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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A CONSERVED PHASE-FIELD SYSTEM WITH MEMORY

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ABSTRACT. We show that any global bounded solution of a conserved phase-field model with memory terms converges to a single stationary state as time goes to infinity. The idea of analyticity plays a key role in our analysis.

1. Introduction. The time evolution of the phase variable $\chi(t,x)$ and the temperature $\vartheta(t,x)$ in the conserved phase-field model proposed by Caginalp [**7**] is governed by the system of differential equations:

$$\tau\partial_t\chi = -\xi^2\Delta(\xi^2\Delta\chi - W'(\chi) + \lambda\vartheta), \tag{1.1}$$

$$\partial_t(\vartheta + \lambda\chi) + \operatorname{div}\mathbf{q} = 0, \tag{1.2}$$

where W is typically a double-well potential, λ is a positive constant representing the latent heat, $\tau > 0$ and $\xi > 0$ stands for a relaxation time and correlation length, respectively, and $\mathbf{q}$ denotes the heat flux. Here we shall assume that $\mathbf{q}$ is determined by the linearized Coleman-Gurtin [**8**] constitutive relation:

$$\mathbf{q} = -k_I\nabla\vartheta - k*\nabla\vartheta, \tag{1.3}$$

where the constant $k_I > 0$ is the instantaneous heat conductivity, k is a suitable dissipative kernel, and the symbol $*$ denotes the time convolution:

$$k*v(t) = \int_0^\infty k(s)v(t-s)\,ds.$$

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The material occupies a bounded regular domain $\Omega \subset \mathbf{R}^3$ and the system (1.1)–(1.3) is complemented by the homogeneous Neumann boundary condition for χ, ϑ and the so-called chemical potential $-\xi^2\Delta\chi + W'(\chi) - \lambda\vartheta$ which can be expressed by

$$\nabla\chi \cdot \mathbf{n}|_{\partial\Omega} = \nabla(\Delta\chi) \cdot \mathbf{n}|_{\partial\Omega} = \nabla\vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$
$$\text{with } \mathbf{n} \text{ the outer normal vector.}$$

For the sake of simplicity, we set the constants τ, ξ and k_I equal to 1. The system (1.1), (1.2) with $\mathbf{q}$ given by (1.3) then reads

$$\partial_t\chi = -\Delta(\Delta\chi - W'(\chi) + \lambda\vartheta) \tag{1.4}$$
$$\partial_t(\vartheta + \lambda\chi) = \Delta\vartheta + k * \Delta\vartheta \tag{1.5}$$
$$\nabla\chi \cdot \mathbf{n} = 0, \quad \nabla(\Delta\chi) \cdot \mathbf{n} = 0, \quad \nabla\vartheta \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

Systems of the same or comparable type, with or without memory terms, as well as nonconserved systems, i.e., systems with the pure chemical potential on the right-hand side of (1.4) have been recently studied by many authors (see Colli et al. [**9**], Grasselli et al. [**15**], Vegni [**25**], Aizicovici and Feireisl [**2**], Novick-Cohen [**20**], etc.) The questions of well-posedness and existence of finite dimensional attractors were considered in [**15**], and the dissipativity of the system was studied in [**25**]. In particular, the long-time behavior of solutions seems to be well understood and the equilibrium (stationary) solutions of the problem

$$\begin{gathered} \Delta(\Delta\chi_\infty - W'(\chi_\infty)) = 0, \\ \nabla\chi_\infty \cdot \mathbf{n} = \nabla(\Delta\chi_\infty) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \\ \vartheta_\infty \equiv 0 \end{gathered} \tag{1.7}$$

have been identified as the only candidates to belong to the ω-limit set of each individual trajectory (cf. [**9**, Theorem 2.2]). For the sake of simplicity, using the fact that both $\int_\Omega \chi(t)\,dx$ and $\int_\Omega \vartheta(t)\,dx$ are conserved quantities, we normalize the initial functions $\chi(0), \vartheta(0)$ so that

$$\int_\Omega \chi(0,x)\,dx = \int_\Omega \vartheta(0,x)\,dx = 0.$$

If the problem (1.7) admits only a finite number of solutions, then any solution $\chi(t), \vartheta(t)$ converges as $t \to \infty$ to a single stationary state. See, e.g., [**1**] for such a result for a nonconserved system in the one-dimensional case. However, the structure of the set of stationary solutions for a general domain may be quite complicated; in particular, the set in question may contain a continuum of nonradial solutions if Ω is a ball or an annulus. If this is the case, it seems highly nontrivial to decide whether or not the solutions converge to a single stationary state. It is well known that this might not happen even for finite-dimensional dynamical systems (cf. Aulbach [**4**]). Similar examples for semilinear parabolic equations were derived by Poláčik and Rybakowski [**21**]. Positive results in this direction were obtained by Aizicovici and Feireisl [**2**] for the nonconserved system (which basically differs from (1.1) and (1.2) because of the second order dynamics for χ) and Feireisl et al. [**11**] for the system (1.1), (1.2) without the memory term.

In 1983, Simon [**24**] developed a method to study the long-time behavior of gradient-like dynamical systems based on deep results from the theory of analytic functions of several variables due to Lojasiewicz [**19**]. More specifically, the following assertion holds (see [**19**, Theorem 4, p. 88]):

Proposition 1.1. *Let $G : U(a) \to C$ be a real analytic function defined on an open neighborhood $U(a)$ of a point $a \in \mathbf{R}^n$. Then there exist $\theta \in (0, 1/2)$ and $\delta > 0$ such that*

$$|\nabla G(z)| \geq |G(z) - G(a)|^{1-\theta} \quad \text{for all } z \in \mathbf{R}^n, \quad |z - a| < \delta.$$

Simon succeeded in proving a generalized version of the above theorem applicable to analytic functionals on Banach spaces. Later on, Jendoubi [**18**] and Haraux and Jendoubi [**16**] simplified considerably Simon's original approach making it accessible for application to a broad class of semilinear problems with variational structure. Related results in this direction were also obtained by Feireisl and Takáč [**13**], Hoffmann and Rybka [**17**], etc. Last, but not least, the same method has been successfully modified to deal with degenerate parabolic equations of porous media type (see [**12**]).

In some cases, Simon's approach can be used to deal with problems with only a partial variational structure. A typical example could be the system (1.1)–(1.3) with the memory term omitted in (1.3) (i.e., for

$k = 0$). Indeed, the "elliptic" part of (1.1) is the variational derivative of the free energy functional with respect to χ while (1.2) is not. On the other hand, since the temperature always tends to zero when time is large, it is possible to modify Simon's method to prove convergence of the phase variable χ to a single stationary state under fairly general conditions imposed on W (see [**11**, Theorem 1.1]). It is the aim of the present paper to show that similar results can be obtained when the memory effects are taken into account in (1.3). Specifically, our main result is the following:

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Suppose that the "free energy" function W is real analytic on $\mathbf{R}$. In addition, assume that the instantaneous heat conductivity k_I is strictly positive and the kernel k satisfies*

$$
\begin{gathered}
k \in L^1(0,\infty), \quad k \text{ is convex on } (0,\infty), \\
dk'(s) + \delta k'(s)\, ds \geq 0 \quad \text{for a certain } \delta > 0.
\end{gathered} \tag{1.8}
$$

Let (χ, ϑ) be a globally defined strong solution of the problem (1.4)–(1.6) *such that*

$$
\sup_{t>0} \Big(\sup_{x\in\Omega} (|\chi(t,x)| + |\vartheta(t,x)|) \Big) < \infty. \tag{1.9}
$$

Then there exists χ_∞, a solution of the stationary problem (1.7)*, such that*

$$
\chi(t) \to \chi_\infty, \vartheta(t) \to 0 \quad \text{in } C(\overline{\Omega}) \quad \text{as } t \to \infty.
$$

Remark 1.1. By a globally defined strong solution we mean that $\chi, \vartheta, \chi_t, \vartheta_t, D_x^4\chi, D_x^2\vartheta$ are in the space $L^r_{\text{loc}}(0,\infty; L^2(\Omega))$ for any $r \geq 1$, and the boundary conditions (1.6) are satisfied for all $t \in (0,\infty)$. Moreover, $\chi(0)$ is supposed to belong to $W^{2,2}(\Omega)$, the past values of ϑ are given for $t \in (-\infty, 0]$ so that (1.6) is satisfied and $\vartheta \in L^\infty((-\infty,0]; W^{2,2}(\Omega))$.

Remark 1.2. The first condition in (1.8), which is sufficient for an existence result, implies that k is nonnegative and nonincreasing. The second condition, which implies the exponential decay of k and $-k'$, plays an essential role in the proof of convergence, when the Lojasiewicz theorem is applied. It is also used in the proof of compactness of trajectories of solutions. A similar assumption appears in [**2**], [**14**] and [**15**].

A typical example of a kernel k satisfying (1.8) is $k(s) = s^{-\alpha}e^{-\beta s}$, $0 \le \alpha < 1$, $\beta > 0$.

Remark 1.3. The assumption of analyticity of W on $\mathbf{R}$ could be slightly relaxed. It is sufficient to assume that W is analytic on an interval $(\underline{z}, \bar{z})$; see Lemma 4.1.

The assumption that (χ, ϑ) is a strong solution of the problem (1.4)–(1.6) is not restrictive. It will be clear from the estimates presented in Section 3 that any weak solution emanating from smooth initial data will be globally defined and regular on the interval $(0, \infty)$. Moreover, those estimates also allow for a broader class of free energy functionals W than those considered in Grasselli, Pata and Vegni [**15**], Vegni [**25**] and Colli et al. [**9**].

2. Preliminaries.

In this section we review some properties of the kernel k that follow from the hypothesis (1.8), and transform the convolution term in (1.5).

Lemma 2.1. *Let k satisfy* (1.8)*. Then*

$$\lim_{s\to 0+} s\,k(s) = \lim_{s\to 0+} s^2 k'(s) = 0, \tag{2.1}$$

$$\lim_{s\to \infty} s\,k(s) = \lim_{s\to \infty} s^2 k'(s) = 0, \tag{2.2}$$

$$s\,k'(s) \in L^1(0,1), \ \int_0^1 s^2\,dk'(s) < \infty, \tag{2.3}$$

$$s^2 k'(s) \in L^1(1,\infty), \ \int_1^\infty s^2\,dk'(s) < \infty, \tag{2.4}$$

and

$$sk(s) \in L^1(0,\infty), \ \int_s^\infty k(t)\,dt \in L^2(0,\infty). \tag{2.5}$$

Proof. For the proof of (2.1)–(2.4), see [**2**]. The rest follows from (2.4), integration by parts and the relation

$$s^2k(s) \le s^2 \int_s^\infty (-k')(t)\,dt \le -\int_s^\infty t^2k'(t)\,dt.$$

We can also use the fact that (1.8) implies $-k'(s) \le Ce^{-\delta s}$ and

$$k(s) = -\int_s^\infty k'(z)\,dz \le \frac{C}{\delta}\,e^{-\delta s} \quad \text{for } s > 1. \qquad \square$$

We will assume that the past history of the temperature and $\chi(0)$ are given such that

$$\vartheta \in L^\infty((-\infty, 0], W^{2,2}(\Omega)) \quad \text{and} \quad \chi(0) \in W^{2,2}(\Omega). \tag{2.6}$$

Following [**10**], [**14**], we introduce the quantity

$$\eta(t,s,x) = \int_{t-s}^t \vartheta(z,x)\,dz, \quad s \ge 0.$$

Accordingly, making use of Lemma 2.1, we can write

$$k * \vartheta(t) = \int_0^\infty k(s)\frac{\partial}{\partial s}\,\eta(t,s)\,ds = -\int_0^\infty k'(s)\eta(t,s)\,ds.$$

We decompose the kernel k as follows:

$$k = k_1 + k_2, \quad k_1(s) = \begin{cases} k(s) & \text{for } 0 < s \le 1 \\ 0 & \text{for } s > 1 \end{cases}, \quad k_2 = k - k_1. \tag{2.7}$$

Assuming that $\Delta\vartheta \in L^2(\sigma, \sigma+1)$ for all $\sigma \in \mathbf{R}$, we arrive at the following estimate:

$$\begin{aligned} &\Big\| \int_0^\infty k_2(s)\Delta\vartheta(t-s)\,ds \Big\|_{L^2(\Omega)} \\ &= \Big\| -k(1)\int_{t-1}^t \Delta\vartheta(z)\,dz + \int_1^\infty (-k')(s)\int_{t-s}^t \Delta\vartheta(z)\,dz\,ds \Big\|_{L^2(\Omega)} \\ &\le k(1)\sup_{\sigma\in R}\left[\int_\sigma^{\sigma+1} \|\Delta\vartheta(z)\,dz\|^2_{L^2(\Omega)}\right]^{1/2} \\ &\quad + \int_1^\infty (-k')(s)(s+1)\sup_{\sigma\in R}\left[\int_\sigma^{\sigma+1} \|\Delta\vartheta(z)\,dz\|^2_{L^2(\Omega)}\right]^{1/2} \\ &\le C\sup_{\sigma\in R}\left[\int_\sigma^{\sigma+1} \|\Delta\vartheta(z)\,dz\|^2_{L^2(\Omega)}\right]^{1/2}. \end{aligned} \tag{2.8}$$

Moreover,

$$\int_\Omega (k*\vartheta)\vartheta\,dx = \frac{1}{2}\bigg[\frac{d}{dt}\int_0^\infty (-k')(s)\|\eta(t,s)\|^2_{L^2(\Omega)}\,ds + \int_0^\infty (-k)'(s)\frac{\partial}{\partial s}\|\eta(t,s)\|^2_{L^2(\Omega)}\,ds\bigg],$$

whence, by virtue of Lemma 2.1,

$$\int_\Omega (k*\nabla\vartheta)\nabla\vartheta\,dx = \frac{1}{2}\bigg[\frac{d}{dt}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds + \int_0^\infty \|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,dk'(s)\bigg], \tag{2.9}$$

$$\int_\Omega (k*\Delta\vartheta)\Delta\vartheta\,dx = \frac{1}{2}\bigg[\frac{d}{dt}\int_0^\infty (-k')(s)\|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\,ds + \int_0^\infty \|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\,dk'(s)\bigg]. \tag{2.10}$$

3. A priori estimates. Asymptotic compactness. This part is meant to convince the reader that, under natural restrictions on W, condition (1.9) is satisfied and trajectories of strong solutions are precompact in $C(\overline{\Omega})$.

We present some a priori estimates of solutions of the problem (1.4)–(1.6) based on more or less standard arguments. Comparable results can be found in the existing literature (cf., e.g., [**10**], [**14**], etc.) As a consequence, we also obtain useful information on the structure of the ω-limit sets related to globally defined solutions.

Throughout this section, the free energy functional $W : \mathbf{R} \to \mathbf{R}$ will be supposed to satisfy the following hypotheses:

$$W(z) \ge 0 \quad \text{for all } z \ge 0 \tag{3.1}$$

$$W'(z)z > 0 \quad \text{for } |z| > R, \tag{3.2}$$

$$W'(z)z \ge c_1 W(z) - c_2 \quad \text{for all } z \in \mathbf{R}, \tag{3.3}$$

$$W''(z) \ge -c_3, \tag{3.4}$$

$$W \in C^{3+\mu}(\mathbf{R}), \quad |W''(z)| \le c_4(1+|z|^{p-1}), \quad 1 \le p < 5, \tag{3.5}$$

where c_j, $j = 1, \dots, 4$ denote positive constants. Remark that the classical double-well potential $W(z) = (z^2 - 1)^2/8$ (cf. [**7**]) satisfies (3.1)–(3.5).

We start with the homogeneous Neumann problem associated with the Laplace equation. Let $g \in L^2(\Omega)$ be such that $\int_\Omega g\,dx = 0$. The unique solution v of the problem

$$\begin{cases} -\Delta v = g & \text{in } \Omega, \\ \nabla v \cdot \vec{n} = 0 & \text{on } \partial\Omega, \quad \int_\Omega v\,dx = 0, \end{cases} \tag{3.6}$$

will be denoted by $v = -\Delta_N^{-1}[g]$.

Multiplying the equation (1.4) by $-\Delta_N^{-1}[\chi_t]$ and integrating the resulting expression by parts, we get

$$\frac{d}{dt}\biggl(\int_\Omega \frac{1}{2}|\nabla\chi|^2 + W(\chi)\biggr) + \int_\Omega |(-\Delta_N)^{-1/2}[\chi_t]|^2\,dx - \lambda\int_\Omega \vartheta\chi_t\,dx = 0. \tag{3.7}$$

Multiplying the equation (1.5) by ϑ and integrating by parts, we obtain

$$\frac{d}{dt}\frac{1}{2}\|\vartheta\|^2_{L^2(\Omega)} + \lambda\int_\Omega \vartheta\chi_t\,dx + \|\nabla\vartheta\|^2_{L^2(\Omega)} + \int_\Omega k * \nabla\vartheta\cdot\nabla\vartheta\,dx = 0. \tag{3.8}$$

Consequently, using (2.9), the relation (3.8) takes the form

$$\frac{d}{dt}\frac{1}{2}\biggl[\|\vartheta\|^2_{L^2(\Omega)} + \int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds\biggr] + \lambda\int_\Omega \vartheta\chi_t\,dx + \|\nabla\vartheta\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\infty \|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,dk'(s) = 0. \tag{3.9}$$

If we add (3.7) and (3.9), we obtain the energy equality

$$\begin{aligned} &\frac{d}{dt}\Bigl[\int_\Omega \Bigl(\frac{1}{2}\,|\nabla\chi(t)|^2 + \frac{1}{2}|\vartheta(t)|^2 + W(\chi(t))\Bigr)\,dx \\ &\qquad + \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds\Bigr] \\ &\qquad + \|(-\Delta)_N^{-1/2}[\chi_t(t)]\|^2_{L^2(\Omega)} + \|\nabla\vartheta(t)\|^2_{L^2(\Omega)} \\ &\qquad + \frac{1}{2}\int_0^\infty \|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,dk'(s) = 0. \end{aligned} \tag{3.10}$$

We thereby arrive at

Lemma 3.1. *Let W satisfy* (3.1)–(3.5). *If, in addition, all the other hypotheses of Theorem* 1.1 *hold, then there exists E_0 depending only on the quantities*

$$\sup_{t\in(-\infty,0]}\|\nabla\vartheta(t)\|_{L^2(\Omega)}, \quad \|\nabla\chi(0)\|_{L^2(\Omega)}, \quad \|\chi(0)\|_{L^\infty(\Omega)}$$

such that

$$\sup_{t>0}\|\vartheta(t)\|_{L^2(\Omega)} + \sup_{t>0}\|\nabla\chi(t)\|_{L^2(\Omega)} \le E_0, \tag{3.11}$$

$$\int_0^\infty \|(-\Delta_N)^{-1/2}[\chi_t(t)]\|^2_{L^2(\Omega)} + \|\vartheta(t)\|^2_{W^{1,2}(\Omega)}\,dt \le E_0, \tag{3.12}$$

$$\int_0^\infty\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds\,dt \le E_0. \tag{3.13}$$

Next we multiply (1.4) by χ and (1.5) by $(-\Delta_N)^{-1}[\vartheta_t]$ to obtain

$$\frac{d}{dt}\frac{1}{2}\|\chi\|^2_{L^2(\Omega)} + \|\Delta\chi\|^2_{L^2(\Omega)} + \int_\Omega W''(\chi)|\nabla\chi|^2\,dx = -\lambda\int_\Omega \vartheta\Delta\chi\,dx$$

$$\|(-\Delta_N)^{-1/2}[\vartheta_t]\|^2_{L^2(\Omega)} + \lambda\int_\Omega (-\Delta_N)^{-1/2}[\vartheta_t]\cdot(-\Delta_N)^{-1/2}[\chi_t]\,dx$$
$$= -\frac{1}{2}\frac{d}{dt}\|\vartheta\|^2_{L^2(\Omega)} + \int_\Omega k*\Delta^{1/2}\vartheta\cdot(-\Delta_N)^{-1/2}[\vartheta_t]\,dx$$

We rewrite $k*\Delta^{1/2}\vartheta$ in the form $\int_0^t k(s)\Delta^{1/2}\vartheta(t-s)\,ds + \int_t^\infty k(s)\Delta^{1/2}\vartheta(t-s)\,ds$ to deduce $k*\|\Delta^{1/2}\vartheta\|_{L^2(\Omega)} \in L^2(0,\infty)$ using (2.5), (2.6) and (3.12). Consequently, (3.11) and the Young inequality imply

$$\int_0^\infty \|(-\Delta_N)^{-1/2}[\vartheta_t]\|^2_{L^2(\Omega)}\,dt \le E_0, \tag{3.14}$$

$$\int_t^{t+1} \|\Delta\chi\|^2_{L^2(\Omega)}\,d\tau \le E_0 \quad \text{for any } t \ge 0, \tag{3.15}$$

provided that $\Delta\chi(0) \in L^2(\Omega)$ and E_0 is large enough.

To improve the estimates on χ, we write (1.4) as an evolutionary equation

$$\frac{\partial \chi}{\partial t} + \Delta^2 \chi = \Delta[W'(\chi)] - \lambda \Delta \vartheta. \tag{3.16}$$

Let p be as in (3.5). We prove first that

$$\chi \in L^r(t, t+1; W^{2,q_1}(\Omega)), \quad t \geq 0$$
$$\text{for any } 1 \leq r < \infty, \quad q_1 = \min\left\{2, \frac{6}{p}\right\}.$$

For this, for all $1 < q < \infty$, we define a linear operator $\Delta_{N,q}$ on the Banach space $L^q(\Omega)$ by

$$\mathcal{D}(\Delta_{N,q}) = \{v \in W^{2,q}(\Omega) \mid \nabla v \cdot \vec{n} = 0 \text{ on } \partial\Omega\}, \quad \Delta_N v = \Delta v,$$

and rewrite (3.16) in the abstract form

$$\chi_t + \Delta_{N,q}^2 \chi = h; \quad h = h_1 + h_2,$$

where

$$h_1 = \Delta[W'(\chi)]; \quad h_2 = -\lambda \Delta \vartheta.$$

From (3.11) we know that h_2 is bounded in $L^\infty(t, t+1; \mathcal{D}(-\Delta_{N,2}^{-1}))$ uniformly for all $t \geq 0$. On the other hand, using (3.11) and the Sobolev imbedding $W^{1,2}(\Omega) \subset L^6(\Omega)$, we have $\chi \in L^\infty(0, \tau; L^6(\Omega))$ for all $\tau > 0$. From (3.5) we get $W'(\chi) \in L^\infty(0, \tau; L^{6/p}(\Omega))$.

Recall that $h = \Delta[W'(\chi) - \lambda\vartheta]$; also, $\Delta_{N,q}^{-1}(\Delta_{N,q})f = f - (1/|\Omega|)\times \int_\Omega f(x)\,dx$ for any $f \in L^q(\Omega)$. Hence for $q_1 = \min\{2, 6/p\}$

$$\begin{aligned}
\|h\|_{\mathcal{D}(\Delta_{N,q_1}^{-1})} &= \|\Delta_{N,q_1}^{-1}[h]\|_{L^{q_1}(\Omega)} \\
&= \|[W'(\chi) - \lambda\vartheta] - \frac{1}{|\Omega|}\int_\Omega [W'(\chi) - \lambda\vartheta]\,dx\|_{L^{q_1}(\Omega)} \\
&\leq C(\|W'(\chi)\|_{L^{q_1}(\Omega)} + \|\vartheta\|_{L^{q_1}(\Omega)}).
\end{aligned}$$

This implies that $\chi \in L^r(t, t+1; W^{2,q_1}(\Omega))$, $r \geq 1$. Consequently, by the Sobolev embedding theorem,

$$\chi \in L^r(t, t+1; L^{q_2}(\Omega)) \quad \text{with } q_2 = \frac{3q_1}{3 - 2q_1}$$
$$\text{if } 2q_1 < 3, \quad q_2 = \infty \quad \text{otherwise.}$$

Next we argue by induction (bootstrap argument). We deduce from (3.5) that

$$W'(\chi) \in L^{r/p}(t, t+1; L^{q_2/p}(\Omega)).$$

Remark that we have

$$\frac{q_2}{p} - q_1 = \frac{6}{p(p-4)} - \frac{6}{p} > 0$$

if $p \in (4, 5)$, $q_2 = \infty$ if $p \leq 4$. Hence, after a finite number of steps, we arrive at the estimate

$$\begin{gathered} \chi \in L^r(t, t+1; W^{2,2}(\Omega)) \subset L^r(t, t+1; L^\infty(\Omega)), \\ t \geq 0 \quad \text{for any } 1 \leq r < \infty. \end{gathered} \tag{3.17}$$

Also, by (3.16), $\chi_t \in L^r(t, t+1; [W^{2,2}]^*(\Omega))$ which implies

$$\begin{gathered} \chi \in C([t, t+1]; (W^{2,2}(\Omega), [W^{2,2}]^*(\Omega))_\theta), \\ \text{with } \theta \text{ satisfying} \quad \theta\Big(1 - \frac{1}{r}\Big) > \frac{1-\theta}{r}, \end{gathered}$$

(that is, $\theta > 1/r$), where $(.,.)_\theta$ denotes the interpolation space (see, e.g., [**23**, Corollary 8, p. 90]). As $r > 1$ is arbitrary, we can choose θ small enough such that $(W^{2,2}(\Omega), [W^{2,2}]^*(\Omega))_\theta \hookrightarrow C(\overline{\Omega})$. Therefore,

$$\sup_{t>0} \|\chi(t)\|_{C(\overline{\Omega})} \leq C_\infty. \tag{3.18}$$

This implies that $W''(\chi)$, $W'''(\chi)$ are bounded, and $\nabla\chi$ is bounded in $L^r(t, t+1; L^6(\Omega))$ for all r, independently of $t > 0$. Then (cf. also (3.15))

$$\int_t^{t+1} \|\Delta W'(\chi(s))\|^2_{L^2(\Omega)}\, ds < C \quad \text{for all } t > 0. \tag{3.19}$$

Moreover, by (3.11) and (3.12),

$$\vartheta \in L^\infty(t, t+1; L^2(\Omega)) \cap L^2(0, \infty; W^{1,2}(\Omega)).$$

It follows, by the same reasoning as above, that

$$\chi \text{ is bounded in } L^2(t, t+1; W^{3,2}(\Omega)), \quad \text{uniformly for } t \geq 0. \tag{3.20}$$

Now we multiply (1.5) by $-\Delta(\vartheta+\lambda\chi)$, integrate by parts and use (2.10) to obtain

$$\frac{1}{2}\frac{d}{dt}\Big[\|\nabla(\vartheta+\lambda\chi)\|^2_{L^2(\Omega)}+\int_0^\infty(-k')(s)\|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\,ds\Big]$$
$$+\|\Delta\vartheta\|^2_{L^2(\Omega)}+\lambda\int_\Omega\Delta\vartheta\Delta\chi\,dx+\frac{1}{2}\int_0^\infty\|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\,dk'(s)$$
$$=\lambda\int_\Omega k*\nabla\vartheta\cdot\nabla\Delta\chi\,dx.$$

If we set

$$F(t)=\|\nabla(\vartheta+\lambda\chi)(t)\|^2_{L^2(\Omega)}+\int_0^\infty(-k')(s)\|\Delta\eta(t,s)\|^2_{L^2(\Omega)}\,ds,$$

employ (1.8) and the Poincaré and Young inequalities, we get

$$\frac{d}{dt}F(t)+aF(t)\le C(1+\|\chi(t)\|^2_{W^{3,2}(\Omega)}+k*\|\nabla\vartheta\|^2_{L^2(\Omega)}(t)),$$

for some small $a>0$. This yields the estimate

$$\begin{aligned}F(t)&\le C\Big(1+\sup_{t>0}\int_t^{t+1}\|\chi(s)\|^2_{W^{3,2}(\Omega)}+k*\|\nabla\vartheta\|^2_{L^2(\Omega)}(s)\,ds\Big)\\&\le C_1,\end{aligned}\tag{3.21}$$

by (3.20) and (3.12). We arrive at the following result:

Lemma 3.2. *Under the hypotheses of Lemma* 3.1*, there exists* E_0 *depending only on the quantities*

$$\sup_{t\in(-\infty,0]}\|\Delta\vartheta(t)\|_{L^2(\Omega)},\quad \|\Delta\chi(0)\|_{L^2(\Omega)},$$

such that

$$\sup_{t\ge0}\|\nabla\vartheta(t)\|_{L^2(\Omega)}\le E_0,\tag{3.22}$$

$$\int_t^{t+1}\|\Delta\vartheta(s)\|^2_{L^2(\Omega)}\,ds\le E_0\quad\textit{for all } t\ge0.\tag{3.23}$$

By virtue of (3.19), (3.23), the phase field variable χ satisfies the equation (3.16) with the right-hand side bounded in $L^2(t,t+1;L^2(\Omega))$ independently of t. Therefore, we obtain

$$\chi \in L^2(t,t+1;W^{4,2}(\Omega)), \quad \chi_t \in L^2(t,t+1;L^2(\Omega)), \quad t>1. \tag{3.24}$$

To obtain a better regularity of ϑ, we rewrite (1.5) as an integrodifferential equation for $e=\vartheta+\lambda\chi$ and use a maximal regularity result [**22**, Theorem 8.7] for the equation

$$\begin{aligned} e_t(t)-\Delta e(t)-\int_0^t k_1(s)\Delta e(t-s)\,ds \\ =\int_t^1 k_1(s)\Delta\vartheta(t-s)\,ds+\int_1^\infty k_2(s)\Delta\vartheta(t-s)\,ds \\ -\lambda\Delta\chi(t)-\lambda\int_0^t k_1(s)\Delta\chi(t-s)\,ds, \end{aligned}$$

where $0\le t\le 1$. The right-hand side of the above equation belongs to the space $L^r(0,1;L^2(\Omega))$ for any $r\ge 1$ by (2.6), (2.8), (3.24) and (3.17). This gives $\Delta e\in L^r(0,1;L^2(\Omega))$ and, consequently, $\Delta\vartheta$ in the same space. Then we argue by induction. If we denote $e^n(t)=e(t+n)$, the right-hand sides of the corresponding equations for e^n belong to $L^r(0,1;L^2(\Omega))$ provided that $\Delta\vartheta^{n-1}\in L^r(0,1;L^2(\Omega))$. Overlapping the intervals $(\sigma,\sigma+1)$ if necessary, we find a uniform $W^{2,2}$ bound for the initial conditions as well as an $L^r(0,1;L^2(\Omega))$ bound for the right-hand sides of all these equations.

This gives $\Delta\vartheta\in L^r(t,t+1;L^2(\Omega))$, and consequently

$$\chi_t\in L^r(t,t+1;L^2(\Omega)), \quad r\ge 1, \quad t>0,$$

and then also

$$\|\vartheta_t\|_{L^r(t,t+1;L^2(\Omega))}\le C \quad \text{for any } r\ge 1, \quad t>0. \tag{3.25}$$

In particular (cf. also [**23**]), we have obtained the following result:

Proposition 3.1. *Let $\Omega\subset\mathbf{R}^3$ be a bounded domain with a sufficiently smooth boundary. Let $W\in C^{3+\mu}(\mathbf{R})$ satisfy the hypotheses* (3.1)–(3.5).

Then for any strong solution (χ, ϑ) *of the problem* (1.4)–(1.6) *on the time interval* $(0,\infty)$*, the trajectories*

$$\cup_{t\geq 1}\chi(t), \quad \cup_{t\geq 1}\vartheta(t)$$

are precompact in the space $C(\overline{\Omega})\cap W^{1,2}(\Omega)$*.*

Remark 3.1. By applying the parabolic regularity theory to equation (3.16), we infer that $\chi \in L^r(t,t+1;W^{4,2}(\Omega))$, $r\geq 1$, which results in compactness of $\Delta\chi$ in $C([t,t+1];C(\overline{\Omega}))$, $t>0$.

4. Long-time behavior. The ω-limit sets. The aim of the present section is to prove the following auxiliary result.

Proposition 4.1. *Let* $\Omega \subset \mathbf{R}^3$ *be a bounded domain with a sufficiently smooth boundary. Let* $W \in C^{3+\mu}(\mathbf{R})$ *be given satisfying the hypotheses* (3.1)–(3.5)*. Let* χ, ϑ *be a globally defined strong solution of the problem* (1.4)–(1.6) *such that*

$$\int_\Omega \chi(\tau)\,dx = \int_\Omega \vartheta(\tau)\,dx = 0 \quad \textit{for a certain } \tau > 0.$$

Then

$$\vartheta(t)\to 0 \quad \textit{in } C(\overline{\Omega})\cap W^{1,2}(\Omega) \quad \textit{as } t\to\infty, \tag{4.1}$$

and any sequence $t_n\to\infty$ *contains a subsequence* (*not relabeled*)*, such that*

$$\chi(t_n)\to\chi_\infty \quad \textit{in } C(\overline{\Omega})\cap W^{1,2}(\Omega) \tag{4.2}$$

where $\chi_\infty = \chi_\infty(x)$ *satisfies*

$$\int_\Omega \nabla\chi_\infty\cdot\nabla\phi + W'(\chi_\infty)\phi\,dx = 0, \tag{4.3}$$

$$\nabla\chi_\infty\cdot\vec{n} = 0 \quad \textit{on } \partial\Omega, \qquad \int_\Omega \chi_\infty\,dx = 0, \tag{4.4}$$

for any test function $\phi\in W^{1,2}(\Omega)$*,* $\int_\Omega \phi\,dx = 0$*.*

Proof. (i) Since ϑ satisfies (3.12), and (3.14), (3.22) hold, (4.1) follows from Proposition 3.1.

(ii) The equation (1.4) can be written in the form

$$\int_\Omega \Delta_N^{-1}[\chi_t]\phi\,dx = \int_\Omega (\nabla\chi\cdot\nabla\phi + W'(\chi)\phi - \lambda\vartheta\phi)\,dx \tag{4.5}$$

for any test function ϕ as in (4.3).

Now let $t_n \to \infty$. In accordance with Proposition 3.1, there exists a subsequence (not relabeled) such that

$$\chi(t_n) \to \chi_\infty \quad \text{in } C(\overline{\Omega}) \cap W^{1,2}(\Omega).$$

Moreover, setting

$$\chi_n(t) = \chi(t_n + t)$$

one has

$$\chi_n \to \tilde{\chi} \quad \text{in } C([0,1]; C(\overline{\Omega}) \cap W^{1,2}(\Omega)) \quad \text{where } \tilde{\chi}(0) = \chi_\infty.$$

Consequently, passing to the limit for $n \to \infty$ in (4.5), we get

$$\int_{t-h}^{t+h}\int_\Omega (\nabla\tilde{\chi}\cdot\nabla\phi + W'(\tilde{\chi})\phi)\,dx\,dt = 0$$

for any $t \in [0,1]$ and any $h > 0$, which yields the relation (4.3). ◻

Let us define the ω-limit set $\omega[\chi]$ as

$$\omega[\chi] = \{\chi_\infty \mid \chi(t_n) \to \chi_\infty \text{ in } C(\overline{\Omega}) \cap W_n^{1,2}(\Omega) \text{ for a certain } t_n \to \infty\}.$$

Remark 4.1. The solutions lying in an ω-limit set $\omega[\chi]$ satisfy certain restrictions. Assume one can find constants $\underline{z}, \bar{z}$ such that

$$\begin{gathered} W'(z_1) < W'(z_2) \quad \text{whenever } z_1 < z_2 \\ \text{and either } z_1 < \underline{z} \quad \text{or} \quad z_2 > \bar{z}. \end{gathered} \tag{4.6}$$

Assume, for instance, that there is a stationary solution χ of the problem

$$\Delta(\Delta\chi - W'(\chi)) = 0 \quad \text{in } \Omega, \quad \nabla\chi \cdot \vec{n} = \nabla(\Delta\chi) \cdot \vec{n} = 0 \quad \text{on } \partial\Omega$$

such that

$$\sup_{x\in\Omega} \chi(x) \geq \bar{z}.$$

Then, necessarily, χ is constant on Ω.

Indeed, the function χ solves the problem

$$-\Delta\chi + W'(\chi) = \frac{1}{|\Omega|} \int_\Omega W'(\chi)\, dx \quad \text{in } \Omega, \quad \nabla\chi \cdot \vec{n}|_{\partial\Omega} = 0.$$

Consider $w(x) = \max_{y\in\overline{\Omega}} \chi(y) - \chi(x)$, which in accordance with the hypothesis (4.6), satisfies

$$-\Delta w + z(x)w \geq 0 \quad \text{on } \Omega$$

for a certain bounded function z. By virtue of the strong maximum principle, either w is strictly positive on $\overline{\Omega}$, which is impossible, or $w \equiv 0$.

A similar result holds when $\inf_{x\in\Omega} \chi(x) \leq \underline{z}$.

Hence, since the integral mean

$$\frac{1}{|\Omega|} \int_\Omega \chi\, dx$$

is time invariant, and consequently, constant on each ω-limit set, we have

Lemma 4.1. *Let W satisfy the hypothesis* (4.6). *Let (χ, ϑ) be a uniformly bounded trajectory of* (1.4), (1.5)*, for $t > T$.*

Then either $\omega[\chi]$ is a singleton, or there exist $\underline{r}, \bar{r}$ such that

$$\underline{z} < \underline{r} \leq \inf_{x\in\Omega} \chi_\infty(x) \leq \sup_{x\in\Omega} \chi_\infty(x) \leq \bar{r} < \bar{z} \quad \text{for all } \chi_\infty \in \omega[\chi].$$

5. A generalized version of the Lojasiewicz theorem. In this section we collect some preparatory material for the proof of Theorem 1.1. To begin with, it is important to observe that the conclusion of Theorem 1.1 holds if $\omega[\chi]$ is a singleton. Consequently, from now on, we shall assume that $\omega[\chi]$ contains at least two different functions. Accordingly, since we are interested in the ω-limit set of one particular trajectory which is uniformly bounded with respect to the χ-component, we are allowed to suppose, without loss of generality, that W has been modified outside of the interval $[-L, L]$, where

$$-L < -L/2 \leq \chi(t) \leq L/2 < L \quad \text{for all } t > T,$$

in such a way that

$$W(z) \quad \text{is real analytic on } (-L, L); \tag{5.1}$$

$$|W'(z)|,\ |W''(z)| \quad \text{are uniformly bounded for } z \in \mathbf{R}. \tag{5.2}$$

We introduce an operator

$$\mathcal{A}[v] = -\Delta v + W'(v), \quad \nabla v \cdot \vec{n} = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega v\, dx = 0.$$

The following result is standard:

Lemma 5.1. *Let W satisfy the hypotheses* (5.2). *Then the operator $\mathcal{A}$ is continuously Fréchet differentiable on the spaces*

$$\mathcal{A} : W^{2,p}_N(\Omega) = \Big\{ v \in W^{2,p}(\Omega) \mid \nabla v \cdot \vec{n}|_{\partial\Omega} = 0, \int_\Omega v\, dx = 0 \Big\} \mapsto L^p(\Omega), \quad p \geq 2, \tag{5.3}$$

and

$$\mathcal{A} : W^{1,2}_N(\Omega) = \Big\{ v \in W^{1,2}(\Omega) \Big| \int_\Omega v\, dx = 0 \Big\} \mapsto [W^{1,2}_N(\Omega)]^*, \tag{5.4}$$

respectively. Its Fréchet derivative has the representation

$$D\mathcal{A}[v]\eta = -\Delta\eta + W''(v)\eta, \quad \eta \in W^{2,p}_N(\Omega)$$
$$\textit{with } D\mathcal{A}[v] \in \mathcal{B}(W^{2,p}_N(\Omega), L^p(\Omega))$$

in the first case, and

$$\langle D\mathcal{A}[v], \eta\rangle = \int_\Omega \nabla v\cdot\nabla\eta + W''(v)\eta\,dx, \quad \eta\in W_N^{1,2}(\Omega),$$

with $D\mathcal{A}[v]\in\mathcal{B}(W_N^{1,2}(\Omega),[W_N^{1,2}(\Omega)]^*)$ *in the second case.*

Remark. Throughout the text, we are using the relation

$$W^{1,2}(\Omega)\subset L^2(\Omega)\approx[L^2(\Omega)]^*\subset[W^{1,2}]^*(\Omega).$$

Now we report the following auxiliary result:

Lemma 5.2. *Under the hypotheses* (5.1), (5.2), *let* $v\in W_N^{2,p}(\Omega)$, $p\geq 2$, *be such that*

$$-L/2 < v(x) < L/2 \quad \textit{for all } x\in\Omega.$$

Then there exists a neighborhood $U(v)$ *in* $W_N^{2,p}(\Omega)$ *such that*

$$\mathcal{A}|_{U(v)} \longmapsto L^p(\Omega) \quad \textit{is analytic.}$$

The proof can be done in exactly the same way as that of [**3**, Lemma 4.2] and will be omitted. We consider the standard definition of analyticity (see, e.g., [**26**, Vol. I, Definition 8.8]); for the definition of analyticity and related results see also [**3**].

Now we are in a position to state the main result of this section, which represents a version of Proposition 1.1 for analytic functionals on a Banach space. The main idea is the same as that of Simon [**24**]; specifically, we derive an infinite dimensional analogue of the Lojasiewicz theorem. Let us define a functional

$$I(v)=\int_\Omega (|\nabla v|^2+W(v))\,dx. \tag{5.5}$$

By virtue of Lemma 5.1, we have

$$I\in C^2(W_N^{1,2}(\Omega)) \quad\text{and}\quad I'(v)=\mathcal{A}(v)\in[W_N^{1,2}]^*.$$

Proposition 5.1. *Let W satisfy the hypotheses* (5.1), (5.2). *Let $w \in W^{2,p}_N$,*

$$-L/2 < w(x) < L/2 \quad \text{for all } x \in \Omega.$$

Then for any $P > 0$ there exist constants $\theta \in (0, 1/2)$, $M(P)$, $\varepsilon(P)$ such that

$$|I(v) - I(w)|^{1-\theta} \le M\| - \Delta v + W'(v)\|_{[W^{1/2}_N(\Omega)]^*} \tag{5.6}$$

holds for any $v \in W^{1,2}_N(\Omega)$ such that

$$\|v - w\|_{L^2(\Omega)} < \varepsilon, \quad |I(v) - I(w)| < P. \tag{5.7}$$

The proof is identical with [**12**, Proposition 6.1], and we omit it.

6. Proof of Theorem 1.1. With the results of the preceding section at hand, we can complete the proof of Theorem 1.1. We start with the following observation proved in [**12**, Lemma 7.1]:

Lemma 6.1. *Let $Z \ge 0$ be a measurable function on $(0, \infty)$ such that*

$$Z \in L^2(0, \infty), \quad \|Z\|_{L^2(0,\infty)} \le Y$$

and there exist $\alpha \in (1, 2)$, $\xi > 0$, and an open set $\mathcal{M} \subset (0, \infty)$ such that

$$\left(\int_t^\infty Z^2(s)\, ds \right)^\alpha \le \xi Z^2(t) \quad \text{for a.a. } t \in \mathcal{M}.$$

Then $Z \in L^1(\mathcal{M})$ and there exists a constant $c = c(\xi, \alpha, Y)$ independent of $\mathcal{M}$ such that

$$\int_{\mathcal{M}} Z(s)\, ds \le c.$$

Now we shall make use of the energy equality (3.10). Denoting by E the *total energy*,

$$\begin{aligned} E(t) = \frac{1}{2} \int_\Omega |\nabla \chi(t)|^2 + 2W(\chi(t))\, dx + \frac{1}{2} \|\vartheta\|^2_{L^2(\Omega)} \\ + \frac{1}{2} \int_0^\infty (-k')(s) \|\nabla \eta(t, s)\|^2_{L^2(\Omega)}\, ds, \end{aligned}$$

we have

$$E(t) \longrightarrow E_\infty \quad \text{as } t \to \infty.$$

Moreover, by virtue of (4.1), one has

$$\|\vartheta(t)\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } t \to \infty, \tag{6.1}$$

and

$$\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds \le \int_0^\tau (-k)'(s)s\int_{t-\tau}^t \|\nabla\vartheta(z)\|^2_{L^2(\Omega)}\,ds + \int_\tau^\infty (-k)'(s)s\sup_{z\in R}\|\nabla\vartheta(z)\|^2_{L^2(\Omega)}\,ds.$$

If $\tau > 0$ is large enough, the second term on the right-hand side of the above inequality is small since $s(-k)'(s)$ is integrable. On the other hand, the first term tends to zero for large t for any fixed τ due to (3.12). Consequently, the right-hand side of the above inequality tends to zero when $t \to \infty$:

$$\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds \longrightarrow 0 \quad \text{as } t \to \infty. \tag{6.2}$$

Note that

$$I(\chi(t)) \to I_\infty = \frac{1}{2}\int_\Omega |\nabla\chi_\infty|^2 + 2W(\chi_\infty)\,dx \quad \text{for any } \chi_\infty \in \omega[\chi].$$

In particular, the energy of all solutions $\chi_\infty \in \omega[\chi]$ equals the same constant I_∞ and, by virtue of (6.2),

$$\int_\Omega \frac{1}{2}|\nabla\chi(t)|^2 + W(\chi(t))\,dx = I(\chi(t)) \to E_\infty = I(\chi_\infty) \quad \text{as } t \to \infty \tag{6.3}$$

for arbitrary $\chi_\infty \in \omega[\chi]$, where I is the functional defined in (5.5). Integrating (3.10) with respect to t and making use of (1.8), one obtains

$$\begin{aligned}
&\int_t^\infty \|-\Delta_N^{-1/2}\chi_t\|^2_{L^2(\Omega)} + \|\nabla\vartheta\|^2_{L^2(\Omega)}\,ds \\
&\qquad + \frac{\delta}{2}\int_t^\infty\int_0^\infty (-k)'(s)\|\nabla\eta(\tau,s)\|^2_{L^2(\Omega)}\,ds\,d\tau \\
&\quad \le I(\chi(t)) - I_\infty + \frac{1}{2}\|\vartheta(t)\|^2_{L^2(\Omega)} \\
&\qquad + \frac{1}{2}\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|^2_{L^2(\Omega)}\,ds.
\end{aligned} \tag{6.4}$$

Now, assume $\chi_\infty \in \omega[\chi]$ and $\omega[\chi]$ is not a singleton, since otherwise there is nothing to prove. In accordance with Lemma 4.1, χ_∞ satisfies the hypotheses of Proposition 5.1. We take

$$\mathcal{M} = \{t \in (0,\infty) \mid \|\chi(t) - \chi_\infty\|_{L^2(\Omega)} < \varepsilon\}$$

where $\varepsilon > 0$ is the same as in Proposition 5.1. Since $I_\infty = I(\chi_\infty)$ we can use Proposition 5.1 to obtain

$$I(\chi(t)) - I_\infty \le C\big\| - \Delta_N^{-1}\chi_t(t) - \lambda\vartheta(t)\big\|_{[W^{1,2}]^*(\Omega)}^{1/(1-\theta)}, \quad \theta \in \Big(0, \frac{1}{2}\Big),$$

which combined with (6.4) and the Poincaré inequality yields the following conclusion:

$$\begin{aligned}
&\int_t^\infty \|(-\Delta_N)^{-1/2}[\chi_t]\|_{L^2(\Omega)}^2 + \|\nabla\vartheta\|_{L^2(\Omega)}^2\, ds \\
&\qquad + \frac{\delta}{2}\int_t^\infty \int_0^\infty (-k)'(s)\|\nabla\eta(\tau,s)\|_{L^2(\Omega)}^2\, ds\, d\tau \\
&\quad \le C\big(\| - \Delta_N^{-1}[\chi_t(t)]\|_{L^2(\Omega)}^2 + \lambda\|\vartheta(t)\|_{L^2(\Omega)}^2\big)^{1/(2-2\theta)} + \frac{1}{2}\|\vartheta(t)\|_{L^2(\Omega)}^2 \\
&\qquad + \frac{1}{2}\int_0^\infty (-k')(s)\|\nabla\eta(t,s)\|_{L^2(\Omega)}^2\, ds
\end{aligned}$$

provided $t \in \mathcal{M}$.

Making use of (6.1), (6.2), we can take

$$\begin{aligned}
Z(t) = \Big[&\|(-\Delta_N)^{-1/2}[\chi_t(t)]\|_{L^2(\Omega)}^2 + \|\nabla\vartheta(t)\|_{L^2(\Omega)}^2 \\
&+ \frac{\delta}{2}\int_0^\infty (-k)'(s)\|\nabla\eta(t,s)\|_{L^2(\Omega)}^2\, ds\Big]^{1/2}
\end{aligned}$$

in Lemma 6.1. We remark that $Z \in L^2(0,\infty)$ by Lemma 3.1. We conclude that

$$\int_{\mathcal{M}} \|(-\Delta_N)^{-1/2}[\chi_t(t)]\|_{L^2(\Omega)}\, dt < \infty.$$

In particular, we have

$$\|\chi(t_1) - \chi(t_2)\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \tag{6.5}$$

provided t_1, t_2 are large enough and the whole interval (t_1, t_2) lies in $\mathcal{M}$. Observe that we have used the boundedness of the trajectory $\chi(t)$ in the $W^{1,2}$-norm interpolated with the $[W^{1,2}]^*$-norm. Since $\chi_\infty \in \omega[\chi]$ we can choose $T_0 > 0$ so that

$$\|\chi(T_0) - \chi_\infty\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \tag{6.6}$$

and, consequently, $[T, \infty) \subset \mathcal{M}$. Indeed, take

$$\bar{t} = \inf\{t > T_0 \mid \|\chi(t) - \chi_\infty\|_{L^2(\Omega)} \geq \varepsilon\}.$$

Clearly, $\bar{t} > T_0$ and

$$\|\chi(\bar{t}) - \chi_\infty\|_{L^2(\Omega)} = \varepsilon$$

provided $\bar{t}$ is finite. On the other hand, by virtue of (6.5), (6.6),

$$\begin{aligned}\|\chi(t) - \chi_\infty\|_{L^2(\Omega)} &\leq \|\chi(t) - \chi(T_0)\|_{L^2(\Omega)} \\ &\quad + \|\chi(T_0) - \chi_\infty\|_{L^2(\Omega)} \leq \frac{2}{3}\,\varepsilon \quad \text{for any } T_0 \leq t < \bar{t}\end{aligned}$$

which yields $\bar{t} = \infty$. This implies the convergence of $\chi(t)$ in $L^2(\Omega)$. Hence, $\omega[\chi]$ is a singleton.

This, together with Proposition 3.1, concludes the proof of Theorem 1.1.

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