_i\sum_\ell\sum_\mu\sum_\nu (y_i'y_\mu)\psi(R_\mu)(y_i'y_\nu)\psi(R_\nu)(y_i'y_\ell)\psi(R_\ell)\psi''(R_i).$$

Again, computing $E(W'A^*W)^2$, the eight subscripts must be equal at least in pairs, and it is not difficult to show that the dominating term is

$$\sum_i\sum_\ell\sum_\mu\sum_\nu \|y_i\|^2\|y_\ell\|^2(y_i'y_\mu)^2(y_\ell'y_\nu)^2 = \sum_i\sum_\ell \|y_i\|^4\|y_\ell\|^4 = \Big(\sum_i\|y_i\|^4\Big)^2.$$

Therefore, by condition X3,

$$|W'A^*W| = \mathcal{O}_p\Big(\sum_i\|y_i\|^4\Big) = \mathcal{O}_p\left(n\cdot\frac{p^2}{n^2}\right) = \mathcal{O}_p\left(\sqrt{p}\cdot\frac{p^{3/2}}{n}\right) = o_p(\sqrt{p}). \qquad \square$$

To obtain a convergence result for $\|\hat{\theta}\|^2$ under a condition weaker than N1 would require that $\|A\|^2/\sqrt{p} \to_P 0$ (under the weaker condition). However, from (3.17) (using the third and fourth terms), it is clear that

$$E\|A\|^2 \ge b\sum_i\|y_i\|^4 \ge b'p^2/n$$

(for some constants b and b'). Thus, $\|A\|^2/\sqrt{p}$ should not be $o_p(p^{3/2}/n)$, and, hence, it would seem that N1 is essentially necessary for a uniform normal approximation in R^p (whether or not ψ is antisymmetric).

4. The CLT for W in R^p. Normal convergence results like those in Theorem 3.3 require a Central Limit Theorem for $W = \Sigma y_i\psi(R_i)$ in R^p when p^2/n may be large. Unfortunately, as shown in Portnoy (1984b), a general result may not hold if p^2/n does not converge to zero (even in symmetric situations). Thus, strong assumptions and use of special features of W are required. Portnoy (1984d) provides an appropriate normal approximation result by assuming that $\{x_i\}$ form a random sample of size n in R^p and by making use of the fact that W depends on a one-dimensional function of R_i. The model for the rows, x_i, of the

design matrix is as follows:
(4.1) Let $(s_1, \dots, s_n)$ be i.i.d. according to a distribution with compact support in $(0, \infty)$; and given $(s_1, \dots, s_n)$, let $(x_1, \dots, x_n)$ be independent with $x_i \sim \mathcal{N}_p(0, s_i I)$. Assume $Es_i = 1$ (without loss of generality by rescaling the original model).

THEOREM 4.1. *Assume conditions* N1 *and* P5 *and let W be defined by* (3.11). *Let* $P(\cdot|x_n)$ *denote the conditional distribution of* $(R_1, \dots, R_n)$ *given the design matrix,* $X = (x_1, \dots, x_n)$. *Let* $\varepsilon_n = \mathcal{O}(1/n^d)$ *for some* $d > 0$ *and let* $A_n \subset R^p$ *be a sequence of sets such that with* $Z \sim \mathcal{N}_p(0, I)$,

$$P\{Z \in A_n(\varepsilon_n) - A_n\} \to 0 \quad \text{as } n \to \infty, \tag{4.2}$$

where $A_n(\varepsilon)$ *is the* ε*-neighborhood of* A_n. *Then there are sets* $B_n \subset R^{pn}$ *such that under* (4.1),

$$P\{X \in B_n\} \to 0 \quad \text{as } n \to \infty, \tag{4.3}$$

and if $(x_1, \dots, x_n) \in B_n$ *then*

$$|P(W \in A_n | x_n) - P\{Z \in A_n\}| \to 0 \quad \text{as } n \to \infty. \tag{4.4}$$

This result is proved in Portnoy (1984d). It should be noted that if N1 and P1 hold, more tedious computations will provide the result of Theorem 4.1 even if ψ is not odd function. Also note that (4.3) is sufficient to obtain unconditional convergence; for (4.3) and (4.4) provide convergence in probability for the conditional probabilities, which suffice to apply the dominated convergence theorem (see Pratt, 1960). Lastly, note that the arguments in Portnoy (1984a) and (1984d) can be extended to show that conditions $X1, \dots, X5$ hold in probability under (4.1).

REFERENCES

HUBER, PETER J. (1981). *Robust Statistics.* John Wiley and Sons, New York.

PRATT, J. W. (1960). On interchanging limits and integrals. *Ann. Math. Statist.* **31** 74–00.

PORTNOY, S. (1984a). Asymptotic behavior of M-estimators of p regression parameters when p^2/n is large. I: Consistency. *Ann. Statist.* **12** 1298–1309.

PORTNOY, S. (1984b). On the CLT in R^p when $p \to \infty$. Unpublished manuscript.

PORTNOY, S. (1984c). Asymptotic behavior of the empiric distribution of M-estimated residuals from a regression model with many parameters. Tentatively accepted by *Ann. Statist.* (August, 1985).

PORTNOY, S. (1984d). A CLT applicable to robust regression in R^p for p large. Unpublished manuscript.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801