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Asymptotic Behavior of Solutions of Even-Order Differential Equations with Several Delays

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Abstract: The higher-order delay differential equations are used in the describing of many natural phenomena. This work investigates the asymptotic properties of the class of even-order differential equations with several delays. Our main concern revolves around how to simplify and improve the oscillation parameters of the studied equation. For this, we use an improved approach to obtain new properties of the positive solutions of these equations.

Keywords: delay differential equation; even order; sufficient conditions; noncanonical case

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1. Introduction

This work investigates the asymptotic and oscillatory properties of solutions of delay differential equation (DDE) of even-order

$$\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)' + \sum_{i=1}^{L} h_i(\mathfrak{k}) f(s(\lambda_i(\mathfrak{k}))) = 0,\ \mathfrak{k} \geq \mathfrak{k}_0, \tag{1}$$

Throughout this study, we assume $\gamma \in \mathbb{Q}^+$ is a ratio of odd numbers, m and L are positive integers, $m \geq 4$ is even, $\beta, h_i \in C^1(I_0)$, $\beta(\mathfrak{k}) > 0$, $h_i(\mathfrak{k}) \geq 0$, $\beta'(\mathfrak{k}) \geq 0$, $\lambda_i \in C(I_0)$, $\lambda_i(\mathfrak{k}) \leq \mathfrak{k}$, $\lambda_i'(\mathfrak{k}) > 0$, $\lim_{\mathfrak{k}\to\infty} \lambda_i(\mathfrak{k}) = \infty$, $I_\vartheta := [\mathfrak{k}_\vartheta, \infty)$, $f \in C(\mathbb{R}, \mathbb{R})$ and $f(s) \geq s^\gamma$ for $s \neq 0$.

By a solution of (1) we denote to a function s in $C^{m-1}([\mathfrak{k}_*, \infty))$ for some $\mathfrak{k}_* \geq \mathfrak{k}_0$, which $\left(\beta \cdot \left(s^{(m-1)}\right)^{\gamma}\right) \in C^1([\mathfrak{k}_*, \infty))$ and s satisfies (1) on $[\mathfrak{k}_*, \infty)$. Moreover, we suppose $\sup\{|s(\varrho)| : \varrho \geq \mathfrak{k}_1\} > 0$ for every $\mathfrak{k}_1$ in $[\mathfrak{k}_*, \infty)$, and

$$\delta_0(\mathfrak{k}_0) := \int_{\mathfrak{k}_0}^{\infty} \frac{1}{\beta^{1/\gamma}(v)} dv < \infty. \tag{2}$$

A solution s of (1) is said to be nonoscillatory if it is eventually positive or eventually negative; otherwise, it is said to be oscillatory.

Delay differential equations as one of the branches of functional differential equations appear when modeling several phenomena in different branches of science, see Hale [1], Arino et al. [2], and Rihan [3]. In mathematical models of basic and applied sciences phenomena, even-order differential equations are frequently encountered. Elasticity difficulties, structural deformation, and soil settling are examples of applications; see [4].

The study of second-order DDEs and their properties has always been a subject of continuous interest by researchers. For more information about the oscillation neutral DDEs of second-order. In [5], Bohner et al. investigated the oscillatory properties of the class of second-order DDE of neutral type. They improved and simplified the results of

Agarwal et al. [6] and Han et al. [7]. The results in [8–12], recently, also contributed to the development of the study of qualitative behavior of DDEs of second-order.

Although higher-order equations are important, higher-order DDEs have not received as much attention as in the case of second-order DDEs. Since 2011, a research movement focused on the study the asymptotic behavior of DDEs of even-order in the noncanonical case (2). Zhang et al. [13–15] established criteria to ensure the oscillation of solutions of a class of DDEs of even-order. Using a different approach, Baculikova et al. [16] studied the asymptotic behavior of even-order DDEs in the canonical and noncanonical cases. For more interesting, very recently, results about oscillation of higher-order DDEs, see [17–19].

In this paper, we obtain the oscillatory properties of the even-order DDE with several delays (1). We extend Bohner's results in [5] to higher order equations in order to improve and simplify previous results in the literature.

Lemma 1. *[20] (Lemma 1.1) Let $v \in C^m(I_0,(0,\infty))$ and $v^{(n)}$ be eventually of one sign for all large $\mathfrak{k}$. Then, there is an integer $a \in [0,m]$, with $m+a$ even for $v^{(m)} \geq 0$, or $m+a$ odd for $v^{(m)} \leq 0$ such that*

$$a > 0 \text{ yields } v^{(\iota)}(\mathfrak{k}) > 0 \text{ for } \iota = 0,1,...,a-1$$

and

$$a \leq m-1 \text{ yields } (-1)^{a+\iota} v^{(\iota)}(\mathfrak{k}) > 0 \text{ for } \iota = a, a+1, ..., m-1,$$

eventually.

Lemma 2. *[21] (Lemma 2.2.3) Assume that $\nu \in C^m(I_0,(0,\infty))$, $\nu^{(m)}$ is of fixed sign and not identically zero on a subray of I_0, and that there is a $\mathfrak{k}_1 \in I_0$ with $\nu^{(m-1)}(\mathfrak{k})\nu^{(m)}(\mathfrak{k}) \leq 0$ for $\mathfrak{k} \in I_1$. If $\lim_{\mathfrak{k}\to\infty} \nu(\mathfrak{k}) \neq 0$, then*

$$\nu \geq \frac{\alpha}{(m-1)!}\mathfrak{k}^{m-1}\left|\nu^{(m-1)}\right|,$$

for every $\alpha \in (0,1)$ and $\mathfrak{k} \in I_\alpha$, $\mathfrak{k}_\alpha \geq \mathfrak{k}_1$.

2. Preliminaries

Let us define

$$\delta_l(\mathfrak{k}) := \int_{\mathfrak{k}}^{\infty} \delta_{l-1}(v)\mathrm{d}v, \text{ for } l = 1,2,...,m-2$$

and

$$\lambda(\mathfrak{k}) := \min\{\lambda_i(\mathfrak{k}) : i = 1,2,...,m\}.$$

Also, our results require the condition

$$\delta_m(\mathfrak{k}) < \infty \text{ for } m = 0,1,...,m-2. \tag{3}$$

Lemma 3. *Let $s \in C(I_0,(0,\infty))$ be a solution of (1). Then $\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)' \leq 0$, and s satisfies and its derivatives satisfy one of the following cases, eventually,*

$$(1)\ s'(\mathfrak{k}) > 0,\ s^{(m-1)}(\mathfrak{k}) > 0,\ s^{(m)}(\mathfrak{k}) \leq 0;$$
$$(2)\ s'(\mathfrak{k}) > 0,\ s^{(m-2)}(\mathfrak{k}) > 0,\ s^{(m-1)}(\mathfrak{k}) < 0;$$
$$(3)\ s'(\mathfrak{k}) < 0,\ s^{(m-2)}(\mathfrak{k}) > 0,\ s^{(m-1)}(\mathfrak{k}) < 0.$$

Proof. Let $s \in C(I_0,(0,\infty))$ be a solution of (1). From (1), we have

$$\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)' \leq -\sum_{i=1}^{L} h_i(\mathfrak{k}) s^{\gamma}(\lambda_i(\mathfrak{k})) \leq 0.$$

From (1) and Lemma 1, we get the three possible cases (1), (2) and (3) for $\mathfrak{k} \geq \mathfrak{k}_1$, $\mathfrak{k}_1$ large enough. The proof is complete. □

Theorem 1. *Let $s \in C(I_0,(0,\infty))$ be a solution of (1). If*

$$\limsup_{\mathfrak{k}\to\infty} \int_{\mathfrak{k}_1}^{\mathfrak{k}} \left(\frac{1}{\beta^{1/\gamma}(\varrho)} \left(\int_{\mathfrak{k}_1}^{\varrho} \sum_{i=1}^{L} h_i(v)\delta_{m-2}^{\gamma}(\lambda_i(v))\mathrm{d}v \right)^{1/\gamma} \right) \mathrm{d}\varrho = \infty, \tag{4}$$

then satisfies Case (2) *of Lemma 3.*

Proof. Let $s \in C(I_0,(0,\infty))$ be a solution of (1). From Lemma 3, we have the cases (1)–(3). First, we assume that Case (3) of Lemma 3 holds on I_1. Since $\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)' \le 0$, we have

$$\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma} \le \beta(\mathfrak{k}_1)\left(s^{(m-1)}(\mathfrak{k}_1)\right)^{\gamma} := -M < 0, \tag{5}$$

which is

$$\beta^{1/\gamma}(\mathfrak{k})s^{(m-1)}(\mathfrak{k}) \le (-M)^{1/\gamma} = -M^{1/\gamma}, \tag{6}$$

since γ is a ratio of two odd integers. If we divide (6) by $\beta^{1/\gamma}$ and integrating from $\mathfrak{k}$ to ϱ, we get

$$s^{(m-2)}(\varrho) \le s^{(m-2)}(\mathfrak{k}) - M^{1/\gamma} \int_{\mathfrak{k}}^{\varrho} \frac{1}{\beta^{1/\gamma}(v)}\mathrm{d}v.$$

Letting $\varrho \to \infty$, we get

$$0 \le s^{(m-2)}(\mathfrak{k}) - M^{1/\gamma}\delta_0(\mathfrak{k}). \tag{7}$$

Integrating (7) $(m-2)$ times from $\mathfrak{k}$ to ∞, we obtain

$$s'(\mathfrak{k}) \le -M^{1/\gamma}\delta_{m-3}(\mathfrak{k}), \tag{8}$$

and

$$s(\mathfrak{k}) \ge M^{1/\gamma}\delta_{m-2}(\mathfrak{k}). \tag{9}$$

From (1) and (9), we have

$$\begin{aligned} (\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma})' &\le -\sum_{i=1}^{L} h_i(\mathfrak{k})s^{\gamma}(\lambda_i(\mathfrak{k})) \\ &\le -M\sum_{i=1}^{L} h_i(\mathfrak{k})\delta_{m-2}^{\gamma}(\lambda_i(\mathfrak{k})). \end{aligned} \tag{10}$$

Integrating (10) from $\mathfrak{k}_1$ to $\mathfrak{k}$, we obtain

$$\begin{aligned} \beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma} &\le \beta(\mathfrak{k}_1)(s^{(m-1)}(\mathfrak{k}_1))^{\gamma} - M\int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v)\delta_{m-2}^{\gamma}(\lambda_i(v))\mathrm{d}v \\ &\le -M\int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v)\delta_{m-2}^{\gamma}(\lambda_i(v))\mathrm{d}v. \end{aligned} \tag{11}$$

Integrating (11) from $\mathfrak{k}_1$ to $\mathfrak{k}$, we get

$$s^{(m-2)}(\mathfrak{k}) \le s^{(m-2)}(\mathfrak{k}_1) - M^{1/\gamma}\int_{\mathfrak{k}_1}^{\mathfrak{k}} \left(\frac{1}{\beta(\varrho)} \int_{\mathfrak{k}_1}^{\varrho} \sum_{i=1}^{L} h_i(v)\delta_{m-2}^{\gamma}(\lambda_i(v))\mathrm{d}v \right)^{1/\gamma} \mathrm{d}\varrho.$$

At $\mathfrak{k} \to \infty$, we get a contradiction with (4).

Now, let Case (1) of Lemma 3 holds on I_1. Also, we find from (4) and (2) that $\int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v)\delta_{m-2}^{\gamma}(\lambda_i(v))dv$ must be unbounded. Moreover, since $\delta'_{m-2}(\mathfrak{k}) < 0$, it is easy to see that

$$\int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v)\mathrm{d}v \to \infty \text{ as } \mathfrak{k} \to \infty. \tag{12}$$

Integrating (1) from $\mathfrak{k}_2$ to $\mathfrak{k}$, we get

$$\begin{aligned}\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma} &\leq \beta(\mathfrak{k}_2)(s^{(m-1)}(\mathfrak{k}_2))^{\gamma} - \int_{\mathfrak{k}_2}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) s^{\gamma}(\lambda_i(v)) \mathrm{d}v \\ &\leq \beta(\mathfrak{k}_2)(s^{(m-1)}(\mathfrak{k}_2))^{\gamma} - \int_{\mathfrak{k}_2}^{\mathfrak{k}} s^{\gamma}(\lambda(v)) \sum_{i=1}^{L} h_i(v) \mathrm{d}v \\ &\leq \beta(\mathfrak{k}_2)(s^{(m-1)}(\mathfrak{k}_2))^{\gamma} - s^{\gamma}(\lambda(\mathfrak{k}_2)) \int_{\mathfrak{k}_2}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) \mathrm{d}v,\end{aligned}$$

which in view of (12) contradicts to the positivity of $s^{(m-1)}$ as $\mathfrak{k} \to \infty$. The proof is complete. □

Theorem 2. *Let $s \in C(I_0, (0, \infty))$ be a solution of (1). If*

$$\limsup_{\mathfrak{k}\to\infty} \delta_{m-2}^{\gamma}(\mathfrak{k}) \int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) \mathrm{d}v > 1, \tag{13}$$

then s satisfies Case (2) *of Lemma 3.*

Proof. Let $s \in C(I_0, (0, \infty))$ be a solution of (1). We have from Lemma 3, the cases (1)–(3) for s and its derivatives.

First, we suppose that Case (3) of Lemma 3 holds on I_1. Then,

$$s^{(m-2)}(\mathfrak{k}) \geq -\int_{\mathfrak{k}}^{\infty} \beta^{-1/\gamma}(v)\beta^{1/\gamma}(v)s^{(m-1)}(v)\mathrm{d}v \geq -\beta^{1/\gamma}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})\delta_0(\mathfrak{k}). \tag{14}$$

Integrating (14) $(m-4)$ times from $\mathfrak{k}$ to ∞, we arrive at

$$s'(\mathfrak{k}) \leq \beta^{1/\gamma}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})\delta_{m-3}(\mathfrak{k}). \tag{15}$$

and

$$s(\mathfrak{k}) \geq -\beta^{1/\gamma}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})\delta_{m-2}(\mathfrak{k}). \tag{16}$$

Integrating (1) from $\mathfrak{k}_1$ to $\mathfrak{k}$, we get

$$\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma} \leq \beta(\mathfrak{k}_1)(s^{(m-1)}(\mathfrak{k}_1))^{\gamma} - \int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) s^{\gamma}(\lambda_i(v)) \mathrm{d}v,$$

since $\lambda'(\mathfrak{k}) > 0$, and $v \leq \mathfrak{k}$, we obtain

$$\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma} \leq -s^{\gamma}(\lambda_i(\mathfrak{k})) \int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) \mathrm{d}v. \tag{17}$$

Since $\lambda(\mathfrak{k}) \leq \mathfrak{k}$, we have

$$\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma} \leq -s^{\gamma}(\mathfrak{k}) \int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) \mathrm{d}v. \tag{18}$$

From (16) and (18), we arrive at

$$\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma} \leq \beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma} \delta_{m-2}^{\gamma}(\mathfrak{k}) \int_{\mathfrak{k}_1}^{\mathfrak{k}} \sum_{i=1}^{L} h_i(v) \mathrm{d}v. \tag{19}$$

Divide both sides of inequality (19) by $\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma}$ and taking the lim sup, we get

$$\limsup_{\mathfrak{k}\to\infty}\delta_{m-2}^{\gamma}(\mathfrak{k})\int_{\mathfrak{k}_1}^{\mathfrak{k}}\sum_{i=1}^{L}h_i(v)\mathrm{d}v\le 1,$$

which contradicts with (13).

Next, let Case (1) of Lemma 3 holds on I_1. Since $\delta_{m-2}(\mathfrak{k})<\infty$, we have, from (13), that (12) holds. Then, we continue the proof as in Theorem 1. Therefore the proof is complete. □

3. Oscillation Criteria

Lemma 4. *Suppose that s satisfies Case (2) of Lemma 3. If there is a $\alpha\in(0,1)$ such that*

$$\int_{\mathfrak{k}_0}^{\infty}\left(\frac{1}{\beta(v)}\int_{\mathfrak{k}_1}^{v}\sum_{i=1}^{L}h_i(v)\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(v)\right)^{\gamma}\mathrm{d}v\right)^{1/\gamma}\mathrm{d}v=\infty, \tag{20}$$

then $\lim_{\mathfrak{k}\to\infty}s^{(m-2)}(\mathfrak{k})=0$.

Proof. Assume that $s(\mathfrak{k})$ is a positive solution of (1), and satisfies Case (2) of Lemma 3. since $s^{(m-2)}(\mathfrak{k})>0$ and $s^{(m-1)}(\mathfrak{k})<0$, thus, we obtain that $\lim_{\mathfrak{k}\to\infty}s^{(m-2)}(\mathfrak{k})=c\ge 0$. We claim that $\lim_{\mathfrak{k}\to\infty}s^{(m-2)}(\mathfrak{k})=0$. Suppose the contrary that $c>0$. Thus, there exists a $\mathfrak{k}_1\ge\mathfrak{k}_0$ such that

$$s^{(m-2)}(\lambda_i(\mathfrak{k}))\ge c \text{ for } \mathfrak{k}\ge\mathfrak{k}_1. \tag{21}$$

From (1), we have

$$\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)'\le-\sum_{i=1}^{L}h_i(\mathfrak{k})s^{\gamma}(\lambda_i(\mathfrak{k})). \tag{22}$$

Using Lemma 2 and the fact that s is a positive increasing function, we get

$$s(\mathfrak{k})\ge\frac{\alpha}{(m-2)!}\mathfrak{k}^{m-2}s^{(m-2)}(\mathfrak{k}), \tag{23}$$

using (23), (22) becomes

$$\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)'\le-\sum_{i=1}^{L}h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(\mathfrak{k})\right)^{\gamma}\left(s^{(m-2)}(\lambda_i(\mathfrak{k}))\right)^{\gamma}, \tag{24}$$

from (21), we get

$$\left(\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma}\right)'\le-c^{\gamma}\sum_{i=1}^{L}h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(\mathfrak{k})\right)^{\gamma}, \tag{25}$$

for $\mathfrak{k}\ge\mathfrak{k}_1$. Integrating (25) twice from $\mathfrak{k}_1$ to $\mathfrak{k}$, we obtain

$$s^{(m-1)}(\mathfrak{k})\le-c\left(\frac{1}{\beta(\mathfrak{k})}\int_{\mathfrak{k}_1}^{\mathfrak{k}}\sum_{i=1}^{L}h_i(v)\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(v)\right)^{\gamma}\mathrm{d}v\right)^{1/\gamma}$$

and

$$s^{(m-2)}(\mathfrak{k})\le s^{(m-2)}(\mathfrak{k}_1)-c\int_{\mathfrak{k}_1}^{\mathfrak{k}}\left(\frac{1}{\beta(v)}\int_{\mathfrak{k}_1}^{v}\sum_{i=1}^{L}h_i(v)\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(v)\right)^{\gamma}\mathrm{d}v\right)^{1/\gamma}\mathrm{d}v.$$

Letting $\mathfrak{k}\to\infty$ and using (20), we obtain that $\lim_{\mathfrak{k}\to\infty}s^{(m-2)}(\mathfrak{k})=-\infty$, which contradicts $s^{(m-2)}(\mathfrak{k})>0$. Thus, the proof is complete. □

Lemma 5. *Suppose that (20) holds, $s(\mathfrak{k}) \in C(I_0,(0,\infty))$ is a solution of (1). If s satisfies Case (2) of Lemma 3, and there is a $\mu \geq 0$ with*

$$\frac{\delta_{m-2}(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})}\left(\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v\right)^{1/\gamma} \geq \mu, \tag{26}$$

for some $\alpha \in (0,1)$, then

$$\left(\frac{s^{(m-2)}(\mathfrak{k})}{\delta^{\mu}_{m-2}(\mathfrak{k})}\right)' \leq 0, \tag{27}$$

where

$$G_i(v) = h_i(v)\left(\frac{\alpha\lambda_i^{m-2}(v)}{(m-2)!}\right)^{\gamma}.$$

Proof. Assume that (1) has a positive solution $s(\mathfrak{k})$ and satisfies Case (2) of Lemma 3. From Lemma 4, we arrive at (24). Integrating (24) from $\mathfrak{k}_1$ to $\mathfrak{k}$, we find

$$\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma} - \beta(\mathfrak{k}_1)\left(s^{(m-1)}(\mathfrak{k}_1)\right)^{\gamma} \leq -\int_{\mathfrak{k}_1}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\left(s^{(m-2)}(\lambda_i(v))\right)^{\gamma}\mathrm{d}v,$$

since $\lambda'(\mathfrak{k}) > 0$, and $v \leq \mathfrak{k}$, we obtain

$$\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma} - \beta(\mathfrak{k}_1)\left(s^{(m-1)}(\mathfrak{k}_1)\right)^{\gamma} \leq -\left(s^{(m-2)}(\lambda_i(\mathfrak{k}))\right)^{\gamma}\int_{\mathfrak{k}_1}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v,$$

and so

$$\begin{aligned}\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma} \leq\;& \beta(\mathfrak{k}_1)\left(s^{(m-1)}(\mathfrak{k}_1)\right)^{\gamma} - \left(s^{(m-2)}(\lambda_i(\mathfrak{k}))\right)^{\gamma}\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v\\ &+\left(s^{(m-2)}(\lambda_i(\mathfrak{k}))\right)^{\gamma}\int_{\mathfrak{k}_0}^{\mathfrak{k}_1}\sum_{i=1}^{L}G_i(v)\mathrm{d}v.\end{aligned} \tag{28}$$

Using Lemma 4, we get that $\lim_{\mathfrak{k}\to\infty} s^{(m-2)}(\mathfrak{k}) = 0$. Thus, there is a $\mathfrak{k}_2 \geq \mathfrak{k}_1$ such that

$$\beta(\mathfrak{k}_1)\left(s^{(m-1)}(\mathfrak{k}_1)\right)^{\gamma} + \left(s^{(m-2)}(\lambda_i(\mathfrak{k}))\right)^{\gamma}\int_{\mathfrak{k}_0}^{\mathfrak{k}_1}\sum_{i=1}^{L}G_i(v)\mathrm{d}v < 0, \text{ for every } \mathfrak{k} \geq \mathfrak{k}_2,$$

thus (28) becomes

$$\begin{aligned}\beta(\mathfrak{k})\left(s^{(m-1)}(\mathfrak{k})\right)^{\gamma} &\leq -\left(s^{(m-2)}(\lambda_i(\mathfrak{k}))\right)^{\gamma}\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v\\ &\leq -\left(s^{(m-2)}(\mathfrak{k})\right)^{\gamma}\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v,\end{aligned} \tag{29}$$

and so

$$s^{(m-1)}(\mathfrak{k}) \leq -\frac{s^{(m-2)}(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})}\left(\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v\right)^{1/\gamma}.$$

Next, we have that

$$\left(\frac{s^{(m-2)}(\mathfrak{k})}{\delta^{\mu}_{m-2}(\mathfrak{k})}\right)' = \frac{\delta^{\mu}_{m-2}(\mathfrak{k})s^{(m-1)}(\mathfrak{k}) + \mu\delta^{\mu-1}_{m-2}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})s^{(m-2)}(\mathfrak{k})}{\delta^{2\mu}_{m-2}(\mathfrak{k})}. \tag{30}$$

This implies

$$\begin{aligned}
&\delta^{\mu}_{m-2}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})+\mu\delta^{\mu-1}_{m-2}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})s^{(m-2)}(\mathfrak{k})\\
&\quad\leq -\delta^{\mu}_{m-2}(\mathfrak{k})\frac{s^{(m-2)}(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})}\left(\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v\right)^{1/\gamma}+\mu\delta^{\mu-1}_{m-2}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})s^{(m-2)}(\mathfrak{k})\\
&\quad\leq\left(\frac{-\delta_{m-2}(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})}\left(\int_{\mathfrak{k}_0}^{\mathfrak{k}}\sum_{i=1}^{L}G_i(v)\mathrm{d}v\right)^{1/\gamma}+\mu\right)\delta^{\mu-1}_{m-2}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})s^{(m-2)}(\mathfrak{k}).
\end{aligned}$$

It follows from (26) that $\delta^{\mu}_{m-2}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})+\mu\delta^{\mu-1}_{m-2}(\mathfrak{k})\delta_{m-3}(\mathfrak{k})s^{(m-2)}(\mathfrak{k})\leq 0$, which, with (30), implies the function $s^{(m-2)}(\mathfrak{k})/\delta^{\mu}_{m-2}(\mathfrak{k})$ is nonincreasing. This completes the proof. □

Theorem 3. *Assume that (20) and (26) hold, $s(\mathfrak{k})\in C(I_0,(0,\infty))$ is a solution of (1) and $\gamma\geq 1$. If there exists a positive function $\eta(\mathfrak{k})\in C^1[\mathfrak{k}_0,\infty)$ such that*

$$\limsup_{\mathfrak{k}\to\infty}\int_{\mathfrak{k}_0}^{\mathfrak{k}}\left(W(v)-\frac{\beta(v)\eta(v)}{(\gamma+1)^{(\gamma+1)}}\left(\frac{\eta'(v)}{\eta(v)}+\frac{1+\gamma}{\beta^{1/\gamma}(v)\delta(v)}\right)^{\gamma+1}\right)\mathrm{d}v=\infty, \tag{31}$$

where

$$W(\mathfrak{k}):=\eta(\mathfrak{k})\sum_{i=1}^{L}h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(\mathfrak{k})\right)^{\gamma}\frac{\delta^{\gamma\mu}_{m-2}(\lambda_i(\mathfrak{k}))}{\delta^{\gamma\mu}_{m-2}(\mathfrak{k})}+(1-\gamma)\frac{\eta(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta^{\gamma+1}(\mathfrak{k})},$$

for some $\alpha\in(0,1)$, then Case 2 does not satisfied.

Proof. Assume the contrary that (1) has a positive solution $s(\mathfrak{k})$ and satisfies Case (2) of Lemma 3. Noting that $\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma}$ is non-increasing, we have

$$\begin{aligned}
s^{(m-2)}(\nu)-s^{(m-2)}(\mathfrak{k}) &= \int_{\mathfrak{k}}^{\nu}\frac{1}{\beta^{1/\gamma}(v)}\left(\beta(v)\left(s^{(m-1)}(v)\right)^{\gamma}\right)^{1/\gamma}\mathrm{d}v\\
&\leq \beta^{1/\gamma}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})\int_{\mathfrak{k}}^{\nu}\frac{1}{\beta^{1/\gamma}(v)}\mathrm{d}v.
\end{aligned}$$

Letting $\nu\to\infty$, we get

$$-s^{(m-2)}(\mathfrak{k})\leq\beta^{1/\gamma}(\mathfrak{k})s^{(m-1)}(\mathfrak{k})\delta(\mathfrak{k}). \tag{32}$$

Define the function $\omega(\mathfrak{k})$ by

$$\omega(\mathfrak{k}):=\eta(\mathfrak{k})\left(\frac{\beta(\mathfrak{k})(s^{(m-1)}(\mathfrak{k}))^{\gamma}}{(s^{(m-2)}(\mathfrak{k}))^{\gamma}}+\frac{1}{\delta^{\gamma}(\mathfrak{k})}\right). \tag{33}$$

From (32), we have $\omega(\mathfrak{k})>0$ for $\mathfrak{k}\geq\mathfrak{k}_1$. Differentiating (33), we obtain

$$\omega'=\frac{\eta'}{\eta}\omega+\eta\left(\frac{(\beta(s^{(m-1)})^{\gamma})'}{(s^{(m-2)})^{\gamma}}-\frac{\gamma\beta(s^{(m-1)})^{\gamma+1}}{(s^{(m-2)})^{\gamma+1}}-\frac{\gamma\delta'}{\delta^{\gamma+1}}\right),$$

which follows from (1) and (33) that

$$\begin{aligned}
\omega' \leq\ & \frac{\eta'}{\eta}\omega-\frac{\eta\sum_{i=1}^{L}h_i s^{\gamma}(\lambda_i)}{(s^{(m-2)}(\mathfrak{k}))^{\gamma}}-\frac{\gamma\eta(\mathfrak{k})}{\beta^{1/\gamma}}\left(\frac{\omega}{\eta}-\frac{1}{\delta^{\gamma}}\right)^{(\gamma+1)/\gamma}\\
&+\frac{\gamma\eta}{\beta^{1/\gamma}\delta^{\gamma+1}}.
\end{aligned} \tag{34}$$

From (23) and (34), we have

$$\begin{aligned}\omega' &\leq \frac{\eta'}{\eta}\omega(\mathfrak{k}) - \frac{\eta}{(s^{(m-2)})^{\gamma}}\sum_{i=1}^{L} h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}\right)^{\gamma}\left(s^{(m-2)}(\lambda_i)\right)^{\gamma} \\ &\quad -\frac{\gamma\eta}{\beta^{1/\gamma}}\left(\frac{\omega}{\eta}-\frac{1}{\delta^{\gamma}}\right)^{(\gamma+1)/\gamma} + \frac{\gamma\eta}{\beta^{1/\gamma}\delta^{\gamma+1}},\end{aligned}$$

using (27), we get

$$\begin{aligned}\omega' &\leq \frac{\eta'}{\eta}\omega - \frac{\eta}{(s^{(m-2)})^{\gamma}}\sum_{i=1}^{L} h_i\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}\right)^{\gamma}\frac{(s^{(m-2)})^{\gamma}\delta_{m-2}^{\gamma\mu}(\lambda_i)}{\delta_{m-2}^{\gamma\mu}} \\ &\quad -\frac{\gamma\eta}{\beta^{1/\gamma}}\left(\frac{\omega}{\eta}-\frac{1}{\delta^{\gamma}}\right)^{(\gamma+1)/\gamma} + \frac{\gamma\eta}{\beta^{1/\gamma}\delta^{\gamma+1}},\end{aligned}$$

that is

$$\begin{aligned}\omega' &\leq \frac{\eta'}{\eta}\omega - \eta\sum_{i=1}^{L} h_i\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}\right)^{\gamma}\frac{\delta_{m-2}^{\gamma\mu}(\lambda_i)}{\delta_{m-2}^{\gamma\mu}} \\ &\quad -\frac{\gamma\eta}{\beta^{1/\gamma}}\left(\frac{\omega}{\eta}-\frac{1}{\delta^{\gamma}}\right)^{(\gamma+1)/\gamma} + \frac{\gamma\eta}{\beta^{1/\gamma}\delta^{\gamma+1}}.\end{aligned} \tag{35}$$

Using the inequality

$$v_1^{(\gamma+1)/\gamma} - (v_1-v_2)^{(\gamma+1)/\gamma} \leq \frac{v_2^{1/\gamma}}{\gamma}[(1+\gamma)v_1 - v_2],\ v_1v_2 \geq 0,$$

with $v_1 = \omega/\eta,\ v_2 = 1/\delta^{\gamma}$, we obtain

$$\begin{aligned}\omega' &\leq \frac{\eta'(\mathfrak{k})}{\eta(\mathfrak{k})}\omega(\mathfrak{k}) - \eta(\mathfrak{k})\sum_{i=1}^{L} h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(\mathfrak{k})\right)^{\gamma}\frac{\delta_{m-2}^{\gamma\mu}(\lambda_i(\mathfrak{k}))}{\delta_{m-2}^{\gamma\mu}(\mathfrak{k})} + \frac{\gamma\eta(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta^{\gamma+1}(\mathfrak{k})} \\ &\quad -\frac{\gamma\eta(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})}\left((\frac{\omega(\mathfrak{k})}{\eta(\mathfrak{k})})^{(\gamma+1)/\gamma} - \frac{1}{\gamma\delta(\mathfrak{k})}\left[(1+\gamma)\frac{\omega(\mathfrak{k})}{\eta(\mathfrak{k})} - \frac{1}{\delta^{\gamma}(\mathfrak{k})}\right]\right),\end{aligned}$$

which is

$$\begin{aligned}\omega'(\mathfrak{k}) &\leq \left(\frac{\eta'(\mathfrak{k})}{\eta(\mathfrak{k})} + \frac{1+\gamma}{\beta^{1/\gamma}(\mathfrak{k})\delta(\mathfrak{k})}\right)\omega(\mathfrak{k}) - \eta(\mathfrak{k})\sum_{i=1}^{L} h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(\mathfrak{k})\right)^{\gamma}\frac{\delta_{m-2}^{\gamma\mu}(\lambda_i(\mathfrak{k}))}{\delta_{m-2}^{\gamma\mu}(\mathfrak{k})} \\ &\quad -\frac{\gamma}{\beta^{1/\gamma}(\mathfrak{k})\eta^{1/\gamma}(\mathfrak{k})}\omega^{(\gamma+1)/\gamma}(\mathfrak{k}) - \frac{\eta(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta^{\gamma+1}(\mathfrak{k})} + \frac{\gamma\eta(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta^{\gamma+1}(\mathfrak{k})}.\end{aligned}$$

By using the inequality

$$\nu\psi - V\psi^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{(\gamma+1)}}\frac{\nu^{\gamma+1}}{V^{\gamma}},\ V > 0,$$

with $\nu = \eta'/\eta + (1+\gamma)/\left(\beta^{1/\gamma}\delta\right),\ V = \gamma/\left(\beta^{1/\gamma}\eta^{1/\gamma}\right)$ and $\psi = \omega$, we find

$$\begin{aligned}\omega'(\mathfrak{k}) &\leq -\eta(\mathfrak{k})\sum_{i=1}^{L} h_i(\mathfrak{k})\left(\frac{\alpha}{(m-2)!}\lambda_i^{m-2}(\mathfrak{k})\right)^{\gamma}\frac{\delta_{m-2}^{\gamma\mu}(\lambda_i(\mathfrak{k}))}{\delta_{m-2}^{\gamma\mu}(\mathfrak{k})} + (\gamma-1)\frac{\eta(\mathfrak{k})}{\beta^{1/\gamma}(\mathfrak{k})\delta^{\gamma+1}(\mathfrak{k})} \\ &\quad +\frac{\beta(\mathfrak{k})\eta(\mathfrak{k})}{(\gamma+1)^{(\gamma+1)}}\left(\frac{\eta'(\mathfrak{k})}{\eta(\mathfrak{k})} + \frac{1+\gamma}{\beta^{1/\gamma}(\mathfrak{k})\delta(\mathfrak{k})}\right)^{\gamma+1}.\end{aligned}$$

Integrating this inequality from $\mathfrak{k}_1$ to $\mathfrak{k}$, we find

$$\int_{\mathfrak{k}_1}^{\mathfrak{k}} \left(W(v) - \frac{\beta(v)\eta(v)}{(\gamma+1)^{(\gamma+1)}} \left(\frac{\eta'(v)}{\eta(v)} + \frac{1+\gamma}{\beta^{1/\gamma}(v)\delta(v)} \right)^{\gamma+1} \right) \mathrm{d}v \leq \omega(\mathfrak{k}_1),$$

which contradicts (31). This completes the proof. □

Theorem 4. *Assume that $\gamma \geq 1$, (20), (26) and (4) hold. If there is a positive $\eta(\mathfrak{k}) \in C^1[\mathfrak{k}_0, \infty)$, $\eta(\mathfrak{k}) > 0$, such that (31) holds, then every solution of (1) is oscillatory.*

Proof. Suppose that $s(\mathfrak{k})$ is a nonoscillatory solution of (1). Then, we have that a $\mathfrak{k}_1 \in [\mathfrak{k}_0, \infty)$ such that $s(\mathfrak{k}) > 0$ and $s(\lambda_i(\mathfrak{k})) > 0$ for $\mathfrak{k} \geq \mathfrak{k}_1$. Using Lemma 3, we have three cases (1), (2) and (3). Using Theorem 1, we have that the condition (4) ensure that solution $s(\mathfrak{k})$ satisfies Case (2) of Lemma 3. But, using Theorem 3, we find that condition (31) contrasts with Case (2) of Lemma 3. Therefore, the proof is complete. □

Theorem 5. *Assume that $\gamma \geq 1$, (20), (26) and (13) hold, If there is a positive $\eta(\mathfrak{k}) \in C^1[\mathfrak{k}_0, \infty)$, $\eta(\mathfrak{k}) > 0$, such that (31) holds, then every solution of (1) is oscillatory.*

Proof. Suppose that $s(\mathfrak{k})$ is a nonoscillatory solution of (1). Then, we have that a $\mathfrak{k}_1 \in [\mathfrak{k}_0, \infty)$ such that $s(\mathfrak{k}) > 0$ and $s(\lambda_i(\mathfrak{k})) > 0$ for $\mathfrak{k} \geq \mathfrak{k}_1$. Using Lemma 3, we have three cases (1), (2) and (3). Using Theorem 1, we have that the condition (13) ensure that solution $s(\mathfrak{k})$ satisfies Case (2) of Lemma 3. But, using Theorem 3, we find that condition (31) contrasts with Case (2) of Lemma 3. Therefore, the proof is complete. □

Example 1. *Consider the DDE*

$$(\mathfrak{k}^6 s'''(\mathfrak{k}))' + h_1 \mathfrak{k}^2 s\left(\frac{\mathfrak{k}}{2}\right) + h_2 \mathfrak{k}^2 s\left(\frac{\mathfrak{k}}{3}\right) = 0, \tag{36}$$

where h_1 and $h_2 > 0$. We note that $\gamma = 1$, $\beta(\mathfrak{k}) = \mathfrak{k}^6$, $\lambda_1(\mathfrak{k}) = \mathfrak{k}/2$, and $\lambda_2(\mathfrak{k}) = \mathfrak{k}/3$. Hence, it is easy to see that

$$\delta_0(\mathfrak{k}) = \frac{1}{5\mathfrak{k}^5},\ \delta_1(\mathfrak{k}) = \frac{1}{20\mathfrak{k}^4} \text{ and } \delta_2(\mathfrak{k}) = \frac{1}{60\mathfrak{k}^3}.$$

If we choose

$$\eta(\mathfrak{k}) = 1/\mathfrak{k}^5$$

and

$$\mu = \alpha\left(\frac{h_1}{120} + \frac{h_2}{270}\right),$$

then (20), (26) and (4) are satisfied, and

$$W(\mathfrak{k}) = \frac{1}{\mathfrak{k}^5}\left(h_1 \mathfrak{k}^2 \left(\frac{\lambda_1 \mathfrak{k}^2}{8}\right) \left(2^3\right)^{\alpha\left(\frac{h_1}{120}+\frac{h_2}{270}\right)} + h_2 \mathfrak{k}^2 \left(\frac{\lambda_2 \mathfrak{k}^2}{18}\right) \left(3^3\right)^{\alpha\left(\frac{h_1}{120}+\frac{h_2}{270}\right)} \right).$$

Now, the condition (31) is satisfied if

$$h_1\left(\frac{\lambda_1}{8}\right)\left(2^3\right)^{\alpha\left(\frac{h_1}{120}+\frac{h_2}{270}\right)} + h_2\left(\frac{\lambda_2}{18}\right)\left(3^3\right)^{\alpha\left(\frac{h_1}{120}+\frac{h_2}{270}\right)} > \frac{25}{4}, \tag{37}$$

for $\lambda_i \in (0,1)$. Thus, by using Theorem 4, we conclude that all solutions of Equation (36) are oscillatory, if (37) satisfied.

Remark 1. *If we consider the special case $(\mathfrak{k}^5 s'''(\mathfrak{k}))' + h_0 \mathfrak{k} s(\mathfrak{k}/2) = 0$, then every solution is oscillatory if $h_0 > 20.518$. While by using Corollary 2.1 in [14] and Corollary 2 in [16], we have* *that every solution is oscillatory if* $h_0 > 32$. *Consequently, Our results ensure that the solutions of the equation* $(\mathfrak{t}^5 s'''(\mathfrak{t}))' + 23\mathfrak{t}s(\mathfrak{t}/2) = 0$ *oscillate, while the other results fail.*

4. Conclusions

The oscillatory behavior of the even-order DDE with multiple delays are obtained in this study. In order to improve and simplify prior results in the literature, we expand Bohner's results in [5] to higher order equations. The new approach uses relation (27) to get a better estimate of the ratio $\frac{s^{(m-2)} \circ \lambda_i}{s^{(m-2)}}$, which distinguishes it from the previously used approach. It is interesting to extend our results to neutral and advanced delay equations.

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