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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A TWO TERM DIFFERENCE EQUATION

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Dedicated to Paul Waltman on the occasion of his 60th birthday

We will be concerned with the $2n$-th order linear difference equation

$$Ly(t) \equiv \Delta^n[p(t-n)\Delta^n y(t-n)] + q(t)y(t) = 0 \tag{1}$$

where $p(t) > 0$ on the discrete interval $[a,\infty) \equiv \{a, a+1, \dots\}$ and where $q(t)$ is defined on the discrete interval $[a+n,\infty)$. Here Δ denotes the forward difference operator, i.e., $\Delta y(t) = y(t+1) - y(t)$. A function y defined on the discrete interval $[a,\infty)$ is a solution of (1), provided (1) holds for $t \geq a+n$.

There has been much recent interest in difference equations. See the recent books [**1, 4** and **7**–**9**] and the many references therein. Discrete time linear systems arise in discrete linear optimal control and filtering problems [**14**]. Cheng [**3**] studied equation (1) with $p(t) \equiv 1$ and $n = 2$. Smith and Taylor [**12**] studied a variation of equation (1) with $p(t) \equiv 1$, $n = 2$, and two additional lower order terms. We are also motivated by [**6**] and [**13**].

We now introduce *quasi-difference operators* so that the Lagrange identity of (1) has a nice form. For $0 \leq i \leq n-1$, define

$$\Delta_i y(t) = \Delta^i y(t),$$

and for $n \leq i \leq 2n-1$, define

$$\Delta_i y(t) = \Delta^{i-n}[p(t-i+n-1)\Delta^n y(t-i+n-1)].$$

One can then prove the Lagrange identity for (1).

Theorem 1. *For y and z defined on $[a,\infty)$,*

$$z(t)Ly(t) - y(t)Lz(t) = \Delta\{z(t); y(t)\}$$

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for $t \geq a+n$, where the Lagrange bracket of $z(t)$ and $y(t)$ is defined by

$$\{z(t);y(t)\} = \sum_{i=0}^{2n-1}(-1)^i\Delta_i z(t)\Delta_{2n-1-i}y(t)$$

for $t \geq a+n$.

Proof. Consider

$$\begin{aligned}
\Delta\{z(t);y(t)\} &= \Delta\Bigg\{\sum_{i=0}^{n-1}(-1)^i\Delta^i z(t)\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)] \\
&\qquad + \sum_{i=1}^{n}(-1)^{n+i-1}\Delta^{i-1}[p(t-i)\Delta^n z(t-i)]\Delta^{n-i}y(t)\Bigg\} \\
&= \sum_{i=0}^{n-1}(-1)^i\Delta^i z(t)\Delta^{n-i}[p(t-n+i)\Delta^n y(t-n+i)] \\
&\quad + \sum_{i=0}^{n-1}(-1)^i\Delta^{i+1}z(t)\Delta^{n-1-i}[p(t-n+i+1) \\
&\qquad\qquad \cdot \Delta^n y(t-n+i+1)] \\
&\quad + \sum_{i=1}^{n}(-1)^{n+i-1}\Delta^{i-1}[p(t-i+1) \\
&\qquad\qquad \cdot \Delta^n z(t-i+1)]\Delta^{n-i+1}y(t) \\
&\quad + \sum_{i=1}^{n}(-1)^{n+i-1}\Delta^i[p(t-i)\Delta^n z(t-i)]\Delta^{n-i}y(t) \\
&= z(t)\Delta^n[p(t-n)\Delta^n y(t-n)] \\
&\quad + \sum_{i=1}^{n-1}(-1)^i\Delta^i z(t)\Delta^{n-i}[p(t-n+i)\Delta^n y(t-n+i)] \\
&\quad - \sum_{i=1}^{n}(-1)^i\Delta^i z(t)\Delta^{n-i}[p(t-n+i)\Delta^n y(t-n+i)] \\
&\quad - \sum_{i=0}^{n-1}(-1)^{n+i-1}\Delta^i[p(t-i)\Delta^n z(t-i)]\Delta^{n-i}y(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n-1}(-1)^{n+i-1}\Delta^i[p(t-i)\Delta^n z(t-i)]\Delta^{n-i}y(t)\\
&-\Delta^n[p(t-n)\Delta^n z(t-n)]y(t)\\
=\;& z(t)\Delta^n[p(t-n)\Delta^n y(t-n)]-(-1)^n\Delta^n z(t)p(t)\Delta^n y(t)\\
&-(-1)^{n-1}p(t)\Delta^n z(t)\Delta^n y(t)\\
&-\Delta^n[p(t-n)\Delta^n z(t-n)]y(t)\\
=\;& z(t)\{\Delta^n[p(t-n)\Delta^n y(t-n)]+q(t)y(t)\}\\
&-y(t)\{\Delta^n[p(t-n)\Delta^n z(t-n)]+q(t)z(t)\}\\
=\;& z(t)Ly(t)-y(t)Lz(t). \qquad \square
\end{aligned}
$$

It is easy to see that there is a unique solution of equation (1) satisfying the conditions

$$\Delta_i y(t_0)=\alpha_i, \qquad 0\le i\le 2n-1$$

where the α_i are given constants. For each fixed $s\in[a,\infty)$, let $y_j(t,s)$, $0\le j\le 2n-1$ be the solution of (1) satisfying the conditions

$$\Delta_i y_j(s,s)=\delta_{ij}$$

$0\le i,j\le 2n-1$, where quasi-differences are with respect to the first variable and where δ_{ij} is the Kronecker delta. Note that if $1\le j\le n$, then

$$y_j(s+i,s)=0, \qquad 0\le i\le j-1$$

and if $n+1\le j\le 2n-1$, then

$$y_j(s+i,s)=0, \qquad n-j\le i\le n-1.$$

In particular, $y_j(t,s)$ has j consecutive zeros starting at s if $1\le j\le n$ and j consecutive zeros starting at $s+n-j$ if $n+1\le j\le 2n-1$. We now obtain formulas relating the quasi-differences for $y_j(t,s)$ and $y_{2n-1-j}(s,t)$ (for the analogous differential equations case see [**11**]).

Theorem 2. *For $0\le i,\ j\le 2n-1$,*

$$\Delta_i y_j(t,s)=(-1)^{i+j}\Delta_{2n-1-j}y_{2n-1-i}(s,t) \tag{2}$$

where the quasi-differences on both sides of the equation are with respect to the first variable.

Proof. Fix integers t_1 and t_2 in $[a+n,\infty)$. By the Lagrange identity

$$\{y_j(t,t_1); y_{2n-1-i}(t,t_2)\} = \text{constant} \tag{3}$$

for $t \geq a+n$. Hence, the left side of (3) is the same when evaluated at t_1 and t_2 which gives us

$$(-1)^j \Delta_{2n-1-j} y_{2n-1-i}(t_1,t_2) = (-1)^i \Delta_i y_j(t_2,t_1).$$

This gives us the desired result with $s = t_1$ and $t = t_2$. □

Define the generalized Wronskian (Casoratian) of $y_{2n-1}(t,s),\dots,$ $y_{2n-j}(t)$ by

$$W[y_{2n-1}(t,s),\dots,y_{2n-j}(t,s)] = \begin{vmatrix} y_{2n-1}(t,s) & \cdots & y_{2n-j}(t,s) \\ \Delta_1 y_{2n-1}(t,s) & \cdots & \Delta_1 y_{2n-j}(t,s) \\ \vdots & \ddots & \cdots \\ \Delta_{j-1} y_{2n-1}(t,s) & \cdots & \Delta_{j-1} y_{2n-j}(t,s) \end{vmatrix}$$

for $1 \leq j \leq 2n$.

Using (2) we obtain the following result (for a similar result for differential equations, see [**5**]).

Corollary 1. *For* $1 \leq j \leq 2n$,

$$W[y_{2n-1}(t,s),\dots,y_{2n-j}(t,s)] = (-1)^j W[y_{2n-1}(s,t),\dots,y_{2n-j}(s,t)].$$

The following result follows immediately from this result.

Corollary 2. *For* $1 \leq k \leq n$ *there is a nontrivial solution* u *of* (1) *satisfying*

$$\begin{aligned} u(s+j) &= 0, \qquad k-n \leq j \leq n-1 \\ u(t+i) &= 0, \qquad 0 \leq i \leq k-1 \end{aligned}$$

where $s+n-1<t$ *if and only if there is a nontrivial solution* v *of* (1) *satisfying*

$$v(s+j)=0, \qquad 0\le j\le k-1$$
$$v(t+i)=0, \qquad k-n\le i\le n-1.$$

For any function y defined on $[a,\infty)$, we define for $t\ge a+n$ the operators E and F by

$$\begin{aligned}Ey(t)=\sum_{\tau=a+n-1}^{t-1}\big\{[\Delta^{n-1}y(\tau-1)+(n-1)\Delta^{n-1}y(\tau)]p(\tau-1)\\ \cdot\Delta^n y(\tau-1)\big\}\\ -(-1)^n\sum_{i=0}^{n-2}(-1)^i(i+1)\Delta^i y(t)\Delta^{n-2-i}[p(t-n+i)\\ \cdot\Delta^n y(t-n+i)]\end{aligned}$$

and

$$\begin{aligned}Fy(t)&=\Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1)\\ &\quad-(-1)^n\sum_{i=0}^{n-2}(-1)^i\Delta^i y(t)\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)].\end{aligned}$$

Here, as is common for the difference calculus, whenever the upper limit of a sum is less than the lower limit of the sum, the sum is understood to be zero.

Lemma 1. *If* y *is defined for* $t\ge a$, *then*

(4) $$\Delta Ey(t)=Fy(t), \qquad t\ge a+n.$$

Further, if y *is a solution of equation* (1), *then*

(5) $$\Delta Fy(t)=p(t-1)[\Delta^n y(t-1)]^2+(-1)^n q(t)y^2(t).$$

In particular, if

(6) $$(-1)^n q(t)\ge 0, \qquad t\ge a+n,$$

then F is nondecreasing along solutions y of equation (1).

Proof. We first show (4)

$$\begin{aligned}\Delta Ey(t) = {} & [\Delta^{n-1}y(t-1)+(n-1)\Delta^{n-1}y(t)]p(t-1)\Delta^n y(t-1)\\ & -(-1)^n\sum_{i=0}^{n-2}(-1)^i(i+1)\Delta^{i+1}y(t)\Delta^{n-2-i}[p(t-n+i+1)\\ & \qquad \cdot\Delta^n y(t-n+i+1)]\\ & -(-1)^n\sum_{i=0}^{n-2}(-1)^i(i+1)\Delta^{i}y(t)\Delta^{n-1-i}[p(t-n+i)\\ & \qquad \cdot\Delta^n y(t-n+i)].\end{aligned}$$

Evaluating the first sum at $n-2$ and reindexing, we obtain

$$\begin{aligned}\Delta Ey(t) = {} & \Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1)\\ & +(n-1)\Delta^{n-1}y(t)p(t-1)\Delta^n y(t-1)\\ & -(n-1)\Delta^{n-1}y(t)p(t-1)\Delta^n y(t-1)\\ & -(-1)^n\sum_{i=1}^{n-2}(-1)^{i-1}(i)\Delta^{i}y(t)\Delta^{n-1-i}[p(t-n+i)\\ & \qquad \cdot\Delta^n y(t-n+i)]\\ & -(-1)^n\sum_{i=0}^{n-2}(-1)^i(i+1)\Delta^{i}y(t)\Delta^{n-1-i}[p(t-n+i)\\ & \qquad \cdot\Delta^n y(t-n+i)]\\ = {} & \Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1)\\ & -(-1)^n\sum_{i=0}^{n-2}(-1)^i\Delta^{i}y(t)\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)]\\ = {} & Fy(t).\end{aligned}$$

Now we will show (5).

$$\begin{aligned}\Delta Fy(t) = {} & \Delta^{n-1}y(t)\Delta[p(t-1)\Delta^n y(t-1)] + p(t-1)[\Delta^n y(t-1)]^2 \\ & - (-1)^n \sum_{i=0}^{n-2}(-1)^i \Delta^{i+1}y(t)\Delta^{n-1-i}[p(t-n+i+1) \\ & \qquad \cdot \Delta^n y(t-n+i+1)] \\ & - (-1)^n \sum_{i=0}^{n-2}(-1)^i \Delta^i y(t)\Delta^{n-i}[p(t-n+i)\Delta^n y(t-n+i)].\end{aligned}$$

Evaluating the first sum at $n-2$ and reindexing, we obtain

$$\begin{aligned}\Delta Fy(t) = {} & \Delta^{n-1}y(t)\Delta[p(t-1)\Delta^n y(t-1)] + p(t-1)][\Delta^n y(t-1)]^2 \\ & - \Delta^{n-1}y(t)\Delta[p(t-1)\Delta^n y(t-1)] \\ & - (-1)^n \sum_{i=1}^{n-2}(-1)^{i-1}\Delta^i y(t)\Delta^{n-i}[p(t-n+i)\Delta^n y(t-n+i)] \\ & - (-1)^n \sum_{i=0}^{n-2}(-1)^i \Delta^i y(t)\Delta^{n-i}[p(t-n+i)\Delta^n y(t-n+i)] \\ = {} & p(t-1)[\Delta^n y(t-1)]^2 - (-1)^n y(t)\Delta^n[p(t-n)\Delta^n y(t-n)] \\ = {} & p(t-1)[\Delta^n y(t-1)]^2 + (-1)^n q(t)y^2(t)\end{aligned}$$

provided y is a solution of equation (1). Also, if (6) holds then $\Delta Fy(t) \geq 0$ on $[a+n, \infty)$. Hence F is nondecreasing along solutions of equation (1) for $t \geq a+n$. □

To obtain another expression for $Fy(t)$, note that

$$\begin{aligned}Fy(t) = {} & \Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1) \\ & - (-1)^n \sum_{i=0}^{n-2}(-1)^i \Delta^i y(t)\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)] \\ = {} & \Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1) \\ & - (-1)^n \sum_{i=0}^{n-2}(-1)^i[\Delta^{i+1}y(t-1) + \Delta^i y(t-1)] \\ & \qquad \cdot \Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)]\end{aligned}$$

Separating the sum, then evaluating the first sum at $n-2$ and reindexing, we obtain

$$\begin{aligned} Fy(t) &= \Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1) - \Delta^{n-1}y(t-1) \\ &\quad \cdot \Delta[p(t-2)\Delta^n y(t-2)] \\ &\quad - (-1)^n \sum_{i=1}^{n-2}(-1)^{i-1}\Delta^i y(t-1)\Delta^{n-i}[p(t-n+i-1) \\ &\qquad \cdot \Delta^n y(t-n+i-1)] \\ &\quad - (-1)^n \sum_{i=0}^{n-2}(-1)^i\Delta^i y(t-1)\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)] \\ &= \Delta^{n-1}y(t-1)\{p(t-1)\Delta^n y(t-1) - \Delta[p(t-2)\Delta^n y(t-2)]\} \\ &\quad - (-1)^n \sum_{i=1}^{n-2}(-1)^i\Delta^i y(t-1)\{\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)] \\ &\qquad - \Delta^{n-i}[p(t-n+i-1)\Delta^n y(t-n+i-1)]\} \\ &\quad - (-1)^n y(t-1)\Delta^{n-1}[p(t-n)\Delta^n y(t-n)]. \end{aligned}$$

Hence,

$$\begin{aligned} Fy(t) &= \Delta^{n-1}y(t-1)p(t-2)\Delta^n y(t-2) \\ &\quad - (-1)^n \sum_{i=1}^{n-2}(-1)^i\Delta^i y(t-1)\Delta^{n-1-i}[p(t-n+i-1) \\ &\qquad \cdot \Delta^n y(t-n+i-1)] \\ &\quad - (-1)^n y(t-1)\Delta^{n-1}[p(t-n)\Delta^n y(t-n)]. \end{aligned} \tag{7}$$

We can form another operator on the set of functions y defined on $[a,\infty)$. We define for $t \geq a+n-1$ the operator $\tilde{F}$

$$\begin{aligned} \tilde{F}y(t) &= \Delta^{n-1}y(t-1)p(t-1)\Delta^n y(t-1) \\ &\quad - (-1)^n \sum_{i=1}^{n-2}(-1)^i\Delta^i y(t)\Delta^{n-1-i}[p(t-n+i)\Delta^n y(t-n+i)] \\ &\quad - (-1)^n y(t)\Delta^{n-1}[p(t-n+1)\Delta^n y(t-n+1)]. \end{aligned}$$

As in the proof of Lemma 1, we can show that

$$\Delta\tilde{F}y(t) = p(t-1)[\Delta^n y(t-1)]^2 + (-1)^n q(t+1)y^2(t+1).$$

When $n = 2$ and $p(t) \equiv 1$ (in this case the middle term is understood to be zero) the form (7) of the operator F is the same as an expression studied by Cheng [**3**]. The corresponding operator studied by Smith and Taylor [**12**] for $n = 2$ and $p(t) \equiv 1$ is the same as our operator $\tilde{F}$. We will primarily be using the two forms of F, and we use $\tilde{F}$ here as an illustration of other possible identities.

If y is a solution of (1) such that $Fy(t) \leq 0$ in a neighborhood of infinity, then we say y is a *type I* solution. Further, if $Fy(t) > 0$ in a neighborhood of infinity, then we say y is a *type II* solution. Smith and Taylor [**12**] show the existence of two linearly independent type I solutions for the case when $n = 2$ and $p(t) \equiv 1$. Note that if (6) holds, then by Lemma 1 all solutions of (1) are type I or type II solutions. We will say y is a *strict type I* solution provided $Fy(t) < 0$ in a neighborhood of infinity.

If y is a solution of (1) on the interval $[a, \infty)$, then we say y has a *generalized zero at* t_0 provided either $y(t_0) = 0$ for $t_0 \geq a$, or for $t_0 > a$ there is an integer $k \in \{1, \dots, t_0 - a\}$ such that $(-1)^k y(t_0 - k) y(t_0) > 0$ where if $k > 1$, $y(t_0 - k + 1) = \cdots = y(t_0 - 1) = 0$.

Theorem 3. *Assume* (6) *holds. Then any nontrivial solution of equation* (1) *with* $n - 1$ *consecutive zeros followed immediately by a generalized zero is a type* II *solution. In particular, the difference equation* (1) *has* n *linearly independent type* II *solutions.*

Proof. Assume y is a nontrivial solution of (1) satisfying

$$y(t_0 + i) = 0, \qquad 0 \leq i \leq n - 2 \tag{8}$$

and y has a generalized zero at $t_0 + n - 1$.

Extend the domain of $p(t)$ and $q(t)$ to the set of integers $(-\infty, \infty)$ by

$$\begin{aligned} p(t) &= p(a), \qquad t \leq a \\ q(t) &= q(a + n), \qquad t \leq a + n. \end{aligned}$$

It suffices to show that equation (1) with these new coefficients satisfies the theorem. Note that $Fy(t)$ is now defined and nondecreasing on $(-\infty, \infty)$.

We first consider the case where $y(t_0 + n - 1) = 0$. Since y is a nontrivial solution of (1), y can have at most $2n - 1$ consecutive zeros. By possibly increasing t_0, we may assume without loss of generality that

$$y(t_0 + n) \neq 0.$$

Then, using (8), we get that

$$\begin{aligned} Fy(t_0 + 2) &= \Delta^{n-1} y(t_0 + 1) p(t_0 + 1) \Delta^n y(t_0 + 1) \\ &\quad - \Delta^{n-2} y(t_0 + 2) \Delta[p(t_0) \Delta^n y(t_0)] \\ &= y(t_0 + n)\{p(t_0 + 1)\Delta^n y(t_0 + 1) - \Delta[p(t_0)\Delta^n y(t_0)]\} \\ &= p(t_0) y(t_0 + n) \Delta^n y(t_0) \\ &= p(t_0) y^2(t_0 + n) > 0. \end{aligned}$$

Hence, by Lemma 1, $Fy(t) > 0$ on $[t_0 + 2, \infty)$ and y is a type II solution of (1).

Now consider the case where (8) holds and y has a generalized zero at $t_0 + n - 1$, but

$$y(t_0 + n - 1) \neq 0.$$

In this case

$$(-1)^n y(t_0 - 1) y(t_0 + n - 1) > 0.$$

Consider

$$\begin{aligned} Fy(t_0 + 1) &= \Delta^{n-1} y(t_0) p(t_0) \Delta^n y(t_0) - \Delta^{n-2} y(t_0 + 1) \\ &\quad \times \Delta[p(t_0 - 1)\Delta^n y(t_0 - 1)] \\ &= y(t_0 + n - 1)\{p(t_0)\Delta^n y(t_0) - \Delta[p(t_0 - 1)\Delta^n y(t_0 - 1)]\} \\ &= p(t_0) y(t_0 + n - 1) \Delta^n y(t_0 - 1) \\ &= p(t_0)[y^2(t_0 + n - 1) + (-1)^n y(t_0 - 1) y(t_0 + n - 1)] > 0. \end{aligned}$$

Hence, by Lemma 1, $Fy(t) > 0$ on $[t_0 + 1, \infty)$ and y is a type II solution of (1).

We now show that there are n linearly independent type II solutions of (1). Let $y_k(t)$, $1 \leq k \leq n$ be the solutions of (1) satisfying

$$y_k(a + i) = 0, \qquad 0 \leq i \leq 2n - 1, \qquad i \neq n + k - 1$$
$$y_k(a + n - k - 1) = 1.$$

Since y_k, $1 \le k \le n$, are nontrivial solutions with n consecutive zeros starting at a, we have by the first part of the proof that y_k, $1 \le k \le n$, are type II solutions. Clearly these solutions are linearly independent. $\square$

Theorem 4. *If* (6) *holds, then the difference equation* (1) *has* n *linearly independent type* I *solutions.*

Proof. For each fixed $s \ge a+n$, let $v_k(t,s)$, $1 \le k \le n$, be a nontrivial solution of equation (1) satisfying the $2n-1$ boundary conditions

$$v_k(a+i,s) = 0, \qquad 0 \le i \le n-1, \qquad i \ne k-1$$
$$v_k(s+i,s) = 0, \qquad 0 \le i \le n-1.$$

Then define

$$u_k(t,s) = \frac{v_k(t,s)}{\sqrt{v_k^2(a,s) + v_k^2(a+1,s) + \cdots + v_k^2(a+2n-1,s)}}$$

for $1 \le k \le n$, $s \ge a+n$. Then $u_k(t,s)$ is a solution of equation (1) satisfying

$$\sum_{i=0}^{2n-1} u_k^2(a+i,s) = 1.$$

Hence, for each k, $1 \le k \le n$, the sequence $\{u_k(a,s), u_k(a+1,s), \dots, u_k(a+2n-1,s)\}_{s=a+n}^{\infty}$ has a convergent subsequence $\{u_k(a,s_{jk}), u_k(a+1,s_{jk}), \dots, u_k(a+2n-1,s_{jk})\}_{j=1}^{\infty}$. Let

$$v_{ik} = \lim_{j\to\infty} u_k(a+i-1, s_{jk})$$

$1 \le i \le 2n$. Then

$$\sum_{i=1}^{2n} v_{ik}^2 = 1.$$

Let y_k, $1 \le k \le n$, be the solutions of equation (1) satisfying

$$y_k(a+i) = v_{i+1,k}$$

$0 \le i \le 2n-1$.

Since

$$Fu_k(s_{jk}+1, s_{jk}) = 0$$

and $Fu_k(t, s_{jk})$ is nondecreasing,

$$Fu_k(t, s_{jk}) \le 0, \qquad \text{on } [a+n, s_{jk}+1].$$

Letting $j \to \infty$, we get that

$$Fy_k(t) \le 0, \qquad t \ge a+n.$$

Hence, y_k, $1 \le k \le n$, are type I solutions of (1).

Note that

$$y_k(a+i) = 0, \qquad 0 \le i \le n-1, \qquad i \ne k-1.$$

If $y_k(a+k-1) = 0$, then y_k would have n consecutive zeros and so by Theorem 3 would be a type II solution. Hence $y_k(a+k-1) \ne 0$, $1 \le k \le n$. It easily follows from this that $y_k(t)$, $1 \le k \le n$, are linearly independent. □

Theorem 5. *If* (6) *holds and* y *is a type* I *solution of equation* (1), *then*

$$\sum_{t=a}^{\infty} p(t)[\Delta^n y(t)]^2 < \infty \tag{9}$$

and

$$\sum_{t=a+n}^{\infty} (-1)^n q(t) y^2(t) < \infty. \tag{10}$$

If $q(t) \ne 0$ *in a neighborhood of infinity, then every nontrivial type* I *solution of equation* (1) *is a strict type* I *solution.*

Let y be a type I solution of (1). Then

$$Fy(t) \le 0, \qquad t \ge a+n.$$

Let

$$M = \lim_{t\to\infty} Fy(t) \le 0.$$

Summing both sides of (5) from $a+n$ to ∞, we get that

$$M - Fy(a+n) = \sum_{t=a+n}^{\infty} \{p(t-1)[\Delta^n y(t-1)]^2 + (-1)^n q(t) y^2(t)\}.$$

Thus (9) and (10) hold.

Now assume $q(t) \neq 0$ in a neighborhood of infinity and v is a nontrivial type I solution of (1). Then $Fv(t) \le 0$ for $t \ge a+n$. Assume there is a $t_0 \in [a+n, \infty)$ such that $Fv(t_0) = 0$. Then $Fv(t) \equiv 0$ on $[t_0, \infty)$. But then $\Delta Fv(t) \equiv 0$ on $[t_0, \infty)$. Hence, from (5) we get that

$$p(t-1)[\Delta^n v(t-1)]^2 + (-1)^n q(t) v^2(t) = 0, \qquad t \ge t_0.$$

Since $q(t) \neq 0$ in a neighborhood of infinity, we get that v is the trivial solution which is not possible. Hence, we must have

$$Fv(t) < 0, \qquad t \ge a+n,$$

which means that v is a strict type I solution of equation (1). □

From Theorems 4 and 5, we obtain the following result, which is related to the recessive solutions of Ahlbrandt and Hooker [**2**].

Corollary 3. *If* (6) *holds and*

$$\liminf_{t\to\infty} (-1)^n q(t) > 0,$$

then equation (1) has n linearly independent type I *solutions v_k, $1 \le k \le n$, satisfying*

$$\lim_{t\to\infty} v_k(t) = 0.$$

A close look at the proof of Theorems 4 and 5 shows one could prove the following result.

Corollary 4. *Assume* (6) *holds and there is an increasing sequence of integers* $\{t_j\}_{j=0}^{\infty} \subset [a+n, \infty)$ *such that*

$$\limsup_{j\to\infty}[t_j - t_{j-1}] < \infty$$

$$\liminf_{j\to\infty} Q_j > 0, \qquad \liminf_{j\to\infty} P_j > 0$$

where

$$Q_{nj+i} = (-1)^n q(t_j + i)$$

for $0 \le i \le n-1$, $j \ge 0$ *and*

$$P_{nj+i} = p(t_j + i - 1)$$

for $0 \le i \le \limsup_{j\to\infty}[t_j - t_{j-1}]$, $j \ge 0$, *then equation* (1) *has* n *linearly independent type* I *solutions* v *satisfying*

$$\lim_{t\to\infty} v(t) = 0.$$

Definition. We say that equation (1) is (n,n)-disconjugate on $[a,\infty)$ provided there is no nontrivial solution y such that

$$y(t_1 + i) = 0, \qquad 0 \le i \le n-2 \tag{11a}$$

$$y(t_2 + i) = 0, \qquad 0 \le i \le n-2 \tag{11b}$$

and y has a generalized zero at both $t_1 + n - 1$ and $t_2 + n - 1$ where $a \le t_1 < t_1 + n \le t_2$.

This definition for (n,n)-disconjugacy is more general than the definition for $(k, m-k)$-disconjugacy given in [**10**] for the case when $k = n$ and $m = 2n$.

Theorem 6. *If* (6) *holds, then equation* (1) *is* (n,n)*-disconjugate on* $[a,\infty)$.

Proof. Assume y is a nontrivial solution of equation (1) which satisfies (11a), (11b) and has a generalized zero at $t_1 + n - 1$. We will consider the three cases: (i) $t_2 = t_1 + n$ and $y(t_1 + n - 1) = 0$, (ii) $t_2 > t_1 + n$

and $y(t_1+n-1)=0$, and (iii) $y(t_1+n-1)\neq 0$. We will show that y cannot have a generalized zero at t_2+n-1.

For case (i) assume $t_2=t_1+n$ and $y(t_1+n-1)=0$. If $t_1=a$ here and y has a generalized zero at t_2+n-1, then $y(t_1+2n-1)=0$. Thus y is the trivial solution; therefore, we assume $t_1>a$. Consider equation (1) evaluated at $t=t_1+n-1$; with (11a) and (11b) we obtain

$$p(t_1+n-1)y(t_1+2n-1)+(-1)^{2n}p(t_1-1)y(t_1-1)=0.$$

But this implies that

$$(-1)^{2n}y(t_1-1)y(t_1+2n-1)<0.$$

That is, y does not have a generalized zero at $t=t_1+2n-1=t_2+n-1$.

For case (ii) assume $t_2>t_1+n$ and $y(t_1+n-1)=0$. By possibly increasing t_1, we can assume without loss of generality that $t=t_1+n-1$ is the last consecutive zero of y beginning with $t=t_1$. So $y(t_1+n)\neq 0$.

Extend the domain of $p(t)$ and $q(t)$ to the set of integers $(-\infty,\infty)$ by

$$\begin{aligned} p(t)&=p(a), \qquad t\le a\\ q(t)&=q(a+n), \qquad t\le a+n. \end{aligned}$$

It suffices to show that equation (1) with these new coefficients is (n,n)-disconjugate on $(-\infty,\infty)$. Note that $Fy(t)$ is now defined and nondecreasing on $(-\infty,\infty)$. Using (11a) we get that

$$\begin{aligned} Fy(t_1+2)&=\Delta^{n-1}y(t_1+1)p(t_1+1)\Delta^n y(t_1+1)-\Delta^{n-2}y(t_1+2)\\ &\quad\cdot\Delta[p(t_1)\Delta^n y(t_1)]\\ &=y(t_1+n)p(t_1+1)\Delta^n y(t_1+1)-y(t_1+n)[p(t_1+1)\\ &\qquad\qquad\cdot\Delta^n y(t_1+1)-p(t_1)\Delta^n y(t_1)]\\ &=p(t_1)y^2(t_1+n)\\ &>0. \end{aligned}$$

Hence

$$Fy(t)>0$$

for $t \geq t_1 + 2$. In particular, $Fy(t_2) > 0$. Evaluating $Fy(t_2)$, we obtain from (11b)

$$\Delta^{n-1} y(t_2 - 1) p(t_2 - 1) \Delta^n y(t_2 - 1) > 0$$

so that

$$(-1)^{n-1} y(t_2 - 1) p(t_2 - 1)[y(t_2 + n - 1) + (-1)^n y(t_2 - 1)] > 0.$$

Hence

$$(-1)^n y(t_2 - 1) y(t_2 + n - 1) < 0,$$

which along with (11b) implies y has no generalized zero (and hence no zero) at $t = t_2 + n - 1$.

For case (iii) assume $(-1)^n y(t_1 - 1) y(t_1 + n - 1) > 0$. As in case (ii) extend the definitions of $p(t)$ and $q(t)$, then note that $Fy(t)$ is defined and nondecreasing on $(-\infty, \infty)$. Using (11a) we get that

$$\begin{aligned} Fy(t_1 + 1) &= \Delta^{n-1} y(t_1) p(t_1) \Delta^n y(t_1) - \Delta^{n-2} y(t_1 + 1) \Delta[p(t_1 - 1) \\ &\qquad \cdot \Delta^n y(t_1 - 1)] \\ &= y(t_1 + n - 1) p(t_1) \Delta^n y(t_1) - y(t_1 + n - 1)[p(t_1) \\ &\qquad \cdot \Delta^n y(t_1) - p(t_1 - 1) \Delta^n y(t_1 - 1)] \end{aligned}$$

$$\begin{aligned} Fy(t_1 + 1) &= y(t_1 + n - 1) p(t_1 - 1)[y(t_1 + n - 1) + (-1)^n y(t_1 - 1)] \\ &= p(t_1 - 1)[y^2(t_1 + n - 1) + (-1)^n y(t_1 + n - 1) y(t_1 - 1)] \\ &> 0. \end{aligned}$$

Hence,

$$Fy(t) > 0$$

for $t \geq t_1 + 1$. In particular, $Fy(t_2) > 0$. Evaluating $Fy(t_2)$, we obtain using (11b)

$$\Delta^{n-1} y(t_2 - 1) p(t_2 - 1) \Delta^n y(t_2 - 1) > 0$$

so that

$$(-1)^{n-1} y(t_2 - 1) p(t_2 - 1)[y(t_2 + n - 1) + (-1)^n y(t_2 - 1)] > 0.$$

Hence,

$$(-1)^n y(t_2 - 1) y(t_2 + n - 1) < 0$$

which, along with (11b), implies that y has no generalized zero (and hence no zero) at $t = t_2 + n - 1$. □

Theorem 7. *Every unbounded solution of* (1) *where*

$$\liminf_{t\to\infty} q(t) > 0 \tag{12}$$

and

$$0 < \liminf_{t\to\infty} p(t) \le \limsup_{t\to\infty} p(t) < \infty, \tag{13}$$

is oscillatory.

Proof. Assume y is an unbounded solution of (1) to show that y is oscillatory. Suppose that y is nonoscillatory, then there is a $t_0 \in [a, \infty)$ such that all values $y(t)$ have the same sign on $[t_0, \infty)$. We may assume $y(t) > 0$ on $[t_0, \infty)$. Since y is an unbounded positive solution of (1), we have by (12) that

$$\Delta^n[p(t)\Delta^n y(t)] = -q(t+n)y(t+n) < 0 \tag{14}$$

on $[t_0, \infty)$, and

$$\liminf_{t\to\infty} \Delta^n[p(t)\Delta^n y(t)] = \liminf_{t\to\infty} -q(t+n)y(t+n) = -\infty. \tag{15}$$

But

$$\Delta^{n-1}[p(t)\Delta^n y(t)] - \Delta^{n-1}[p(t_0)\Delta^n y(t_0)] = \sum_{s=t_0}^{t-1} \Delta^n[p(s)\Delta^n y(s)]. \tag{16}$$

Hence, by expressions (14), (15) and (16), we have

$$\liminf_{t\to\infty} \Delta^{n-1}[p(t)\Delta^n y(t)] = -\infty. \tag{17}$$

Furthermore, by expression (14)

$$\begin{aligned}\Delta^{n-1}[p(t+1)\Delta^n y(t+1)] &= \Delta^n[p(t)\Delta^n y(t)] + \Delta^{n-1}[p(t)\Delta^n y(t)] \\ &< \Delta^{n-1}[p(t)\Delta^n y(t)]\end{aligned}$$

on $[t_0, \infty)$. Thus, by (17), there is a $t_1 \in [t_0, \infty)$ such that

$$\Delta^{n-1}[p(t)\Delta^n y(t)] < 0$$

on $[t_1, \infty)$.

By continuing in this fashion of summing each expression it is easily shown that

$$\liminf_{t\to\infty} \Delta^i[p(t)\Delta^n y(t)] = -\infty,$$

for $i = n-2, n-3, \dots, 0$, and using (13)

$$\liminf_{t\to\infty} \Delta^i y(t) = -\infty,$$

for $i = n, n-1, \dots, 0$. Thus

$$\liminf_{t\to\infty} y(t) = -\infty.$$

But this contradicts the assumption that $y(t) > 0$ on $[t_0, \infty)$. Hence if (12) and (13) hold, then every unbounded solution y of (1) is oscillatory. □

The following theorem demonstrates that type II solutions are unbounded for the special case when $n = 2$ and $p(t) \equiv 1$. We believe, but have been unable to show, that, for the more general case, type II solutions are unbounded for any n is also true with the added assumption

$$0 < \liminf_{t\to\infty} p(t) \le \limsup_{t\to\infty} p(t) < \infty.$$

For the following theorem, we consider equation (1) with $n = 2$ and $p(t) \equiv 1$ that is the fourth order linear difference equation

$$\Delta^4 y(t-2) + q(t)y(t) = 0, \qquad t \ge a+2 \tag{18}$$

where $q(t) \ge 0$ on $[a+2, \infty)$. Let y be defined on $[a, \infty)$, then for $t \ge a+2$ operator F becomes

$$Fy(t) = \Delta y(t-1)\Delta^2 y(t-1) - y(t)\Delta^3 y(t-2) \tag{19}$$

and take a different antidifference to redefine the operator E by

$$Ey(t) = [\Delta y(t-1)]^2 - y(t)\Delta^2 y(t-2). \tag{20}$$

Theorem 8. *If* (6) *holds, then type* II *solutions of* (18) *are unbounded.*

Assume that y is a type II solution of (18), i.e., there is a $t_0 \in [a+2,\infty)$ such that $Fy(t_0) > 0$. As in Lemma 1, by (6) F is nondecreasing along each solution y of (18). Hence, by

$$\Delta Ey(t) = Fy(t) > 0$$

and by (5)

$$\begin{aligned}\Delta^2 Ey(t) &= \Delta Fy(t)\\ &= [\Delta^2 y(t-1)]^2 + q(t)y^2(t)\\ &\geq 0\end{aligned}$$

on $[t_0,\infty)$. Hence, we get that

$$\lim_{t\to\infty} Ey(t) = \infty.$$

By the way E is defined in (20) if y is bounded, then so is Ey. But Ey is unbounded, thus y must be unbounded. Hence, all type II solutions of (18) are unbounded. □

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