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Asymptotic behavior of one-dimensional random dynamical systems

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0. Introduction.

Let S be a measurable space and let $\{\tau_s\}_{s\in S}$ be a family of transformations from the unit interval I into itself which are nonsingular with respect to the Lebesgue measure m on I. Given a measure preserving transformation σ and an S-valued random variable ξ on a probability space $(\Omega, \mathcal{F}, P)$, consider a model of a random dynamical system whose time evolution is given by

$$x_{n+1}=\tau_{\xi_{n+1}(\omega)}(x_n) \qquad \text{for} \quad n\geqq 1,$$

where $\xi_n=\xi\circ\sigma^{n-1}$.

Following S. Kakutani [4], we introduce a skew product transformation T on $I\times\Omega$ which is defined by

$$T(x, \omega)=(\tau_{\xi_1(\omega)}x, \sigma\omega) \qquad \text{for} \quad (x, \omega)\in I\times\Omega .$$

Since $\mathrm{proj}_I\circ T^n(x, \omega)=\tau_{\xi_n(\omega)}\tau_{\xi_{n-1}(\omega)}\cdots\tau_{\xi_1(\omega)}x$, we investigate the asymptotic behavior of the dynamical system $(T, m\times P)$ instead of the above random dynamical system. This was done in [3], [6], and [7], in the simplest case when ξ_n's are independent and identically distributed. The aim of this paper is to show that even if $\{\xi_n\}_{n=1}^{\infty}$ is a stationary sequence of dependent random variables, the skew product transformation T has $(m\times P)$-absolutely continuous invariant measures under some mild conditions and T admits various spectral decompositions according to the ergodic property of $\{\xi_n\}_{n=1}^{\infty}$. To do this as in previous paper [6], we introduce the so-called Perron-Frobenius operator $\mathcal{L}$ of T and investigate the asymptotic property of $\mathcal{L}^n\Phi$ for $\Phi\in L^1(m\times P)$. However, instead of estimating $\mathcal{L}^n\Phi$ itself we here estimate $\int_B\mathcal{L}^n\Phi d(m\times P)$ for $B\in\mathcal{B}(I)\times\mathcal{F}$. In Basic Lemma we will give a fundamental inequality on which the whole proof depends heavily (see section 3).

In section 1 we will give the definitions of some basic concepts. The main results are collected in Theorem 2.1. Sections 3 and 4 are devoted to the proof of Theorem 2.1. In section 5 we will give an application of the theorem to random stochastic matrices.

1. Preliminaries.

We call the collection $(X, \mathcal{B}, \mu, T)$ (or (T, μ) if there occurs no confusion) a *dynamical system* if $(X, \mathcal{B}, \mu)$ is an abstract Lebesgue space and T is a μ-nonsingular transformation on X. In particular we call it a *one-dimensional dynamical system* if $X=I$: the unit interval, $\mathcal{B}=\mathcal{B}(I)$: the topological Borel field on I and $\mu=m$: the Lebesgue measure on I.

Let $(S, \mathcal{B}(S))$ be a Polish space and let $\{(I, \mathcal{B}(I), m, \tau_s)\}_{s\in S}$ be a family of one-dimensional dynamical systems. Assume that the map $S\times I\ni(s, x)\to\tau_s x\in I$ is $\mathcal{B}(S\times I)|\mathcal{B}(I)$-measurable. Given another dynamical system $(\Omega, \mathcal{F}, P, \sigma)$ where P is σ-invariant and given an S-valued random variable ξ, consider the following time evolution:

$$x_{n+1}=\tau_{\xi_{n+1}(\omega)}(x_n) \qquad n\geqq 1, \quad \text{where} \quad \xi_n=\xi\circ\sigma^{n-1}. \tag{1.1}$$

The sequence of random iterations $\{\tau_{\xi_n}\tau_{\xi_{n-1}}\cdots\tau_{\xi_1}\}_{n=1}^{\infty}$ is called a *random dynamical system.*

We define a skew product transformation $T: I\times\Omega\to I\times\Omega$ by

$$T(x, \omega)=(\tau_{\xi_1(\omega)}(x), \sigma\omega). \tag{1.2}$$

Since τ_s's are m-nonsingular, $(I\times\Omega, \mathcal{B}(I)\times\mathcal{F}, m\times P, T)$ becomes a dynamical system. Note that

$$\mathrm{proj}_I\circ T^n(x, \omega)=\tau_{\xi_n(\omega)}\tau_{\xi_{n-1}(\omega)}\cdots\tau_{\xi_1(\omega)}x. \tag{1.3}$$

Next we introduce the Perron-Frobenius operator which plays an important role in this paper. Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system. The *Perron-Frobenius operator* $\mathcal{L}_{T,\mu}$ *of* T *with respect to* μ is defined by

$$\mathcal{L}_{T,\mu}\phi=\frac{d}{d\mu}\int_{T^{-1}(\cdot)}\phi d\mu \qquad \text{for each} \quad \phi\in L^1(\mu). \tag{1.4}$$

We list some basic properties of the Perron-Frobenius operator which will be used later:

(1) $\mathcal{L}_{T,\mu}$ is characterized by the following identity

$$\int\psi\mathcal{L}_{T,\mu}\phi d\mu=\int\psi\circ T\phi d\mu \qquad \text{for any} \quad \psi\in L^\infty(\mu) \quad \text{and for any} \quad \phi\in L^1(\mu). \tag{1.5}$$

(2) Let $(\tilde{T}, \mu)$ be another dynamical system. Then we have

$$\mathcal{L}_{T\circ\tilde{T},\mu}=\mathcal{L}_{T,\mu}\circ\mathcal{L}_{\tilde{T},\mu}. \tag{1.6}$$

(3) For $\phi\in L^1(\mu)$, $\mathcal{L}_{T,\mu}\phi=\phi$ if and only if $\phi\mu$ is T-invariant.

(4) Let ν be a μ-absolutely continuous T-invariant probability measure with density h. Then we have

$$h\mathcal{L}_{T,\nu}\phi=\mathcal{L}_{T,\mu}(h\phi) \qquad \mu\text{-a.e.} \tag{1.7}$$

2. The main results.

We begin with the definition of the space $\mathscr{D}$ of *nondegenerate piecewise* C^2 *transformations.* We say $\tau\in\mathscr{D}$ if there is a partition $0=a_0<a_1<\cdots<a_k=1$ such that the restriction $\tau|_{(a_{i-1},a_i)}$ is a C^2 function and can be extended to $[a_{i-1}, a_i]$ as a C^2 function for each i and

$$d(\tau)=\inf_{x\neq a_i}\left|\frac{d\tau}{dx}\right|>0. \tag{2.1}$$

We say that a partition $0=a_0<a_1<\cdots<a_k=1$ is the *minimal partition for* τ if it is minimal in the sense of refinement among all partitions satisfying the above. If τ belongs to $\mathscr{D}$ and $0=a_0<a_1<\cdots<a_k=1$ is the minimal partition for τ, it is obvious that $\tau|_{(a_{i-1},a_i)}$ is strictly increasing or strictly decreasing. For $\tau\in\mathscr{D}$ put

$$\beta(\tau)=d(\tau)^{-1}\left\{\max_{1\leq i\leq k}((a_i-a_{i-1})^{-1})+\max_{1\leq i\leq k}\frac{\sup|(\tau|^{-1}_{(a_{i-1},a_i)})''|}{\inf|(\tau|^{-1}_{(a_{i-1},a_i)})'|}\right\} \tag{2.2}$$

where $0=a_0<a_1<\cdots<a_k=1$ is the minimal partition for τ.

From now on we assume that the family $\{\tau_s\}_{s\in S}$ is contained in $\mathscr{D}$ and put

$$\alpha(s)=d(\tau_s)^{-1} \tag{2.3}$$

and for each n and for $s_1, s_2, \cdots, s_n\in S$, put

$$\beta_n(s_1, s_2, \cdots, s_n)=\beta(\tau_{s_1}\tau_{s_2}\cdots\tau_{s_n}). \tag{2.4}$$

To get the results we need the following two assumptions.

(A.1) $\quad M_0=\sup\left\{M\in[-\infty, \infty]\,;\ \sum_{n=1}^{\infty}P\left\{\frac{1}{n}\sum_{i=1}^{n}\log\alpha(\xi_i)\geqq -M\right\}<\infty\right\}>0.$

(A.2) There is a positive constant K such that $\sup_s\alpha(s)\leqq K$, $\sup_s\beta_1(s)\leqq K$ and for some integer $N>M_0^{-1}\log 2$ (if $M_0=\infty$ we regard $M_0^{-1}\log 2$ as 0), $\sup_{s_1,s_2,\cdots,s_N\in S}\beta_N(s_1, s_2, \cdots, s_N)\leqq K.$

The assumption (A.1) implies that there is a constant $c>1$ such that $d(\tau_{\xi_n(\omega)}\tau_{\xi_{n-1}(\omega)}\cdots\tau_{\xi_1(\omega)})\geqq c^n$ eventually for P almost all $\omega\in\Omega$. The condition (A.2) is rather technical and it is automatically satisfied if S is a finite set. If ξ_n's are independent, (A.1) and (A.2) can be replaced with simpler assumptions

(A.1)′ $\quad \int\log\alpha(\xi_1)dP<0.$

(A.2)′ For some $N>(-\log 2)\left(\int\log\alpha(\xi_1)dP\right)^{-1}$, $\beta_N(\xi_1, \xi_2, \cdots, \xi_N)\in L^1(P)$ and $\beta_1(\xi_1)\in L^1(P)$.

It is easy to see that (A.1)′ implies (A.1) if one investigates the speed of convergence in the strong law of large numbers.

Here we give some examples in which the assumption (A.1) is satisfied.

EXAMPLE 1. Suppose that $\{\tau_s\}_{s\in S}$ is uniformly expanding i. e. $d(\tau_s)=\alpha(s)^{-1}\geqq c$ for some constant $c>1$ for all $s\in S$. It is easy to show that M_0 is not smaller than $\log c>0$.

EXAMPLE 2. Let $\Omega=\{0, 1, \cdots, p-1\}$, $P\{i\}=1/p$ for $i=0, 1, \cdots, p-1$ and $\sigma:\Omega\to\Omega$ is defined by $\sigma(i)=i+1 \bmod p$. Let $S=\Omega$ and ξ be the identity map. (A.1) is satisfied if $\prod_{i=0}^{p-1}\alpha(i)<1$.

EXAMPLE 3. Let $\Omega=\{\omega\in \boldsymbol{C}; |\omega|=1\}$, P be the Haar measure on Ω, and $\sigma:\Omega\to\Omega$ be an irrational rotation, i. e. $\sigma\omega=a\omega$ where a is not a root of 1. And let $S=\Omega$ and ξ be the identity map. (A.1) is satisfied if there is a continuous function ϕ on Ω such that $\alpha\leqq\phi$ and $\int\log\phi dP<0$. This can be verified as follows: Put $e_k(\omega)=\omega^k$, $k\in\boldsymbol{Z}$, then it is clear that $e_k=(a^k-1)^{-1}(e_k\circ\sigma-e_k)$. We have

$$\left|\frac{1}{n}\sum_{i=1}^{n}e_k\circ\sigma^i-\int e_k dP\right|\leqq\frac{2}{|e^k-1|n}.$$

Thus for any $\delta>0$, there exists n_0 such that $n\geqq n_0$ implies

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}e_k\circ\sigma^i-\int e_k dP\right|>\delta\right\}=0.$$

Without loss of generality we may assume that ϕ is bounded from below by some positive constant. Then $f=\log\phi$ is also continuous on Ω. By the Stone-Weierstrass theorem, for any $\varepsilon>0$, there exists f_ε, a finite linear combination of e_k's such that $\sup_\omega|f(\omega)-f_\varepsilon(\omega)|<\varepsilon$. Hence if $\varepsilon<\left|\int fdP\right|/4$,

$$\begin{aligned}
&P\left\{\frac{1}{n}\sum_{i=1}^{n}\log\alpha\circ\sigma^i>\frac{1}{2}\int fdP\right\}\\
&\leqq P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}f\circ\sigma^i-\int fdP\right|>\frac{1}{2}\left|\int fdP\right|\right\}\\
&\leqq P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}f_\varepsilon\circ\sigma^i-\int f_\varepsilon dP\right|>\frac{1}{2}\left|\int fdP\right|-2\varepsilon\right\}=0
\end{aligned}$$

if n is large enough. This implies that (A.1) is satisfied.

EXAMPLE 4. Let $\Omega=S^N$, $\sigma:\Omega\to\Omega$ be the shift and ξ be the projection on the first coordinate. (A.1) is satisfied if one of the following conditions is valid.

(i) $\{\xi_n\}_{n=1}^\infty$ is strongly mixing with mixing coefficients $\{\phi(n)\}_{n=1}^\infty$ satisfying $\sum_{n=1}^\infty n\phi(n)<\infty$ and $\int\log\alpha dP<0$.

(ii) $\{\xi_n\}_{n=1}^\infty$ is uniformly mixing with mixing coefficients $\{\phi(n)\}_{n=1}^\infty$ satisfying $\sum_{n=1}^\infty \phi(n)<\infty$ and $\int\log\alpha dP<0$.

For the proof see [2, Ch. 18].

Now we are ready to state our results.

THEOREM 2.1. *Let* $\{(I, \mathcal{B}(I), m, \tau_s)\}_{s\in S}$ *be a family of one-dimensional dynamical systems such that* $\{\tau_s\}_{s\in S}$ *is a subset of* $\mathcal{D}$ *and the map* $S\times I\ni(s, x)\rightarrow \tau_s x\in I$ *is* $\mathcal{B}(S\times I)|\mathcal{B}(I)$*-measurable. Let* $(\Omega, \mathcal{F}, P, \sigma)$ *be a dynamical system such that* $(\Omega, \mathcal{F}, P)$ *is an abstract Lebesgue space and* P *is* σ*-invariant. Let* ξ *be an* S*-valued random variable on* $(\Omega, \mathcal{F}, P)$ *and* T *the skew product transformation defined by* (1.2). *Assume that* (A.1) *and* (A.2) *are satisfied. Then,*

(1) *There exists at least one* $(m\times P)$*-absolutely continuous* T*-invariant finite measure. In the rest of this paper such a measure will be abbreviated as an a.c.i. measure of* $(T, m\times P)$ *etc.*

(2) *If the dynamical system* (σ, P) *is ergodic, there exists a finite number of a.c.i. probability measures* $Q_1, Q_2, \cdots, Q_l$ *of* $(T, m\times P)$ *such that*

(i) *for each* $i=1, 2, \cdots, l$, *the dynamical system* (T, Q_i) *is ergodic;*

(ii) *if* Q *is an a.c.i. countably additive set function of* $(T, m\times P)$, *then* Q *can be written as a linear combination of* Q_i*'s.*

(3) *If* (σ, P) *is totally ergodic and* Q_i *is one of the probability measures stated above, there is an integer* N_i *and a collection of disjoint sets* $L_{i,0}, L_{i,1}, \cdots, L_{i,N_i-1}\subset\mathcal{B}(I)\times\mathcal{F}$ *such that*

(i) $TL_{i,j}=L_{i,j+1}$ $(0\leqq j<N_i-1)$, $TL_{i,N_i-1}=L_{i,0}$;

(ii) *for each* $j=0, 1, \cdots, N_i-1$, *the dynamical system* $(T^{N_i}, Q_{i,j})$ *is totally ergodic where* $Q_{i,j}=N_iQ_i|_{L_{i,j}}$.

(4) *Moreover, if the dynamical system* (σ, P) *is exact, so is the dynamical system* $(T^{N_i}, Q_{i,j})$ *which is stated above. For the definition of exactness see* [8].

3. Existence of $(m\times P)$-absolutely continuous T-invariant measures.

In this section we prove the statement (1) of Theorem 2.1. To begin with we need a lemma due to Lasota and Yorke [5]. Put $\boldsymbol{BV}=\{\phi\in L^1(m); \vee\phi=\inf\{\tilde{\vee}\tilde{\phi}; \phi=\tilde{\phi}\ m\text{-a.e.}\}<\infty\}$ where $\tilde{\vee}\tilde{\phi}$ denotes the usual total variation of $\tilde{\phi}$ on I. $\vee\phi$ is called the total variation of ϕ belonging to $L^1(m)$. For $\tau\in\mathcal{D}$, let $\mathcal{L}_{\tau,m}$ be the Perron-Frobenius operator of τ with respect to m.

LEMMA 3.1. *If* τ *is in* $\mathcal{D}$, *then we have*

$$\vee\mathcal{L}_{\tau,m}\phi\leqq 2d(\tau)^{-1}\vee\phi+\beta(\tau)\|\phi\|_{1,m} \quad \textit{for all} \quad \phi\in\boldsymbol{BV}. \tag{3.1}$$

Let δ be a positive constant such that

$$2e^{-NM_0}<\delta<1, \tag{3.2}$$

where N and M_0 are the values appearing in the assumptions (A.1) and (A.2). For any $p\in\boldsymbol{N}$ and for any $n\geqq p$ put

(3.3) $$\Omega_p^n=\bigcup_{j=p-1}^{n-1}\{\alpha(\xi_n)\alpha(\xi_{n-1})\cdots\alpha(\xi_{n-j})\geqq(2^{-1}\delta)^{(j+1)/N}\}.$$

The basic lemma throughout the paper is the following:

BASIC LEMMA. *Suppose that* (A.1) *and* (A.2) *are satisfied. Let* Φ *be in* $L^1(m\times P)$ *with* $\overline{\vee}(\Phi)=\sup_{\omega\in\Omega}\vee\Phi(\cdot,\omega)<\infty$. *Then, for any* $p\in N$, *there is a positive constant* K_p *which is independent of* Φ *such that*

(3.4) $$\int_B \mathcal{L}^n\Phi d(m\times P)$$
$$\leqq\int_{\Omega_p^n}\|\Phi\|_{1,m}dP+K_p(m\times P_n(\Phi))(B)+C\rho^n\overline{\vee}(\Phi)(m\times P)(B)$$

for every $n\geqq p$, *where* $\mathcal{L}=\mathcal{L}_{T,m\times P}$, $C=2^{N-1}K^{N-1}$, $\rho=\delta^{1/(2N)}$ *and* $P_n(\Phi)$ *is the measure on* $(\Omega,\mathcal{F})$ *defined by*

(3.5) $$P_n(\Phi)(\Gamma)=\int_{\sigma^{-n}\Gamma}\|\Phi\|_{1,m}dP \qquad \text{for} \quad \Gamma\in\mathcal{F}.$$

PROOF. Without loss of generality we may assume that $\Phi\geqq 0$. Put $[s_1,s_2,\cdots,s_n]=\{\omega\in\Omega;\xi_i(\omega)=s_i,\ i=1,2,\cdots,n\}$. Since $(S,\mathcal{B}(S))$ is a polish space and $(\Omega,\mathcal{F},P)$ is an abstract Lebesgue space, the partition $\eta_n=\{[s_1,s_2,\cdots,s_n];s_i\in S,1\leqq i\leqq n\}$ becomes a measurable partition of Ω. Let $(\Omega_{\eta_n},\mathcal{F}_{\eta_n},P_{\eta_n})$ be the factor space of $(\Omega,\mathcal{F},P)$ with respect to η_n and let $\{P_{[s_1,s_2,\cdots,s_n]}\}_{[s_1,s_2,\cdots,s_n]\in\eta_n}$ be the Rohlin decomposition of P corresponding to η_n. Then for any $A\in\mathcal{B}(I)$ and $\Gamma\in\mathcal{F}$, we have

(3.6) $$\int_{A\times\Gamma}\mathcal{L}^n\Phi d(m\times P)$$
$$=\int dP_{\eta_n}\int dP_{[s_1,\cdots,s_n]}1_{(\sigma^{-n}\Gamma)\cap\Omega_p^n}\int_A\mathcal{L}_{s_n}\cdots\mathcal{L}_{s_1}\Phi dm$$
$$+\int dP_{\eta_n}\int dP_{[s_1,\cdots,s_n]}1_{(\sigma^{-n}\Gamma)\cap(\Omega\setminus\Omega_p^n)}\int_A\mathcal{L}_{s_n}\cdots\mathcal{L}_{s_1}\Phi dm$$

for $n\geqq p$, where $\mathcal{L}_{s_i}=\mathcal{L}_{\tau_{s_i},m}$. The first term of the above equation is dominated by $\int_{\Omega_p^n}\|\Phi\|_{1,m}dP$. So we have to estimate the second term. We can write $p=qN+r$ $(0\leqq r<N)$ and $n=jN+k$ $(0\leqq k<N)$. Using Lemma 3.1 repeatedly we see that

$$\vee\mathcal{L}_{s_{jN+k}}\cdots\mathcal{L}_{s_1}\Phi$$
$$\leqq\sum_{l=0}^{j-1}2^l\alpha(s_{jN+k})\cdots\alpha(s_{(j-l)N+k+1})$$
$$\times\beta_N(s_{(j-l)N+k},\cdots,s_{(j-l-1)N+k+1})\|\Phi\|_{1,m}$$
$$+2^j\alpha(s_{jN+k})\cdots\alpha(s_{k+1})\vee\mathcal{L}_{s_k}\cdots\mathcal{L}_{s_1}\Phi$$

and

$$\vee\mathcal{L}_{s_k}\cdots\mathcal{L}_{s_1}\Phi\leqq 2^k\alpha(s_k)\cdots\alpha(s_1)\vee\Phi+\{\beta_1(s_k)+2\alpha(s_k)\beta_1(s_{k-1})+\cdots$$
$$+2^{k-1}\alpha(s_k)\cdots\alpha(s_2)\beta_1(s_1)\}\|\Phi\|_{1,m}.$$

If $\omega\in\Omega\setminus\Omega_p^n$, by (A.2) and (3.3) we can see

$$2^l\alpha(s_{jN+k})\cdots\alpha(s_{(j-l)N+k+1})\leqq\delta^l \qquad \text{for} \quad l>q.$$

So we have

$$2^l\alpha(s_{jN+k})\cdots\alpha(s_{(j-l)N+k+1})\beta_N(s_{(j-l)N+k},\ \cdots,\ s_{(j-l-1)N+k+1})$$
$$\leqq\begin{cases}\delta^l K & l>q\\ 2^l K^{Nl+1} & l\leqq q.\end{cases}$$

Thus

$$\vee\mathcal{L}_{s_n}\cdots\mathcal{L}_{s_1}\Phi\leqq\Big\{\frac{K(2^{q+1}K^{(q+1)N}-1)}{2K^N-1}+\frac{\delta^{q+1}K(1-\delta^{j-q-1})}{1-\delta}$$
$$+\frac{K(2^{N-1}K^{N-1}-1)}{2K-1}\delta^j\Big\}\|\Phi\|_{1,m}$$
$$+2^{N-1}K^{N-1}\delta^j\bigtriangledown(\Phi).$$

Put

$$K_p=\frac{K(2^{q+1}K^{(q+1)N}-1)}{2K^N-1}+\frac{K}{1-\delta}+\frac{K(2^{N-1}K^{N-1}-1)}{2K-1}+1.$$

Since $\|\phi\|_{\infty,m}\leqq\vee\phi+\|\phi\|_{1,m}$ for $\phi\in\boldsymbol{BV}$, it follows that the second term of the right hand side in (3.6) is not larger than

$$K_pm(A)\int_{\sigma^{-n}\Gamma}\|\Phi\|_{1,m}dP+C\rho^n\bigtriangledown(\Phi)m(A)P(\Gamma),$$

where $C=2^{N-1}K^{N-1}$ and $\rho=\delta^{1/(2N)}$. Therefore

$$\int_{A\times\Gamma}\mathcal{L}^n\Phi d(m\times P)\leqq\int_{\Omega_p^n}\|\Phi\|_{1,m}dP+K_pm(A)\int_{\sigma^{-n}\Gamma}\|\Phi\|_{1,m}dP$$
$$+C\rho^n\bigtriangledown(\Phi)m(A)P(\Gamma).$$

From this inequality, it is easy to prove the inequality (3.4).

Now we are ready to prove the statement (1) in Theorem 2.1.

PROOF OF (1) IN THEOREM 2.1. To begin with we note that for every $\Phi\in L^1(m\times P)$ with $\bigtriangledown(\Phi)<\infty$, $\{(1/n)\sum_{i=0}^{n-1}\mathcal{L}^i\Phi\}_{n=1}^\infty$ is weakly sequentially compact in $L^1(m\times P)$. In fact,

$$P(\Omega_p^n)\leqq\sum_{j=p-1}^{n-1}P\{\alpha(\xi_n)\alpha(\xi_{n-1})\cdots\alpha(\xi_{n-j})\geqq(2^{-1}\delta)^{(j+1)/N}\} \tag{3.7}$$
$$\leqq\sum_{j=p-1}^{\infty}P\{\alpha(\xi_{j+1})\alpha(\xi_j)\cdots\alpha(\xi_1)\geqq(2^{-1}\delta)^{(j+1)/N}\}$$
$$+\sum_{j=p}^{\infty}P\Big\{\frac{1}{j}\sum_{i=1}^{j}\log\alpha(\xi_i)\geqq\log(2^{-1}\delta)^{1/N}\Big\}.$$

But $\log(2^{-1}\delta)^{1/N}>-M_0$, since $2e^{-NM_0}<\delta<1$. So from the assumption (A.1) $\sup_nP(\Omega_p^n)\to0$ as $p\to\infty$. Now given $\varepsilon>0$, take sufficiently large p so that $\int_{\Omega_p^n}\|\Phi\|_{1,m}dP<\varepsilon$ and fix it. It is easy to see that $m\times P_n(\Phi)(B)<\varepsilon$ and

$C\rho^n \bigvee(\Phi)(m\times P)(B)<\varepsilon$ if $(m\times P)(B)$ is small enough. Thus we have

$$\lim_{\varepsilon\downarrow 0}\sup_n \sup_{B:(m\times P)(B)<\varepsilon}\left|\int_B \mathcal{L}^n\Phi d(m\times P)\right|=0$$

from Basic Lemma. Therefore

$$\lim_{\varepsilon\downarrow 0}\sup_n \sup_{B:(m\times P)(B)<\varepsilon}\left|\int_B \frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}^i\Phi d(m\times P)\right|=0.$$

Hence $\{(1/n)\sum_{i=0}^{n-1}\mathcal{L}^i\Phi\}_{n=1}^{\infty}$ is weakly sequentially compact in $L^1(m\times P)$ (see [**1**, p. 294]). By the Kakutani-Yosida Theorem, $(1/n)\sum_{i=0}^{n-1}\mathcal{L}^i\Phi$ converges strongly in $L^1(m\times P)$. From the formula (1.5)

$$\int\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}^i\Phi d(m\times P)=\int\Phi d(m\times P).$$

Therefore take Φ in $L^1(m\times P)$ such that $\Phi\geqq 0$ and $\int\Phi d(m\times P)=1$, then $\Phi^*=\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}\mathcal{L}^i\Phi$ satisfies $\mathcal{L}\Phi^*=\Phi^*$ and $\int\Phi^* d(m\times P)=1$. This implies that $\Phi^*(m\times P)$ is an a.c.i. probability measure of $(T, m\times P)$.

4. The spectral decomposition.

As before we always assume (A.1) and (A.2).

LEMMA 4.1. *For $p\in\boldsymbol{N}$, put*

$$\gamma_p=\sup_k \limsup_{n\to\infty}\frac{1}{n}\sum_{i=p}^{n-1}P(\Omega_p^{ik}) \qquad \textit{and} \qquad \varepsilon_p=\limsup_{n\to\infty}P(\Omega_p^n)$$

where Ω_p^n is the set defined by (3.3).

(1) *Let k be a positive integer. If the dynamical system (σ^k, P) is ergodic, we have*

$$\left|\int_B\Phi^* d(m\times P)\right|\leqq\{\gamma_p+K_p(m\times P)(B)\}\,\|\Phi^*\|_{1,m\times P} \tag{4.1}$$

for all $B\in\mathcal{B}(I)\times\mathcal{F}$, where K_p is the constant which appeared in Basic Lemma, and Φ^* is any $\mathcal{L}^k$-invariant function of $L^1(m\times P)$.

(2) *If the dynamical system (σ, P) is exact, we have*

$$\limsup_{n\to\infty}\left|\int_{B_n}\mathcal{L}^n\Phi d(m\times P)\right| \leqq\{\varepsilon_p+K_p\limsup_{n\to\infty}(m\times P)(B_n)\}\,\|\Phi\|_{1,m\times P} \tag{4.2}$$

for any $\Phi\in L^1(m\times P)$ and any sequence $\{B_n\}_{n=1}^{\infty}$ of sets in $\mathcal{B}(I)\times\mathcal{F}$.

PROOF. (1) First of all we assume that $\bigvee(\Phi)<\infty$. Then by Basic Lemma for fixed $p\in\boldsymbol{N}$ and any $n\geqq p$,

$$\left|\int_B\frac{1}{n}\sum_{i=p}^{n-1}\mathcal{L}^{ik}\Phi d(m\times P)\right|$$

$$\leqq \frac{1}{n}\sum_{i=p}^{n-1}\int_{\Omega_p^{ik}} \|\Phi\|_{1,m} dP + K_p \frac{1}{n}\sum_{i=p}^{n-1}(m\times P_{ik}(\Phi))(B) + \frac{C}{n}\sum_{i=p}^{n-1}\rho^{ik}\nabla(\Phi)(m\times P)(B).$$

Therefore

$$\left|\int_B \Phi^* d(m\times P)\right| \leqq \limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{i=p}^{n-1}\int_{\Omega_p^{ik}}\|\Phi\|_{1,m}dP + K_p\frac{1}{n}\sum_{i=p}^{n-1}(m\times P_{ik}(\Phi))(B)\Big), \tag{4.3}$$

where $\Phi^*=\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}\mathcal{L}^{ik}\Phi$ whose existence is guaranteed by the result in the previous section. Since the set of all Φ with $\nabla(\Phi)<\infty$ is dense in $L^1(m\times P)$, the inequality (4.3) holds for all $\Phi\in L^1(m\times P)$. Note that $P_n(\Phi)(\Gamma) =\int_{\sigma^{-n}\Gamma}\|\Phi\|_{1,m}dP$ if $\Gamma\in\mathcal{F}$. Since (σ^k, P) is ergodic,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}P_{ik}(\Phi)(\Gamma)=P(\Gamma)\int_\Omega\|\Phi\|_{1,m}dP = P(\Gamma)\|\Phi\|_{1,m\times P}$$

which proves that if $B\in\mathcal{B}(I)\times\mathcal{F}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(m\times P_{ik}(\Phi))(B)=(m\times P)(B)\|\Phi\|_{1,m\times P}. \tag{4.4}$$

Next we prove that

$$\|\Phi\|_{1,m}(\omega)=\|\Phi\|_{1,m\times P} \qquad P\text{-a.e.} \tag{4.5}$$

if Φ is $\mathcal{L}^k$-invariant. Indeed, we have

$$\begin{aligned}
&\int_\Gamma\|\Phi\|_{1,m}(\omega)dP(\omega)\\
&=\int_{I\times\Gamma}|\Phi|(x,\omega)dm(x)dP(\omega)\\
&=\int_{I\times\Gamma}(\mathcal{L}^k|\Phi|)(x,\omega)d(m\times P)(x,\omega)\\
&=\int_{\Gamma\times\sigma^{-k}\Gamma}|\Phi|(x,\omega)d(m\times P)(x,\omega)\\
&=\int_{\sigma^{-k}\Gamma}\|\Phi\|_{1,m}(\omega)dP(\omega),
\end{aligned}$$

which shows that $\|\Phi\|_{1,m}(\omega)P(d\omega)$ is a σ^k-invariant measure. Since (σ^k, P) is ergodic, $\|\Phi\|_{1,m}(\omega)=\|\Phi\|_{1,m\times P}$. By (4.3), (4.4), and (4.5) we have (4.1).

(2) Since (σ, P) is exact, it is known that

$$\lim_{n\to\infty}\int_\Omega|E(\|\Phi\|_{1,m}|\sigma^{-n}\mathcal{F})-\|\Phi\|_{1,m\times P}|dP=0. \tag{4.6}$$

If $B\in\mathcal{B}(I)\times\mathcal{F}$,

$$(4.7) \qquad (m\times P_n(\Phi))(B)=\int_I P_n(\Phi)(B_x)dm(x)$$
$$=\int_I\int_{\sigma^{-n}(B_x)}\|\Phi\|_{1,m}(\omega)dP(\omega)dm(x)$$
$$=\int_I\int_{\sigma^{-n}(B_x)}E(\|\Phi\|_{1,m}|\sigma^{-n}\mathcal{F})dP(\omega)dm(x)$$

where $B_x=\{\omega\in\Omega\,;(x,\omega)\in B\}$. Thus we have

$$|(m\times P_n(\Phi))(B)-(m\times P)(B)\|\Phi\|_{1,m\times P}|$$
$$\leqq\int_I\left|\int_{\sigma^{-n}(B_x)}(E(\|\Phi\|_{1,m}|\sigma^{-n}\mathcal{F})-\|\Phi\|_{1,m\times P})dP(\omega)\right|dm(x).$$

By (4.6) and (4.7) we have

$$(4.8) \qquad \lim_{n\to\infty}(m\times P_n(\Phi))(B)=(m\times P)(B)\|\Phi\|_{1,m\times P}$$

uniformly in $B\in\mathcal{B}(I)\times\mathcal{F}$. Next we can see that

$$(4.9) \qquad \lim_{n\to\infty}\left|\int_\Gamma\|\mathcal{L}^n|\Phi|\|_{1,m}(\omega)dP-P(\Gamma)\|\Phi\|_{1,m\times P}\right|=0$$

uniformly in $\Gamma\in\mathcal{F}$. In fact

$$\int_\Gamma\|\mathcal{L}^n|\Phi|\|_{1,m}dP$$
$$=\int_{I\times\Gamma}(\mathcal{L}^n|\Phi|)(x,\omega)d(m\times P)(x,\omega)$$
$$=\int_{T^{-n}(I\times\Gamma)}|\Phi|(x,\omega)d(m\times P)(x,\omega)$$
$$=\int_{\sigma^{-n}\Gamma}\|\Phi\|_{1,m}(\omega)dP(\omega)$$
$$=\int_{\sigma^{-n}}E(\|\Phi\|_{1,m}|\sigma^{-n}\mathcal{F})dP$$

and so from (4.6) we see (4.9). From Basic Lemma and (4.8) we have

$$\limsup_{n\to\infty}\left|\int_{B_n}\mathcal{L}^n\Phi d(m\times P)\right|\leqq\limsup_{n\to\infty}\int_{\Omega_p^n}\|\Phi\|_{1,m}dP$$
$$+K_p\|\Phi\|_{1,m\times P}\limsup_{n\to\infty}(m\times P)(B_n)$$

for all $\Phi\in L^1(m\times P)$. Substituting $\mathcal{L}^l\Phi$ for Φ in the above we have

$$\limsup_{n\to\infty}\left|\int_{B_n}\mathcal{L}^n\Phi d(m\times P)\right|\leqq\limsup_{n\to\infty}\int_{\Omega_p^n}\|\mathcal{L}^l\Phi\|_{1,m}dP$$
$$+K_p\|\Phi\|_{1,m\times P}\limsup_{n\to\infty}(m\times P)(B_n).$$

By (4.9) and the above we get the inequality (4.2).

Now we can prove the statements (2), (3), and (4) in Theorem 2.1.

PROOF OF THE STATEMENT (2). Assume that (σ, P) is ergodic. In order to prove this statement it is enough to show that there are only finitely many a.c.i. probability measures of $(T, m\times P)$ which are mutually singular. Suppose that there were infinitely many such measures. Then we could find a sequence of $\mathcal{L}$-invariant functions $\{\Phi_n\}_{n=1}^{\infty}\subset L^1(m\times P)$ and a sequence of sets $\{B_n\}_{n=1}^{\infty}\subset\mathcal{B}(I)\times\mathcal{F}$ such that B_n are mutually disjoint, each Φ_n are supported in B_n, and $\int\Phi_n d(m\times P)=1$. It is obvious that $(m\times P)(B_n)\to 0$ $(n\to\infty)$. Since it is clear that $\lim_{p\to\infty}\gamma_p=0$ from (3.7) in the proof of the statement (1), we can choose p to be so large that $\gamma_p<1/2$. On the other hand, since (σ, P) is ergodic, from the inequality (4.1)

$$1=\int_{B_n}\Phi_n d(m\times P)\leqq\gamma_p+K_p(m\times P)(B_n)\leqq\frac{1}{2}+K_p(m\times P)(B_n)$$

for each $n\in N$, which is a contradiction. Hence the statement (2) of Theorem 2.1 is valid.

PROOF OF THE STATEMENT (3). If the statement (3) were not true, we could find sequences $\{k_n\}_{n=1}^{\infty}\subset N$, $\{\Phi_n\}_{n=1}^{\infty}\subset L^1(m\times P)$ and $\{B_n\}_{n=1}^{\infty}\subset\mathcal{B}(I)\times\mathcal{F}$ satisfying that (i) k_n is a divisor of k_{n+1}, (ii) Φ_n is $\mathcal{L}^{k_n}$-invariant, supported in B_n and $\int\Phi_n d(m\times P)=1$, (iii) $B_n\supset B_{n+1}$ and $(m\times P)(B_n)\to 0$ $(n\to\infty)$. Applying the inequality (4.1) to (σ^{k_n}, P), Φ_n and B_n we can reach a contradiction as in the preceding argument.

PROOF OF THE STATEMENT (4). Suppose that (σ, P) is exact. First we claim that if Q is an a.c.i. probability measure of $(T, m\times P)$ with density Ψ, the Borel field $\mathcal{D}_\infty=\bigcap_{n=1}^{\infty}T^{-n}(\mathcal{B}(I)\times\mathcal{F})$ has only finitely many atoms with respect to Q. If this is not the case, we could find a sequence $\{B_k\}_{k=1}^{\infty}\subset\mathcal{D}_\infty$ such that $B_k\subset\{\Psi>0\}$, $(m\times P)(B_k\cap B_l)=0$ if $k\neq l$, and $0<Q(B_k)\to 0$ $(k\to\infty)$. Put $\Phi_k=Q(B_k)^{-1}1_{B_k}$. Then from the formula (1.7), we have

$$\int_{T^nB_k}\mathcal{L}^k(\Phi_k\Psi)d(m\times P)=\int_{T^nB_k}\mathcal{L}^n_{T,Q}(\Phi_k)dQ$$

$$=\int_{T^{-n}T^nB_k}\Phi_k dQ$$

$$=1$$

since $T^{-n}T^nB_k=B_k$. Therefore from the inequality (4.2) we have

$$1=\limsup_{n\to\infty}\int_{T^nB_k}\mathcal{L}^n(\Phi_k\Psi)d(m\times P)\leqq\{\varepsilon_p+K_p\limsup_{n\to\infty}(m\times P)(T^nB_k)\}.$$

On the other hand

$$\lim_{k\to\infty}\sup_n(m\times P)(T^nB_k)=0$$

since $m\times P$ and Q are equivalent to each other on $\{\Psi>0\}$, and $Q(T^nB_k)$

$=Q(T^{-n}T^nB_k)=Q(B_k)\to 0$ $(k\to\infty)$. Consequently, we have $1\leqq\varepsilon_p$ for all $p\in N$. This is a contradiction since $\varepsilon_p\to 0$ as $p\to\infty$.

Since $L_{i,j}\in\mathcal{D}_\infty$, without loss of generality we may assume that $Q=Q_{i,j}$, i. e. (T, Q) is already totally ergodic. From the above claim $\mathcal{D}_\infty$ is generated by a finitely many sets $B_1, B_2, \cdots, B_r$ with $Q(B_i)>0$ and $Q(B_i\cap B_j)=0$ if $i\neq j$, $i, j=1, 2, \cdots, r$. We may assume that $Q(B_1)=\min\{Q(B_i)\,;1\leqq i\leqq r\}$. Then one can easily show that $T^{-j}B_1=B_1$ for some $1\leqq j\leqq r$. But the ergodicity of (T^j, Q) implies that $Q(B_1)=1$ and $j=1$. Hence we have $r=1$. This completes the proof.

5. An application to random stochastic matrices.

In this section let S denote the totality of $k\times k$ stochastic matrices. Let $(\Omega, \mathcal{F}, P, \sigma)$ be a dynamical system where P is σ-invariant and ξ be an S-valued random variable. As before the S-valued stationary sequence is defined by $\xi_n=\xi\circ\sigma^{n-1}$ for $n\geqq 1$. For matrix valued random variable ξ, $E\xi$ denotes the matrix $(E\xi_{ij})$. We are concerned with the asymptotic behavior of the expectation of random products $\xi_1\xi_2\cdots\xi_n$. If ξ_n's are independent and identically distributed, then $E(\xi_1\xi_2\cdots\xi_n)=E(\xi_1)^n$ and the asymptotic behavior of $E(\xi_1)^n$ is well studied in the theory of Markov chain. For the general case we can see the following but do not have a detailed description of its behavior.

PROPOSITION. *$E(\xi_1\xi_2\cdots\xi_n)$ converges in the sense of arithmetic mean.*

PROOF. For each $s=(s_{ij})$, consider the following one-dimensional transformation

$$\tau_s(x)=\frac{4}{s_{ij}}\Big(x-\frac{i-1}{k}-\frac{q}{4k}-\frac{1}{4k}\sum_{l=1}^{j-1}s_{il}\Big)+\frac{j-1}{k} \tag{5.1}$$

$$\text{if } \frac{i-1}{k}+\frac{q}{4k}+\frac{1}{4k}\sum_{l=1}^{j-1}s_{il}\leqq x<\frac{i-1}{k}+\frac{q}{4k}+\frac{1}{4k}\sum_{l=1}^{j}s_{il},$$

with $s_{ij}\neq 0$, $1\leqq i\leqq k$, and $1\leqq j\leqq k$, $q=0, 1, 2, 3$.

Then it is easy to see that the map $(s, x)\to\tau_s x$ is $\mathcal{B}(S\times I)\,|\,\mathcal{B}(I)$-measurable and

$$k^{-1}(s_1s_2\cdots s_n)_{ij}=m\Big((\tau_{s_n}\tau_{s_{n-1}}\cdots\tau_{s_1})^{-1}\Big[\frac{j-1}{k}, \frac{j}{k}\Big]\cap\Big[\frac{i-1}{k}, \frac{i}{k}\Big]\Big).$$

Thus

$$k^{-1}\frac{1}{n}\sum_{l=1}^{n}(E\xi_1\xi_2\cdots\xi_l)_{ij}$$

$$=\frac{1}{n}\sum_{l=1}^{n}\int m\Big((\tau_{\xi_l}\tau_{\xi_{l-1}}\cdots\tau_{\xi_1})^{-1}\Big[\frac{j-1}{k}, \frac{j}{k}\Big]\cap\Big[\frac{i-1}{k}, \frac{i}{k}\Big]\Big)dP$$

$$=\frac{1}{n}\sum_{l=1}^{n}\int 1_{[(j-1)/k,\,j/k]\times\Omega}\circ T^l 1_{[(i-1)/k,\,i/k]\times\Omega}\,dm\,dP$$

$$=\frac{1}{n}\sum_{l=1}^{n}\int_{[(j-1)/k,j/k]\times\Omega}\mathcal{L}_{T,m\times P}^{l}1_{[(i-1)/k,i/k]\times\Omega}dm\,dP.$$

Therefore we can complete the proof if the strong convergence of $(1/n)\sum_{l=0}^{n-1}\mathcal{L}_{T,m\times P}^{l}\Phi$ is established for $\Phi\in L^{1}(m\times P)$.

Since $d_{\tau_s}\geqq 4$, (A.1) is satisfied. But since $\beta_1(s)\geqq(k/4)\max_{i,j}s_{ij}^{-1}$, the assumption (A.2) is not satisfied. But in this case we can prove that

$$\bigvee\mathcal{L}_s\phi\leqq 2d_{\tau_s}^{-1}\bigvee\phi+k\|\phi\|_{1,m}\qquad\text{for}\quad\phi\in\boldsymbol{BV},$$

using the special property of τ_s defined in (5.1). In other words the inequality (3.1) is valid with $\beta(\tau)$ replaced by k. Using the above inequality and noting that $2d(\tau_s)^{-1}<1/2$, we can show that the proof of Basic Lemma does work with $N=1$. Thus all the results in the preceding sections are valid in our case. In particular, $(1/n)\sum_{i=0}^{n-1}\mathcal{L}_{T,m\times P}^{i}\Phi$ converges strongly in $L^{1}(m\times P)$ for $\Phi\in L^{1}(m\times P)$.

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