Asymptotic Behavior of Solutions of Parabolic Equations

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Introduction. Consider the parabolic equation

$$(1)\qquad Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = f(x, t)$$

in a cylinder D with a bounded n-dimensional base B ($D = \{(x, t); x \in B, t > 0\}$). Let $u(x, t)$ be a solution of (1) in D with the boundary condition

$$(2)\qquad u(x, t) = h(x, t) \quad \text{for} \quad x \in \dot{B}, \quad t > 0 \qquad (\dot{B} = \text{boundary of } B).$$

It was proved in [1; Theorem 2], under very simple assumptions on L and D, that if h, f and the coefficients a_{ij} , b_i , c of L tend to limits h^0, f^0, a_{ij}^0 , b_i^0 , c^0 as $t \to \infty$, then $u(x, t)$ tends to a limit $u^0(x)$ which satisfies the elliptic equation

$$(3)\qquad \sum a_{ij}^0(x) \frac{\partial^2 u^0(x)}{\partial x_i \, \partial x_j} + \sum b_i^0(x) \frac{\partial u^0(x)}{\partial x_i} + c^0(x)u^0(x) = f^0(x) \qquad (x \in B)$$

and the boundary condition

$$(4)\qquad u^0(x) = h^0(x) \qquad (x \in \dot{B}).$$

The purpose of this paper is to use the above theorem in order to get precise information about the asymptotic behavior of the solutions $u(x, t)$, provided that precise information about the asymptotic behavior of f, h and the coefficients of L is given. Although the considerations in this paper are quite simple, the results obtained here on the asymptotic behavior of the solutions might serve as a quite useful tool in practical calculations.

This paper should be considered as a continuation of [1], and in what follows we shall make free use of the results and notations which appear in [1; Part I]. It is only because [1] is already in the process of being printed that this paper appears separately.

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1. Statement of the theorem. In addition to the assumptions made in [1; Theorem 2] we shall need the following assumptions:

(C_m) The coefficients of L satisfy (uniformly in $x \in \bar{B}$, as $t \to \infty$)

$$a_{ij}(x, t) = a_{ij}^0(x) + \frac{1}{t} a_{ij}^1(x) + \cdots + \frac{1}{t^m} a_{ij}^m(x) + o\left(\frac{1}{t^m}\right),$$

$$b_i(x, t) = b_i^0(x) + \frac{1}{t} b_i^1(x) + \cdots + \frac{1}{t^m} b_i^m(x) + o\left(\frac{1}{t^m}\right),$$

$$c(x, t) = c^0(x) + \frac{1}{t} c^1(x) + \cdots + \frac{1}{t^m} c^m(x) + o\left(\frac{1}{t^m}\right),$$

and all the functions $a_{ij}^k(x)$, $b_i^k(x)$, $c^k(x)$ are Hölder continuous (exponent λ) in $\bar{B}$.

(F_m) $f(x, t)$ is Hölder continuous in compact subsets of $\bar{D}$,

$$f(x, t) = f^0(x) + \frac{1}{t} f^1(x) + \cdots + \frac{1}{t^m} f^m(x) + o\left(\frac{1}{t^m}\right)$$

uniformly with respect to $x \in \bar{B}$, as $t \to \infty$, and the functions $f^k(x)$ are Hölder continuous (exponent λ) in $\bar{B}$.

(H_m) $h(x, t)$ is a continuous functions on $\dot{D}$ and

$$h(x, t) = h^0(x) + \frac{1}{t} h^1(x) + \cdots + \frac{1}{t^m} h^m(x) + o\left(\frac{1}{t^m}\right)$$

uniformly with respect to $x \in \dot{B}$, as $t \to \infty$; the functions $h^k(x)$ are defined and have second Hölder continuous (exponent λ) derivatives in $\bar{B}$.

We shall use the notation

$$M^k v = \sum a_{ij}^k(x) \frac{\partial^2 v}{\partial x_i \, \partial x_j} + \sum b_i^k(x) \frac{\partial v}{\partial x_i} + c^k(x) v.$$

Theorem. *Assume that L satisfies* (A), (B), (C_m) *and that* $c^0(x) \leqq 0$. *Assume further that $\dot{B}$ satisfies* (D′), *that f satisfies* (F_m) *and that h satisfies* (H_m). *If $u(x, t)$ is a solution in D of* (1) *and if it satisfies the boundary condition* (2), *then*

$$u(x, t) = u^0(x) + \frac{1}{t} u^1(x) + \cdots + \frac{1}{t^m} u^m(x) + o\left(\frac{1}{t^m}\right) \tag{5}$$

uniformly with respect to $x \in \bar{B}$ as $t \to \infty$. The functions $u^k(x)$ are defined successively as solutions of the elliptic Dirichlet systems

$$M^0 u^k(x) = f^k(x) - (k - 1)u^{k-1}(x) - \sum_{i=1}^{k} M^i u^{k-i}(x) \qquad (x \in B), \tag{6}$$

$$u^k(x) = h^k(x) \qquad (x \in \dot{B}) \tag{7}$$

(*if $k = 0$, the right side of* (6) *is replaced by* $f^0(x)$).

2. A lemma. To prove the theorem we shall need the following lemma.

Lemma. *Assume that L, defined by* (1), *is uniformly parabolic and with bounded continuous coefficients in $\bar{D}$ and that* $\limsup c(x, t) \leqq 0$ *as $t \to \infty$ (uniformly in $x \in \bar{B}$). Then the following maximum-principle type estimates hold*:

(a) *If $\psi(x, t)$ satisfies*

$$L\psi = F(x, t) \quad in \quad D - D_\sigma ,$$
$$\psi = 0 \quad on \quad B_\sigma \quad and \ on \quad \dot{D} - \dot{D}_\sigma ,$$

then

$$\underset{D-D_\sigma}{\text{l.u.b.}} |\psi| \leqq K_1 \underset{D-D_\sigma}{\text{l.u.b.}} |F|$$

where K_1 depends only on M, M' and the diameter of B, provided σ is sufficiently large (independently of ψ).

(b) *If $\psi(x, t)$ satisfies*

$$L\psi = 0 \quad in \quad D - D_\sigma ,$$
$$|\psi| < \epsilon \quad on \quad B_\sigma \quad and \ on \quad \dot{D} - \dot{D}_\sigma ,$$

then

$$\underset{D-D_\sigma}{\text{l.u.b.}} |\psi| \leqq K_2 \epsilon$$

where K_2 depends only on M, M' and the diameter of B, provided σ is sufficiently large (independently of ψ).

(c) *If $\psi(x, t)$ satisfies*

$$L\psi = 0 \quad in \quad D - D_\sigma ,$$
$$\psi = 0 \quad on \quad \dot{D} - \dot{D}_\sigma$$

and if σ is sufficiently large (independently of ψ), then for any $\epsilon > 0$ there exists T depending on ϵ, M, M', the diameter of B and on l.u.b. $|\psi(x, \sigma)|$ $(x \in B)$ *such that*

$$|\psi(x, t)| < \epsilon \quad in \quad D - D_T .$$

Proof of (a). Without loss of generality we may assume that all the points $x = (x_1 , \cdots , x_n)$ of $\bar{B}$ satisfy $0 \leqq x_1 < R$. Consider the function

$$v(x) = e^{\alpha R} - e^{\alpha x_1}$$

used also in [3]. If α is sufficiently large then

$$Lv(x) < 0 \quad \text{in} \quad \bar{D}$$

provided $c(x, t) \leqq \delta$ and δ is a sufficiently small positive number depending on M, M' and R. Take σ such that $c(x, t) \leqq \delta$ in $D - D_\sigma$. Then the function

$$w(x) = C_1(\underset{D-D_\sigma}{\text{l.u.b.}} |F|)v(x)$$

(where C_1 is such that $C_1 Lv < -1$) satisfies

$$Lw < -\operatorname*{l.u.b.}_{D-D_\sigma} |F|$$

(since, as we may assume, $\text{l.u.b.}_{D-D_\sigma} |F| > 0$) and

$$w > 0 \geqq \psi \quad \text{on} \quad B_\sigma \quad \text{and on} \quad \dot{D} - \dot{D}_\sigma .$$

Since $L\psi \geqq -\text{l.u.b.} |F|$, we can apply a lemma of WESTPHAL-PRODI [6], [4] (see also [2]) and thus conclude that $\psi < w$ in $D - D_\sigma$. Similarly $-\psi < w$, and the proof of (a) follows.

Proof of (b). Let C_2 be a constant which satisfies $C_2 v(x) > 1$ for all $x \in \bar{B}$ and define $w(x) = C_2 \epsilon v(x)$. Comparing, as in the previous proof, $\pm \psi$ with w by means of the lemma of WESTPHAL-PRODI, the proof of (b) is easily completed.

Proof of (c). The function

$$z(x, t) = (N - e^{\beta x_1}) \exp \{-\delta(t - \sigma)\},$$

introduced by NARASIMHAN [2] and used also in [1], satisfies (for appropriate N, β, δ), as one can easily prove, $Lz < 0$ provided $c(x, t) \leqq \gamma$ and γ is positive and sufficiently small. Hence if σ is sufficiently large then $Lz < 0$ in $D - D_\sigma$. Taking C_3 to be such that $C_3 z(x, \sigma) > |\psi(x, \sigma)|$ for $x \in \bar{B}$ and applying the lemma of WESTPHAL-PRODI, the proof of (c) is easily completed.

3. Proof of the theorem. We first remark that [1; Theorem 1] (about the convergence to zero of solutions of (1), (2), with $f \to 0$, $h \to 0$ as $t \to \infty$) remains true if the assumption $c(x, t) \leqq 0$ made in that theorem is replaced by the weaker assumption $\limsup c(x, t) \leqq 0$. Indeed, reviewing the proof of [1; Theorem 1] we see that the functions w_0, w' and w'' can be estimated as in [1] under the present weaker assumption on $c(x, t)$ with the aid of the parts (a), (b) and (c), respectively, of the lemma.

Using the above remark we next observe that the proof of [1; Theorem 2] also remains valid if the assumption $c(x, t) \leqq 0$ is replaced by the assumption $\lim c(x, t) = c^0(x) \leqq 0$. This last fact will be used in what follows.

We immediately use it to conclude that $u(x, t) = u^0(x) + o(1)$. Proceeding to prove (5) by induction we assume that

$$u(x, t) = u^0(x) + \frac{1}{t} u^1(x) + \cdots + \frac{1}{t^{m-1}} u^{m-1}(x) + o\left(\frac{1}{t^{m-1}}\right) \tag{8}$$

where the u^k satisfy (6), (7), and we shall prove (5) with $u^m(x)$ satisfying the system (6), (7) with $k = m$; this will complete the proof of the theorem.

Using the definition of M^k and writing

$$u(x, t) = u^0(x) + \frac{1}{t} u^1(x) + \cdots + \frac{1}{t^{m-1}} u^{m-1}(x) + \frac{1}{t^m} v(x, t), \tag{9}$$

equation (1) can be written in the form

$$\left[\left(M^0 - \frac{\partial}{\partial t}\right) + \sum_{k=1}^{m} \frac{1}{t^k} M^k + \sum o\left(\frac{1}{t^m}\right) \frac{\partial^2}{\partial x_i \partial x_j} + \sum o\left(\frac{1}{t^m}\right) \frac{\partial}{\partial x_i} + o\left(\frac{1}{t^m}\right)\right] \tag{10}$$
$$\cdot\left[u^0(x) + \frac{1}{t} u^1(x) + \cdots + \frac{1}{t^{m-1}} u^{m-1}(x) + \frac{1}{t^m} v(x, t)\right]$$
$$= f^0(x) + \frac{1}{t} f^1(x) + \cdots + \frac{1}{t^m} f^m(x) + o\left(\frac{1}{t^m}\right).$$

Using (6) we find that $f^k(x)$ $(0 \leqq k \leqq m - 1)$ is equal to the coefficient of t^{-k} on the left side of (10) which is (not counting a possible contribution from $v(x, t)$)

$$M^0 u^k(x) + (k - 1) u^{k-1}(x) + \sum_{i=1}^{k} M^i u^{k-i}(x).$$

Thus $v(x, t)$ satisfies the equation

$$\left(M^0 - \frac{\partial}{\partial t}\right) v + \sum o(1) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum o(1) \frac{\partial v}{\partial x_i} + o(1) v \tag{11}$$
$$- f^m(x) - (m - 1) u^{m-1}(x) - \sum_{k=1}^{m} M^k u^{m-k} + o(1).$$

Here we made use of the fact that for each $k \leqq m - 1$

$$\left|\frac{\partial u^k(x)}{\partial x_i}\right| \leqq \text{const.}, \qquad \left|\frac{\partial^2 u^k(x)}{\partial x_i \partial x_j}\right| \leqq \text{const.} \qquad (x \in \bar{B}). \tag{12}$$

The truth of (12) follows from the Schauder theory [5] (see also [3]); one can prove by induction on k that each $u^k(x)$ has second Hölder continuous (exponent λ) derivatives in $\bar{B}$. Here we make use of the regularity assumptions on the $h^k(x)$.

Substituting in the boundary condition (2) the function $u(x, t)$ in its form (9) and using (7) for $k = 0, 1, \cdots, m - 1$, we conclude that $v(x, t)$ satisfies

$$v(x, t) = h^m(x) + o(1) \quad \text{for} \quad x \in \dot{B}. \tag{13}$$

We now apply to the function $v(x, t)$ [2; Theorem 2] in its strong version, namely, [2; Theorem 2] for Lv such that the coefficient of v in Lv is not necessarily non-positive, but tends to a non-positive limit as $t \to \infty$. We then conclude

$$u^m(x) = \lim_{t\to\infty} v(x, t) \quad \text{exists uniformly in} \quad x \in \bar{B}$$

and $u^m(x)$ satisfies (6), (7) with $k = m$. Substituting $v(x, t) = u^m(x) + o(1)$ in (9), the proof is completed.

Added in proof. Using the remark "Added in proof" of [1] it follows that the theorem holds also in case $f(x, t)$ is not assumed to be Hölder continuous in compact subsets of $\bar{D}$ but merely continuous.

4. ***Remark 1.*** In the case $m = 0$ we assumed only that $h^0(x)$ is continuous [1]. For $m > 0$ it is not enough to assume merely the continuity (or even existence of second continuous derivatives) of the $h^k(x)$, since this will not ensure the existence of solutions $u^k(x)$ of the system (6), (7). *The theorem, however, will remain true if we weaken the assumption* (H_m) *by assuming that* $h^m(x)$ *is merely continuous on* $\dot{B}$. Indeed, the proof for $k < m$ remains the same. For $k = m$ we first approximate $h^m(x)$ by a polynomial $\tilde{h}^m(x)$ and then conclude (as in [1]) that the corresponding solution $\tilde{u}^m(x)$ ($\tilde{u}^m(x)$ satisfies (6) with $k = m$ in $\bar{B}$ and is equal to $\tilde{h}^m(x)$ on $\dot{B}$) approximates $v(x, t)$ as $t \to \infty$. Since $\tilde{u}^m(x)$ also approximates $u^m(x)$ (by the maximum principle), we conclude that $\lim_{t\to\infty} v(x, t) = u^m(x)$.

Remark 2. Clearly, the theorem remains true also for $m = \infty$. In that case we obtain for $u(x, t)$ the asymptotic expansion

$$u(x, t) \sim \sum_{k=0}^{\infty} u^k(x) t^{-k}$$

which, however, need not converge.

Remark 3. Consider the equation $Lu = f(x, t, u)$ where f is nonlinear in u. Assume that for large t

$$f(x, t, u) = f^0(x, u) + \frac{1}{t} f^1(x, u) + \cdots + \frac{1}{t^m} f^m(x, u) + o\left(\frac{1}{t^m}\right)$$

uniformly with respect to $x \in \bar{B}$ and $|u| < A$ (for any fixed A). Under certain conditions on the functions $f^k(x, u)$ one can extend [1; Theorem 4] (which is the analogue of [1; Theorem 2] for the equation $Lu = f(x, t, u)$) and obtain an asymptotic expansion for $u(x, t)$.

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