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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF COAGULATION-FRAGMENTATION MODELS

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Abstract. In this paper we study the evolution of the number density $n(x,c,t)$ of droplets of volume x with chemical concentration c subject to both coalescence (coagulation) and rupture (fragmentation, breakage). It is proved that as $t \to \infty$ the concentration tends to a limit value c_∞ and the measure $xn(x,c,t)dxdc$ converges to $\delta(c-c_\infty)dL(x)$ where $\delta(y)$ is the Dirac measure and dL is a measure satisfying the appropriate equilibrium equation.

1. The model. Consider a mixture of a large number of incompressible spherical droplets of varying volume size x in a solution (e.g., water or air). When the mixture is well stirred the distribution of droplet size can be taken to be homogeneous in space. We may then introduce the notion of density number $n(x)$:

$$n(x)dx = \text{ number of droplets in unit volume}$$
$$\text{whose volume size lies between } x \text{ and } x+dx;$$

$n(x)$ is independent of the location of the volume element.

Since the droplets are in continuous motion, they may collide and, as a result, coalesce. The motion may also cause rupture. We assume that if two droplets of volumes x and ξ coalesce, they will form a droplet of volume $x+\xi$. The rate of coalescence is given by a function $K(x,\xi)$, which is called the *coalescence* or *coagulation kernel.*

We also introduce the *fragmentation* or *breakage* kernel $B(x,\xi)$ which is the rate that a droplet of volume x breaks into two droplets of volumes ξ and $x-\xi$.

Throughout this paper we assume that

$$\begin{aligned}
&K(x,\xi) \text{ is continuous and } \geq 0 \text{ for } x,\xi \geq 0,\\
&K(x,\xi) = K(\xi,x),\\
&K(x,\xi) \leq C_0,\\
&B(x,\xi) \text{ is continuous and } \geq 0 \text{ for } x \geq \xi \geq 0,\\
&B(x,\xi) = B(x,x-\xi),\\
&\int_0^x B(x,\xi)d\xi \leq C_1,\\
&\int_0^x \xi B(x,\xi)d\xi \leq C_2 x.
\end{aligned} \tag{1.1}$$

Remark 1.1. Note that if volume is preserved under rupture, then

$$\int_0^x B(x,\xi)\xi d\xi = x,$$

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so that the last condition in (1.1) is very natural.

Due to coagulation and fragmentation the number density will actually vary with time, and we introduce

$$n(x,t) = \text{ number density at time } t.$$

The following conservation law holds:

$$\begin{aligned}\frac{d}{dt}n(x,t)dx = &\text{ number gained through coalescence}\\ &+ \text{number gained through rupture}\\ &- \text{number lost through coalescence}\\ &- \text{number lost through rupture.}\end{aligned}$$

This can be expressed by the evolution equation:

$$\begin{aligned}\frac{d}{dt}n(x,t) &= \frac{1}{2}\int_0^x K(x-\xi,\xi)n(x-\xi,t)n(\xi,t)d\xi + \int_x^\infty B(\xi,x)n(\xi,t)d\xi\\ &-n(x,t)\int_0^\infty K(x,\xi)n(\xi,t)d\xi - n(x,t)\int_0^x \frac{\xi}{x}B(x,\xi)d\xi.\end{aligned} \tag{1.2}$$

Here the underlying assumption is that, although $n(x,t)$ varies with t, the mixture is maintained homogeneous in space at any moment (due to continuous fast stirring).

Together with (1.2) we prescribe an initial condition

$$n(x,0) = n_0(x), \quad n_0(x) \geq 0. \tag{1.3}$$

There is substantial literature on coagulation and coagulation-fragmentation models. Melzak [10] proved a general existence, uniqueness and positivity of a solution for all $t > 0$. For other results we refer to [2], [4], [9] and the references therein.

We shall actually be interested in a situation where each droplet carries a chemical species which is uniformly distributed within it; the concentration c of the chemical species is a variable quantity. We then introduce

$$\begin{aligned}n(x,c,t)dxdc = &\text{ number of droplets at time } t\\ &\text{with volume size in the interval } (x, x+dx)\\ &\text{and concentration (of the chemical}\\ &\text{species) in the interval } (c, c+dc).\end{aligned}$$

Analogously to (1.2), the number density $n(x,c,t)$ evolves as follows:

$$
\begin{aligned}
\frac{d}{dt}n(x,c,t) &= \frac{1}{2}\int_0^x\int_0^\infty K(x-\xi,\xi)n\left(x-\xi,\frac{xc-\xi\gamma}{x-\xi},t\right)n(\xi,\gamma,t)\frac{x}{x-\xi}d\gamma d\xi \\
&+\int_x^\infty B(\xi,x)n(\xi,c,t)d\xi \\
&-n(x,c,t)\int_0^\infty\int_0^\infty K(x,\xi)n(\xi,\gamma,t)d\gamma d\xi \\
&-n(x,c,t)\int_0^x \frac{\xi}{x}B(x,\xi)d\xi,
\end{aligned} \tag{1.4}
$$

where we define

$$n(x,c',t)=0 \quad \text{if } c'<0.$$

Here we used the fact that rupture does not change the concentration. On the other hand, if particles (ξ,γ) and (η,β) coalesce to produce (x,c), then

$$x=\xi+\eta, \quad xc=\xi\gamma+\eta\beta$$

so that

$$\eta=x-\xi, \quad \beta=\frac{xc-\xi\gamma}{x-\xi}$$

and $d\gamma d\xi d\eta d\beta = d\gamma d\xi \frac{x}{x-\xi}dxdc$, since the Jacobian determinant

$$\begin{vmatrix} \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial c} \\ \frac{\partial\beta}{\partial x} & \frac{\partial\beta}{\partial c} \end{vmatrix} \quad \text{is equal to } \frac{x}{x-\xi};$$

this explains the appearance of $x/(x-\xi)$ in the first integral on the right hand side of (1.4).

Remark 1.2. The last integral on the right-hand side of (1.2), or (1.4), can also be written in the form

$$\frac{1}{2}\int_0^x B(x,\xi)d\xi$$

since

$$
\begin{aligned}
\frac{1}{2}\int_0^x B(x,\xi)d\xi &= \frac{1}{2}\frac{1}{x}\int_0^x \left[\xi B(x,\xi)+(x-\xi)B(x,\xi)\right]d\xi \\
&= \frac{1}{2}\frac{1}{x}\left[\int_0^x \xi B(x,\xi)d\xi + \int_0^x (x-\xi)B(x,x-\xi)d\xi\right] \\
&= \frac{1}{x}\int_0^x \xi B(x,\xi)d\xi.
\end{aligned}
$$

Together with (1.4) we prescribe an initial condition

$$n(x,c,0)=n_0(x,c), \quad n_0(x,c)\geq 0. \tag{1.5}$$

The model (1.4) arises in the process of making a photographic film. The film is made of several emersion layers. One of them consists of oil droplets in aqueous solution. The mixture, prior to being coated on the film substrate, is subjected to coalescence and rupture by means of stirring. A key problem is to determine the rate of coalescence (which depends on parameters, such as the amount of surfactant added at the initial time). One of the methods to determine coalescence is based on adding chemilumenescent species to the mixture:

The optical signal, which is the number of photons emitted per unit time from a droplet, is a function, $f(x,c)$, of the species concentration c in the droplet and the droplet volume x. One can measure the total signal

$$S(t)=\int_0^\infty\int_0^\infty f(x,c)n(x,c)dxdc, \tag{1.6}$$

and this is used to to evaluate the rate of coalescence [8, Chap. 6].

Note that if two drops with parameters (x_1,c_1) and (x_2,c_2) coalesce to form a drop with parameters $(\tilde{x},\tilde{c})$ then

$$\tilde{x}=x_1+x_2, \quad \tilde{c}=\frac{x_1c_1+x_2c_2}{x_1+x_2}.$$

Typically $f(x,c)=xc^2$, and then

$$f(\tilde{x},\tilde{c})<f(x_1,c_1)+f(x_2,c_2),$$

so that the total signal should decrease.

The model (1.4) was developed by David Ross; see [8, Chap. 6]. He conjectured that as $t\to\infty$ the concentrations c will all tend to a constant, say c_∞. With this in mind, one defines the *degree of coalescence*

$$D(t)=\int_0^\infty\int_0^\infty (c-c_\infty)^2 xn(x,c,t)dxdc \tag{1.7}$$

and one would like to prove rigorously that

$$D(t)\to 0 \ \text{ if } \ t\to\infty. \tag{1.8}$$

Consider next the measures $d\mu_t$ defined by

$$d\mu_t(x,c)=xn(x,c,t)dxdc \tag{1.9}$$

If (1.8) is true then we may go one step further and try to prove that as $t\to\infty$

$$d\mu_t(x,c)\to\delta(c-c_\infty)dL(x) \tag{1.10}$$

in the sense of weak convergence of measures, where $\delta(y)$ is the Dirac measure and

(1.11) dL is a solution of the equilibrium equation (1.2).

In this paper we shall prove, under mild additional conditions on K and B that (1.8) holds. We shall also establish (1.10), (1.11) under the additional condition that

$$\begin{aligned} &K(x,\xi) \le \eta(x+\xi), \quad \gamma_0 \equiv \inf_{x>0} \frac{1}{x} \int_0^x \xi B(x,\xi)d\xi > 0, \text{ where} \\ &2\eta \int_0^\infty \int_0^\infty x n_0(x,c)dxdc < \gamma_0. \end{aligned} \tag{1.12}$$

The structure of the paper is as follows:

In §2 we establish existence, uniqueness and positivity of the solution of (1.4), (1.5) for all $t > 0$. Assuming that

$$n_0(x) = \int_0^\infty n_0(x,c)dc \tag{1.13}$$

we also prove that the function

$$n(x,t) = \int_0^\infty n(x,c,t)dc \tag{1.14}$$

is the unique solution of (1.2), (1.3). In §3 we prove conservation of mass

$$\int_0^\infty x n(x,t)dx = \text{const.}$$

We also show that if $n_0(x,c) = 0$ whenever $c > c_*$ (c_* constant) then $n(x,c,t) = 0$ for $c > c_*$. Using these results we derive in §4 a formula for dS/dt (see (4.2)) when S is defined in (1.6) and $f(x,c)$ is any function satisfying $|f(x,c)| \le C(1+x)$. This formula is the key to the proofs of both (1.8) (in §5) and (1.10), (1.11) (in §6).

We finally remark that a coalescence problem for $n(x,c,t)$ was studied in [1], with c being the concentration of surfactant which inhibits coalescence. The limit behavior of the solution and the mathematical method for deriving it are entirely different from the limit behavior and the methods of the present paper.

2. Existence and uniqueness. We shall need the following assumptions:

$$\begin{aligned} &n_0(x) \text{ is continuous for } x \ge 0, \\ &0 \le n_0(x) \le A_0, \quad \int_0^\infty n_0(x)dx \le A_0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &n_0(x,c) \text{ is continuous for } x \ge 0,\ c \ge 0, \\ &0 \le n_0(x,c) \le A_1, \quad \int_0^\infty \int_0^\infty n_0(x,c)dxdc \le A_1 \end{aligned} \tag{2.2}$$

where A_0, A_1 are positive constants.

THEOREM 2.1. *If (1.1), (2.1) hold then there exists a unique solution of (1.2), (1.3) with $n(x,t)$, $\partial n(x,t)/\partial t$ continuous for $x \geq 0$, $t \geq 0$, such that, for each $T > 0$,*

$$\sup_{x\geq 0}|n(x,t)| + \int_0^\infty |n(x,t)|dx \leq C(T) < \infty \text{ if } 0 \leq t \leq T. \tag{2.3}$$

The solution has the following additional properties:

$$n(x,t) \text{ is analytic in } t, \quad t \geq 0, \tag{2.4}$$

$$n(x,t) \geq 0. \tag{2.5}$$

This theorem is due to Melzak [10]. We note that if K and B have compact support and $n_0(x) > 0$ for all $x > 0$, then the proof of (2.5) is much simpler than the proof given in [10] (or [3]). Indeed, if x is large then from (1.2) we see that $dn(x,t)/dt = 0$ so that $n(x,t) = n_0(x) > 0$. Hence if (2.5) is not true, with strict inequality, then there is a smallest $t = t_0 > 0$ such that $n(x,t) > 0$ if $t < t_0$ for all $x \geq 0$ and $n(x_0,t_0) = 0$ for a finite point x_0. From (1.2) we then deduce that

$$\frac{dn(x_0,t_0)}{dt} > 0,$$

which is a contradiction. Finally, the solution $n(x,t)$ for general K, B, n_0, can be obtained as a pointwise limit of solutions with compact K, B and strictly positive $n_0(x)$ (cf. [3] or [4]), so that (2.5) is true in general.

THEOREM 2.2. *If (1.1), (2.2) hold then there exists a unique solution of (1.4), (1.5) with $n(x,c,t)$, $\partial n(x,c,t)/\partial t$ continuous for $x \geq 0$, $c \geq 0$, $t \geq 0$, such that, for each $T > 0$,*

$$\sup_{x\geq 0, c\geq 0}|n(x,c,t)| + \int_0^\infty\int_0^\infty |n(x,c,t)|dxdc \leq C(T) < \infty \text{ if } 0 \leq t \leq T. \tag{2.6}$$

The solution has the following additional properties:

$$n(x,c,t) \text{ is analytic in } t, \quad t \geq 0, \tag{2.7}$$

$$n(x,c,t) \geq 0. \tag{2.8}$$

Proof. The proof is similar to the proof of Theorem 2.1 provided we can treat the first term on the right-hand side of (1.4) analogously to the first term on the right-hand side of (1.2); this means that we have to control the factor $x/(x-\xi)$ in the integrand.

Introduce the operator

$$J(m,n) = \frac{1}{2}\int_0^x\int_0^\infty K(x-\xi,x)m\left(x-\xi, \frac{xc-\xi\gamma}{x-\xi}\right)\frac{x}{x-\xi}n(\xi,\gamma)d\gamma d\xi$$

and break up the integral into

$$\int_0^{x/2}\int_0^\infty + \int_{x/2}^x\int_0^\infty$$

In the second integral substitute

$$\gamma \to \gamma' = \frac{xc - \xi\gamma}{x - \xi}, \quad \text{or } \gamma = \frac{xc - (x-\xi)\gamma'}{\xi}, \quad d\gamma = -\frac{x-\xi}{\xi} d\gamma'.$$

We then obtain

$$\begin{aligned} J(m,n) &= \frac{1}{2} \int_0^{x/2} \int_0^{\infty} K(x-\xi,\xi) m\left(x-\xi, \frac{xc-\xi\gamma}{x-\xi}, t\right) n(\xi,\gamma,t) \frac{x}{x-\xi} d\gamma d\xi \\ &-\frac{1}{2} \int_{x/2}^{x} \int_0^{\infty} K(x-\xi,\xi) n\left(\xi, \frac{xc-(x-\xi)\gamma'}{\xi}, t\right) m(x-\xi,\gamma',t) \frac{x}{\xi} d\gamma' d\xi. \\ &= J_1 - J_2. \end{aligned}$$

Since the factors $\frac{x}{x-\xi}$ and $\frac{x}{\xi}$ in J_1 and J_2 respectively are uniformly bounded, we can easily establish, as in [10], the bound

$$|J(m,n)|_{L^\infty} \le C|n|_{L^\infty} \int\int |m| + C|m|_{L^\infty} \int\int |n| \tag{2.9}$$

where

$$\begin{aligned} |f|_{L^\infty} &= \sup_{x\ge 0, c\ge 0} |f(x,c,t)|, \\ \int\int |f| &= \int_0^\infty \int_0^\infty |f(x,c,t)| dc dx. \end{aligned}$$

Next we estimate

$$\int\int |J(m,n)| dc dx$$

by substituting

$$c \to c' = \frac{xc - \xi\gamma}{x-\xi}, \quad dc' = \frac{x}{x-\xi} dc.$$

we obtain

$$\int\int |J(m,n)| \le C \int\int |m| \cdot \int\int |n| \tag{2.10}$$

With (2.9), (2.10) at hand, we can now proceed as in [10], with just minor changes, to establish Theorem 2.2. The positivity of $n(x,c,t)$ can be proved, more simply, by the argument following Theorem 2.1.

THEOREM 2.3. *If (1.1), (2.1), (2.2) hold and, in addition,*

$$n_0(x) = \int_0^\infty n_0(x,c) dc, \tag{2.11}$$

then

$$\int_0^\infty n(x,c,t)dc = n(x,t). \tag{2.12}$$

where $n(x,c,t)$ and $n(x,t)$ are the solutions of (1.4), (1.5) and (1.2), (1.3), respectively.

Proof. We first proceed formally to integrate (1.4) with respect to c. In the first integral on the right-hand side we substitute

$$c \to c' = \frac{xc - \xi\gamma}{x - \xi}, \quad dc' = \frac{x}{x-\xi}dc \tag{2.13}$$

to get

$$\frac{1}{2}\int_0^\infty K(x-\xi,\xi)n(x-\xi,t)n(\xi,t)d\xi. \tag{2.14}$$

where $n(x,t)$ is the function defined by the left-hand side of (2.12). All the other integrals yield immediately the corresponding integrals of (1.2). Hence, by uniqueness, the left-hand side of (2.12) is the solution of (1.2), (1.3).

To prove the theorem rigorously we rewrite (1.4) in integrated form:

$$n(x,c,t) - n_0(x,c) = \int_0^t dt \left\{ \frac{1}{2}\int_0^x \int_0^\infty K \cdots + \cdots \right\}. \tag{2.15}$$

Since

$$\int_0^\infty \int_0^\infty n(x,c,t)dcdx \le C(T) < \infty \text{ if } 0 \le t \le T,$$

the integral

$$\int_0^\infty n(x,c,t)dc$$

is finite a.e. in x, for each t, and we may integrate (2.15) with respect to c, $0 \le c < \infty$. Each of the four multiple integrals thus obtained is well defined and is finite. We now proceed with the substitution (2.13) to obtain the term (2.14, and the proof of the theorem then follows as before.

3. Conservation and invariant laws.

THEOREM 3.1. *(Conservation law). If (1.1), (2.1) hold and*

$$\int_0^\infty x n_0(x)dx < \infty, \tag{3.1}$$

then

$$\int_0^\infty xn(x,t)dx = \int_0^\infty xn_0(x)dx \tag{3.2}$$

for all $t > 0$.

Proof. The assertion (3.2) means that

$$\frac{d}{dt}\int_0^\infty xn(x,t)dx = 0. \tag{3.3}$$

This relation follows formally as a special case of Theorem 4.1 with $f(x,c) = x$. However, in order to apply the proof rigorously we need to know a priori that, for any $T > 0$,

$$\int_0^\infty xn(x,t)dx \le C(T) < \infty \quad \text{if } \ 0 \le t \le T. \tag{3.4}$$

If K, B and n_0 have compact support then (3.4) is true and then so is (3.2). We now approximate general K, B, n_0 by K_j, B_j, $n_{0,j}$ with compact support and denote the corresponding solutions by $n_j(x,t)$. Then, on one hand we have

$$\int_0^\infty xn_j(x,t)dx = \int_0^\infty xn_{0,j}(x)dx \le C$$

for all j, and on the other hand $n_j(x,t) \to n(x,t)$ pointwise (cf. [4]). It follows that $n(x,t)$ satisfies (3.4) and consequently (3.3), or (3.2), holds.

THEOREM 3.2. *If (1.1), (2.2) hold and*

$$n_0(x,c) = 0 \quad \textit{for } x \ge 0,\ c > c_* \ (c_* > 0), \tag{3.5}$$

then

$$n(x,c,t) = 0 \quad \textit{for } x \ge 0,\ c \ge c_*\ t \ge 0. \tag{3.6}$$

Proof. If $c > c_*$ then in the first integral on the right-hand side of (1.4)

$$\text{either } \gamma > c_* \text{ or else } \frac{xc - \xi\gamma}{x - \xi} > c_*.$$

Assuming that K has compact support and using (2.3), we conclude that the integral is bounded by $CN(t)$, where

$$N(t) = \sup_{x \ge 0, c \ge c_*} n(x,c,t).$$

If B has compact support then the second integral on the right-hand side of (1.4) is bounded by

$$C \sup_{\xi > x} n(\xi, c, t), \qquad \text{or by } \ CN(t).$$

From (1.4) it then follows that

$$\frac{dn(x,c,t)}{dt} \leq CN(t).$$

If $x \geq 0$, $c > c_*$, so that

$$N(t) \leq N(0) + C\int_0^t N(t')dt'.$$

Since $N(0) = 0$, the assertion (3.6) follows.

Having proved (3.6) in the case where K and B have compact support, we can now establish the same result for general K, B by approximation (cf. the proof of Theorem 3.1).

For convenience we collect all the assumptions made so far on $n_0(x,c)$ and $n_0(x)$:

$$\text{(3.7)}\qquad \begin{cases} n_0(x) \text{ and } n_0(x,c) \text{ are continuous functions for } x \geq 0,\ c \geq 0, \\ 0 \leq n_0(x) \leq A_0, \quad \int_0^\infty (1+x)n_0(x)dx < \infty, \\ 0 \leq n_0(x,c) \leq A_1, \quad \int_0^\infty n_0(x,c)dc = n_0(x), \\ n_0(x,c) = 0 \ \text{ if } \ c \geq c_* \ (c_* > 0). \end{cases}$$

COROLLARY 3.3. *If (1.1) and (3.7) are satisfied then (3.2) and (3.6) hold.*

4. A formula for dS/dt. Let $f(x,c)$ be any function satisfying

$$\text{(4.1)}\qquad |f(x,c)| \leq C(1+x) \quad \text{if} \quad x \geq 0,\ \ c \leq c_*$$

and arbitrary otherwise. By (2.6) and Corollary 3.3 it follows that the integral

$$\text{(4.2)}\qquad S(t) = \int_0^\infty\int_0^\infty f(x,c)n(x,c,t)dxdc$$

exists for all $t \geq 0$.

THEOREM 4.1. *If (1.1), (3.7) hold and f satisfies (4.1), then*

$$\text{(4.3)}\qquad \begin{aligned} \frac{dS(t)}{dt} &= \frac{1}{2}\int_0^\infty dx\int_0^\infty d\xi\int_0^\infty dc\int_0^\infty d\gamma K(x,\xi)n(x,c,t)n(\xi,\gamma,t) \\ &\qquad\times\left[f\left(x+\xi, \frac{xc+\xi\gamma}{x+\xi}\right) - f(x,c) - f(\xi,\gamma)\right] \\ &\quad+\int_0^\infty dx\int_0^\infty dc\int_x^\infty B(\xi,x)n(\xi,c,t)\left[f(x,c) - \frac{x}{\xi}f(\xi,c)\right]d\xi. \end{aligned}$$

Note that, by (2.6) and Corollary 3.3, all the integrals on the right-hand side of (4.3) are absolutely convergent.

Proof. Formally

$$\frac{dS(t)}{dt} = \int_0^\infty \int_0^\infty f(x,c)\frac{dn(x,c,t)}{dt}dxdc. \tag{4.4}$$

By substituting dn/dt from (1.4) and using (2.6), (3.2), (4.1), we find that each of the resulting four integrals is absolutely convergent.

A standard argument then shows that $S(t)$ is differentiable and dS/dt is given by (4.4).

We can next write

$$\frac{dS}{dt} = I_1 + I_2 \tag{4.5}$$

where

$$\begin{aligned} I_1 &= \int_0^\infty dc \left[\int_0^\infty f(x,c)dx \int_x^\infty B(\xi,x)n(\xi,c,t)d\xi \right. \\ &\quad \left. - \int_0^\infty f(x,c)n(x,c,t)dx \int_0^x \frac{\xi}{x}B(x,\xi)d\xi \right], \\ I_2 &= \frac{1}{2}\int_0^\infty dx \int_0^\infty f(x,c)dc \int_0^x \int_0^\infty K(x-\xi,\xi)n\left(x-\xi, \frac{xc-\xi\gamma}{x-\xi}, t\right) n(\xi,\gamma,t)\frac{x}{x-\xi}d\gamma d\xi \\ &\quad - \int_0^\infty dc \int_0^\infty dx \int_0^\infty n(x,c,t)K(x,\xi)n(\xi,\gamma,t)f(x,c)d\gamma d\xi \\ &= I_{21} - I_{22}. \end{aligned}$$

To evaluate I_1 we change the notation of the variables $x \to \xi$ and $\xi \to x$ in the second integral in $[\cdots]$ to get

$$\int_0^\infty d\xi \int_0^\xi \frac{x}{\xi}B(\xi,c)f(\xi,c)n(\xi,c,t)dx.$$

This, in turn, is equal, by changing the order of integration, to

$$\int_0^\infty dx \int_x^\infty \frac{x}{\xi}B(\xi,x)f(\xi,c)n(\xi,c,t)d\xi.$$

Hence

$$I_1 = \int_0^\infty dc \int_0^\infty dx \int_x^\infty B(\xi,x)\left[f(x,c) - \frac{x}{\xi}f(\xi,c)\right] n(\xi,c,t)d\xi \tag{4.6}$$

Next we turn to I_{21} and change variables $x \to x'$ where $x - \xi = x'$ (for each fixed ξ) and $c \to c'$ by

$$c = \frac{x'c' + \xi\gamma}{x' + \xi}, \quad \text{or} \quad \frac{xc - \xi\gamma}{x - \xi} = c'.$$

We get

$$I_{21} = \frac{1}{2} \int_0^\infty dx' \int_0^\infty d\xi \int_0^\infty dc' \int_0^\infty d\gamma K(x', \xi) n(x', c, t) n(\xi, \gamma, t) f\left(\xi + x', \frac{x'c' + \xi\gamma}{x' + \xi}\right).$$

We shall use the relations

$$\begin{aligned} \int_0^\infty dc' \int_0^\infty d\gamma \Phi &= \int_0^\infty dc' \int_0^{c'} d\gamma \Phi + \int_0^\infty dc' \int_{c'}^\infty d\gamma \Phi \\ &= \int_0^\infty dc' \int_0^{c'} d\gamma \Phi + \int_0^\infty d\gamma \int_0^\gamma dc' \Phi, \end{aligned} \tag{4.7}$$

which is valid with any integrand Φ. Since after integration, in I_{21}, with respect to x' and ξ we get an expression $\Psi(\gamma, c')$ which is symmetric in γ, c', we can write

$$\int_0^\infty d\gamma \int_0^\gamma dc' \Psi(\gamma, c') = \int_0^\infty dc' \int_0^{c'} d\gamma \Psi(\gamma, c').$$

Hence for the purpose of calculating I_{21} we may replace the right-hand side of (4.7) by

$$2 \int_0^\infty dc' \int_0^{c'} d\gamma \Phi.$$

Consequently

$$I_{21} = \int_0^\infty dx' \int_0^\infty d\xi \int_0^\infty dc' \int_0^{c'} d\gamma K(x', \xi) n(x', c', t) n(\xi, \gamma, t) f\left(x' + \xi, \frac{x'c' + \xi\gamma}{x' + \xi}, t\right). \tag{4.8}$$

Next we can write

$$\begin{aligned} I_{22} &= \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^c d\gamma K(x, \xi) n(x, c, t) n(\xi, \gamma, t) f(x, c) \\ &+ \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma K(x, \xi) n(x, c, t) n(\xi, \gamma, t) f(x, c) \\ &= L_1 + L_2 \end{aligned} \tag{4.9}$$

In L_2 we write

$$\int_0^\infty dc \int_c^\infty d\gamma = \int_0^\infty d\gamma \int_0^\gamma dc$$

to get

$$L_2 = \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty d\gamma \int_0^\gamma dc K(x, \xi) n(x, c, t) n(\xi, \gamma, t) f(x, c).$$

Changing the notation of variables $\gamma \leftrightarrow c$ and $x \leftrightarrow \xi$ we find, using the symmetry of $K(x,\xi)$, that

$$L_2 = \int_0^\infty d\xi \int_0^\infty dx \int_0^\infty dc \int_0^c d\gamma K(x,\xi) n(\xi,\gamma,t) n(x,c,t) f(\xi,c).$$

We substitute this expression into (4.9) and, upon renaming x', c' in (4.8) by x, c, obtain

$$\begin{aligned} I_2 = I_{21} - I_{22} &= \int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^c d\gamma K(x,\xi) n(x,c,t) n(\xi,\gamma,t) \\ &\times \left[f\left(x+\xi, \frac{xc+\xi\gamma}{x+\xi}\right) - f(x,c) - f(\xi,\gamma) \right]. \end{aligned}$$

Next we write in I_2

$$\cdots \int_0^\infty dc \int_0^c d\gamma \quad \text{as} \quad \cdots \int_0^\infty d\gamma \int_\gamma^\infty dc$$

and change notation $c \leftrightarrow \gamma$, $x \leftrightarrow \xi$. We find that I_2 is equal to a similar integral in which

$$\int_0^\infty dc \int_0^c d\gamma \text{ is replaced by } \int_0^\infty dc \int_c^\infty d\gamma$$

(here we again use the symmetry of $K(x,\xi)$). It follows that I_2 is equal to $\frac{1}{2}$ times the first integral on the right-hand side of (4.3). Recalling also (4.6), (4.5), the assertion (4.3) follows.

Formula (4.3) will play a key role in the subsequent sections. Note that the choice $f = x$ gives the conservation law (3.2) which was already proved. On the other hand the choice $f = xc$ gives a new conservation law:

$$\int_0^\infty \int_0^\infty xcn(x,c,t)dxcdc = \int_0^\infty \int_0^\infty xcn_0(x,c)dxdc \text{ for all } t \geq 0. \tag{4.10}$$

COROLLARY 4.2. *If $g(c)$ is a convex function then*

$$\frac{d}{dt} \int_0^\infty \int_0^\infty g(c) xn(x,c,t) dxdc \leq 0.$$

Indeed, the second integral on the right-hand side of (4.3) is zero whereas the first integral is ≤ 0

COROLLARY 4.3. *If $n_0(x,c) = 0$ for $0 \leq c \leq \tilde{c}$, $x \geq 0$ then $n(x,c,t) = 0$ for $0 \leq c \leq \tilde{c}$, $x \geq 0$, $t \geq 0$.*

Indeed, taking $g(c)$ to be the convex function $(\tilde{c} - c)^+$ we conclude that

$$\int_0^\infty \int_0^{\tilde{c}} (\tilde{c} - c)^+ xn(x,c,t) dxdc \leq \int_0^\infty \int_0^{\tilde{c}} (\tilde{c} - c)^+ xn_0(x,c) dxdc = 0$$

and the assertion follows.

The proof of Theorem 4.1 can also be applied (or, rather, specialized) to functions of the form

$$\tilde{S}(t) = \int_0^\infty f(x)n(x,t)dx. \tag{4.11}$$

We then get:

THEOREM 4.4. *If (1.1), (3.7) hold and $f(x)$ satisfies*

$$|f(x)| \le C(1+x), \tag{4.12}$$

then

$$\begin{aligned} \frac{d\tilde{S}(t)}{dt} &= \frac{1}{2}\int_0^\infty\int_0^\infty K(x,\xi)n(x,t)n(\xi,t)\left[f(x+\xi) - f(x) - f(\xi)\right]dxd\xi \\ &+ \int_0^\infty dx \int_x^\infty B(\xi,x)n(\xi,t)\left[f(x) - \frac{x}{\xi}f(\xi)\right]d\xi. \end{aligned} \tag{4.13}$$

5. $D(t) \to 0$ if $t \to \infty$. In this section we impose additional mild assumptions on K, B and n_0:

$$\int_0^\infty x^2 n_0(x)dx < \infty \tag{5.1}$$

and

$$\begin{aligned} &\text{(a)} && K(x,\xi) \le C(x+\xi); \\ &\text{(b)} && \partial_x K(x,\xi) \ge -C; \\ &\text{(c)} && \exists \text{ a positive constant } \gamma_1 \text{ such that, for any } N \gg 1, \\ & && K(x,\xi) \ge \frac{\gamma_1}{N}(x+\xi) \text{ if } 0 \le x, \xi \le N; \\ &\text{(d)} && \int_0^\xi B(\xi,x)x(\xi - x)dx \ge \gamma_2\xi^2 \;\; \forall \xi > 0, \text{ for some } \gamma_2 > 0. \end{aligned} \tag{5.2}$$

Remark 1.1 indicates that condition (5.2)(d) is quite natural. Note also that (5.2)(c) is satisfied if

$$K(x,\xi) \ge \begin{cases} \delta(x+\xi) & \text{for } x+\xi < 1 \\ \delta & \text{for } x+\xi > 1 \end{cases}$$

where δ is a small positive constant.

Set

$$M_0 = \int_0^\infty\int_0^\infty x n_0(x,c)dxdc, \quad M_1 = \int_0^\infty\int_0^\infty xc n_0(x,c)dxdc, \tag{5.3}$$

$$c_\infty = \frac{M_1}{M_0}. \tag{5.4}$$

In this section we prove:

THEOREM 5.1. *If (1.1), (3.7) and (5.1), (5.2) hold, then the function*

$$D(t) \equiv \int_0^\infty \int_0^\infty x(c-c_\infty)^2 n(x,c,t)\,dx\,dc \tag{5.5}$$

converges to zero as $t \to \infty$.

We need several lemmas.

LEMMA 5.2. *There exists a constant C independent of t such that*

$$\int_0^\infty x^2 n(x,t)\,dx \le C \quad \forall t > 0. \tag{5.6}$$

Proof. Consider first the case where

$$K, B \text{ and } n_0 \text{ have compact support.} \tag{5.7}$$

Then, for any $T > 0$,

$$\tilde{S}(t) \equiv \int_0^\infty x^2 n(x,t)\,dx \le C(T) < \infty \quad \text{if} \quad 0 \le t \le T,$$

and we may apply (4.13) with $f(x) = x^2$ (The proof is the same as that of Theorem 4.4.). We get

$$\begin{aligned}\frac{d\tilde{S}}{dt} &\le C \int_0^\infty \int_0^\infty n(x,t)n(\xi,t)x\xi\,dx\,d\xi \\ &+ \int_0^\infty n(\xi,t)\,d\xi \int_0^\xi B(\xi,x)(x^2 - x\xi)\,dx,\end{aligned}$$

since $K(x,\xi) \le C$. Using Theorem 3.1 and (5.2) (d) we obtain

$$\frac{d\tilde{S}}{dt} \le C - \gamma_2 \int_0^\infty \xi^2 n(\xi,t)\,d\xi = C - \gamma_2 \tilde{S}(t).$$

Consequently

$$\tilde{S}(t) \le \tilde{S}(0)e^{-\gamma_2 t} + \frac{C}{\gamma_2}. \tag{5.8}$$

and (5.6) follows with a constant which is independent of the assumptions in (5.7). Approximating general K, B, n_0 by functions with compact support and applying (5.8) to the corresponding solutions, we obtain, in the limit, the assertion (5.6).

Introduce the function

$$S_2(t) = \int_0^\infty \int_0^\infty xc^2 n(x,c,t)\,dx\,dc. \tag{5.9}$$

Then

$$D(t) = S_2(t) - 2c_\infty M_1 + C_\infty^2 M_0. \tag{5.10}$$

LEMMA 5.3. *The function $S_2(t)$ satisfies:*

$$\frac{dS_2(t)}{dt} \leq 0, \tag{5.11}$$

$$\frac{d^2 S_2(t)}{dt^2} \leq C \tag{5.12}$$

for all $t > 0$, and

$$\frac{dS_2(t)}{dt} \to 0 \quad \textit{if} \quad t \to \infty. \tag{5.13}$$

Proof. Using (4.3) with $f(x,c) = xc^2$ and noting that

$$f\left(x+\xi, \frac{xc+\xi\gamma}{x+\xi}\right) - f(x,c) - f(\xi,\gamma) = -\frac{x\xi}{x+\xi}(c-\gamma)^2, \tag{5.14}$$

we get

$$\frac{dS_2}{dt} = -\frac{1}{2}\int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma \frac{K(x,\xi)}{x+\xi} x\xi (c-\gamma)^2 n(x,c,t) n(\xi,\gamma,t), \tag{5.15}$$

so that $dS_2/dt \leq 0$.

To prove (5.12) we want to apply Theorem 4.1 with

$$f(x,c) = \frac{K(x,\xi)}{x+\xi} x(c-\gamma)^2; \quad \xi, \gamma \text{ are parameters.} \tag{5.16}$$

for simplicity we first take $B = 0$. Then

$$\begin{aligned}\frac{d^2 S_2}{dt^2} &= -\frac{1}{2}\int_0^\infty dx \int_0^\infty d\xi \int_0^\infty dc \int_0^\infty d\gamma \xi n(\xi,\gamma,t) \int_0^\infty d\zeta \int_0^\infty d\sigma K(x,\xi) n(x,c,t)\, n(\xi,\sigma,t) \\ &\quad \times \left[\frac{K(x+\zeta,\xi)}{x+\zeta+\xi}(x+\zeta)\left(\frac{xc+\zeta\sigma}{x+\zeta} - \gamma\right)^2 - \frac{K(x,\xi)}{x+\xi}x(c-\gamma)^2 \right. \\ &\quad \left. - \frac{K(\zeta,\xi)}{\zeta+\xi}\zeta(\sigma-\gamma)^2\right].\end{aligned}$$

Using

$$\frac{xc+\zeta\sigma}{x+\zeta} - \gamma = \frac{x(c-\gamma)+\zeta(\sigma-\gamma)}{x+\zeta}$$

and a calculation similar to (5.14), we find that the expression in brackets is equal to

$$\begin{aligned}\frac{K(x+\zeta,\xi)}{x+\zeta+\xi}\left(\frac{-x\zeta(c-\sigma)^2}{x+\zeta}\right) &+ \left(\frac{K(x+\zeta,\xi)}{x+\zeta+\xi} - \frac{K(x,\xi)}{x+\xi}\right) x(c-\gamma)^2 \\ &+ \left(\frac{K(x+\zeta,\xi)}{x+\zeta+\xi} - \frac{K(\zeta,\xi)}{\zeta+\xi}\right) \zeta(\sigma-\gamma)^2.\end{aligned}$$

By (5.2)(a), $K(x,\zeta)$ times the first term is bounded by $cx\zeta$, and, by (5.2)(a) and (5.2)(b), $K(x,\zeta)$ times each of the remaining two terms is bounded from below by $-Cx\zeta$.

It follows that d^2S_2/dt^2 is bounded from above by

$$C\left(\int_0^\infty\int_0^\infty xn(x,c,t)dxdt\right)^3 \leq \text{const.}$$

If $B \neq 0$ then we have to estimate the corresponding positive contribution to d^2S_2/dt^2:

$$\frac{1}{2}\int_0^\infty dx\int_0^\infty d\xi\int_0^\infty dc\int_0^\infty d\gamma\xi n(\xi,\gamma,t)f(x,c)n(x,c,t)\int_0^x \frac{\zeta}{x}B(x,\zeta)d\zeta \tag{5.17}$$

where $f(x,c)$ is defined as in (5.16). Since $K(x,\xi) \leq C(x+\xi)$, we have $f(x) \leq Cx$. Recalling also the last condition in (1.1) we find that the expression in (5.17) is bounded by

$$C\left(\int_0^\infty\int_0^\infty xn(x,c,t)dxdc\right)^2 \leq \text{const.}$$

This completes the proof of (5.12).

It remains to prove (5.13). If the assertion is not true then there is a sequence $t_n \to \infty$ such that

$$0 \geq S_2'(t_n) \to -\alpha < 0 \text{ if } n \to \infty.$$

Using (5.12) we get

$$S_2(t_n+\epsilon) - S_2(t_n) = \epsilon S_s'(t_n) + \frac{\epsilon^2}{2}S_2''(\tilde{t}_n) \leq -\frac{\alpha\epsilon}{2}$$

if ϵ is positive and small enough. It follows that we have $f(x,c) \leq Cx$.

$$S_2(t_n+\epsilon) \leq S_2(t_1) - \frac{n\epsilon}{2} < 0$$

if n is sufficiently large, a contradiction.

For any $N \gg 1$ introduces the quantities

$$\begin{aligned} M_{0,N} &= \int_0^N dx\int_0^\infty xn(x,c,t)dc, \\ M_{1,N}(x) &= \int_0^N dx\int_0^\infty cxn(x,c,t)dc \end{aligned}$$

and

$$S_{2,N}(t) = \int_0^n dx\int_0^\infty c^2xn(x,c,t)dc.$$

By Lemma 5.2,

$$(5.18)\qquad \begin{cases} |S_2(t) - S_{2,N}(t)| \leq \dfrac{C}{N} \text{ for all } t \geq 0, \\ |M_{0,N} - M_0| + |M_{1,N} - M_1| \leq \dfrac{C}{N}. \end{cases}$$

LEMMA 5.4. *There holds:*

$$(5.19)\qquad \int_0^\infty dx \int_0^\infty dc \int_0^\infty d\xi \int_0^\infty d\gamma x\xi(c-\gamma)^2 n(x,c,t)n(\xi,\gamma,t) = 2\left(S(t)M_0 - M_1^2\right)$$

and, in particular,

$$(5.20)\qquad S(t)M_0 \geq M_1^2.$$

The proof is obvious.

Proof of Theorem 5.1. by (5.15)

$$\frac{dS_2(t)}{dt} = -\frac{1}{2} \iiiint\limits_{x<N,\,\xi<N} dx dc d\xi d\gamma \frac{K(x,\xi)}{x+\xi} x\xi(c-\gamma)^2 n(x,c,t)n(\xi,\gamma,t) + J_N$$

where, since $K(x,\xi) \leq C$,

$$|J_N| \leq C \iiiint\limits_{\xi>N} dx dc d\xi d\gamma x n(x,c,t)n(\xi,\gamma,t) \leq \frac{C}{N^2};$$

the last inequality follows by Lemma 5.2. Using (5.2)(c) we conclude that

$$\frac{dS_2}{dt} \leq -\frac{\gamma_1}{N} \iiiint\limits_{x<N,\xi<N} dx dc d\xi d\gamma x\xi(c-\gamma)^2 n(x,c,t)n(\xi,\gamma,t) + \frac{C}{N^2}.$$

Evaluating the last integral as in (5.19), we get

$$\frac{dS_2}{dt} \leq -\frac{2\gamma_1}{N}\left(S_{2,n}(t)M_{0,N} - M_{1,N}^2\right) + \frac{C}{N^2}$$

so that

$$S_{2.N}(t)M_{0,N} - M_{1,N}^2 \leq -\frac{N}{2\gamma_1}S_2'(t) + \frac{C}{N}.$$

Using (5.18) we deduce that

$$S_2(t)M_0 - M_1^2 \leq -\frac{N}{2\gamma_1}S_2'(t) + \frac{C}{N}.$$

Taking $t \to \infty$ and using (5.13) we find that

$$\overline{\lim_{t\to\infty}}\left(S(t)M_0 - M_1^2\right) \leq \frac{C}{N}$$

Taking $N \to \infty$ and recalling (5.20), we find that

$$S(t)M_0 - M_1^2 \to 0 \quad \text{if } t \to \infty.$$

Consequently, as $t \to \infty$,

$$D(t) \to \frac{M_1^2}{M_0} - 2C_\infty M_1 + C_\infty^2 M_0 = 0, \qquad \text{by (5.10) and (5.4)},$$

and the proof of Theorem 5.1 is complete

Remark 5.5. Dubovskii and Stewart [4] proved existence and uniqueness for (1.2), (1.3) also in case $K(x,\xi)$ is unbounded, provided

$$\begin{aligned} K(x,\xi) &\le C(x+\xi), \\ n_0(x) &\le Ce^{-\lambda x} \quad (\lambda > 0); \end{aligned}$$

The solution is exponentially decreasing, and uniqueness is in the class of exponentially decreasing functions. Their result can be extended to (1.4), (1.5). Suppose now that K also satisfies:

$$K(x,\xi) \ge \gamma_3(x+\xi), \quad \gamma_3 > 0.$$

Then the proof of Theorem 5.1 remains valid without actually splitting integrals by $x \gtrless N$ or $\xi \gtrless N$. Thus we get, in this case,

$$\frac{dS_2}{dt} \le -\gamma_3 \left(S_2(t)M_0 - M_1^2 \right).$$

The non-negative function

$$T(t) = S_2(t) - \frac{M_1^2}{M_0}$$

then satisfies

$$\frac{dT}{dt} + 2\gamma_3 T \le 0$$

so that

$$0 \le T(t) \le T(0)e^{-2\gamma_2 t}.$$

We conclude that, as $t \to \infty$,

$$D(t) \to 0 \text{ exponentially fast.}$$

Note that in this case we do not require the assumption (5.2)(d).

6. Asymptotic behavior of $xn(x,c,t)dxdc$. In this section we add one more assumption on K, B, n_0:

$$K(x,\xi) \le \eta(x+\xi) \text{ where } 2M_0\eta < \gamma_2, \quad \gamma_2 \text{ as in (5.2)(d).} \tag{6.1}$$

THEOREM 6.1. *Let (1.1), (3.7), (5.1), (5.2) and (6.1) hold. Then there exists a non-negative measure $dL(x)$ such that*

$$\lim_{t\to\infty}\int_0^\infty\int_0^\infty g(x,c)xn(x,c,t)dxdc = \int_0^\infty g(x,c_\infty)dL(x) \tag{6.2}$$

for any bounded continuous function $g(x)$.

Introducing the measures

$$d\mu_t(x,c) = xn(x,c,t)dxdc$$

we can restate (6.2) in the form:

$$d\mu_t(x,c) \to \delta(c-c_\infty)dL(x) \text{ as } t\to\infty, \text{ in the sense of weak convergence,} \tag{6.3}$$

where $\delta(y)$ is the Dirac measure.

The proof requires several lemmas.

LEMMA 6.2. *Under the assumptions of Theorem 6.1.*

$$\lim_{t\to\infty}\int_0^\infty g(x)xn(x,t)dx \textit{ exists} \tag{6.4}$$

for any bounded continuous function $g(x)$.

Proof. Consider first the function

$$S_0(t) = \int_0^\infty x^2 n(x,t)dx$$

By (4.13)

$$\begin{aligned}
\frac{dS_0}{dt} &= \int_0^\infty\int_0^\infty K(x,\xi)n(x,t)n(\xi,t)x\xi dxd\xi \\
&+ \int_0^\infty dx\int_x^\infty B(\xi,x)n(\xi,t)(x^2-x\xi)d\xi \\
&\le \eta\int_0^\infty\int_0^\infty (x+\xi)n(x,t)n(\xi,t)x\xi dxd\xi \\
&\qquad + \int_0^\infty n(\xi,t)d\xi\int_0^\infty B(\xi,x)(x^2-x\xi)dx \\
&\le 2\eta S_0(t)M_0 - \gamma_2 S_0(t)
\end{aligned}$$

so that

$$\frac{dS_0}{dt} \le -\delta S_0(t), \quad \delta = \gamma_2 - 2\eta M_0 > 0. \tag{6.5}$$

Suppose now that $g(x)$ is a uniformly Lipschitz continuous function and write

$$\begin{aligned}\int_0^\infty g(x)xn(x,t)dx &= \int_0^\infty (g(x)x + Nx^2)n(x,t)dx \\ &\quad -N\int_0^\infty x^2 n(x,t)dx \equiv \tilde{S}_1(t) - NS_0(t).\end{aligned} \tag{6.6}$$

By (6.5), $S_0(t)$ is monotone decreasing, and hence

$$\lim_{t\to\infty} S_0(t) \text{ exists.} \tag{6.7}$$

Next, by Theorem 4.4 with $f(x) = xg(x)$,

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty g(x)xn(x,t)dx \\ &\le \frac{1}{2}\int_0^\infty dx \int_0^\infty d\xi \eta(x+\xi)n(x,t)n(\xi,t)x\xi \\ &\qquad \times \left[\frac{g(x+\xi)-g(x)}{\xi} + \frac{g(x+\xi)-g(\xi)}{x}\right] \\ &\quad + \int_0^\infty dx \int_x^\infty B(\xi,x)n(\xi,t)x[g(x)-g(\xi)]d\xi.\end{aligned}$$

The first integral on the right-hand side is bounded by

$$C\int_0^\infty x^2 n(x,t)dx \cdot \int_0^\infty \xi n(\xi,t)d\xi$$

and the second integral is bounded by

$$\int_0^\infty n(\xi,t)d\xi \int_0^\xi B(\xi,x)xC|x-\xi|dx \le C\int_0^\infty \xi^2 n(\xi,t)d\xi.$$

Recalling (6.5), it follows that

$$\frac{d\tilde{S}_1}{dt} \le -\delta NS_0(t) + C_0 S_0(t) < 0 \quad \text{if } N > \frac{C_0}{\delta},$$

and consequently

$$\lim_{t\to\infty} \tilde{S}_1(t) \quad \text{exists.}$$

Combining this with (6.7) we deduce from (6.6) that (6.4) holds for any uniformly Lipschitz continuous function $g(x)$.

Consider next the case where $g(x)$ is any bounded continuous function and let g_ϵ be a mollifier of g:

$$|g_\epsilon(x) - g(x)| < \epsilon \text{ and } |g'_\epsilon(x)| \le \frac{C}{\epsilon} \quad \text{for all } x \ge 0.$$

Write

$$\begin{aligned}\int_0^\infty g(x)xn(x,t)dx &= \int_0^\infty g_\epsilon(x)xn(x,t)dx \\ &+ \int_0^\infty (g(x) - g_\epsilon(x))\,xn(x,t)dx \equiv T_1(t) + T_2(t).\end{aligned}$$

Clearly

$$|T_2(t)| < \epsilon.$$

By what was already proved above

$$A_\epsilon \equiv \lim_{t\to\infty} T_1(t) \text{ exists.}$$

Hence

$$\lim_{t\to\infty}\left|\int_0^\infty g(x)xn(x,t)dx - A_\epsilon\right| \le \epsilon.$$

Since the A_ϵ are uniformly bounded, we can take a sequence $\epsilon = \epsilon_m \to 0$ such that $A_{\epsilon_m} \to A$, and obtain

$$\overline{\lim_{t\to\infty}}\left|\int_0^\infty g(x)xn(x,t)dx - A\right| = 0.$$

This completes the proof of Lemma 6.2.

Lemma 6.3. *Under the assumptions of Theorem 6.1,*

$$\lim_{t\to\infty}\int_0^\infty\int_0^\infty g(x,c)xn(x,c,t)dxdc \text{ exists} \tag{6.8}$$

for any bounded continuous function $g(x,c)$.

The proof is similar to the proof of Lemma 6.2 and will be omitted. Note that here we work (cf.(6.6)) with

$$(g(x,c) + Nx^2) \text{ and } -Nx^2.$$

The limit in (6.8) is a bounded linear functional Φ on the space of bounded continuous functions g equipped with the L^∞ norm. By [6, pp. 261–262] we can write

$$\begin{aligned}&\lim_{t\to\infty}\int_0^\infty\int_0^\infty g(x,c)xn(x,c,t)dxdc \\ &= \int\int g(x,c)dN(x,c) \equiv \Phi(g)\end{aligned} \tag{6.9}$$

where dN is a bounded finitely additive regular set function. It will be also useful to view the right-hand side of (6.9) as a distribution, and use the properties

$$
\begin{aligned}
|\Phi(g)| &\leq C|g|_{L^\infty}, \\
\Phi(g) &\geq 0 \ \text{ if } \ g \geq 0.
\end{aligned}
\tag{6.10}
$$

From Theorem 5.1 it follows that

$$\int\int (c - c_\infty)^2 dN(x,c) = 0$$

and therefore also

$$\int\int g(x,c)(c - c_\infty)^2 dN(x,c) = 0 \tag{6.11}$$

for any bounded continuous function $g(x,c)$.

If $g(x,c)$ vanishes in a neighborhood of the line $\{c = c_\infty\}$ then we can apply (6.11) to the function

$$\tilde{g}(x,c) = \begin{cases} \dfrac{g(x,c)}{(c-c_\infty)^2} & \text{on } \operatorname{supp} g \\ 0 & \text{elsewhere} \end{cases}$$

and conclude that

$$\Phi(g) = \int\int \tilde{g}(x,c)(c - c_\infty)^2 dN(x,c) = 0.$$

It follows that the support of the distribution Φ lies in the set $\{c = c_\infty\}$. By a standard result in the theory of distributions (e.g. [7, p. 69, Theorem 31], Φ must then have the form

$$\Phi(g) = \sum_{0 \leq i,j \leq l} \frac{\partial^{i+j}}{\partial c^i \partial x^j} \int g(x, c_\infty) dk_{ij}(x)$$

where the derivatives $\partial^{i+j}/\partial c^i \partial x^j$ are taken in the sense of distributions, and dk_{ij} are measures. In view of (6.10) we must have $l = 0$ and, furthermore, dk_{00} is a non-negative measure. This completes the proof of Theorem 6.1.

The equilibrium equation (1.1) is

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty K(x-\xi,\xi) n_\infty(x-\xi) n_\infty(\xi) d\xi + \int_x^\infty B(\xi,x) n_\infty(\xi) d\xi \\
&-n_\infty(x)\int_0^\infty K(x,\xi) n_\infty(\xi) d\xi - n_\infty(x)\int_0^x \frac{\xi}{x} B(x,\xi) d\xi = 0.
\end{aligned}
\tag{6.12}
$$

If $n_\infty(x)$ is a bounded, non-negative, $L^1(0,\infty)$-solution of (6.12), then the measure $dL(x) = x n_\infty(x) dx$ satisfies:

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty dL(\xi)\int_0^\infty dL(x) K(x,\xi)\left[\frac{g(x+\xi)-g(x)}{\xi} + \frac{g(x+\xi)-g(\xi)}{x}\right] \\
&+\int_0^\infty dL(\xi)\int_0^\xi B(\xi,x)\frac{x}{\xi}(g(x)-g(\xi)) dx = 0
\end{aligned}
\tag{6.13}
$$

for any Lipschitz function g. Indeed, this follows from Theorem 4.4 applied to $f(x) = xg(x)$.

Definition. A bounded non-negative measure $dL(x)$ is called a *weak solution* of (6.12) if (6.13) holds for any continuously differentiable function $g(x)$ with uniformly bounded derivative.

Theorem 6.4. *Under the assumptions of Theorem 6.1 the measure $dL(x)$ is a weak solution of (6.12).*

Proof. We first establish (6.13) in case

$$|g'(x)| + |g''(x)| \leq C, \quad g(x) \text{ monotone increasing.} \tag{6.14}$$

The starting point is the assertion that

$$\frac{d}{dt} \int_0^\infty xg(x)n(x,t) \to 0 \text{ if } t \to \infty. \tag{6.15}$$

To prove it we set

$$\tilde{S}_2(t) = \int_0^\infty xg(x)n(x,t)dx$$

and proceed as in the proof of Lemma 5.3. By Lemma 6.2

$$\lim_{t\to\infty} \tilde{S}_2(t) \text{ exists.} \tag{6.16}$$

We next claim that

$$\frac{d^2\tilde{S}_2(t)}{dt^2} \leq C. \tag{6.17}$$

To prove it we use Theorem 4.4 with $f(x) = xg(x)$ to get:

$$\begin{aligned} \frac{d\tilde{S}_2(t)}{dt} &= \frac{1}{2}\int_0^\infty dx \int_0^\infty d\xi K(x,\xi)x\xi n(x,t)n(\xi,t) \\ &\quad \times \left[\frac{g(x+\xi)-g(x)}{\xi} + \frac{g(x+\xi)-g(\xi)}{x}\right] \\ &\quad + \int_0^\infty dx \int_x^\infty B(\xi,x)n(\xi,t)x(g(x)-g(\xi))d\xi \equiv J_1(t)+J_2(t), \end{aligned} \tag{6.18}$$

and continue to differentiate once more in t Consider first the case $B = 0$. Then we can proceed similarly to the proof of (5.13) with $f(x,c)/x$ in (5.16) replaced by

$$f(x) = K(x,\xi)\left[\frac{g(x+\xi)-g(x)}{\xi} + \frac{g(x+\xi)-g(\xi)}{x}\right]$$

where ξ is viewed as a parameter. Using (5.2)(b) and the assumptions on g in (6.14) we easily deduce that

$$\frac{d^2\tilde{S}_2}{dt^2} \leq C.$$

If $B(x,\xi) \not\equiv 0$ then the additional terms in $d^2\check{S}_2/dt^2$ are also bounded, as can easily be shown.

Having proved (6.16), (6.17), we can now quickly prove (6.15). Indeed, if (6.15) is not true then there is a sequence $t_n \to \infty$ such that

$$\beta_n \equiv \frac{d\check{S}_2(t_n)}{dt} \to \beta \neq 0 \text{ if } n \to \infty.$$

It follows that

$$\begin{aligned} \check{S}_2(t_n - \beta_n\epsilon) - \check{S}_2(t_n) &= -\beta_n^2\epsilon + \frac{1}{2}\beta_n^2\epsilon^2\check{S}_2''(\tilde{t}_n) \\ &\leq -\beta_n^2\epsilon + C\epsilon^2 < -\frac{1}{2}\beta^2\epsilon \end{aligned}$$

if ϵ is positive and small and n is large; this is a contradiction to (6.16).

So far we have proved that the left-hand side of (6.18) converges to zero as $t \to \infty$. Next we evaluate the right-hand side of (6.18) as $t \to \infty$. By Theorem 6.1

$$\begin{aligned} J_2(t) &= \int_0^\infty \xi n(\xi,t)d\xi \left[\frac{1}{\xi}\int_0^\xi B(\xi,x)x(g(x)-g(\xi))dx\right] \\ &\to \int_0^\infty dL(\xi)\int_0^\xi B(\xi,x)\frac{x}{\xi}(g(x)-g(\xi))dx. \end{aligned} \tag{6.19}$$

To evaluate $J_1(t)$ introduce the functions

$$\begin{aligned} F(x,t) &= \int_0^\infty d\xi K(x,\xi)\xi n(\xi,t)\frac{g(x+\xi)-g(x)}{\xi}, \\ F_\infty(x) &= \int_0^\infty dL(\xi)K(x,\xi)\frac{g(x+\xi)-g(x)}{\xi}. \end{aligned}$$

The family of functions $\{F(x,t), t > 0\}$ is uniformly bounded and equicontinuous, and, by Theorem 6.1,

$$F(x,t) \to F_\infty(x) \text{ if } t \to \infty.$$

It follows that the convergence is uniform in any boundary interval $0 \leq x \leq R$. Using also Lemma 5.2, we easily deduce that

$$\int_0^\infty xn(x,t)\left[F(x,t) - F_\infty(x)\right]dx \to 0 \text{ if } n \to \infty.$$

Since, by Theorem 6.1,

$$\int_0^\infty dxn(x,t)F_\infty(x) \to \int_0^\infty dL(x)F_\infty(x)$$

we conclude that

$$\int_0^\infty xn(x,t)F(x,t)dx \to \int_0^\infty dL(x)F_\infty(x)$$

if $t \to \infty$. Combining this result with (6.19), we deduce that the right-hand side of (6.18) converges to the left-hand side of (6.13), and since the left-hand side of (6.18) converges to zero, the equality (6.13) follows.

So far we have assumed that g satisfies (6.14). However by approximation we can establish (6.13) for any monotone increasing g with first bounded and continuous derivative. Finally, if g is not monotone increasing, we can decompose it into a difference of two monotone functions, and this completes the proof of Theorem 6.4.

Remark 6.1. If $n_0(x)$ is a stationary solution then $n(x,t) = n_0(x)$ and so

$$\int_0^\infty n(x,c,t)dc = n_0(x).$$

It follows that

$$\int_0^\infty \int_0^\infty g(x)xn(x,c,t)dxdc = \int_0^\infty g(x)xn_0(x)dx$$

and

$$xn(x,c,t)dxdc \to \delta(c-c_\infty)(xn_0(x)dx).$$

Remark 6.2. It was observed by Carr [2] that if K and B are related by

$$K(x,y)Q(x)Q(y) = B(x+y,y)Q(x+y)$$

where Q is a positive function, then

$$n_\infty(x) = e^{\lambda x}Q(x)$$

is an equilibrium solution. Dubovskii and Stewart [5] proved that if $K(x,y)$ and $B(x,y)$ are linear functions then $n(x,t)$ converges to an equilibrium solution $n_\infty(x)$; their methods are entirely different from the methods of the present paper.

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